Relative Gröbner and Involutive Bases For Ideals In Quotient Rings

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Abstract. In this paper, we firstly extend the concept of Gröbner bases to relative Gröbner bases for ideals in and modules over quotient rings. We develop a "relative" variant of both Buchberger's criteria and Schreyer's theorem for syzygies. We then introduce the new notion of relative involutive bases and present an algorithm for their construction. Finally, we define the new notion of relatively quasi-stable ideals and exploit it for the construction of coordinates in which finite relative Pommaret bases exist.

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1. Introduction

The concept of *Gröbner bases* along with the first algorithm to compute them was introduced by Buchberger in his PhD thesis [4, 5]. Since then, many interesting applications of these bases have been found in mathematics, science, and engineering. Due to this wide range of applications of Gröbner bases, many improvements of the original algorithm and alternative approaches have been developed. For example, Buchberger himself proposed two criteria to improve his algorithm by removing superfluous reductions [3]. Based on effective linear algebra methods and by applying these two criteria, Faugère described the F_4 algorithm [8]. He also proposed a signature-based algorithm known as F_5 algorithm [9]. As a final example, Gao et al. [10] presented a new approach to compute simultaneously the Gröbner bases of an ideal and of its syzygy module.

Involutive bases (which are a special kind of Gröbner bases with additional combinatorial properties) have their origin in the works by Janet [18] on the analysis of systems of (linear) partial differential equations. Zharkov and Blinkov [29] introduced the notion of *involutive polynomial bases* inspired by works of Pommaret [23]. Then, Gerdt and Blinkov [12] introduced the general concepts of *involutive divisions* and *involutive bases* for polynomial ideals and derived with them an alternative algorithm for computing Gröbner bases. Involutive bases arise via a restriction of the usual divisibility relation of monomials to an involutive division. Gerdt [11] proposed an efficient algorithm to compute these bases. For an implementation of this algorithm and extensive benchmarks, we refer to the website `http://invo.jinr.ru`. For a comprehensive study and applications of the theory of involutive bases to commutative algebra and the geometric theory of partial differential equations, we refer to [26].

Let $\mathcal{P} = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring over a field $\mathbb{K}$. A well-known application of Gröbner bases as well as involutive bases is the construction of free resolutions of finitely generated

$\mathcal{P}$-modules, see e. g. [2, 6, 20, 24, 25]. As already the title of Buchberger's thesis [4] indicates, Gröbner bases are also used for effective computations in the quotient ring $\mathcal{P}/\mathcal{I}$ where $\mathcal{I} \lhd \mathcal{P}$ is an ideal. In this work, we are interested in basic tools for the construction of free resolutions of finitely generated $\mathcal{P}/\mathcal{I}$-modules. For this purpose, we are firstly concerned with extending the concept of Gröbner bases to ideals in and modules over $\mathcal{P}/\mathcal{I}$ (we refer to these new bases as *relative Gröbner bases*). Such an extension is not new. Some ideas can already be found in textbooks like [1]. La Scala and Stillman [20] sketched the necessary theoretical background and implemented procedures in MACAULAY2 not only for computing Gröbner bases, but also for free resolutions. Nevertheless, we believe that it is worth while to take a closer look at the details of such an extension for further generalisations. We also present a variant of Buchberger's algorithm for the computation of relative Gröbner bases. However, our main contribution in this paper consists of introducing the concept of *relative involutive bases* for ideals in and modules over $\mathcal{P}/\mathcal{I}$ and designing an algorithm for their construction. In this context, we will generalise the notion of well-known combinatorial notion of a quasi-stable (monomial) ideal to *relative quasi-stable ideals* for the computation of relative Pommaret bases in $\mathcal{P}/\mathcal{I}$. We note that most of the algorithms proposed in this work have been implemented in MAPLE and their codes are available at the website `https://amirhashemi.iut.ac.ir/softwares`.

This paper is structured as follows. Section 2 recalls the basic notations and definitions used throughout. In Section 3, we introduce relative Gröbner bases and establish the analogy to the Schreyer construction for ideals in quotient rings. In Section 4, we provide the basics for the construction of relative Gröbner bases developing criteria analogous to Buchberger's criterion and Buchberger's (first and second) criteria to improve the computation of relative Gröbner bases. Section 5 is devoted to the study of relative involutive bases. We introduce the notion of a relative involutive division and study the basic properties of relative involutive bases. In Section 6, we thoroughly investigate the required properties for the construction of relative involutive bases. Section 7 provides a study of some combinatorial properties of finite relative Pommaret bases. We introduce the new notion of a relative quasi-stable ideal and apply it to propose a deterministic algorithm for the construction of finite relative Pommaret bases. We conclude with some remarks on future research.

2. Preliminaries

Let $n \geq 1$ be a natural number. We write $\mathcal{P} := \mathbb{K}[X] = \mathbb{K}[x_1, \dots, x_n]$ for the polynomial ring on n variables over a field $\mathbb{K}$. Every polynomial $f \in \mathcal{P}$ is a (finite) linear combination of *terms* cx^μ, where $c \in \mathbb{K} \setminus \{0\}$, $\mu \in \mathbb{Z}^n_{\geq 0}$ and $x^\mu = x_1^{\mu_1} \cdots x_n^{\mu_n}$ is a *monomial*. We write $\operatorname{supp}(f)$ for the finite set of monomials appearing in f. The set of all monomials in $\mathcal{P}$ defines a monoid for the multiplication which we denote by the symbol $\mathcal{M}$.

If an ideal $\mathcal{I} \unlhd \mathcal{P}$ is generated over the ring $\mathcal{P}$ by a subset $S \subseteq \mathcal{P}$, we write $\mathcal{I} = \langle S \rangle_{\mathcal{P}}$. An ideal $\mathcal{I} \unlhd \mathcal{P}$ is called a *monomial ideal*, if it can be generated by monomials. Such an ideal possesses a unique finite minimal generating set $\mathrm{G}(\mathcal{I}) \subseteq \mathcal{M}$ consisting of those monomials of $\mathcal{I}$ which have no divisor in $\mathcal{I}$ other than themselves. An *order ideal* is a subset $\mathcal{O} \subseteq \mathcal{M}$ such that for each monomial $x^\mu \in \mathcal{O}$, all divisors of x^μ are also contained in $\mathcal{O}$. In other words, $\mathcal{O}$ is an order ideal, if and only if there exists a monomial ideal $\mathcal{I}$ such that $\mathcal{M} \setminus \mathcal{O} = \mathcal{I} \cap \mathcal{M}$.

A *monomial ordering* is a well-ordering $\prec$ on $\mathcal{M}$ which respects the multiplication of monomials, that is, $1 \prec x^\mu$ for all $x^\mu \neq 1$ and if $x^\mu \prec x^\nu$, then $x^\mu \cdot x^\rho \prec x^\nu \cdot x^\rho$ for all $x^\rho \in \mathcal{M}$. Given a monomial ordering $\prec$ and a polynomial $f \in \mathcal{P} \setminus \{0\}$, we denote the *leading monomial of* f by $\operatorname{lm}(f) := \max_\prec\{\operatorname{supp}(f)\}$ where $\operatorname{supp}(f)$ stands for the set of all monomials appearing in f. Also, we write $\operatorname{lc}(f)$ for the coefficient of $\operatorname{lm}(f)$ in f. The leading term is then written as $\operatorname{lt}(f) := \operatorname{lc}(f)\operatorname{lm}(f)$. For each subset $F \subset \mathcal{P}$, we denote by $\operatorname{lm}(F)$ the set $\{\operatorname{lm}(f) \mid f \in F\}$.

For each ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ and each monomial ordering $\prec$, there exists a finite subset $G \subseteq \mathcal{I}$ such that $\mathrm{lm}(G)$ generates the monomial ideal $\mathrm{lm}(\mathcal{I}) := \langle \mathrm{lm}(f) \mid f \in \mathcal{I} \rangle_{\mathcal{P}}$. Such a subset G of $\mathcal{I}$ is called a *Gröbner basis* of $\mathcal{I}$ for the monomial ordering $\prec$. Note that every Gröbner basis of $\mathcal{I}$ is in particular also a generating set of $\mathcal{I}$. Gröbner bases are not unique, but every ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ has for each monomial ordering $\prec$ a unique *reduced Gröbner basis* which satisfies additionally that for each $g \in G$ (i) $\mathrm{lc}(g) = 1$ and (ii) no monomial of g lies in $\langle \mathrm{lm}(G \smallsetminus \{g\}) \rangle$. The monomials in the order ideal $\mathcal{M} \smallsetminus \mathrm{lm}(\mathcal{I})$ form a $\mathbb{K}$-linear basis of the quotient ring $\mathcal{P}/\mathcal{I}$ and each polynomial $f \in \mathcal{P}$ has a unique *normal form* with respect to the Gröbner basis G which is a linear combination of monomials of $\mathcal{M} \smallsetminus \mathrm{lm}(\mathcal{I})$. We denote this normal form by $\mathrm{NF}_G(f)$. If $G = \{g_1, \dots, g_t\}$ is a Gröbner basis of the ideal $\mathcal{I}$ for $\prec$, then there exists for each ideal member $f \in \mathcal{I}$ a representation $f = q_1 g_1 + \dots + q_t g_t$ with $q_i \in \mathcal{P}$ and where for each index i with $q_i \neq 0$ we have $\mathrm{lm}(q_i g_i) \preceq \mathrm{lm}(f)$. Such a representation is called a *standard representation* with respect to G for f; it is generally not unique.

The concept of monomial orderings and Gröbner bases can straightforwardly be extended to submodules of free $\mathcal{P}$-modules. Let $s \geq 1$ be a positive integer and view the elements of the free $\mathcal{P}$-module $\mathcal{P}^s$ as row vectors. Moreover, let $\{\mathbf{e}_1, \dots, \mathbf{e}_s\}$ be the standard basis of $\mathcal{P}^s$. Then, every vector $\mathbf{f} \in \mathcal{P}^s$ is a finite $\mathbb{K}$-linear combination of *module monomials* $x^\mu \mathbf{e}_i$ with $x^\mu \in \mathcal{M}$ and $i \in \{1, \dots, s\}$. A *module monomial ordering* $\prec$ is a total ordering and well-ordering on the set of all module monomials such that, for all i and for all $x^\mu \in \mathcal{M}$, if $x^\mu \mathbf{e}_i \prec x^\nu \mathbf{e}_j$, then $x^\mu x^\rho \mathbf{e_i} \prec x^\nu x^\rho \mathbf{e}_j$ for all $x^\rho \in \mathcal{M}$. Similar to the polynomial case, any element $\mathbf{f} \in \mathcal{P}^s$ can be written as a linear combination of module monomials and one is able to define the notions of module leading coefficient, module leading monomial and module leading term for $\mathbf{f}$ which are denoted by $\mathbf{lc}(\mathbf{f})$, $\mathbf{lm}(\mathbf{f})$ and $\mathbf{lt}(\mathbf{f})$, respectively. If $\mathbf{u} := x^\mu \mathbf{e}_i$ and $\mathbf{v} := x^\nu \mathbf{e}_j$ are two module monomials in $\mathcal{P}^s$, then we say that $\mathbf{u}$ divides $\mathbf{v}$, and write $\mathbf{u} \mid \mathbf{v}$ if $i = j$ and x^μ divides x^ν. If $\mathbf{u}$ divides $\mathbf{v}$, then the quotient $\mathbf{v}/\mathbf{u}$ is defined to be $x^\nu / x^\mu \in \mathcal{M}$. Based on these definitions, one is able to build a theory of Gröbner bases for submodules of $\mathcal{P}^s$ similar to the one for ideals in $\mathcal{P}$. For a description of the division algorithm, Buchberger's algorithm to compute Gröbner bases and further details on their theory, we refer to standard textbooks like [1, 7, 6].

We finally recall some basic notions around involutive bases. More details on them, corresponding algorithms and applications can be found in [26]. Involutive bases are a special form of Gröbner bases with additional combinatorial properties. The main point is that to each generator h in a basis H a subset $L(h, H) \subseteq X$ of multiplicative variables is assigned and that one considers only linear combinations of the generators where each generator $h \in H$ is multiplied by a coefficient depending only on the variables in $L(h, H)$. In contrast to Gröbner bases, not every monomial basis of a monomial ideal is automatically an involutive basis.

The rule for the assignment of the multiplicative variables is called an *involutive division*. Given a finite set H of monomials, the involutive division L assigns to each monomial $h \in H$ the *multiplicative variables* $L(h, H)$ such that the corresponding *involutive cones* $\mathcal{C}_{L,H}(h) := h\mathbb{K}[L(h, H)] \cap \mathcal{M}$ satisfy the following conditions:

1. If H contains two monomials h, h' such that $\mathcal{C}_{L,H}(h) \cap \mathcal{C}_{L,H}(h') \neq \varnothing$, then either $h \in \mathcal{C}_{L,H}(h')$ or $h' \in \mathcal{C}_{L,H}(h)$.
2. If H contains two monomials h, h' such that $h \in \mathcal{C}_{L,H}(h')$, then $\mathcal{C}_{L,H}(h) \subseteq \mathcal{C}_{L,H}(h')$.
3. If $H' \subset H$ are two sets both containing the monomial h, then $\mathcal{C}_{L,H}(h) \subseteq \mathcal{C}_{L,H'}(h)$.

Given a finite set H of monomials and an involutive division L, we call H a *weak L-involutive basis* of the monomial ideal $\mathcal{I} = \langle H \rangle$, if $\bigcup_{h \in H} \mathcal{C}_{L,H}(h)$ generates $\mathcal{I}$ as a $\mathbb{K}$-linear space. It is a *(strong) L-involutive basis*, if in addition the involutive cones $\mathcal{C}_{L,H}(h)$ are pairwise disjoint. The L-involutive basis H is *minimial*, if any other L-involutive basis H' of $\mathcal{I}$ contains H as subset. L is called *noetherian*, if every monomial ideal $\mathcal{I}$ possesses an L-involutive basis. One can show that for a *constructive* (see [26] for a definition), noetherian division every monomial ideal $\mathcal{I}$ has a unique minimal L-involutive basis.

Given a finite set H of polynomials, a monomial ordering $\prec$ and an involutive division L, we call H a *weak L-involutive basis* of the ideal $\mathcal{I} = \langle H\rangle$, if $\operatorname{lm} H$ is a weak L-involutive basis of $\operatorname{lm}\mathcal{I}$. For a *(strong) L-involutive basis*, we require in addition that $\operatorname{lm} H$ is a strong L-involutive basis and that all generators $h \in H$ have pairwise disjoint leading monomials. We assign to each polynomial $h \in H$ the multiplicative variables $L(\operatorname{lm}(h), \operatorname{lm}(H))$ and define the involutive cone $\mathcal{C}_{L,H,\prec}(h) := h\mathbb{K}[L(h,H)]$. A strong involutive basis H of an ideal $\mathcal{I}$ induces then a disjoint decomposition $\mathcal{I} = \bigoplus_{h\in H} \mathcal{C}_{L,H,\prec}(h)$ as $\mathbb{K}$-linear spaces. H is a *minimal* L-involutive basis of $\mathcal{I}$, if $\operatorname{lm} H$ is a minimal L-involutive basis of $\operatorname{lm}\mathcal{I}$.

For most purposes, two involutive divisions are particularly important. For the *Pommaret division* P, the assignment rule is very simple. Given a monomial x^μ, the *class* of x^μ, denoted by $\operatorname{cls}(x^\mu)$, is defined as $k = \min\{i \mid \mu_i \neq 0\}$. Let $P(x^\mu) = \{x_1, \dots, x_k\}$. It is a so-called global division where the assignment is independent of any ambient set H. For the *Janet division*, we introduce for a finite set $H \subset \mathcal{M}$ the following subsets: $(\nu_i, \dots, \nu_n) = \{x^\mu \in H \mid \forall j \geq i : \mu_j = \nu_j\}$; note that $() = H$. We now have that $x_i \in J(x^\mu, H)$, if $\mu_i = \max\{\nu_i \mid x^\nu \in (\mu_{i+1}, \dots, \mu_n)\}$.

In contrast to the Janet division, the Pommaret division is not noetherian. However, one can show that this is only a problem of the used coordinates: after a generic linear change of variables any ideal possesses a Pommaret basis provided the coefficient field $\mathbb{K}$ is large enough (see [15] for an extensive discussion and a deterministic algorithm for finding a suitable change of variables). As generally a monomial ideal does not remain monomial after a linear change of variables, Pommaret bases exist only for a special class of monomial ideals. For Pommaret bases, we will always consider the degree reverse lexicographical ordering $\prec$ with $x_1 \prec \cdots \prec x_n$.

Definition 2.1. A monomial ideal $\mathcal{I}$ is called *quasi-stable*, if for any monomial $x^\mu \in \mathcal{I}$ and for any index i with $\operatorname{cls}(x^\mu) < i \leq n$ an exponent $s \geq 0$ exists such that $x_i^s x^\mu / x_{\operatorname{cls}(x^\mu)} \in \mathcal{J}$. A polynomial ideal $\mathcal{I}$ is in *quasi-stable position*, if $\operatorname{lt}(\mathcal{I})$ is quasi-stable.

Quasi-stable ideals appear in many places (and are known under many different names like *ideals of Borel type*, *ideals of nested type* or *weakly stable ideals*). Besides the above combinatorial definition, they can be characterised by many algebraic properties. For our purposes, the following characterisation is relevant.

Proposition 2.2 ([26, Prop. 5.3.4])**.** *A monomial ideal $\mathcal{I}$ possesses a finite Pommaret basis, if and only if it is quasi-stable.*

3. Relative Gröbner Bases and Syzygies

A basic building block of the theory of Gröbner basis is polynomial division. Since we are interested in establishing an analogous theory for ideals in a quotient ring $\mathcal{P}/\mathcal{I}$, we need a division algorithm that takes the ideal $\mathcal{I}$ into account as well. Suppose we are given a reduced Gröbner basis G of an ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ with respect to a given monomial ordering $\prec$. Additionally, let $h_1, \dots, h_r \in \mathcal{P}$ be polynomials which are reduced with respect to G, i. e. $\operatorname{NF}_G(h_i) = h_i$ for all $1 \leq i \leq r$. Finally, we are given a polynomial $f \in \mathcal{P}$ which we want to divide by the set $H = \{h_1, \dots, h_r\}$ modulo $\mathcal{I}$. The result is then a polynomial $\tilde{f}$, reduced with respect to G and with no monomial in its support divisible by any monomial in $\operatorname{lm}(H)$. Algorithmically, this result can be achieved by repeatedly applying the normal form operation NF_G followed by a classical polynomial division step with respect to H. Algorithm 1 is a formalisation of this idea.

Remark 3.1. The support of the quotient polynomial q_k belonging to h_k computed during the course of Algorithm 1 is contained in the order ideal $\mathcal{M} \setminus (\operatorname{lm}(\mathcal{I}) : \operatorname{lm}(h_k))$. Since $H \cup G$ need not be a Gröbner basis of $\langle H\rangle_{\mathcal{P}} + \mathcal{I}$, the polynomial p in the output of Algorithm 1 is not uniquely determined by the input, but depends on the chosen polynomials g and h_i, resp., in the various reduction steps.

Algorithm 1: Relative Polynomial Division

Data: A monomial ordering $<$, an ideal $\mathcal{I} \trianglelefteq \mathcal{P}$, a Gröbner basis G of $\mathcal{I}$, a set of polynomials $H = \{h_1, \dots, h_r\} \subset \mathcal{P}$ with $\mathrm{NF}_G(h_i) = h_i$ for all i and $f \in \mathcal{P}$

Result: A polynomial $p \in \mathcal{P}$ with support disjoint from $\langle \mathrm{lm}(\mathcal{I}), \mathrm{lm}(H) \rangle$, polynomials $q_1, \dots, q_r \in \mathcal{P}$ with $f - p - \sum_{i=1}^{r} q_i h_i \in \mathcal{I}$

begin
- $\tilde{f} \longleftarrow f; \quad p \longleftarrow 0$
- **for** $i = 1, \dots, r$ **do**
  - $q_i \longleftarrow 0$
- **while** $\tilde{f} \neq 0$ **do**
  - **if** $\mathrm{lm}(\tilde{f}) \in \langle \mathrm{lm}(G) \rangle$ **then**
    - Choose $g \in G$ with $\mathrm{lm}(g) | \mathrm{lm}(\tilde{f})$
    - $\tilde{f} \longleftarrow \tilde{f} - \frac{\mathrm{lt}(\tilde{f})}{\mathrm{lt}(g)} g$
  - **else if** $\mathrm{lm}(\tilde{f}) \in \langle \mathrm{lm}(H) \rangle$ **then**
    - Choose $h_i \in H$ with $\mathrm{lm}(h_i) | \mathrm{lm}(\tilde{f})$
    - $q_i \longleftarrow q_i + \frac{\mathrm{lt}(\tilde{f})}{\mathrm{lt}(h_i)}; \quad \tilde{f} \longleftarrow \tilde{f} - \frac{\mathrm{lt}(\tilde{f})}{\mathrm{lt}(h_i)} h_i$
  - **else**
    - $p \longleftarrow p + \mathrm{lt}(\tilde{f}); \quad \tilde{f} \longleftarrow \tilde{f} - \mathrm{lt}(\tilde{f})$
- **return** $(p, q_1, \dots, q_r)$

Definition 3.2. If the polynomial p is a possible output of Algorithm 1 for input $f, H, \mathcal{I}, <$, then we write $f \longrightarrow^{*}_{H,\mathcal{I},<} p$ and say that f *reduces to* p *with respect to* H *modulo* $\mathcal{I}$. We omit the reference to the monomial ordering $<$ if no confusion can arise.

Given an ideal $\mathcal{I} \lhd \mathcal{P}$, we are interested in defining something like Gröbner bases for ideals in the quotient ring $\mathcal{P}/\mathcal{I}$. As it makes no sense to speak of monomials in this ring, a direct approach does not appear meaningful. Instead, we exploit the well-known fact that any ideal in $\mathcal{P}/\mathcal{I}$ is of the form $\mathcal{J}/\mathcal{I}$ for an ideal $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{P}$. Therefore our basic idea is to determine suitable bases of $\mathcal{J}$ "relative" to $\mathcal{I}$ which may be interpreted as Gröbner bases of $\mathcal{J}/\mathcal{I}$.

From now on, we fix a monomial ordering $\leq$ on $\mathcal{P}$ and leading monomials, terms, coefficients, Gröbner bases etc. will always be determined with respect to it. In particular, the leading ideal $\mathrm{lm}(\mathcal{I})$ is thus fixed. Every coset[1] $[f]_{\mathcal{I}} = f + \mathcal{I} \in \mathcal{P}/\mathcal{I}$ contains then a unique representative $\tilde{f} = \mathrm{NF}_{\mathcal{I}}(f)$ with $\mathrm{supp}(\tilde{f}) \cap \mathrm{lm}(\mathcal{I}) = \varnothing$ (it can be easily determined as the normal form of f with respect to an arbitrary Gröbner basis of $\mathcal{I}$). If not explicitly stated otherwise, we will in the sequel always assume that each coset $[f]$ is described by this unique representative. This allows us to define $\mathrm{lm}([f]) = \mathrm{lm}(f)$ and accordingly $\mathrm{lc}([f])$, $\mathrm{lt}([f])$. For an ideal $\mathcal{J}/\mathcal{I} \trianglelefteq \mathcal{P}/\mathcal{I}$, we then find $\mathrm{lm}(\mathcal{J}/\mathcal{I}) = \mathrm{lm}(\mathcal{J}) \setminus \mathrm{lm}(\mathcal{I})$. Finally, we denote by π the canonical projection $\mathcal{P} \to \mathcal{P}/\mathcal{I}$.

Definition 3.3. Let $\mathcal{I} \subseteq \mathcal{J} \lhd \mathcal{P}$ be ideals. The finite subset $H \subset \mathcal{J}$ is called a *Gröbner basis of* $\mathcal{J}$ *relative to* $\mathcal{I}$, if $\langle \mathrm{lm}(H) \rangle + \mathrm{lm}(\mathcal{I}) = \mathrm{lm}(\mathcal{J})$. A finite subset $\hat{H} = \{[h_1], \dots, [h_r]\} \subset \hat{\mathcal{J}} = \mathcal{J}/\mathcal{I} \lhd \mathcal{P}/\mathcal{I}$ is a *Gröbner basis* of $\hat{\mathcal{J}}$, if $\{h_1, \dots, h_r\}$ is a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$ or equivalently if $\langle \mathrm{lm}(\hat{H}) \rangle + \mathrm{lm}(\mathcal{I}) = \mathrm{lm}(\mathcal{J})$.

Relative Gröbner bases exist, since every Gröbner basis of $\mathcal{J}$ is also a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$. Given a relative Gröbner basis of $\mathcal{J}$ with respect to $\mathcal{I}$, we can extend it trivially to a Gröbner basis of $\mathcal{J}$. Relative Gröbner bases can be characterised similarly to the classical case.

[1] We omit the index $\mathcal{I}$, if it is clear from the context by which ideal we factor.

Proposition 3.4. *Let $H = \{h_1, \dots, h_t\} \subset \mathcal{J}$ be a finite set and G a Gröbner basis of $\mathcal{I}$. Then the following statements are equivalent:*

- *H is a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$.*
- *$H \cup G$ is a Gröbner basis of $\mathcal{J}$.*
- *For any $f \in \mathcal{J}$, we have $f \longrightarrow^*_{H,\mathcal{I},\prec} 0$.*
- *Any $f \in \mathcal{J}$ has a* relative standard representation *of the form $f = g + \sum_{i=1}^{t} q_i h_i$ where $g \in \mathcal{I}$ and $\mathrm{lm}(q_i h_i) \preceq \mathrm{lm}(f)$ for each i with $q_i \neq 0$.*

Proof. By definition of a relative Gröbner basis, we know that $\langle \mathrm{lm}(H)\rangle + \langle \mathrm{lm}(G)\rangle = \mathrm{lm}(\mathcal{J})$. Thus, if $f \in \mathcal{I}$, then $\mathrm{lm}(f)$ is divisible by some $\mathrm{lm}(g)$ with $g \in G$ and if $f \in \mathcal{J} \setminus \mathcal{I}$, then $\mathrm{lm}(f)$ is divisible by some $\mathrm{lm}(h)$ with $h \in H$. Thus, H being a relative Gröbner basis of $\mathcal{J}$ is equivalent to $H \cup G$ being a Gröbner basis of $\mathcal{J}$. The last two statements follow by classical properties of Gröbner bases. □

As a consequence, the classical Buchberger algorithm provides us already with a basic procedure to compute relative Gröbner bases. More precisely, we have the following observation.

Proposition 3.5. *Let $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{P}$ be two polynomial ideals and F a finite generating set of $\mathcal{J}$. Let furthermore G be a Gröbner basis of $\mathcal{I}$ and call $H_{\mathrm{Buchberger}}$ the Gröbner basis of $\mathcal{J}$ obtained by applying Buchberger's algorithm to the set $F \cup G$. Then $H := H_{\mathrm{Buchberger}} \setminus \mathcal{I}$ is a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$.*

Proof. Since $H_{\mathrm{Buchberger}}$ is a Gröbner basis of $\langle F, G\rangle_{\mathcal{P}} = \mathcal{J}$, it is of course also a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$ and we can discard all elements belonging to $\mathcal{I}$, as their leading monomials do not divide any monomial in $\mathrm{lm}(\mathcal{J}) \setminus \mathrm{lm}(\mathcal{I})$. □

Assume, again, that $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{P}$ are polynomial ideals. If F generates $\mathcal{J}$, and G is a Gröbner basis of $\mathcal{I}$, then $\mathrm{NF}_G(F) \cup G$ also generates $\mathcal{J}$. Applying Proposition 3.5 and observing that each element that is added during the course of Buchberger's algorithm is reduced with respect to G, i. e. in normal form with respect to $\mathcal{I}$, we get a Gröbner basis H of $\mathcal{J}$ relative to $\mathcal{I}$ with $H = \mathrm{NF}_G(H)$. Iteratively discarding any element of H whose leading monomial is divisible by the leading monomial of another element of H, then performing a full auto-reduction and finally normalising leading monomials, we get a *reduced Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$*, that is, a set H with $\mathrm{NF}_G(H) = H$ with the additional properties

- $|H| = |\,\mathrm{G}(\mathrm{lm}(\mathcal{J})) \setminus \mathrm{lm}(\mathcal{I})|$,
- $\{\mathrm{lm}(h) \mid h \in H\} = \mathrm{G}(\mathrm{lm}(\mathcal{J})) \setminus \mathrm{lm}(\mathcal{I})$,
- $\forall h \in H : \mathrm{lc}(h) = 1$,
- $\forall h \in H : \mathrm{supp}(h - \mathrm{lt}(h)) \subseteq \mathcal{M} \setminus \mathrm{lm}(\mathcal{J})$.

Proposition 3.6. *Let H be a Gröbner basis (resp., the reduced Gröbner basis) of $\mathcal{J} \supseteq \mathcal{I}$ and let G be a Gröbner basis of $\mathcal{I}$. Then $\overline{H} := \mathrm{NF}_G(H)$ is a Gröbner basis (resp., the reduced Gröbner basis) of $\mathcal{J}$ relative to $\mathcal{I}$.*

Proof. Let $x^\mu \in \mathrm{G}(\mathrm{lm}(\mathcal{J})) \setminus \mathrm{lm}(\mathcal{I})$. Then there exists a polynomial $h \in H$ with $\mathrm{lm}(h) = x^\mu$. Now, since the leading monomial of h cannot be reduced modulo $\mathcal{I}$ and since reduction modulo $\mathcal{I}$ (as, indeed, reduction modulo any set) does not introduce higher monomials than the monomials that are eliminated by the reduction, we have $\mathrm{lm}(\mathrm{NF}_G(h)) = x^\mu$. The claim follows. □

We may extend the above theory to modules. This requires a bit care with the used orderings. We continue to assume that we are given an ideal $\mathcal{I} \lhd \mathcal{P}$ and a monomial ordering $\prec$ defining the monomial ideal $\mathrm{lm}(\mathcal{I})$. Then we fix on the free module $\mathcal{P}^r$ with the standard basis $\{\mathbf{e}_1, \dots, \mathbf{e}_r\}$ a module monomial ordering $\prec$ which must be *compatible* to $\prec$ in the sense that if $x^\mu \prec x^\nu$ then also $x^\mu \mathbf{e}_i \prec x^\nu \mathbf{e}_i$ for any index $1 \leq i \leq r$ (obviously, any POT or TOP lift of $\prec$ will be compatible to $\prec$, but also any Schreyer ordering based on $\prec$).

If we write an element of $(\mathcal{P}/\mathcal{I})^r$ as a vector of cosets, then we use again the convention that for each coset the unique representative in normal form with respect to $\mathcal{I}$ has been chosen. As in the scalar case, this convention allows us to extend the notions of leading module monomial, leading coefficient etc. to vectors of cosets. Denoting again by π the extension of the canonical projection onto the cosets to the projection $\mathcal{P}^r \to (\mathcal{P}/\mathcal{I})^r$, we associate with any $\mathcal{P}/\mathcal{I}$-submodule $\widehat{\mathcal{N}} \subseteq (\mathcal{P}/\mathcal{I})^r$ the $\mathcal{P}$-submodule $\mathcal{N} = \pi^{-1}(\widehat{\mathcal{N}})$ and find then analogously to the scalar case that $\mathbf{lm}(\mathcal{N}) = \mathbf{lm}(\widehat{\mathcal{N}}) + \sum_{i=1}^{r} \operatorname{lm}(\mathcal{I})\mathbf{e}_i$.

Definition 3.7. Let $\mathcal{I} \lhd \mathcal{P}$ be a polynomial ideal and $\mathcal{N} \subset \mathcal{P}^r$ a $\mathcal{P}$-submodule containing $\mathcal{I}^r$. A finite set $B \subset \mathcal{N}$ disjoint from $\mathcal{I}^r$ is called a *Gröbner basis of* $\mathcal{N}$ *relative to* $\mathcal{I}$, if $\langle \mathbf{lm}(B) \rangle + \sum_{i=1}^{r} \operatorname{lm}(\mathcal{I})\mathbf{e}_i = \mathbf{lm}(\mathcal{N})$. A finite subset $\{[\mathbf{h}_1], \dots, [\mathbf{h}_s]\}$ of a $\mathcal{P}/\mathcal{I}$-submodule $\widehat{\mathcal{N}} \subseteq (\mathcal{P}/\mathcal{I})^r$ is a *Gröbner basis* of $\widehat{\mathcal{N}}$, if $\{\mathbf{h}_1, \dots, \mathbf{h}_s\}$ is a Gröbner basis of $\mathcal{N}$ relative to $\mathcal{I}$.

Note that, if G is a Gröbner basis of $\mathcal{I}$, then B is a relative Gröbner basis of $\mathcal{N}$ with respect to $\mathcal{I}$ if and only if the set $B \cup \{g\mathbf{e}_i \mid g \in G, 1 \leq i \leq r\}$ is a Gröbner basis of $\mathcal{N}$ because of the assumed compatibility of the monomial orderings $\prec$ and $\prec$. With this definition, we can now analyse relative Gröbner bases of syzygy modules. Schreyer's construction [24] allows us to compute a Gröbner basis of a syzygy module of a set of polynomials, if this set is a Gröbner basis of the ideal it generates. A decisive step in Schreyer's construction is the definition of a module monomial ordering adapted to the given Gröbner basis. By a characteristic property of Gröbner bases, all S-polynomials of the given set reduce to zero, which yields syzygies whose leading module monomials encode the monomials where the cancellations defining the S-polynomials happen. We now introduce the necessary notation to be able to adapt it to the relative case.

Construction 3.8. *Let* $G = \{g_1, \dots, g_s\}$ *be a Gröbner basis of the ideal* $\mathcal{I} \trianglelefteq \mathcal{P}$. *We write* $\operatorname{Syz}(G) = \operatorname{Syz}(g_1, \dots, g_s)$*; recall that* $(p_1, \dots, p_s) \in \operatorname{Syz}(G)$ *if and only if* $\sum_{i=1}^{s} p_i g_i = 0$. *On the module* $\mathcal{P}^s$, *define the* Schreyer module monomial ordering $\prec_S$ *by*

$$x^\mu \mathbf{e}_i \prec_S x^\nu \mathbf{e}_j \iff x^\mu \operatorname{lm}(g_i) \prec x^\nu \operatorname{lm}(g_j) \vee \big(x^\mu \operatorname{lm}(g_i) = x^\nu \operatorname{lm}(g_j) \wedge j < i\big). \tag{1}$$

For $1 \leq i < j \leq s$, *the* S-polynomial *of the generators* g_i *and* g_j *is defined to be* $\operatorname{S}(g_i, g_j) := \frac{\operatorname{lcm}(\operatorname{lm}(g_i), \operatorname{lm}(g_j))}{\operatorname{lt}(g_i)} g_i - \frac{\operatorname{lcm}(\operatorname{lm}(g_i), \operatorname{lm}(g_j))}{\operatorname{lt}(g_j)} g_j$. *By Buchberger's criterion,* $\operatorname{S}(g_i, g_j)$ *reduces to zero with respect to* G *for each* i, j, *which entails that it has a standard representation* $\sum_{\ell=1}^{s} q_\ell g_\ell$, *where the polynomials* $q_\ell \in \mathcal{P}$ *are such that* $\operatorname{lm}(q_\ell)\operatorname{lm}(g_\ell) \preceq \operatorname{lm}(\operatorname{S}(g_i, g_j))$ *for all* $\ell \in \{1, \dots, s\}$ *with* $q_\ell \neq 0$. *By definition of the Schreyer ordering, the leading module monomial of the resulting syzygy*

$$\mathbf{S}_{ij} := \frac{\operatorname{lcm}(\operatorname{lm}(g_i), \operatorname{lm}(g_j))}{\operatorname{lt}(g_i)} \mathbf{e}_i - \frac{\operatorname{lcm}(\operatorname{lm}(g_i), \operatorname{lm}(g_j))}{\operatorname{lt}(g_j)} \mathbf{e}_j - \sum_{\ell=1}^{s} q_\ell \mathbf{e}_\ell \tag{2}$$

is $\frac{\operatorname{lcm}(\operatorname{lm}(g_i), \operatorname{lm}(g_j))}{\operatorname{lt}(g_i)} \mathbf{e}_i$, *and one can show that the set* $\{\mathbf{S}_{ij} \mid 1 \leq i < j \leq s\}$ *is a Gröbner basis of* $\operatorname{Syz}(G)$ *with respect to the Schreyer module monomial ordering adapted to* G. *Below, we refer to* $\mathbf{S}_{ij}$ *as the* S-syzygy *corresponding to* g_i *and* g_j.

Consider again two ideals $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{P}$. Let $H = \{h_1, \dots, h_r\}$ be a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$ and let $G = \{g_1, \dots, g_s\}$ be a Gröbner basis of $\mathcal{I}$. We may as well assume both bases to be reduced, but this is not strictly necessary. But we will always assume that $\operatorname{NF}_G(H) = H$, i. e. H is given by polynomials in normal form with respect to the ideal $\mathcal{I}$ and the given monomial ordering. To apply Schreyer's construction, we need to choose an enumeration of the polynomials involved. We choose to give precedence to the polynomials in H over the polynomials in G. This will be useful later as a kind of elimination ordering when we look at the syzygies of H relative to $\mathcal{I}$.

In the Schreyer construction of $\operatorname{Syz}(H, G) := \operatorname{Syz}(h_1, \dots, h_r, g_1, \dots, g_s)$ – the set $H \cup G$ is a Gröbner basis of $\mathcal{J}$ by Proposition 3.4 – we have a certain degree of freedom in that for each S-polynomial, we may choose one of the various available standard representations with respect to

$H \cup G$. Specifically, every time a term belonging to a monomial in $\mathrm{lm}(\mathcal{I})$ needs to be reduced, we can choose a reduction by an element of G; note that Algorithm 1 implements just this kind of reduction. This way, reductions with respect to H are only performed for terms belonging to monomials of the order ideal $\mathcal{M} \setminus \mathrm{lm}(\mathcal{I})$. This has the effect that the quotient polynomial q belonging to an element $h \in H$ is built up exclusively of terms belonging to monomials not in $\mathrm{lm}(\mathcal{I})$. Divisors of monomials in $\mathcal{M} \setminus \mathrm{lm}(\mathcal{I})$ are again not in $\mathrm{lm}(\mathcal{I})$, since $\mathcal{M} \setminus \mathrm{lm}(\mathcal{I})$ is an order ideal.

The canonical projection π onto the cosets induces furthermore a projection map from $\mathrm{Syz}(H, G)$ to $\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}([h_1], \dots, [h_r])$ which we continue to call π. Let $\mathbf{p} \in \mathcal{P}^r$ and $\mathbf{q} \in \mathcal{P}^s$. Then we define

$$\pi : \mathrm{Syz}(h_1, \dots, h_r, g_1, \dots, g_s) \longrightarrow \mathrm{Syz}_{\mathcal{P}/I}([h_1], \dots, [h_r]), \quad (\mathbf{p}, \mathbf{q}) \longmapsto [\mathbf{p}], \tag{3}$$

where by $[\mathbf{p}]$ we denote the vector obtained from $\mathbf{p}$ by taking cosets in each component. Of course, we still need to prove that π has the properties that one would expect from a projection map. For this, we shall need the definition of an A-polynomial.

Definition 3.9. With the above notations, an S-polynomial of a pair $(h_i, g_\alpha) \in H \times G$ is called an *A-polynomial* and is denoted by $\mathrm{A}(h_i, g_\alpha)$. As in Construction 3.8, we also introduce the corresponding notion of an *A-syzygy* denoted by $\mathbf{A}_{i\alpha}$.

To justify the notations introduced in Definition 3.9, let us note that $\mathbf{S}_{ij}$ is the syzygy induced by the S-polynomial of two generators $h_i, h_j \in H$, whereas $\mathbf{A}_{i\alpha}$ is the syzygy induced by annihilating the leading monomial of h_i modulo $\mathrm{lm}(I)$. As it is well-known, the letter "S" is an abbreviation for "syzygy", whereas "A" refers to "annihilator" inspired by the work of Norton and Salagean [22] on Gröbner bases over principal ideal rings (see also [19]).

In the rest of this section, for the sake of simplicity, we will use the subindices $r+1, \dots, r+s$ for the polynomials $g_1, \dots, g_s$; i.e. we define $h_\alpha := g_{\alpha - r}$ for $\alpha = r+1, \dots, r+s$. Thus, we will consider the set $\{h_1, \dots, h_{r+s}\}$ and we will study the syzygy module $\mathrm{Syz}(H, G)$ of these polynomials in $\mathcal{P}^{r+s}$. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_{r+s}\}$ be the standard basis of $\mathcal{P}^{r+s}$. By $\mathbf{S}_{ij}$ (respectively $\mathbf{A}_{i\alpha}$), we mean the syzygy module element corresponding to the S-polynomial $\mathrm{S}(h_i, h_j)$ with $1 \le i < j \le r$ (respectively to the A-polynomial $\mathrm{A}(h_i, h_j)$ with $1 \le i \le r$ and $r+1 \le \alpha \le r+s$) involving the module elements $\{\mathbf{e}_1, \dots, \mathbf{e}_{r+s}\}$, see (2). However, when we write $\mathbf{S}_{\alpha\beta}$ we mean the syzygy module element corresponding to the S-polynomial $\mathrm{S}(h_\alpha, h_\beta)$ with $r+1 \le \alpha < \beta \le r+s$ containing merely the module elements $\{\mathbf{e}_{r+1}, \dots, \mathbf{e}_{r+s}\}$. With these notation we can state the following proposition.

Proposition 3.10. *The map π defined in (3) is a $\mathcal{P}$-linear surjective map. The syzygy module $\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}([h_1], \dots, [h_r])$ is generated as a $\mathcal{P}/\mathcal{I}$-module by the image of the subset*

$$\{\mathbf{S}_{ij} \mid 1 \le i < j \le r\} \cup \{\mathbf{A}_{i\alpha} \mid 1 \le i \le r,\ r+1 \le \alpha \le r+s\} \subset \mathrm{Syz}(H, G)$$

under the map π.

Proof. We first show that π is well-defined. Let $(\mathbf{p}, \mathbf{q}) \in \mathrm{Syz}(H, G)$ with $\mathbf{p} = (p_1, \dots, p_r)$ and $\mathbf{q} = (q_1, \dots, q_s)$. Then $\sum_{i=1}^r p_i h_i + \sum_{\alpha=1}^s q_\alpha g_\alpha = 0$, so $\sum_{i=1}^r p_i h_i = -\sum_{\alpha=1}^s q_\alpha g_\alpha \in I$, which implies $\sum_{i=1}^r [p_i][h_i] = [0]$ modulo I. This means that $[\mathbf{p}] \in \mathrm{Syz}_{\mathcal{P}/I}([h_1], \dots, [h_r])$. Hence, π is well-defined. The $\mathcal{P}$-linearity of π follows from the $\mathcal{P}$-linearity of the canonical projection $\mathcal{P} \to \mathcal{P}/\mathcal{I}$.

To show surjectivity, let $([p_1], \dots, [p_r]) \in \mathrm{Syz}_{\mathcal{P}/\mathcal{I}}([h_1], \dots, [h_r])$ be an arbitrary syzygy. Then $\sum_{i=1}^r [p_i][h_i] = [0]$ and thus $\sum_{i=1}^r p_i h_i \in \mathcal{I}$. A standard representation with respect to G yields polynomials $q_1, \dots, q_s$ with $\sum_{i=1}^r p_i h_i + \sum_{\alpha=1}^s q_\alpha g_\alpha = 0$. Hence $(p_1, \dots, p_r, q_1, \dots, q_s) \in \mathrm{Syz}(h_1, \dots, h_r, g_1, \dots, g_s)$ is a preimage of $([p_1], \dots, [p_r])$ and π is surjective.

Since π is $\mathcal{P}$-linear and surjective, any generating set of $\mathrm{Syz}(H, G)$ is mapped to a generating set of the module $\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}([h_1], \dots, [h_r])$. By Schreyer's construction, we know that the syzygies $\mathbf{S}_{ij}, \mathbf{A}_{i\alpha}, \mathbf{S}_{\alpha\beta}$ form a generating set of $\mathrm{Syz}(H, G)$. The syzygies $\mathbf{S}_{\alpha\beta}$ with $r+1 \le \alpha < \beta \le r+s$ are mapped to 0 under π, as our special choice of a standard representation for $\mathrm{S}(h_\alpha, h_\beta)$ implies that $\mathbf{S}_{\alpha\beta}$ has its first r components equal to 0. Hence we can omit them and our claim follows. $\square$

This result motivates the introduction of some special notations to reflect the two somewhat different subsets making up the constructed generating set of $\mathrm{Syz}(H,G)$, in particular, as the two subsets will be treated quite differently at many places.

Definition 3.11. Let $G = \{g_1,\dots,g_s\}$ be a Gröbner basis of the ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ and let $H = \{h_1,\dots,h_r\}$ be a set disjoint from $\mathcal{I}$ such that $\mathrm{NF}_G(H) = H$ and $H \cup G$ generates the ideal $\mathcal{J} \supseteq \mathcal{I}$. Keeping the above notations, we define the set of all S-syzygies of H relative to G by

$$\mathcal{S}(H,G) = \{\mathbf{S}_{ij} \mid 1 \le i < j \le r\} \tag{4}$$

and the set of all *A-syzygies* of H relative to G by

$$\mathcal{A}(H,G) = \{\mathbf{A}_{i\alpha} \mid 1 \le i \le r,\ r+1 \le \alpha \le r+s\}. \tag{5}$$

Construction 3.8 can be extended to $\mathcal{P}/\mathcal{I}$-submodules of $(\mathcal{P}/\mathcal{I})^r$. Thus, we are able to show that the generating set obtained in Proposition 3.10 is even a Gröbner basis.

Theorem 3.12. *Let $G = \{g_1,\dots,g_s\}$ be the reduced Gröbner basis of the ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ and let $H = \{h_1,\dots,h_r\}$ be a Gröbner basis of the ideal $\mathcal{J} \supseteq \mathcal{I}$ relative to $\mathcal{I}$ such that $\mathrm{NF}_G(H) = H$ and let π be the projection map of the corresponding syzygy modules defined in (3). Then the set $\pi(\mathcal{S}(H,G)) \cup \pi(\mathcal{A}(H,G))$ is a Gröbner basis of the syzygy module $\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}([h_1],\dots,[h_r])$ for the Schreyer ordering $\prec_S$.*

Proof. From Proposition 3.10 above, we know already that the set $\pi(\mathcal{S}(H,G)) \cup \pi(\mathcal{A}(H,G))$ generates $\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}([h_1],\dots,[h_r])$ as a $\mathcal{P}/\mathcal{I}$-module. Thus, we must only show that for each syzygy $[\mathbf{p}] \in \mathrm{Syz}_{\mathcal{P}/\mathcal{I}}([h_1],\dots,[h_r])$ there exists a generator $[\mathbf{q}] \in \pi(\mathcal{S}(H,G)) \cup \pi(\mathcal{A}(H,G))$ such that $\mathbf{lm}([\mathbf{q}])$ divides $\mathbf{lm}([\mathbf{p}])$ with the leading monomials taken with respect to the Schreyer ordering $\prec_S$. Let $[\mathbf{p}] = ([p_1],\dots,[p_r])$. By the proof of Proposition 3.10, there exists a syzygy $\mathbf{S} = (p_1,\dots,p_r,q_1,\dots,q_s) \in \pi^{-1}([\mathbf{p}])$ which implies that the polynomial $g := \sum_{i=1}^r p_i h_i$ lies in the ideal $\mathcal{I}$. Since G is a Gröbner basis of $\mathcal{I}$, there exists a standard representation $g = \sum_{\alpha=1}^s q'_\alpha g_\alpha$ entailing that $\max_\prec\{\mathrm{lm}(q'_1 g_1),\dots,\mathrm{lm}(q'_s g_s)\} \le \mathrm{lm}(g)$. This shows that the preimage $\pi^{-1}([\mathbf{p}])$ also contains the syzygy $\mathbf{S}' = (p_1,\dots,p_r,q'_1,\dots,q'_s)$ which satisfies

$$\max_\prec \left\{\mathrm{lm}(q'_1 g_1),\dots,\mathrm{lm}(q'_s g_s)\right\} \le \max_\prec \left\{\mathrm{lm}(p_1 h_1),\dots,\mathrm{lm}(p_r h_r)\right\}.$$

Assume that $\mathrm{lm}(p_i h_i) = \max_\prec\{\mathrm{lm}(p_1 h_1),\dots,\mathrm{lm}(p_r h_r)\}$ where i is minimal with this property. Then $\mathrm{lm}(p_i)\mathbf{e}_i$ is the module leading monomial of $\mathbf{S}'$ with respect to $\prec_S$. Now, two cases may occur: If there exists $j \in \{1,\dots,r\} \setminus \{i\}$ such that $\mathrm{lm}(p_i h_i) = \mathrm{lm}(p_j h_j)$ then $\mathbf{lm}([\mathbf{p}])$ is divisible by $\mathbf{lm}(\pi(\mathbf{S}_{ij}))$. Otherwise, there exists $\alpha \in \{r+1,\dots,r+s\}$ such that $\mathrm{lm}(p_i h_i) = \mathrm{lm}(q'_\alpha g_\alpha)$. It follows that $\mathbf{lm}([\mathbf{p}])$ is divisible by $\mathbf{lm}(\pi(\mathbf{A}_{i\alpha}))$ and this completes the proof. □

4. Computation of Relative Gröbner Bases

In the previous section, we showed how to compute relative Gröbner bases using the most basic version of Buchberger's algorithm. In this section, we will develop "relative" criteria analogous to Buchberger's S-polynomial criterion for a Gröbner basis as well as Buchberger's (first and second) criteria for recognising unnecessary reductions. As starting point, we recall a result of Möller et al. [21] relating the computation of Gröbner bases of polynomial ideals to Gröbner bases of syzygy modules of sets of monomials.

Theorem 4.1 ([21, Thm 2.7])**.** *Let $G = \{g_1,\dots,g_s\} \subset \mathcal{P}$ be a set of polynomials and B a Gröbner basis of the submodule $\mathrm{Syz}(\mathrm{lm}(g_1),\dots,\mathrm{lm}(g_s)) \subset \mathcal{P}^s$. Then G is a Gröbner basis of the ideal it generates, if and only if for all $\mathbf{b} = (b_1,\dots,b_r) \in B$ we have $\sum_{i=1}^s b_i g_i \longrightarrow_G^+ 0$.*

To obtain an analogous result in the context of relative Gröbner bases, we shall need the next proposition inspired by [22, Thm 4.6].

Proposition 4.2. *Let $\mathcal{I} \lhd \mathcal{P}$ be a monomial ideal and $H = \{x^{\mu_1}, \dots, x^{\mu_r}\} \subset \mathcal{M} \setminus \mathcal{I}$ a set of standard monomials. With $x^{\mu_{ij}} := \mathrm{lcm}(x^{\mu_i}, x^{\mu_j})$, the syzygy module $\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}([x^{\mu_1}], \dots, [x^{\mu_r}])$ is generated as $\mathcal{P}/\mathcal{I}$-module by*

$$B := \left\{ \left[\frac{x^{\mu_{ij}}}{x^{\mu_i}}\right]\mathbf{e}_i - \left[\frac{x^{\mu_{ij}}}{x^{\mu_j}}\right]\mathbf{e}_j \mid 1 \le i < j \le r \right\} \cup \bigcup_{i=1}^{r} [\mathrm{G}(\mathcal{I} : x^{\mu_i})]\,\mathbf{e}_i\ .$$

Proof. Let $G = \{x^{\nu_1}, \dots, x^{\nu_s}\}$ be the minimal generating set of $\mathcal{I}$. It is clear that G (resp. H) is a Gröbner basis for $\mathcal{I}$ (resp. the ideal it generates) with respect to any monomial ordering. Thus, applying Theorem 3.12 to the sets G and H, we see that the first subset in B consists of the projection of the S-polynomials between all pairs of elements of H and the second component comes from the projection of the A-polynomials between all pairs in $H \times G$.

To be more precise for the second component, we show that

$$\langle \pi(\mathcal{A}(H,G)) \rangle_{\mathcal{P}/\mathcal{I}} = \left\langle \bigcup_{i=1}^{r} [\mathrm{G}(\mathcal{I} : x^{\mu_i})]\,\mathbf{e}_i \right\rangle_{\mathcal{P}/\mathcal{I}}\ .$$

By definition, $mx^{\mu_i} \in \mathcal{I}$ for any index i and any monomial $m \in \mathrm{G}(\mathcal{I} : x^{\mu_i})$. Hence there exists an index α such that $x^{\nu_\alpha} \mid mx^{\mu_i}$. It follows that there exists a monomial $u \in \mathcal{P}$ such that $mx^{\mu_i} = u\,\mathrm{lcm}(x^{\mu_i}, x^{\nu_\alpha})$ and the module element $\mathbf{A}_{i\alpha} = \mathrm{lcm}(x^{\mu_i}, x^{\nu_\alpha})/x^{\mu_i}\mathbf{e}_i - \mathrm{lcm}(x^{\mu_i}, x^{\nu_\alpha})/x^{\nu_\alpha}\mathbf{e}_\alpha$ satisfies $u\mathbf{A}_{i\alpha} = m\mathbf{e}_i - mx^{\mu_i}/x^{\nu_\alpha}\mathbf{e}_\alpha$. We conclude that $[m\mathbf{e}_i] = [u]\pi(\mathbf{A}_{i\alpha}) \in \langle \pi(\mathcal{A}(H,G)) \rangle_{\mathcal{P}/\mathcal{I}}$ by the definition of π.

Conversely, let us consider the element $\pi(\mathbf{A}_{i\alpha}) \in \pi(\mathcal{A}(H,G))$ where $\mathbf{A}_{i\alpha}$ is the syzygy corresponding to the A-polynomial between x^{μ_i} and x^{ν_α}. Write $\mathbf{A}_{i\alpha} = \frac{x^\theta}{x^{\mu_i}}\mathbf{e}_i - \frac{x^\theta}{x^{\nu_\alpha}}\mathbf{e}_\alpha$ where $x^\theta = \mathrm{lcm}(x^{\mu_i}, x^{\nu_\alpha})$. By the definition of π, we have $\pi(\mathbf{A}_{i\alpha}) = [\frac{x^\theta}{x^{\mu_i}}]\mathbf{e}_i$. There may also exist $x^{\nu_\beta} \in G$ such that $\mathbf{lm}(\mathbf{A}_{i\beta})$ divides $\mathbf{lm}(\mathbf{A}_{i\alpha}) = \frac{x^\theta}{x^{\mu_i}}\mathbf{e}_i$. Without loss of generality, we may assume that $\mathbf{A}_{i\beta}$ is the minimal element satisfying this property. Let $x^\eta = \mathrm{lcm}(x^{\mu_i}, x^{\nu_\beta})$. Thus, $\pi(\mathbf{A}_{i\beta}) = [\frac{x^\eta}{x^{\mu_i}}]\mathbf{e}_i$. We have $x^\eta/x^{\mu_i} \in \mathcal{I} : x^{\mu_i}$. To finish the proof, it is enough to prove that x^η/x^{μ_i} belongs to the minimal generating set of $\mathcal{I} : x^{\mu_i}$. Suppose, by reductio ad absurdum, that $u \mid x^\eta/x^{\mu_i}$ where $u \in \mathrm{G}(\mathcal{I} : x^{\mu_i})$ and $u \ne x^\eta/x^{\mu_i}$. Thus, $ux^{\mu_i} \in \mathcal{I}$ and $x^{\nu_\gamma} \mid ux^{\mu_i} \mid x^\eta$ for some γ. This entails that $\mathrm{lcm}(x^{\mu_i}, x^{\nu_\gamma})$ divides properly x^η, leading to a contradiction with the choice of $\mathbf{A}_{i\beta}$. Since $\pi(\mathbf{A}_{i\beta})$ divides $\mathbf{lm}(\mathbf{A}_{i\alpha})$, we must have $\mathbf{lm}(\mathbf{A}_{i\alpha}) \in \langle \bigcup_{i=1}^{r} [\mathrm{G}(\mathcal{I} : x^{\mu_i})]\,\mathbf{e}_i \rangle_{\mathcal{P}/\mathcal{I}}$. □

Remark 4.3. If $H = \{c_1x^{\mu_1}, \dots, c_\mu x^{\mu_r}\}$ is a set of terms, then Proposition 4.2 remains essentially true: B is a generating set for $\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}([x^{\mu_1}], \dots, [x^{\mu_r}])$ provided that in the first component of B, $\left[\frac{x^{\mu_{ij}}}{x^{\mu_i}}\right]\mathbf{e}_i - \left[\frac{x^{\mu_{ij}}}{x^{\mu_j}}\right]\mathbf{e}_j$ is replaced by $\left[\frac{x^{\mu_{ij}}}{c_i x^{\mu_i}}\right]\mathbf{e}_i - \left[\frac{x^{\mu_{ij}}}{c_j x^{\mu_j}}\right]\mathbf{e}_j$.

Gröbner bases can be characterised using various properties, among them we mention, besides Buchberger's criterion, that a set G is a Gröbner basis, if and only if any polynomial in $\langle G \rangle$ has a standard representation. Furthermore, G is a Gröbner basis, if and only if any syzygy of $\mathrm{lm}(G)$ can be lifted to a syzygy of G and vice versa. We give below similar characterisations for relative Gröbner bases as in [21, Thm 2.7].

Theorem 4.4. *Let $\prec$ be a monomial ordering on $\mathcal{P}$. Let $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{P}$ be polynomial ideals, let $H = \{h_1, \dots, h_r\} \subset \mathcal{J}$ be a relative generating set of $\mathcal{J}$, that is, $\langle H \rangle + \mathcal{I} = \mathcal{J}$. Then, the following statements are equivalent.*

(1) *H is a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$.*
(2) *For all $\mathbf{b} = ([b_1], \dots, [b_r]) \in \mathrm{Syz}_{\mathcal{P}/\mathrm{lm}(\mathcal{I})}(\mathrm{lt}(H))$, we have $\sum_{i=1}^{r} b_i h_i \longrightarrow^*_{H,\mathcal{I}} 0$.*
(3) *For any generating set B of $\mathrm{Syz}_{\mathcal{P}/\mathrm{lm}(\mathcal{I})}(\mathrm{lt}(H))$ and any $\mathbf{b} = ([b_1], \dots, [b_r]) \in B$, it holds $\sum_{i=1}^{r} b_i h_i \longrightarrow^*_{H,\mathcal{I}} 0$.*
(4) *For all $h \in \mathcal{J}$, there exist polynomials $g \in \mathcal{I}$ and $q_i \in \langle \mathcal{M} \setminus (\mathrm{lm}(\mathcal{I}) : \mathrm{lm}(h_i)) \rangle_{\mathbb{K}}$ for $1 \le i \le r$, such that $h = g + \sum_{i=1}^{r} q_i h_i$ and $\mathrm{lm}(q_i h_i) \preceq \mathrm{lm}(h)$ for all i with $q_i \ne 0$.*

Proof. $(1) \Longrightarrow (2)$. Let G be a Gröbner basis of $\mathcal{I}$. Since $\sum_{i=1}^{r} b_i h_i \in \mathcal{J}$ and $H \cup G$ is a Gröbner basis of $\mathcal{J}$, the claim follows from Proposition 3.4.

$(2) \Longrightarrow (3)$. This is obvious.

$(3) \Longrightarrow (4)$. Let $h \in \mathcal{J}$. We first claim that there exist $g \in \mathcal{I}$ and $q_1, \dots, q_r \in \mathcal{P}$ such that $h = g + \sum_{i=1}^{r} q_i h_i$ and in addition $\mathrm{lm}(q_i h_i) \preceq \mathrm{lm}(h)$ for each i with $q_i \neq 0$. Arguing by reductio ad absurdum, suppose that for each choice of $g \in \mathcal{I}$ and $q_1, \dots, q_r \in \mathcal{P}$ there exists i such that $\mathrm{lm}(q_i h_i) \succ \mathrm{lm}(h)$. Among all such representations of h, we pick a representation $h = g + \sum_{i=1}^{r} q_i h_i$ such that $X := \max\{\mathrm{lm}(q_1 h_1), \dots, \mathrm{lm}(q_r h_r)\}$ is minimal with respect to $\prec$. Without loss of generality, we may assume that $X = \mathrm{lm}(q_1 h_1) = \cdots = \mathrm{lm}(q_k h_k)$ and $\mathrm{lm}(q_i h_i) \prec X$ for each $i > k$. In addition, since $h - \sum_{i=1}^{r} q_i h_i \in \mathcal{I}$, we have $\mathrm{lm}(g) \preceq X$. It follows that $\sum_{i=1}^{k} \mathrm{lt}(q_i)\,\mathrm{lt}(h_i) \in \mathrm{lt}(\mathcal{I})$ and in turn $([\mathrm{lt}(q_1)], \dots, [\mathrm{lt}(q_k)], [0], \dots, [0]) \in \mathrm{Syz}_{\mathcal{P}/\mathrm{lm}(\mathcal{I})}(\mathrm{lt}(H))$ can be written as a combination of the elements in B. From (3) and using the fact that the operation of computing remainders on division by a set is linear, we obtain $\sum_{i=1}^{k} \mathrm{lt}(q_i) h_i \longrightarrow^{*}_{H,\mathcal{I}} 0$. Thus, there exist $\tilde{q}_1, \dots, \tilde{q}_r \in \mathcal{P}$ such that $\sum_{i=1}^{k} \mathrm{lt}(q_i) h_i = \tilde{g} + \sum_{i=1}^{r} \tilde{q}_i h_i$ such that $\mathrm{lm}(\tilde{q}_i h_i) \prec X$ and $\mathrm{lm}(\tilde{g}) \preceq X$ with $\tilde{g} \in \mathcal{I}$. This yields a new representation for h of the form

$$g' + \sum_{i=1}^{r} q'_i h_i := g + \sum_{i=1}^{r} (q_i - \mathrm{lt}(q_i)) h_i + \tilde{g} + \sum_{i=1}^{r} \tilde{q}_i h_i$$

with $g' \in \mathcal{I}$ and $\max_{\prec}\{\mathrm{lm}(q'_1 h_1), \dots, \mathrm{lm}(q'_r h_r)\} \prec X$. As this contradicts our assumptions, our claim is proven. Thus, we are able to find a representation $g + \sum_{i=1}^{r} q_i h_i$ for h such that $\mathrm{lm}(q_i h_i) \preceq \mathrm{lm}(h)$ for each i. Now, if there exists i such that $\mathrm{lm}(q_i h_i)$ is reducible by G, then we can perform this reduction and in consequence we may assume that in the representation $h = g + \sum_{i=1}^{r} q_i h_i$ we have $q_i \in \langle \mathcal{M} \setminus (\mathrm{lm}(\mathcal{I}) : \mathrm{lm}(h_i)) \rangle_{\mathbb{K}}$ for each i and this proves (4).

$(4) \Longrightarrow (1)$. Let $x^{\mu} \in \mathrm{lm}(\mathcal{J}) \setminus \mathrm{lm}(\mathcal{I})$. There exists an element $h \in \mathcal{J}$ with $\mathrm{lm}(h) = x^{\mu}$. From (4), write $h = g + \sum_{k=1}^{r} q_k h_k$. Since $h - \sum_{k=1}^{r} q_k h_k \in \mathcal{I}$, we may assume that $\mathrm{lm}(g) \preceq \mathrm{lm}(h)$. From the choice of x^{μ}, we conclude that $\mathrm{lm}(g) \prec \mathrm{lm}(h)$. Additionally, we know that for all i, $\mathrm{lm}(q_i h_i) \preceq \mathrm{lm}(h)$. Consequently, there exists i with $\mathrm{lm}(q_i h_i) = \mathrm{lm}(h)$ and this shows that H is a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$. □

As a consequence of Propositions 3.4 and 4.2 and Theorem 4.4(3), we get the next theorem.

Theorem 4.5 (Relative Buchberger criterion). *Let $\prec$ be a monomial ordering on $\mathcal{P}$. Let $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{P}$ be two polynomial ideals and $G = \{g_1, \dots, g_t\}$ a Gröbner basis of $\mathcal{I}$. Let $H = \{h_1, \dots, h_r\} \subset \mathcal{J}$ with $\langle H \rangle + \mathcal{I} = \mathcal{J}$. Then, H is a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$ if and only if we have $\mathrm{A}(h_i, g_\alpha) \longrightarrow^{*}_{H,\mathcal{I}} 0$ and $\mathrm{S}(h_i, h_j) \longrightarrow^{*}_{H,\mathcal{I}} 0$ for all indices i, j, α.*

Based on this theorem, we are now able to provide the relative variant of Buchberger's algorithm to compute relative Gröbner bases, i. e. Algorithm 2. For making it more efficient, we recall first Buchberger's criteria which may be applied in Buchberger's algorithm to avoid some superfluous reductions in the course of Gröbner bases computation, for more details see [1, pages 222-225].

Lemma 4.6 (Buchberger's first criterion). *Let $f_i, f_j \in \mathcal{P}$ be two polynomials such that we have $\mathrm{lcm}(\mathrm{lm}(f_i), \mathrm{lm}(f_j)) = \mathrm{lm}(f_i)\,\mathrm{lm}(f_j)$. Then, $\mathrm{S}(f_i, f_j)$ is reduced to zero modulo $\{f_1, f_2\}$.*

Lemma 4.7 (Buchberger's second criterion). *Let $F \subset \mathcal{P}$ be finite and $p, f_i, f_j \in \mathcal{P}$ three polynomials such that the followings conditions hold:*

- $\mathrm{lm}(p)$ *divides* $\mathrm{lcm}(\mathrm{lm}(f_i), \mathrm{lm}(f_j))$,
- $\mathrm{S}(p, f_i)$ *and* $\mathrm{S}(p, f_j)$ *have standard representations with respect to F.*

Then, $\mathrm{S}(f_i, f_j)$ has a standard representation with respect to F.

It is worth noting that these two criteria are applicable in the relative setting using the algorithm described in [1, pages 232]. To apply these criteria in Algorithm 2, we must use also the *relative*

Algorithm 2: Relative Buchberger

Data: A monomial ordering $\prec$, a Gröbner basis $G = \{g_1, \dots, g_t\}$ of $\mathcal{I} \trianglelefteq \mathcal{P}$, a finite set of polynomials $H = \{h_1, \dots, h_r\} \subset \mathcal{P}$ with $\mathrm{NF}_G(h_i) = h_i$ for all i

Result: A Gröbner basis of $\langle H \rangle + \mathcal{I}$ relative to $\mathcal{I}$

begin

$T \longleftarrow H$; $P \longleftarrow \{\{h_i, h_j\}, \{h_i, g\} \mid 1 \le i < j \le r, g \in G\}$

while $P \neq \varnothing$ **do**

Select and remove a critical pair $\{f_i, f_j\}$ from P

Reduce $\mathrm{S}(f_i, f_j) \longrightarrow^*_{T,G} p$

if $p \neq 0$ **then**

$P := P \cup \{\{p, h\}, \{p, g\} \mid h \in T, g \in G\}$; $T := T \cup \{p\}$

return T

normal selection strategy. By this, we mean that when we want to select a pair from P, we pick a pair $\{f_i, f_j\} \in P$ such that $\mathrm{lcm}(\mathrm{lm}(f_i), \mathrm{lm}(f_j))$ is as small as possible. In addition, if there are several pairs sharing the same least common divisor, we select a pair $\{f_i, f_j\} \in P$ such that $\{f_i, f_j\} \cap G \neq \varnothing$, if any. The main idea to prove Buchberger's second criterion is that using the mentioned conditions, one is able to write the S-syzygy corresponding to the pair $\{f_1, f_2\}$ as a combination of the S-syzygies corresponding to the pairs $\{p, f_1\}$ and $\{p, f_2\}$, see [1]. Applying this idea, and beside to the above criteria, we can state the next improvement applicable to the computation of relative Gröbner bases.

Proposition 4.8. *Assume that in Algorithm 2 the pair $\{f_i, f_j\}$ with $f_i, f_j \in T$ is considered. If $\mathrm{lcm}(\mathrm{lm}(f_i), \mathrm{lm}(f_j)) \in \mathrm{lm}(\mathcal{I})$, then this pair is superfluous.*

Proof. Let us first fix some notations. Let $\mathrm{lm}(f_\ell) = x^{\mu_\ell}$ for $\ell = i, j$ and $x^{\mu_{ij}} = \mathrm{lcm}(x^{\mu_i}, x^{\mu_j})$. Assume that $x^{\nu_\alpha} := \mathrm{lm}(g_\alpha) \mid x^{\mu_{ij}}$ for some $g_\alpha \in G$. By assumption, there exist monomials x^γ and x^η such that $x^{\mu_{ij}} = x^\gamma \, \mathrm{lcm}(x^{\mu_i}, x^{\nu_\alpha})$ and $x^{\mu_{ij}} = x^\eta \, \mathrm{lcm}(x^{\mu_j}, x^{\nu_\alpha})$. Thus, we can write

$$\frac{x^{\mu_{ij}}}{\mathrm{lt}(f_i)}\mathbf{e}_i - \frac{x^{\mu_{ij}}}{\mathrm{lt}(f_j)}\mathbf{e}_j =$$
$$x^\gamma \left(\frac{\mathrm{lcm}(x^{\mu_i}, x^{\nu_\alpha})}{\mathrm{lt}(f_i)}\mathbf{e}_i - \frac{\mathrm{lcm}(x^{\mu_i}, x^{\nu_\alpha})}{\mathrm{lt}(g_\alpha)}\mathbf{e}_\alpha \right) - x^\eta \left(\frac{\mathrm{lcm}(x^{\mu_j}, x^{\nu_\alpha})}{\mathrm{lt}(f_j)}\mathbf{e}_j - \frac{\mathrm{lcm}(x^{\mu_j}, x^{\nu_\alpha})}{\mathrm{lt}(g_\alpha)}\mathbf{e}_\alpha \right) .$$

Our selection strategy ensures that at the time we choose the pair $\{f_i, f_j\}$, the A-polynomials $\mathrm{A}(f_i, g_\alpha)$ and $\mathrm{A}(f_j, g_\alpha)$ have already relative standard representations and therefore the S-polynomial $\mathrm{S}(f_i, f_i)$ has a relative standard representation, too, which implies our claim. □

Corollary 4.9. *In Proposition 4.2, one can replace B by*

$$B := \left\{ \left[\frac{x^{\mu_{ij}}}{x^{\mu_i}} \right] \mathbf{e}_i - \left[\frac{x^{\mu_{ij}}}{x^{\mu_j}} \right] \mathbf{e}_j \mid 1 \le i < j \le r \wedge x^{\mu_{ij}} \notin \mathcal{I} \right\} \cup \bigcup_{i=1}^{r} [\mathrm{G}(\mathcal{I} : x^{\mu_i})] \, \mathbf{e}_i \, .$$

5. Relative Involutive Bases

We adapt now the basic definitions from the theory of involutive bases to the situation that we work relative to an ideal $\mathcal{I}$. The basic idea is to require that the usual axioms hold only outside of $\mathcal{I}$. This yields the following extension of the definition of an involutive division which for $\mathcal{I} = 0$ coincides with the standard one. Note that "relative cones" are not necessarily cones in the usual sense, but cones parts of which have been removed.

Definition 5.1. Let $\mathcal{I} \trianglelefteq \mathcal{P}$ be a monomial ideal with minimal generating set $\mathrm{G}(\mathcal{I})$. An *involutive division L relative to $\mathcal{I}$* is a rule which assigns to any monomial $x^\mu \in \mathcal{M} \setminus \mathcal{I}$ which is contained in a finite set $H \subset \mathcal{M} \setminus \mathcal{I}$ of monomials a subset of variables $L(x^\mu, H)$, called *L-multiplicative variables of* $x^\mu \in H$, such that the following conditions are satisfied for the *relative involutive cones* $\mathcal{C}_{L,H,\mathcal{I}}(x^\mu) := x^\mu \cdot K[L(x^\mu, H)] \setminus \mathcal{I}$:

1. If the set H contains two monomials x^μ and x^ν such that $\mathcal{C}_{L,H,\mathcal{I}}(x^\mu) \cap \mathcal{C}_{L,H,\mathcal{I}}(x^\nu) \neq \varnothing$, then either $x^\mu \in \mathcal{C}_{L,H,\mathcal{I}}(x^\nu)$ or $x^\nu \in \mathcal{C}_{L,H,\mathcal{I}}(x^\mu)$.
2. If the set H contains two monomials x^μ and x^ν such that $x^\mu \in \mathcal{C}_{L,H,\mathcal{I}}(x^\nu)$, then $\mathcal{C}_{L,H,\mathcal{I}}(x^\mu) \subseteq \mathcal{C}_{L,H,\mathcal{I}}(x^\nu)$.
3. If $H_1 \subset H_2$ are two sets containing the monomial x^μ, then $\mathcal{C}_{L,H_2,\mathcal{I}}(x^\mu) \subseteq \mathcal{C}_{L,H_1,\mathcal{I}}(x^\mu)$.

Next we show that any classical involutive division induces a relative one (and thus provide many concrete instances of relative involutive divisions). The key question here is how one treats directions leading into $\mathcal{I}$, as different plausible possibilities exist. It turns out that the one chosen here is for many purposes the most convenient one.

Definition 5.2. Let $\mathcal{I} \trianglelefteq \mathcal{P}$ be a monomial ideal with minimal generating set $\mathrm{G}(\mathcal{I})$ and let L be an involutive division on $\mathcal{M}$. Then the following rule defines an associated relative division $L_\mathcal{I}$ relative to $\mathcal{I}$: If a finite monomial set $H \subset \mathcal{M} \setminus \mathcal{I}$ is given, then for each variable x_i ($i \in \{1, \dots, n\}$) and for each $x^\mu \in H$,

$$x_i \in L_\mathcal{I}(x^\mu, H) \quad \Longleftrightarrow \quad \big(x_i \in L(x^\mu, H) \vee x_i x^\mu \in \mathcal{I}\big)\,. \tag{6}$$

Proposition 5.3. *If L is an involutive division on $\mathcal{M}$ and $\mathcal{I} \trianglelefteq \mathcal{P}$, then the rule $L_\mathcal{I}$ defined by* (6) *is an involutive division relative to $\mathcal{I}$.*

Proof. For all monomials $x^\mu \in H$, it is clear by definition that $\mathcal{C}_{L,H,\mathcal{I}}(x^\mu) = \mathcal{C}_{L,H}(x^\mu) \setminus \mathcal{I}$. Now, if x^μ and x^ν are elements of H such that $\mathcal{C}_{L,H,\mathcal{I}}(x^\mu) \cap \mathcal{C}_{L,H,\mathcal{I}}(x^\nu) \neq \varnothing$, then also the classical involutive cones $\mathcal{C}_{L,H}(x^\mu)$ and $\mathcal{C}_{L,H}(x^\nu)$ intersect nontrivially, implying, without loss of generality, $x^\mu \in \mathcal{C}_{L,H}(x^\nu)$. But since $x^\mu \notin \mathcal{I}$, this implies $x^\mu \in \mathcal{C}_{L,H,\mathcal{I}}(x^\nu)$, proving that the first defining property of relative involutive divisions is satisfied by the rule $L_\mathcal{I}$. In the same situation, the inclusion $\mathcal{C}_{L,H}(x^\mu) \subseteq \mathcal{C}_{L,H}(x^\nu)$ must hold for the classical involutive cones, which immediately implies the same inclusion for the corresponding $L_\mathcal{I}$-cones. If, finally, $H_1 \subset H_2$ are two monomial sets disjoint from $\mathcal{I}$ and if $x^\mu \in H_1$, then we have the inclusion $\mathcal{C}_{L,H_2}(x^\mu) \subseteq \mathcal{C}_{L,H_1}(x^\mu)$ for the classical involutive cones, which again immediately implies the same inclusion for the $L_\mathcal{I}$-cones. □

Now we can define relative involutive bases. As in the classical case and as for relative Gröbner bases, we begin by considering the monomial case, before we proceed to general polynomial ideals.

Definition 5.4. Let $\mathcal{I} \trianglelefteq \mathcal{P}$ be a monomial ideal and let L be an involutive division relative to $\mathcal{I}$. Let $H \subset \mathcal{M} \setminus \mathcal{I}$ be a finite monomial set disjoint from $\mathcal{I}$ and set $\mathcal{J} := \langle H \rangle + \mathcal{I}$. We call H a *weak L-involutive basis of $\mathcal{J}$ relative to* $\mathcal{I}$, if the $\mathbb{K}$-spans of the sets $\bigcup_{x^\mu \in H} \mathcal{C}_{L,\mathcal{I},H}(x^\mu)$ and $\mathcal{J} \setminus \mathcal{I}$ coincide. H is called *(strong) involutive basis of $\mathcal{J}$ relative to* $\mathcal{I}$, if it is a weak involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$ and the relative involutive cones $\mathcal{C}_{L,H,\mathcal{I}}(x^\mu)$ for $x^\mu \in H$ are pairwise disjoint.

Example 5.5. Let $\mathcal{P} = \mathbb{K}[x_1, x_2]$ be the polynomial ring in two variables, let the monomial ideal $\mathcal{I}$ be minimally generated by the set $\mathrm{G}(\mathcal{I}) = \{x_2^3, x_1^2x_2^2, x_1^3\}$ and consider $H = \{x_1^2x_2, x_2, x_1\}$. We analyse this constellation of monomial sets first by using the relative involutive division induced by the Pommaret division and the ideal $\mathcal{I}$ and then by using the relative involutive division induced by the Janet division and the ideal $\mathcal{I}$.

1. For the Pommaret division $P_\mathcal{I}$ relative to the ideal $\mathcal{I}$, we find that $P_\mathcal{I}(x_1^2x_2, H) = \{x_1, x_2\}$, as $\mathrm{cls}(x_1^2x_2) = 1$ and $x_2(x_1^2x_2) = x_1^2x_2^2 \in \mathcal{I}$. Furthermore, $P_\mathcal{I}(x_2, H) = \{x_1, x_2\}$ as $\mathrm{cls}(x_2) = 2$ and $P_\mathcal{I}(x_1, H) = \{x_1\}$ as $\mathrm{cls}(x_1) = 1$ and $x_2(x_1) = x_1x_2 \notin \mathcal{I}$. One can now easily see that H is a weak Pommaret basis of $\mathcal{J} = \langle H \rangle + \mathcal{I}$ relative to $\mathcal{I}$. But it is not a strong relative Pommaret

basis, because $\mathcal{C}_{P,H,\mathcal{I}}(x_1^2x_2) \subset \mathcal{C}_{P,H,\mathcal{I}}(x_2)$. But of course an autoreduction yields the strong relative Pommaret basis $H \setminus \{x_1^2x_2\} = \{x_1, x_2\}$.
2. For the Janet division $J_\mathcal{I}$ relative to $\mathcal{I}$, we find that $J_\mathcal{I}(x_1^2x_2, H) = \{x_1, x_2\}$ and $J_\mathcal{I}(x_2, H) = \{x_2\}$ as both monomials contain x_2 linearly and $J_\mathcal{I}(x_1, H) = \{x_1\}$. Since the monomial x_1x_2 does not lie in any of the three relative cones, H is not a weak Janet basis of $\mathcal{J}$ relative to $\mathcal{I}$. Nevertheless, one can easily see that $H \setminus \{x_1^2x_2\}$ is a strong Janet basis of $\mathcal{J}$ relative to $\mathcal{I}$.

Definition 5.6. Let $\mathcal{I} \trianglelefteq \mathcal{P}$ be a polynomial ideal, G a Gröbner basis of $\mathcal{I}$ and L an involutive division relative to $\mathcal{I}$. Let $H \subset \mathcal{P}$ be a finite set satisfying $\mathrm{NF}_G(H) = H$ and set $\mathcal{J} := \langle H \rangle + \mathcal{I}$. We call H a *weak L-involutive basis of $\mathcal{J}$ relative to* $\mathcal{I}$, if $\mathrm{lm}(H)$ is a weak involutive basis of $\mathrm{lm}(\mathcal{J})$ relative to $\mathrm{lm}(\mathcal{I})$. H is called a *(strong) L-involutive basis of $\mathcal{J}$ relative to* $\mathcal{I}$, if $\mathrm{lm}(H)$ is a strong involutive basis of $\mathrm{lm}(\mathcal{J})$ relative to $\mathrm{lm}(\mathcal{I})$ and the mapping $h \mapsto \mathrm{lm}(h)$ is a bijection from H to $\mathrm{lm}(H)$.

Via a relative involutive polynomial division, any strong relative involutive basis of an ideal $\mathcal{J} \supseteq \mathcal{I}$ induces a finite direct sum decomposition of $\mathcal{J}/\mathcal{I}$ as a $\mathbb{K}$-linear space provided one uses the right definition of relative involutive cones in the polynomial case. To make this remark precise, we introduce first Algorithm 3 for the division and then define relative involutive cones using normal forms with respect to G.

Algorithm 3: Relative Involutive Division

Data: Ideal $\mathcal{I} \trianglelefteq \mathcal{P}$; Gröbner basis G of $\mathcal{I}$; set of polynomials $H = \{h_1, \dots, h_m\} \subset \mathcal{P}$ with $\mathrm{NF}_G(H) = H$; involutive division L relative to $\mathcal{I}$; polynomial $f \in \mathcal{P}$
Result: Polynomial $r \in \mathcal{P}$ with $\mathrm{supp}(r) \subseteq \mathcal{M} \setminus \big(\mathrm{lm}(\mathcal{I}) \cup \mathcal{C}_{L,\mathrm{lm}(H),\mathrm{lm}(\mathcal{I})}(\mathrm{lm}(H))\big)$, polynomials $q_1, \dots, q_m \in \mathcal{P}$ with $f - r - \sum_{k=1}^m q_k h_k \in \mathcal{I}$ and $q_k \in \mathbb{K}[L(\mathrm{lm}(h_k), \mathrm{lm}(H))]$

```
begin
    f̃ ⟵ f;   r ⟵ 0
    for k = 1, ..., m do
        q_k ⟵ 0
    while f̃ ≠ 0 do
        if lm(f̃) ∈ ⟨lm(G)⟩ then
            Choose g ∈ G with lm(g) | lm(f̃)
            f̃ ⟵ f̃ − (lt(f̃)/lt(g)) g
        else if lm(f̃) ∈ C_{L,lm(H),lm(I)}(lm(H)) then
            Choose index k such that lm(f̃) ∈ C_{L,lm(H),lm(I)}(lm(h_k))
            q_k ⟵ q_k + lt(f̃)/lt(h_k);   f̃ ⟵ f̃ − (lt(f̃)/lt(h_k)) h_k
        else
            r ⟵ r + lt(f̃);   f̃ ⟵ f̃ − lt(f̃)
    return (r, q_1, ..., q_m)
```

Definition 5.7. Let $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{P}$ be two polynomial ideals, G a Gröbner basis of $\mathcal{I}$ and L an involutive division relative to $\mathrm{lm}(\mathcal{I})$. Let $H \subset \mathcal{J} \setminus \mathcal{I}$ be a finite set satisfying $\mathrm{NF}_G(H) = H$, whose elements have pairwise distinct leading monomials. For $h \in H$ define its *L-involutive cone relative to* $\mathcal{I}$ to be the following K-vector space:

$$\mathcal{C}_{L,\mathrm{lm}(H),\mathrm{lm}(\mathcal{I})}(h) := \big\langle \mathrm{NF}_G(x^\rho h) \mid x^\rho\, \mathrm{lm}(h) \in \mathcal{C}_{L,\mathrm{lm}(H),\mathrm{lm}(\mathcal{I})}(\mathrm{lm}(h)) \big\rangle_\mathbb{K} \,. \tag{7}$$

Theorem 5.8. *Let $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{P}$ be polynomial ideals, L an involutive division relative to $\mathcal{I}$ and $H \subset \mathcal{J} \setminus \mathcal{I}$ a strong L-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$. Then we have the following finite direct sum decomposition of the ideal $\mathcal{J}$ as a $\mathbb{K}$-vector space*

$$\mathcal{J} = \left(\bigoplus_{h \in H} \mathcal{C}_{L,\mathrm{lm}(H),\mathrm{lm}(\mathcal{I})}(h) \right) \oplus \mathcal{I}\,. \tag{8}$$

Proof. Let us refer to the first summand in (8) as A. We show first that A is indeed a direct sum. For this it suffices to show that for any two distinct basis elements $h_1, h_2 \in H$ and any polynomials $f_1 \in \mathcal{C}_{L,\mathrm{lm}(H),\mathrm{lm}(\mathcal{I})}(h_1)$, $f_2 \in \mathcal{C}_{L,\mathrm{lm}(H),\mathrm{lm}(\mathcal{I})}(h_2)$ we have $\mathrm{lm}(f_1) \neq \mathrm{lm}(f_2)$. Indeed, if $h \in H$ is any basis element, then for each $f \in \mathcal{C}_{L,\mathrm{lm}(H),\mathrm{lm}(\mathcal{I})}(h)$ there exists a polynomial $p \in K[L(\mathrm{lm}(h), \mathrm{lm}(H))]$ with $\mathrm{supp}(p) \subseteq \frac{1}{\mathrm{lm}(h)} \mathcal{C}_{L,\mathrm{lm}(H),\mathrm{lm}(\mathcal{I})}(\mathrm{lm}(h))$ such that $f = \mathrm{NF}_G(ph)$. But since the leading monomial $\mathrm{lm}(ph) = \mathrm{lm}(p)\,\mathrm{lm}(h) \notin \mathrm{lm}(\mathcal{I})$, we have $\mathrm{lm}(f) = \mathrm{lm}(ph) \in \mathcal{C}_{L,\mathrm{lm}(H),\mathrm{lm}(\mathcal{I})}(\mathrm{lm}(h))$. These relative monomial L-cones are pairwise disjoint when h varies through H, because H is a strong relative L-involutive basis. This proves that A is a direct sum. This argument also entails that $\mathrm{lm}(A) \cap \mathrm{lm}(\mathcal{I}) = \varnothing$, proving that $A \cap \mathcal{I} = \{0\}$.

Now we show that $A + \mathcal{I} = \mathcal{J}$. Let $f \in \mathcal{J} \setminus \{0\}$. Since H is a strong L-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$, Algorithm 3 applied to f yields the remainder $r = 0$ and we can write $f = g + \sum_{h \in H} q_h h$ with $g \in \mathcal{I}$ and $\mathrm{supp}(q_h) \subseteq \frac{1}{\mathrm{lm}(h)} \mathcal{C}_{L,\mathrm{lm}(H),\mathrm{lm}(\mathcal{I})}(\mathrm{lm}(h))$ for all h. Taking normal forms modulo $\mathcal{I}$ via a Gröbner basis, we get $\mathrm{NF}_{\mathcal{I}}(f) = \sum_{h \in H} \mathrm{NF}_{\mathcal{I}}(q_h h)$, and consequently $f = \tilde{g} + \sum_{h \in H} \mathrm{NF}_{\mathcal{I}}(q_h h)$ for some $\tilde{g} \in \mathcal{I}$. This finishes the proof. □

For the remainder of this section, let us analyse relative syzygy modules $\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}(H)$ where H is a strong L-involutive basis of $\langle H \rangle + \mathcal{I}$ relative to $\mathcal{I}$ for some involutive division L relative to $\mathcal{I}$. The goal is to find relative involutive bases also for these syzygy modules. Since all relative involutive bases are *a fortiori* also relative Gröbner bases, we can build on the work done in previous sections. We need to describe carefully how the combinatorial structure of H carries over to the syzygy module. The distinction of S- and A-polynomials as building blocks of the syzygy modules will be the key for this. Let G be a Gröbner basis of $\mathcal{I}$. As in Proposition 3.10 and Theorem 3.12, we impose an ordering on $H \cup G$ where the elements of H get smaller indices than those of G. Additionally, we impose an *L-ordering* on the elements of H, which means that if for some $h_1, h_2 \in H$ there exists a non-multiplicative variable $x_i \notin L(\mathrm{lm}(h_1), \mathrm{lm}(H))$ such that $x_i\,\mathrm{lm}(h_1) \in \mathcal{C}_{L,\mathrm{lm}(H),\mathcal{I}}(\mathrm{lm}(h_2))$, then h_1 precedes h_2 in the L-ordering. The fact that a linear ordering of H can be achieved which is also an L-ordering follows from the acyclicity of the *L-graph* of H. This can be shown for relative involutive divisions induced by classical continuous divisions completely analogously to the case of classical involutive bases. For further details we refer to [26, Lemma 5.4.5] and the references therein. As a first step, we now analyse the S-polynomials $\mathcal{S}(H, G)$.

Proposition 5.9. *Let $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{P}$ be polynomial ideals, G a Gröbner basis of $\mathcal{I}$, $L_{\mathcal{I}}$ an involutive division relative to $\mathcal{I}$ induced by a continuous involutive division L on $\mathcal{M}$ and $H = \{h_1, \dots, h_r\}$ a strong L-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$ ordered according to an $L_{\mathcal{I}}$-ordering. Then for each S-polynomial $\mathbf{S}_{ij} \in \mathcal{S}(H, G)$ satisfying $\mathrm{lcm}(\mathrm{lm}(h_i), \mathrm{lm}(h_j)) \notin \mathrm{lm}(\mathcal{I})$, we have $\mathrm{lm}(\mathbf{S}_{ij}) \in \langle \{x_1, \dots, x_n\} \setminus L_{\mathcal{I}}(\mathrm{lm}(h), \mathrm{lm}(H)) \rangle \mathbf{e}_i$.*

Proof. Note that, by definition of S-polynomials, $1 \leq i < j \leq r$. Let us write $\mathrm{lm}(h_i) = x^{\mu_i}$, $\mathrm{lm}(h_j) = x^{\mu_j}$, and $\mathrm{lcm}(\mathrm{lm}(h_i), \mathrm{lm}(h_j)) = x^{\mu_{ij}}$. Again by definition of S-polynomials and by the proof of Theorem 3.12, we know that $\mathrm{lm}(\mathbf{S}_{ij}) = \frac{x^{\mu_{ij}}}{x^{\mu_i}} \mathbf{e}_i$. We have to show that the monomial $\frac{x^{\mu_{ij}}}{x^{\mu_i}} \notin \mathbb{K}[L_{\mathcal{I}}(x^{\mu_i}, \mathrm{lm}(H))]$. Assume this was the case. Since the $L_{\mathcal{I}}$-cones $\mathcal{C}_{L_{\mathcal{I}},\mathrm{lm}(H),\mathrm{lm}(\mathcal{I})}(\mathrm{lm}(h))$ for the generators $h \in H$ disjointly decompose $\mathrm{lm}(\mathcal{J}) \setminus \mathrm{lm}(\mathcal{I})$ by Theorem 5.8, it is then impossible that $\frac{x^{\mu_{ij}}}{x^{\mu_j}} \in \mathbb{K}[L_{\mathcal{I}}(x^{\mu_j}, \mathrm{lm}(H))]$, too. Thus there exists a non-multiplicative variable $x_a \notin L_{\mathcal{I}}(x^{\mu_j}, \mathrm{lm}(H))$ such that $x_a | \frac{x^{\mu_{ij}}}{x^{\mu_j}}$. $\mathrm{lm}(H)$ contains then a unique leading monomial $x^{\mu_{k_a}} = \mathrm{lm}(h_{k_a})$

such that $x_a x^{\mu_j} \in \mathcal{C}_{L_{\mathcal{I}},\mathrm{lm}(H),\mathrm{lm}(\mathcal{I})}(x^{\mu_{k_a}})$. By the defining property of $L_{\mathcal{I}}$-orderings, $j < k_a$. Now, if $\frac{x^{\mu_{ij}}}{x_a x^{\mu_j}} \in \mathbb{K}[L_{\mathcal{I}}(x^{\mu_{k_a}}, \mathrm{lm}(H))]$ were true, then $x^{\mu_{ij}} \in \mathcal{C}_{L_{\mathcal{I}},\mathrm{lm}(H),\mathrm{lm}(\mathcal{I})}(x^{\mu_{k_a}})$, which entails $h_i = h_{k_a}$, which is not possible since $i < j < k_a$. So there must necessarily exist a non-multiplicative variable $x_b \notin L_{\mathcal{I}}(x^{\mu_{k_a}}, \mathrm{lm}(H))$ such that $x_b | \frac{x^{\mu_{ij}}}{x_a x^{\mu_j}}$. An iteration of this argument yields an infinite sequence of monomials $x^{\mu_{k_a}}, x^{\mu_{k_b}}, x^{\mu_{k_c}}, \dots$ in $\mathrm{lm}(H)$ belonging to basis elements $h_{k_a}, h_{k_b}, h_{k_c}, \dots$ with indices strictly monotonically increasing, which is not possible. Consequently, the assumption $\frac{x^{\mu_{ij}}}{x^{\mu_i}} \in \mathbb{K}[L_{\mathcal{I}}(x^{\mu_i}, \mathrm{lm}(H))]$ was false and there must necessarily exist a non-multiplicative variable for x^{μ_i} dividing the polynomial part of the leading module monomial of the S-polynomial $\mathbf{S}_{ij}$. □

Proposition 5.9 helps to identify among the set of S-polynomials $\mathcal{S}(H, G)$ an irredundant subset $\mathcal{S}_{L_{\mathcal{I}}}(H, G)$ of S-polynomials induced by non-multiplicative prolongations.

Lemma 5.10. *In the situation of Proposition 5.9, for each basis element $h_i \in H$ and for each non-multiplicative variable $x_k \notin L_{\mathcal{I}}(\mathrm{lm}(h_i), \mathrm{lm}(H))$, there exists an S-polynomial $\mathbf{S}_{ij} \in \mathcal{S}(H, G)$ such that $\mathrm{lm}(\mathbf{S}_{ij}) = x_k \mathbf{e}_i$.*

Proof. There is a unique basis element $h_j \in H$ such that $i < j$ and such that $x_k \,\mathrm{lm}(h_i)$ is in the $L_{\mathcal{I}}$-cone $\mathcal{C}_{L_{\mathcal{I}},\mathrm{lm}(H),\mathrm{lm}(\mathcal{I})}(\mathrm{lm}(h_j))$. Note that, trivially, $x_k \,\mathrm{lm}(h_i) \mid \mathrm{lcm}(\mathrm{lm}(h_i), \mathrm{lm}(h_j))$. But since $\mathrm{lm}(h_i) \neq \mathrm{lm}(h_j)$, it follows that $x_k \,\mathrm{lm}(h_i) = \mathrm{lcm}(\mathrm{lm}(h_i), \mathrm{lm}(h_j))$. This induces, by Construction 3.8, an S-polynomial $\mathbf{S}_{ij}$ with the desired properties. □

Definition 5.11. In the situation of Proposition 5.9 and Lemma 5.11, denote by $\mathcal{S}_{L_{\mathcal{I}}}(H, G)$ the set of all S-polynomials induced by non-multiplicative prolongations of elements from H.

Having analysed the part of the syzygy module induced by the S-polynomials, we now turn to the A-polynomials. Since our goal is to obtain, in each module component of the relative syzygy module $\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}(H)$, a relative involutive basis of the ideal in $\mathcal{P}/\mathcal{I}$ associated to this module component, we need an additional structure for the Gröbner basis G of $\mathcal{I}$. More concretely, we want to achieve that the leading monomials of the A-polynomials $\mathbf{S}_{i\alpha}$ associated to the i-th module component of $\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}(H)$ form part of an involutive basis of the leading ideal of the ideal associated to this i-th module component. To achieve this, a natural assumption on G is for it to be a (strong) L-involutive basis of $\mathcal{I}$, where L is the continuous involutive division on $\mathcal{M}$ inducing the relative involutive division $L_{\mathcal{I}}$.

Definition 5.12. Let $\mathcal{I} \trianglelefteq \mathcal{P}$ be a polynomial ideal and $L_{\mathcal{I}}$ an involutive division relative to $\mathrm{lm}(\mathcal{I})$ induced by a continuous involutive division L on $\mathcal{M}$. We say that $L_{\mathcal{I}}$ is of *Schreyer type* if, whenever H is a strong $L_{\mathcal{I}}$-involutive basis of $\langle H \rangle + \mathcal{I}$ relative to $\mathcal{I}$ and G is a strong L-involutive basis of $\mathcal{I}$, we have that for all $x^\mu \in \mathrm{lm}(H)$ the monomial set

$$B = \left(\left\{ \frac{\mathrm{lcm}(x^\nu, x^\mu)}{x^\mu} \mid x^\nu \in \mathrm{lm}(G) \right\} \setminus \mathrm{lm}(\mathcal{I}) \right) \cup \Big(\{x_1, \dots, x_n\} \setminus L_{\mathcal{I}}\big(x^\mu, \mathrm{lm}(H)\big) \Big) \qquad (9)$$

is an $L_{\mathrm{lm}(\mathcal{I})}$-involutive basis of the ideal $\langle B \rangle + \mathrm{lm}(\mathcal{I})$ relative to $\mathrm{lm}(\mathcal{I})$.

Theorem 5.13. *Let $\mathcal{I} \trianglelefteq \mathcal{P}$ be a polynomial ideal and $L_{\mathcal{I}}$ an involutive division relative to $\mathrm{lm}(\mathcal{I})$ of Schreyer type. Furthermore, let G be a strong L-involutive basis of $\mathcal{I}$, where L is the continuous involutive division on $\mathcal{M}$ inducing $L_{\mathcal{I}}$, and H a strong $L_{\mathcal{I}}$-involutive basis of $\langle H, \mathcal{I} \rangle$ relative to $\mathcal{I}$. Then, the set $\pi(\mathcal{A}(H, G)) \cup \pi(\mathcal{S}_{L_{\mathcal{I}}}(H, G))$, where π is defined as in* (3)*, is an $L_{\mathcal{I}}$-involutive basis of the relative syzygy module $\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}(H)$.*

Proof. By Theorem 3.12, the set $\mathcal{A}(H, G) \cup \mathcal{S}(H, G)$ is mapped by π to a Gröbner basis of the relative syzygy module $\mathrm{Syz}_{\mathcal{P}/\mathcal{I}}(H)$. A closer inspection of the proof of Theorem 3.12 shows that in fact the subset of all A-polynomials and S-polynomials with a leading module monomial whose polynomial part does not belong to $\mathrm{lm}(\mathcal{I})$ suffices. Then, by Proposition 5.9, among the remaining S-polynomials, the subset $\mathcal{S}_{L_{\mathcal{I}}}(H, G)$ suffices. Among the remaining A-polynomials $\mathbf{S}_{i\alpha}$, note that, if x^μ is the leading monomial of the basis element $h_i \in H$, then all monomials from the set $\{\frac{\operatorname{lcm}(x^\nu,x^\mu)}{x^\mu} \mid x^\nu \in \operatorname{lm}(G)\} \setminus \operatorname{lm}(\mathcal{I})$ appear as polynomial part of $\operatorname{lm}(\mathbf{S}_{i\alpha})$ for some index α. And, since for all i, the minimal generators of the quotient ideal $\operatorname{lm}(\mathcal{I}) : x^\mu$ are included in this set, collecting the corresponding A-polynomials and the S-polynomials from $\mathcal{S}_{L_\mathcal{I}}(H,G)$, we get a Gröbner basis of the relative syzygy module by projection via π. Since $L_\mathcal{I}$ is of relative Schreyer type, it is even a relative involutive basis. □

The notion of quasi-stability is well-behaved with respect to standard ideal operations such as sum, intersection, and quotient of given ideals, see [26, Lemma 5.3.5]. Thus, one expects that the Pommaret division P induces a relative involutive division of Schreyer type with respect to a quasi-stable ideal I.

Proposition 5.14. *Let $\mathcal{I} \trianglelefteq \mathcal{P}$ be a quasi-stable polynomial ideal and P the Pommaret division on $\mathcal{M}$. Then the relative involutive division $P_{\operatorname{lm}(\mathcal{I})}$ induced by P is of Schreyer type.*

Proof. Let G be a strong Pommaret basis of $\mathcal{I}$ and H a strong $P_\mathcal{I}$-basis of the ideal $\mathcal{J} := \langle H\rangle + \mathcal{I}$ relative to $\mathcal{I}$. Let $x^\mu \in \operatorname{lm}(H)$ be the leading monomial of a generator $h_i \in H$ and analyse the monomial set $A := \{\frac{\operatorname{lcm}(x^\nu,x^\mu)}{x^\mu} \mid x^\nu \in \operatorname{lm}(G)\}$. Since A contains the minimal generators of the colon ideal $\operatorname{lm}(\mathcal{I}) : x^\mu$, it is a monomial generating set of it.

We now prove that A is an involutive set for the Pommaret division, i. e., it is a Pommaret basis of its $\mathcal{P}$-span. Let $x^\rho \in \langle A\rangle$ be any monomial in the ideal generated by A. Then $x^\mu x^\rho \in \operatorname{lm}(\mathcal{I})$, so that there exists $x^\nu \in \operatorname{lm}(G)$ with $x^\mu x^\rho \in \mathcal{C}_P(x^\nu)$. From this it follows that $\operatorname{lcm}(x^\nu,x^\mu) \mid x^\mu x^\rho$ and $\frac{x^\mu x^\rho}{\operatorname{lcm}(x^\nu,x^\mu)} \mid \frac{x^\mu x^\rho}{x^\nu} \in \mathbb{K}[P(x^\nu)]$. Note that $\operatorname{cls}(\frac{\operatorname{lcm}(x^\nu,x^\mu)}{x^\mu}) \geq \operatorname{cls}(x^\nu)$. In other words, every variable that is Pommaret multiplicative for x^ν is also Pommaret multiplicative for $\frac{\operatorname{lcm}(x^\nu,x^\mu)}{x^\mu}$. Hence, $x^\rho \in \mathcal{C}_P(\frac{\operatorname{lcm}(x^\nu,x^\mu)}{x^\mu})$, proving the involutivity of the set A with respect to the Pommaret division.

We now turn to an analysis of the set $V := \{x_1,\dots,x_n\} \setminus P_{\operatorname{lm}(\mathcal{I})}(x^\mu)$. It contains exactly those variables x_j with index $j \geq \operatorname{cls}(x^\mu)$ for which additionally $x_j x^\mu \notin \operatorname{lm}(\mathcal{I})$. Let us take a closer look at the variables x_j for which $x_j x^\mu \in \operatorname{lm}(\mathcal{I})$. For such a variable, there necessarily exists a leading monomial $x^\nu \in \operatorname{lm}(G)$ such that $x_j x^\mu \in \mathcal{C}_P(x^\nu)$. Since x^μ is an element of the order ideal $\mathcal{M} \setminus \operatorname{lm}(\mathcal{I})$, it follows immediately that $x_j x^\mu = \operatorname{lcm}(x^\mu, x^\nu)$, and so, $x_j = \frac{\operatorname{lcm}(x^\mu,x^\nu)}{x^\mu} \in V$. Consequently, $\langle A,V\rangle = \langle A\rangle + \langle \{x_1,\dots,x_n\} \setminus P(x^\mu)\rangle$, and since both A and $\{x_1,\dots,x_n\} \setminus P(x^\mu)$ are (weak) Pommaret bases of the monomial ideals they generate and the Pommaret division is global, by applying [26, Rem. 3.1.13], we have that $A \cup V$, which is equal to $A \cup (\{x_1,\dots,x_n\} \setminus P(x^\mu))$, is a weak Pommaret basis of $\langle A,V\rangle$.

Finally, by the equivalence (6) in Definition 5.2, the set of multiplicative variables $P_{\operatorname{lm}(\mathcal{I})}(x^\gamma)$ for any $x^\gamma \notin \operatorname{lm}(\mathcal{I})$ is a superset of $P(x^\gamma)$, the set of Pommaret multiplicative variables. This proves that $(A \cup V) \setminus \operatorname{lm}(\mathcal{I})$ is a weak $P_{\operatorname{lm}(\mathcal{I})}$-involutive basis of $\langle (A \cup V) \setminus \operatorname{lm}(\mathcal{I}), \operatorname{lm}(\mathcal{I})\rangle$ relative to $\operatorname{lm}(\mathcal{I})$. Since obviously $V \cap \operatorname{lm}(\mathcal{I}) = \varnothing$, we have proved that $P_{\operatorname{lm}(\mathcal{I})}$ is of Schreyer type. □

The natural question is now whether the Janet division relative to a monomial ideal $\mathcal{I}$ is also of Schreyer type. It turns out that it is not; if one takes the minimal Janet basis for $\mathcal{I}$ and the minimal relative Janet basis of $\mathcal{J} \supset \mathcal{I}$ (for a definition, see Section 6), one cannot expect to obtain relative Janet bases when forming sets B defined as in (9). Here is a concrete counterexample.

Example 5.15. Let the monomial ideal $\mathcal{I} \trianglelefteq \mathcal{P} = K[x,y,z]$ be minimally generated by $\mathrm{G}(\mathcal{I}) = \{x^2y^2z\}$. Since $\mathcal{I}$ is a principal ideal, $\mathrm{G}(\mathcal{I})$ is also the minimal Janet basis of $\mathcal{I}$. Let $\mathcal{J} = \langle x,y\rangle$; clearly, $\mathcal{J} \supset \mathcal{I}$. Moreover, $\{x,y\}$ is the minimal relative Janet basis of $\mathcal{J}$ with respect to $\mathcal{I}$. For y, every variable is $J_\mathcal{I}$-multiplicative. For the generator x, only the variable y is non-multiplicative. Now, if one forms the set B as defined in Equation 9 for the generator x, one obtains $B = \{y, xy^2z\}$, whose first element is induced by the non-multiplicative variable, the second element being $\frac{\operatorname{lcm}(x^2y^2z,x)}{x}$. This set is autoreduced in the classical sense, so no subset of it is a basis of $\mathcal{J}$ relative to $\mathcal{I}$ in any sense – involutive or not. Furthermore, the variable z is $J_{\mathcal{I}}$-non-multiplicative for y, and so the monomial yz is not contained in the relative Janet span of B. Hence, we need to perform an involutive completion on the set B to obtain a relative Janet basis. This example proves that the relative Janet division $J_{\mathcal{I}}$ is not of Schreyer type.

In Example 5.15, an important aspect is that we chose *minimal* Janet bases as generating sets. In a sense that will be made more precise in the following discussion, the minimal bases used in Example 5.15 are not enough adapted to one another. But one can find supersets of both sets which, joined together, form a Janet basis of the larger ideal $\mathcal{J}$ in the classical sense; moreover the sets B constructed as in (9) are then always relative Janet bases.

Lemma 5.16. *Let $\mathcal{I} \trianglelefteq \mathcal{P}$ be a monomial ideal generated by a set $G \subset \mathcal{M}$ and $x^\omega = \operatorname{lcm}(G)$ the least common multiple of all generators. Then $\mathcal{I}$ possesses a Janet basis $H \subset \mathcal{I}$ such that all basis elements $x^\mu \in H$ are divisors of x^ω and such that for all $x^\mu, x^\nu \in H$ we have $\operatorname{lcm}(x^\mu, x^\nu) \in H$, i.e., H is closed under the operation of least common multiple.*

Proof. The set $Z = \{x^\mu \in \mathcal{I} : x^\mu \mid x^\omega\}$ is a finite Janet basis of $\mathcal{I}$ (see for instance [13, Prop. 4.5]). Since $G \subseteq Z$, Z can be regarded as a completion of G. □

Remark 5.17. For a generating set G of $\mathcal{I}$, there may in some cases exist Janet bases of $\mathcal{I}$ closed under least common multiples and containing G which are smaller than the set Z introduced in the proof of Lemma 5.16. They can be constructed via a completion algorithm which alternates between the addition of non-multiplicative prolongations and the addition of new least common multiples. A termination proof of such a procedure can be obtained by noting that the sets constructed by these additions always remain subsets of the completion Z.

Proposition 5.18. *Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{P}$ be two polynomial ideals and F a monomial Janet basis of $\operatorname{lm}(\mathcal{J})$ such that $\operatorname{G}(\operatorname{lm}(\mathcal{J})) \cup \operatorname{G}(\operatorname{lm}(\mathcal{I})) \subseteq F$ and F is closed under least common multiples. Then $\mathcal{J}$ possesses a strong $J_{\operatorname{lm}(\mathcal{I})}$-involutive basis H relative to $\mathcal{I}$ such that $\operatorname{lm}(H) = F \smallsetminus \mathcal{I}$. Moreover, if H is ordered according to a $J_{\mathcal{I}}$-ordering, then for any $i \in \{1, \dots, |H|\}$, the ith component $\operatorname{lm}_i(\operatorname{Syz}_{\mathcal{P}/\mathcal{I}}(H))$ of the module of leading monomials of the relative syzygy module of H has the set $B_i = \left\{\frac{x^\mu}{\operatorname{lm}(h_i)} : x^\mu \in F \wedge \operatorname{lm}(h_i) \mid x^\mu \wedge \operatorname{lm}(h_i) \neq x^\mu\right\}$ as a $J_{\operatorname{lm}(\mathcal{I})}$-involutive basis.*

Proof. The assumption that F is a Janet basis of $\operatorname{lm}(\mathcal{J})$ implies trivially that the set $\tilde{H} := F \smallsetminus \mathcal{I}$ is a $J_{\operatorname{lm}(\mathcal{I})}$-involutive basis of $\operatorname{lm}(\mathcal{J})$ relative to $\operatorname{lm}(\mathcal{I})$. For each $x^\mu \in \tilde{H}$ choose a monic polynomial $h_\mu \in \mathcal{J}$ with $\operatorname{lm}(h) = x^\nu$; then $H := \{\operatorname{NF}_{\mathcal{I}}(h_\mu) \mid x^\mu \in \tilde{H}\}$ is a strong Janet basis of $\mathcal{J}$ relative to $\mathcal{I}$. For any index $i \in \{1, \dots, |H|\}$, the monomial ideal $\operatorname{lm}_i(\operatorname{Syz}_{\mathcal{P}/\mathcal{I}}(H))$ is generated by $\operatorname{G}(\mathcal{I} : \operatorname{lm}(h_i))$ together with the non-multiplicative variables $x_k \notin J_{\mathcal{I}}(\operatorname{lm}(h_i), \operatorname{lm}(H))$. The set $\operatorname{G}(\mathcal{I} : \operatorname{lm}(h_i)) = \left\{\frac{\operatorname{lcm}(x^\mu, \operatorname{lm}(h_i))}{\operatorname{lm}(h_i)} \mid x^\mu \in \operatorname{G}(\mathcal{I})\right\}$ is contained in B_i, since F is closed under least common multiples. Moreover, for each non-multiplicative variable $x_k \notin J_{\mathcal{I}}(\operatorname{lm}(h_i), \operatorname{lm}(H))$, the prolongation $x_k \operatorname{lm}(h_i)$ is in the involutive cone of some other leading monomial $\operatorname{lm}(h_j)$ and hence $x_k \operatorname{lm}(h_i) = \operatorname{lcm}(\operatorname{lm}(h_i), \operatorname{lm}(h_j))$. This implies that $x_k \in B_i$. Thus, the set B_i is a basis of the ideal $\operatorname{lm}_i(\operatorname{Syz}_{\mathcal{P}/\mathcal{I}}(H))$.

We still need to show that B_i is a $J_{\mathcal{I}}$-involutive basis. For this it suffices to show that B_i is a Janet basis of $\langle B_i \rangle$ in the classical sense. By the homotheticity of the Janet division [28, p.265], we have for each $x^\nu \in B_i$ the equality $J(x^\nu, B_i) = J(\operatorname{lm}(h_i)x^\nu, \operatorname{lm}(h_i)B_i)$. Since $\operatorname{lm}(h_i)B_i \subseteq F$, Axiom (3) of the definition of involutive divisions implies $J(x^\nu \operatorname{lm}(h_i), F) \subseteq J(\operatorname{lm}(h_i)x^\nu, \operatorname{lm}(h_i)B_i)$. We claim that this inclusion is in fact an equality. Indeed, let $x_k \notin J(x^\nu \operatorname{lm}(h_i), F)$ be any non-multiplicative variable of $x^\alpha := x^\nu \operatorname{lm}(h_i) \in F$. Then there exists some $x^\rho \in F$ with $\rho_k > \alpha_k$. But then also $x^\beta := \operatorname{lcm}(x^\rho, x^\alpha) \in F$, and obviously, $x^\beta \in B_i \operatorname{lm}(h_i)$. This implies that $x_k \notin J(\operatorname{lm}(h_i)x^\nu, \operatorname{lm}(h_i)B_i)$, because also x^β causes x_k to be non-multiplicative. Thus, we have shown that for each $x^\nu \in B_i$, $J(x^\nu, B_i) = J(x^\nu \operatorname{lm}(h_i), F)$. But it is easy to see that the Janet cones of $B_i \operatorname{lm}(h_i)$ with respect to F yield the whole ideal $\mathcal{J} \cap \langle \operatorname{lm}(h_i) \rangle$. This finishes the proof. □

Example 5.19. Let us take up again Example 5.15. An lcm-closed basis of $\mathcal{J} = \langle x, y, x^2y^2z \rangle$ is given by $F = \{x, y, x^2y^2z, xy, xz, xyz, y^2z, xy^2z, yz\}$. If we order the eight relative generators as $H = \{x, y, xy, xz, yz, xyz, y^2z, xy^2z\}$ (this is indeed a $J_\mathcal{I}$-ordering), we get the following relative Janet bases B_i for the ideals $\operatorname{lm}_i(\operatorname{Syz}_{\mathcal{P}/\mathcal{I}}(H))$, where $1 \le i \le 8$:

$$
\begin{array}{lll}
B_1 = \{y, z, yz, y^2z, xy^2z\}, & B_2 = \{x, xz, yz, xyz, z, x^2yz\}, & B_3 = \{z, yz, xyz\}, \\
B_4 = \{y, y^2, xy^2\}, & B_5 = \{x, y, xy, x^2y\}, & B_6 = \{y, xy\}, \\
B_7 = \{x, x^2\}, & B_8 = \{x\}. &
\end{array}
$$

6. Computation of Relative Involutive Bases

If one wants to compute a relative involutive basis for an ideal $\mathcal{J} \supseteq \mathcal{I}$ by going over to the respective leading ideals, one sees that a necessary condition is that $\operatorname{lm}(\mathcal{J})$ has a finite involutive basis relative to $\operatorname{lm}(\mathcal{I})$. If one chooses a *noetherian* involutive division L, every monomial ideal $\mathcal{Q} \trianglelefteq \mathcal{P}$ has a finite strong L-involutive basis. Thus, a natural choice of a relative involutive division for which one can expect to be able to obtain strong relative involutive bases is a relative division of the form $L_\mathcal{I}$, where $\mathcal{I} \trianglelefteq \mathcal{P}$ is a monomial ideal and L is a classical noetherian involutive division.

Lemma 6.1. *If L is a noetherian involutive division, $\mathcal{I} \trianglelefteq \mathcal{P}$ a monomial ideal, and $L_\mathcal{I}$ the involutive division induced by L relative to $\mathcal{I}$, then every monomial ideal $\mathcal{J} \supseteq \mathcal{I}$ possesses a strong $L_\mathcal{I}$-involutive basis relative to $\mathcal{I}$.*

Proof. There exists a strong monomial L-involutive basis $G \subset \mathcal{J}$ of $\mathcal{J}$. The L-involutive span of the set $G \setminus \mathcal{I}$ is a superset of $\mathcal{J} \setminus \mathcal{I}$, since the L-involutive span of G is a superset of $\mathcal{J}$ and by deletion of elements from G the remaining elements cannot lose multiplicative variables. Going over to $L_\mathcal{I}$, the elements of $G \setminus \mathcal{I}$ may be assigned additional multiplicative variables, but no variable multiplicative with respect to L can become non-multiplicative. Consequently, $G \setminus \mathcal{I}$ is a weak involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$. Performing an $L_\mathcal{I}$-involutive autoreduction, we arrive at s strong involutive basis $H \subseteq G \setminus \mathcal{I}$ of $\mathcal{J}$ relative to $\mathcal{I}$. □

Next to the question of existence of a finite involutive basis, there is also the question whether there exists an algorithmic procedure to compute such a finite involutive basis in a finite number of steps. This is known to be true for classical involutive divisions that are *constructive*. We will not go into the details of this technical definition here, but rather recall a very important algorithmic property that constructive involutive divisions have (see the above cited literature on involutive bases for more details and proofs).

Definition 6.2. Let $H \subset \mathcal{M}$ be a finite monomial set, L be any involutive division (possibly relative to some monomial ideal $\mathcal{I}$) and x^μ a monomial in H (in the relative case we assume $H \cap \mathcal{I} = \emptyset$). For any variable $x_i \notin L(x^\mu, H)$, the monomial $x_i x^\mu$ is called a *non-multiplicative prolongation* of x^μ with respect to the division L and the set H.

Theorem 6.3. *Let $G \subset \mathcal{M}$ be a finite monomial set and L a constructive involutive division. Then G is a weak involutive basis of $\langle G \rangle$, if and only if all non-multiplicative prolongations of all elements of G possess an L-involutive divisor in G. Moreover, given any finite monomial set $G \subset \mathcal{M}$, a weak L-involutive basis $\overline{G} \supseteq G$ of $\langle G \rangle$ can be computed in a finite number of steps by adding to G non-multiplicative prolongations which do not possess involutive divisors.*

Now consider a monomial ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ and a relative involutive division $L_{\mathcal{I}}$ induced by a constructive noetherian division L. For obtaining an algorithm for the $L_{\mathcal{I}}$-involutive completion of a monomial set $H \subset (\mathcal{M} \setminus \mathcal{I})$, we want to use a monomial completion algorithm for L. For this, we need a relative version of local involution.

Proposition 6.4. *Let $\mathcal{I} \trianglelefteq \mathcal{P}$ be a monomial ideal and $H \subset (\mathcal{M} \setminus \mathcal{I})$ a finite set of monomials disjoint from $\mathcal{I}$. Furthermore, let L be a noetherian constructive involutive division and $L_{\mathcal{I}}$ the relative involutive division induced by L and $\mathcal{I}$. Then H is a weak $L_{\mathcal{I}}$-involutive basis of $\mathcal{J} := \langle H\rangle+\mathcal{I}$ relative to $\mathcal{I}$, if and only if for all $x^\mu \in H$ and all $x_k \notin L_{\mathcal{I}}(x^\mu, H)$ the non-multiplicative prolongation $x_k x^\mu$ possesses an $L_{\mathcal{I}}$-involutive divisor in H. Moreover, this criterion of local involution translates into an algorithm which computes for any such set H a superset $H \subseteq \overline{H} \subset (\mathcal{M} \setminus \mathcal{I})$ such that $\overline{H}$ is a weak $L_{\mathcal{I}}$-involutive basis of J relative to $\mathcal{I}$.*

Proof. Let H be such a finite set and set $\mathcal{J} := \langle H\rangle + \mathcal{I}$. Consider a monomial $x^\nu \in \mathcal{J} \setminus \mathcal{I}$. By definition of $L_{\mathcal{I}}$, $x^\nu \in \mathcal{C}_{L_{\mathcal{I}}}(H)$ if and only if $x^\nu \in \mathcal{C}_L(H)$. This means that local involution of H with respect to $L_{\mathcal{I}}$ implies that for all $x^\mu \in H$ and all $x_k \notin L_{\mathcal{I}}(x^\mu, H)$, the non-multiplicative prolongation $x_k x^\mu$ is an element of $\mathcal{C}_L(H)$. Let now $x^\nu \in \mathcal{J} \setminus \mathcal{I}$ be any monomial not contained in $\mathcal{C}_L(H)$. If, whenever $x^\mu \in H$ is a divisor of x^ν and $x_k \notin L(x^\mu, H)$ is a non-multiplicative variable, the non-multiplicative prolongation $x_k x^\mu$ is contained in $\mathcal{C}_L(H)$, then one can construct – just as in the proof of [26, Prop. 4.1.4] – an infinite sequence of elements of H consisting of divisors of x^ν satisfying certain division properties, contradicting the assumption that L is continuous. Hence we can conclude that $x^\nu \in \mathcal{C}_L(H)$, and so, *a fortiori*, also $x^\nu \in \mathcal{C}_{L_{\mathcal{I}}}(H)$. In other words, H is a weak $L_{\mathcal{I}}$-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$. But since $x^\nu \notin \mathcal{I}$ and $\mathcal{M} \setminus \mathcal{I}$ is an order ideal, the necessary containments of non-multiplicative prolongations of divisors of x^ν are indeed given under our assumptions.

To see that this relative local involution criterion translates to a completion algorithm, note that H is not locally involutive relative to $\mathcal{I}$, if and only if there is a classical L-non-multiplicative prolongation which is contained in $\mathcal{J} \setminus \mathcal{I}$ but not in $\mathcal{C}_L(H)$. Now, the existence of such an algorithm follows from the fact that in the classical monomial involutive completion algorithm, we are free to choose a selection strategy for the analysis of non-multiplicative prolongations, and so, we may give preference to those non-multiplicative prolongations which are not contained in $\mathcal{I}$. In other words, if we run the classical involutive monomial completion algorithm on the set H with this special selection strategy for the non-multiplicative prolongations, then a certain intermediate step will yield a weak involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$. And until this stage, no elements of $\mathcal{I}$ will have been added to the prospective involutive basis in the course of the algorithm at all. □

Proceeding to the more general case of two polynomial ideals $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{P}$, an involutive completion algorithm becomes more complex, since one also has to consider the A-polynomials. But note that if the input set H which generates $\mathcal{J}$ relative to $\mathcal{I}$ is already a Gröbner basis of $\mathcal{J}$ relative to $\mathcal{I}$, then all A-polynomials reduce to zero, and an involutive completion procedure is now again largely equivalent to the combinatorial task of monomial relative involutive completion. To overcome the difficulties posed by inputs which are not relative Gröbner bases, it is useful to keep in mind that we are only interested in a combinatorial decomposition of the part of $\mathcal{J}$ that is disjoint from $\mathcal{I}$ and do not care about any decomposition of $\mathcal{I}$. This suggests to treat S-polynomials – in this context represented by non-multiplicative prolongations – differently than A-polynomials. Concretely, for the non-multiplicative prolongations, we use relative involutive reductions and for A-polynomials the usual relative reductions. The candidate set for an involutive basis will then only be enlarged by normal forms of A-polynomials, if they introduce a completely new leading monomial. Hence, in a suitable terminating completion algorithm, A-polynomials will cease to contribute new elements to the candidate set after a finite number of steps, and the algorithm will only add non-multiplicative prolongations to the candidate set from that point on. These considerations lead to

Algorithm 4 which is adapted from [26, Algo. 4.5] which in turn is a slight reformulation of the algorithm originally introduced by Gerdt and Blinkov [13]. The two used reduction algorithms are identical with the division Algorithms 1 and 3 but return only the remainder.

Algorithm 4: Relative Involutive Basis

Data: Gröbner basis G of $\mathcal{I} \coloneqq \langle G \rangle \trianglelefteq \mathcal{P}$, finite set $F \subset \mathcal{P}$ with $F \cap \mathcal{I} = \varnothing$, $\mathrm{NF}_G(F) = F$, constructive noetherian involutive division L and its induced relative division $L_{\mathcal{I}}$.

Result: $L_{\mathcal{I}}$-involutive basis of $\mathcal{J} \coloneqq \langle F \rangle + \mathcal{I}$ relative to $\mathcal{I}$

begin

- $H \longleftarrow$ `InvolutiveHeadAutoreduction`$(F, L_{\mathcal{I}})$
- $A \longleftarrow \Big\{ x^\alpha h \mid h \in H,\ x^\alpha \in \mathrm{G}\big(\mathrm{lm}(\mathcal{I}) : \mathrm{lm}(h)\big) \Big\}$
- $S \longleftarrow \Big\{ xh \mid h \in H,\ x \notin L_{\mathcal{I}}\big(\mathrm{lm}(h), \mathrm{lm}(H)\big) \Big\}$
- **while** $A \cup S \neq \varnothing$ **do**
  - **if** $A \neq \varnothing$ **then**
    - choose $p \in A$ with $\mathrm{lm}(p)$ minimal in $\mathrm{lm}(A)$; $A \longleftarrow A \smallsetminus \{p\}$
    - $p \longleftarrow$ `RelativeReduction`(p, H, G)
  - **else**
    - choose $p \in S$ with $\mathrm{lm}(p)$ minimal in $\mathrm{lm}(S)$; $S \longleftarrow S \smallsetminus \{p\}$
    - $p \longleftarrow$ `RelativeInvolutiveReduction`$(p, H, G, L_{\mathcal{I}})$
  - **if** $p \neq 0$ **then**
    - $H \longleftarrow$ `InvolutiveHeadAutoreduction`$(H \cup \{p\}, L_{\mathcal{I}})$
    - $A \longleftarrow \Big\{ x^\alpha h \mid h \in H,\ x^\alpha \in \mathrm{G}\big(\mathrm{lm}(\mathcal{I}) : \mathrm{lm}(h)\big) \Big\}$
    - $S \longleftarrow \Big\{ xh \mid h \in H,\ x \notin L_{\mathcal{I}}\big(\mathrm{lm}(h), \mathrm{lm}(H)\big) \Big\}$
- **return** H

Theorem 6.5. *Algorithm 4 is correct and terminates.*

Proof. Introduce the notation $H_0 \coloneqq F$ and let H_k denote the set H after the kth time iteration of the **while** loop with $p \neq 0$. The set H_{k+1} is thus constructed from H_k by first adding a polynomial and then performing an involutive head autoreduction which implies that $\langle \mathrm{lm}(H_k) \rangle \subseteq \langle \mathrm{lm}(H_{k+1}) \rangle$. Since the polynomial ring $\mathcal{P}$ is noetherian, there exists an index ℓ such that $\langle \mathrm{lm}(H_k) \rangle = \langle \mathrm{lm}(H_\ell) \rangle$ for all $k \geq \ell$. Whenever H_{k+1} arises from H_k via the addition of the remainder r of an A-polynomial, $\mathrm{lm}(r)$ does not lie in $\langle \mathrm{lm}(H_k), \mathrm{lm}(\mathcal{I}) \rangle$ and hence $\langle \mathrm{lm}(H_k) \rangle \subsetneq \langle \mathrm{lm}(H_{k+1}) \rangle$. After the ℓth time the **while** loop has produced a further generator, therefore only remainders stemming from non-multiplicative prolongations are added and these remainders do not enlarge the leading ideal.

Let $p \in S$ be the non-multiplicative prolongation that is checked for the construction of H_{k+1} with $k \geq \ell$ and let $r \neq 0$ be its remainder after the relative involutive reduction. If $\mathrm{lm}(r) \neq \mathrm{lm}(p)$, then $\mathrm{lm}(r) \prec \mathrm{lm}(p)$. Since $\mathrm{lm}(r)$ is not $L_{\mathcal{I}}$-involutively reducible by the current set $\mathrm{lm}(H_k)$ of leading monomials and also $\mathrm{lm}(r) \in \mathrm{lm}(H_k) = \mathrm{lm}(H_\ell)$, we see with an argument like in the proof of Proposition 6.4 that there must exist a generator $h \in H_k$ and a non-multiplicative variable $x_i \notin L_{\mathcal{I}}(\mathrm{lm}(h), \mathrm{lm}(H_k))$ such that $x_i \,\mathrm{lm}(h) \mid \mathrm{lm}(r)$ and $x_i \,\mathrm{lm}(h) \notin \mathcal{C}_{L_{\mathcal{I}}}(\mathrm{lm}(H_k))$. Hence $x_i h$ cannot reduce to zero in a relative involutive reduction with respect to H_k and $\mathcal{I}$. But this contradicts the normal selection strategy used in Algorithm 4: the non-multiplicative prolongation $x_i h$ must have already been treated at this stage, since $\mathrm{lm}(x_i h) \prec \mathrm{lm}(p)$. Hence, $\mathrm{lm}(p) = \mathrm{lm}(r)$. This means that after the ℓth time the **while** loop has produced a new generator, the sets H_k are modified in such a way that the effect on the corresponding sets $\mathrm{lm}(H_k)$ is a monomial involutive completion with intercalated involutive autoreductions – a process which terminates, see [26, Rem. 4.2.2]. Hence, Algorithm 4 terminates on all inputs and we call the output set H.

We still have to prove the *correctness* of Algorithm 4. When the set H is returned, the sets S and A must be empty. Since S is empty, the set $\mathrm{lm}(H)$ is locally $L_{\mathcal{I}}$-involutive and $L_{\mathcal{I}}$-involutively autoreduced. Hence, it is a strong $L_{\mathcal{I}}$-involutive basis of $\langle \mathrm{lm}(H)\rangle + \mathrm{lm}(\mathcal{I})$ relative to $\mathrm{lm}(\mathcal{I})$. At this point, however, we have not yet proven that $\langle \mathrm{lm}(H)\rangle + \mathrm{lm}(\mathcal{I}) = \mathrm{lm}(\mathcal{J})$. To this end, enumerate the sets H and $\mathrm{lm}(H)$ according to an $L_{\mathcal{I}}$-ordering on $\mathrm{lm}(H)$. The monomial set $\mathrm{lm}(H) \subset \mathcal{M} \setminus \mathrm{lm}(\mathcal{I})$ constitutes a Gröbner basis of $\langle \mathrm{lm}(H)\rangle + \mathrm{lm}(\mathcal{I})$ relative to $\mathrm{lm}(\mathcal{I})$. By Theorem 3.12, the sets $\mathcal{S}(\mathrm{lm}(H), \mathrm{G}(\mathrm{lm}(\mathcal{I})))$ and $\mathcal{A}(\mathrm{lm}(H), \mathrm{G}(\mathrm{lm}(\mathcal{I})))$ of S-syzygies and A-syzygies induce a Gröbner basis of the relative syzygy module $\mathrm{Syz}_{\mathcal{P}/\mathrm{lm}(\mathcal{I})}(\mathrm{lm}(H))$ via the projection mapping π defined in (3). Applying Proposition 5.9 and Lemma 5.10 to $\mathrm{lm}(\mathcal{I}) \subseteq \langle \mathrm{lm}(H)\rangle + \mathrm{lm}(\mathcal{I})$, we see that by comparing module leading monomials, we can replace the set $\mathcal{S}(\mathrm{lm}(H), \mathrm{G}(\mathrm{lm}(\mathcal{I})))$ of all S-polynomials by the smaller set $\mathcal{S}_{L_{\mathcal{I}}}(\mathrm{lm}(H), \mathrm{G}(\mathrm{lm}(\mathcal{I})))$ introduced in Definition 5.11. Let $\mathbf{b} \in (\mathcal{P}/\mathrm{lm}(\mathcal{I}))^{|\mathrm{lm}(H)|}$ be a vector with entries b_i. If $\mathbf{b} \in \mathcal{S}_{L_{\mathcal{I}}}(\mathrm{lm}(H), \mathrm{G}(\mathrm{lm}(\mathcal{I})))$, then the fact that $S = \varnothing$ at the end of Algorithm 4 implies that $\sum_{i=1}^{|\mathrm{lm}(H)|} b_i \cdot h_i \longrightarrow^*_{H,\mathcal{I}} 0$. If $\mathbf{b} \in \mathcal{A}(\mathrm{lm}(H), \mathrm{G}(\mathrm{lm}(\mathcal{I})))$, then it follows analgously from $A = \varnothing$ at the end of Algorithm 4 that $\sum_{i=1}^{|\mathrm{lm}(H)|} b_i \cdot h_i \longrightarrow^*_{H,\mathcal{I}} 0$. By Theorem 4.4, H is thus a Gröbner basis of $\langle H\rangle + \mathcal{I} = \langle F\rangle + \mathcal{I} = \mathcal{J}$ relative to $\mathcal{I}$. Together with the involutive head reducedness of H and the involutivity of $\mathrm{lm}(H)$, this implies that H is a strong $L_{\mathcal{I}}$-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$, finishing the proof of correctness of Algorithm 4. □

Example 6.6. Let $\mathcal{P} = K[x_1, x_2, x_3]$ be a polynomial ring in three variables endowed with the degree reverse lexicographical ordering $\prec$ with $x_3 \prec x_2 \prec x_1$. Let $\mathcal{I} = \langle g\rangle = \langle x_1x_2 + x_2^2\rangle$ be a principal ideal generated by the Gröbner basis $\{g\}$ and consider $\tilde{F} = \{x_1^2x_2 - x_2^3 + x_3^3,\ x_1x_2^2 - x_1^3 + x_1x_2x_3\} \subset \mathcal{P}$. The elements of $\tilde{F}$ are not yet in normal form with respect to $\mathcal{I}$. Applying a normal form algorithm, we get the set $F = \{x_3^3,\ -x_2^3 - x_1^3 - x_2^2x_3\}$ with $\langle F\rangle + \mathcal{I} = \langle \tilde{F}\rangle + \mathcal{I}$.

The data F, $\{g\}$, $\prec$ together with the Janet division $J_{\mathrm{lm}(\mathcal{I})}$ relative to $\mathrm{lm}(\mathcal{I})$ form a valid input for Algorithm 4. For the Janet division every monomial set is Janet autoreduced. This property carries over to the relative Janet division $J_{\mathrm{lm}(\mathcal{I})}$ and hence we can ignore the involutive head autoreductions in Algorithm 4. At first, set $H = \{h_1, h_2\}$ with $h_1 = x_3^3$ and $h_2 = -x_2^3 - x_1^3 - x_2^2x_3$. The A-polynomial with minimal leading term is $x_2 \cdot x_3^3$. It can be ignored, because $\mathrm{lcm}(\mathrm{lm}(g), \mathrm{lm}(x_3^3)) = \mathrm{lm}(g) \cdot \mathrm{lm}(x_3^3)$. At this stage, only the A-polynomial $x_2 \cdot (-x_2^3 - x_1^3 - x_2^2x_3)$ is left to check. Its normal form with respect to $\mathcal{I}$ is $-x_2^3x_3$, and this polynomial is reduced with respect to $H \cup \{g\}$. So it is added to H: $h_3 := -x_2^3x_3$. This yields the new A-polynomial $x_1 \cdot (h_3)$. Its normal form with respect to $\mathcal{I}$ is $x_2^4x_3$ and this is a multiple of h_3, so it reduces to zero.

This is the first time that no A-polynomials are left to check ($A = \varnothing$), so we turn to the $J_{\mathrm{lm}(\mathcal{I})}$-nonmultiplicative prolongations. The variables x_1 and x_2 are multiplicative for all elements of $\mathrm{lm}(H)$, only x_3 is nonmultiplicative for $\mathrm{lm}(h_2)$ and $\mathrm{lm}(h_3)$. Our selection strategy is to choose the $\prec$-minimal prolongation. This is $x_3 \cdot h_3$, which is already involutively reduced and immediately yields the new element $h_4 := -x_2^3x_3^2$. Its A-polynomial, $x_1 \cdot h_4$, has the normal form $x_2^4x_3^2$ with respect to $\mathcal{I}$, which is again just a multiple of h_4 and thereby reduces to zero. Again, $A = \varnothing$, so we are asked to consider nonmultiplicative prolongations. The multiplicative variables of $\mathrm{lm}(h_1), \mathrm{lm}(h_2), \mathrm{lm}(h_3)$ are not altered by the addition of $\mathrm{lm}(h_4)$. This entails that we do not need to check $x_3 \cdot h_3$ again at this time. x_3 is the only nonmultiplicative variable of $\mathrm{lm}(h_4)$. This is also the $\prec$-minimal prolongation, so we check $x_3 \cdot h_4$. It reduces to zero involutively via h_1. There is only the prolongation $x_3 \cdot (h_2) = -x_2^3x_3 - x_1^3x_3 - x_2^2x_3^2$ left to check. It reduces involutively to $h_5 := -x_1^3x_3 - x_2^2x_3^2$ with $\mathrm{lm}(h_5) = x_1^3x_3$. The A-polynomial of h_5 is $x_2 \cdot h_5$ and its normal form with respect to $\mathcal{I}$ is $x_2^4x_3 - x_2^3x_3^2$. This reduces to zero via h_3 and h_4.

We are again asked to consider nonmultiplicative prolongations and since the multiplicative variables of $\operatorname{lm}(h_1), \dots, \operatorname{lm}(h_4)$ are not altered by the addition of $\operatorname{lm}(h_5)$, only the prolongation $x_3 \cdot h_5$ remains to be checked. (Note that $x_2 \in J_{\operatorname{lm}(\mathcal{I})}(\operatorname{lm}(h_5), \operatorname{lm}(H))$, because $x_2 \operatorname{lm}(h_5) \in \operatorname{lm}(\mathcal{I})$.) $x_3 \cdot h_5 = -x_1^3 x_3^2 - x_2^2 x_3^3$ reduces to $h_6 := -x_1^3 x_3^2$ involutively via h_1. The A-polynomial of h_6 is $x_2 \cdot h_6$ and its normal form with respect to $\mathcal{I}$ is $x_2^4 x_3^2$, which reduces to zero via h_3. So we are again asked to consider nonmultiplicative prolongations, and this time no non-zero involutive remainders are computed. Therefore Algorithm 4 returns the strong relative $J_{\operatorname{lm}(\mathcal{I})}$-involutive basis $H = \{x_3^3,\ -x_2^3 - x_1^3 - x_2^2 x_3,\ -x_2^3 x_3,\ -x_2^3 x_3^2,\ -x_1^3 x_3 - x_2^2 x_3^2,\ -x_1^3 x_3^2\}$.

From the theory of involutive bases in $\mathcal{P}$, it is known that for a given constructive noetherian division L every monomial ideal $\mathcal{I} \trianglelefteq \mathcal{P}$ possesses a unique *minimal* L-involutive basis. The proof of this fact is algorithmic in the sense that one can show that the monomial completion algorithm using the addition of non-multiplicative prolongations, applied to the minimal generating set $\mathrm{G}(\mathcal{I})$, always terminates with this unique minimal basis. This fact, in its turn, is proven by showing that each of the prolongations added during the course of the completion algorithm must necessarily be contained in *every* L-involutive basis of $\mathcal{I}$. In view of Proposition 6.4, which shows that the monomial completion procedure can be adapted to the relative situation, this motivates the following definition, which generalises [14, Def. 4.2] to the relative case.

Definition 6.7. Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{P}$ be two ideals and L a constructive noetherian involutive division on $\mathcal{P}$. If $\mathcal{I}$ and $\mathcal{J}$ are monomial ideals and if $H \subseteq \mathcal{J} \smallsetminus \mathcal{I}$ is an $L_{\mathcal{I}}$-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$, then we say that H is a *minimal relative $L_{\mathcal{I}}$-involutive basis*, if $H \subseteq \tilde{H}$ for all $L_{\mathcal{I}}$-involutive bases $\tilde{H}$ of $\mathcal{J}$ relative to $\mathcal{I}$. More generally, we say that a subset $H \subset \mathcal{J} \smallsetminus \mathcal{I}$ is a minimal involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$, if H is a strong $L_{\operatorname{lm}(\mathcal{I})}$-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$ and $\operatorname{lm}(H)$ is a minimal $L_{\operatorname{lm}(\mathcal{I})}$-involutive basis of $\operatorname{lm}(\mathcal{J})$ relative to $\operatorname{lm}(\mathcal{I})$.

Proposition 6.8. *Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{P}$ be two ideals and L a constructive noetherian involutive division on $\mathcal{P}$. Then there exists a unique $L_{\operatorname{lm}(\mathcal{I})}$-involutively autoreduced minimal $L_{\operatorname{lm}(\mathcal{I})}$-involutive basis of $\mathcal{J}$ relative to $\mathcal{I}$.*

Proof. The general case, for polynomial ideals, follows immediately from the monomial case. The monomial case can be proven by a straightforward adaption of the proofs for L-involutive bases. We sketch here only the main argument which implies that the relevant proofs can be adapted to the relative case. A key point in the classical monomial completion algorithm is the selection strategy for non-multiplicative prolongations, which says that exactly those prolongations which do not possess a strict (non-involutive) divisor among the set of eligible prolongations are valid choices for the next element to be added. By Proposition 6.4, for monomial ideals $\mathcal{I}$ and $\mathcal{J}$, a relative $L_{\mathcal{I}}$-involutive basis of $\mathcal{J}$ can be found by applying the L-involutive completion algorithm to $\mathrm{G}(\mathcal{J}) \smallsetminus \mathcal{I}$, choosing prolongations which do not lie in $\mathcal{I}$ as long as possible. Now, if there exists any eligible non-multiplicative prolongation which does not lie in $\mathcal{I}$, then there obviously also exists a prolongation which does not lie in $\mathcal{I}$ and which possesses no strict (non-involutive) divisor among all eligible prolongations. This means the selection strategy can be adapted to the relative case, and the proof of existence and uniqueness of minimal relative $L_{\mathcal{I}}$-involutive bases is thereby reduced to the respective results for L-involutive bases. □

Algorithm 5 combines the ideas behind Algorithm 4 with the classical TQ algorithm for the construction of minimal involutive bases introduced by Gerdt and Blinkov [14] following the formulation given in [26, Algorithm 4.6]. We omit an explicit proof of its termination and correctness, as it is obvious from the corresponding proofs for the two underlying algorithms.

Algorithm 5: Minimal Relative Involutive Basis

Data: Gröbner basis G of $\mathcal{I} := \langle G\rangle \trianglelefteq \mathcal{P}$, finite set $F \subset \mathcal{P}$ with $F \cap \mathcal{I} = \varnothing$, $\mathrm{NF}_G(F) = F$, constructive noetherian involutive division L and its induced relative division $L_{\mathcal{I}}$. Also, $\mathrm{lm}(F)$ is $L_{\mathcal{I}}$-involutively autoreduced.

Result: Minimal $L_{\mathcal{I}}$-involutive basis of $\mathcal{J} := \langle F\rangle + \mathcal{I}$ relative to $\mathcal{I}$

```
begin
    H ⟵ ∅;   Q ⟵ F
    A ⟵ { x^α h | h ∈ H ∪ Q, x^α ∈ G(lm(I) : lm(h)) }
    while A ∪ Q ≠ ∅ do
        if A ≠ ∅ then
            choose p ∈ A with minimal lm(p) in A;   A ⟵ A ∖ {p}
            p ⟵ RelativeReduction(p, H ∪ Q, G)
            if p ≠ 0 then
                Q ⟵ Q ∪ {p};   A ⟵ A ∪ {x^α p | x^α ∈ G(lm(I) : lm(p))}
        else
            choose q ∈ Q with lm(q) minimal in lm(Q);   Q ⟵ Q ∖ {q}
            q ⟵ RelativeInvolutiveReduction(q, H, G, L_I)
            if q ≠ 0 then
                H' ⟵ {h ∈ H | lm q ≺ lm h};   H ⟵ (H ∪ {q}) ∖ H'
                Q ⟵ Q ∪ H' ∪ {xh | h ∈ H, x ∉ L_I(lm h, lm H)}
                A ⟵ { x^α h | h ∈ H ∪ Q, x^α ∈ G(lm(I) : lm(h)) }
    return H
```

7. Relative Quasi-stable Position

It is well-known that Pommaret bases exist only in generic coordinates, more precisely, for ideals in quasi-stable position – see [25, 26]. In [17] a first algorithm for the deterministic construction of such coordinates was developed in the context of differential equations and in [16] it was extended to polynomial ideals. It was based on a comparison of the Janet and Pommaret multiplicative variables of the given basis. Later, an alternative approach to various kinds of stable position based on their combinatorial characterisations was presented in [15]. We will now extend some of these results to the relative setting.

Definition 7.1. Let $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{P}$ be two monomial ideals. We say that $\mathcal{J}$ is *quasi-stable relative to* $\mathcal{I}$, if for all monomials $x^\mu \in \mathcal{J} \setminus \mathcal{I}$ and for all indices i with $\mathrm{cls}(x^\mu) < i \leq n$ there exists an exponent $s \geq 0$ such that either $x_i^s x^\mu \in \mathcal{I}$ or $x_i^s x^\mu / x_{\mathrm{cls}(x^\mu)} \in \mathcal{J}$.

Remark 7.2. Similar to [27, Lemma 3.4], one can show that it suffices to consider in Definition 7.1 the monomials in $\mathrm{G}(\mathcal{J}) \setminus \mathcal{I}$. Quasi-stability relative to $\mathcal{I} = \{0\}$ corresponds to the classical notion of quasi-stability. For $\mathcal{J} \supset \mathcal{I}$ to be quasi-stable relative to $\mathcal{I}$, neither $\mathcal{I}$ nor $\mathcal{J}$ need to be quasi-stable in the classical sense. As as simple example, consider in the ring $\mathcal{P} = \mathbb{K}[x_1, x_2]$ the ideals $\mathcal{I} = \langle x_1^2 x_2, x_1 x_2^2\rangle$ and $\mathcal{J} = \langle x_1 x_2\rangle$. One sees readily that $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$, however, neither $\mathcal{J}$ nor $\mathcal{I}$ contains a monomial of class 2, so both ideals are not quasi-stable.

However, we have the following result which is immediately implied by the definitions.

Lemma 7.3. *Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{P}$ be two monomial ideals. If $\mathcal{J}$ is quasi-stable, then $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$. If $\mathcal{I}$ is quasi-stable and $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$, then $\mathcal{J}$ is quasi-stable.*

Proposition 7.4. *Let $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{P}$ be two monomial ideals. Then $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$, if and only if $\mathcal{J}$ possesses a finite Pommaret basis relative to $\mathcal{I}$.*

Proof. Suppose that $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$. Consider the set

$$H := \Big\{x^\rho \cdot x^\mu \mid x^\mu \in \mathrm{G}(\mathcal{J}) \setminus \mathcal{I} \wedge x^\rho \in \mathbb{K}[x_{\mathrm{cls}(x^\mu)+1}, \dots, x_n] \wedge \frac{x^\rho \cdot x^\mu}{x_{\mathrm{cls}(x^\mu)}} \notin \mathcal{J}\Big\}. \tag{10}$$

By Definition 7.1, it is not difficult to see that H is finite. Thus, it suffices to show that H is a weak Pommaret basis for $\mathcal{J}$ relative to $\mathcal{I}$. Consider a monomial $x^\lambda \in \mathcal{J} \setminus \mathcal{I}$. We decompose it as $x^\lambda = x^\rho x^\sigma x^\mu$ where $x^\mu \in \mathrm{G}(\mathcal{J}) \setminus \mathcal{I}$ is a minimal generator, x^σ contains only multiplicative variables for x^μ with respect to the relative Pommaret division and x^ρ only non-multiplicative ones. If $x^\rho x^\mu \in H$, then we are done, as $\mathrm{cls}(x^\mu) = \mathrm{cls}(x^\rho x^\mu)$ and x^σ contains only multiplicative variables for $x^\rho x^\mu$. If $x^\rho x^\mu \notin H$, then we choose among all monomials $x^\lambda \in \mathcal{J} \setminus \mathcal{I}$ with this property one having the same class and the smallest degree in $x_{\mathrm{cls}(x^\mu)}$. Without loss of generality, assume that our given x^λ is such an element. Therefore, from the definition of H, we conclude that $u := x^\rho \cdot x^\mu / x_{\mathrm{cls}(x^\mu)} \in \mathcal{J}$. Now, two cases may occur. If $u \in \mathcal{I}$, then $x^\lambda \in \mathcal{I}$ in contradiction to our assumptions. Otherwise, we have $u \in \mathcal{J} \setminus \mathcal{I}$ and the degree of u in $x_{\mathrm{cls}(x^\mu)}$ is less than that of x^λ. By our minimality assumption, there exists $v \in H$ which involutively divides u for the Pommaret division relative to $\mathcal{I}$. Thus v also involutively divides x^λ for this division, as $x_{\mathrm{cls}(x^\mu)} x^\sigma$ contains only multiplicative variables for v.

Conversely, suppose that $\mathcal{J}$ has a finite Pommaret basis H relative to $\mathcal{I}$. Arguing by reductio ad absurdum, suppose there exists a monomial $x^\mu \in \mathcal{J} \setminus \mathcal{I}$ with $\mathrm{cls}(x^\mu) < n$ and $j > \mathrm{cls}(x^\mu)$ such that $x_j^s x^\mu \notin \mathcal{I}$ and $x_j^s x^\mu / x_{\mathrm{cls}(x^\mu)} \notin \mathcal{J}$ for all $s \in \mathbb{N}$. Consider the set $\{x_j^s x^\mu \mid s \in \mathbb{N}\} \subset \mathcal{J} \setminus \mathcal{I}$. Since it is infinite and H is a finite Pommaret basis of $\mathcal{J}$ relative to $\mathcal{I}$, there exists a generator $x^\nu \in H$ involutively dividing infinitely many of its elements for the Pommaret division relative to $\mathcal{I}$. Let us pick one of these elements, say $x_j^{s_0} x^\mu$. By the mentioned property, x_j must be multiplicative for x^ν and hence $\mathrm{cls}(x^\nu) > \mathrm{cls}(x^\mu)$. But then x^ν must divide $x_j^{s_0} x^\mu / x_{\mathrm{cls}(x^\mu)}$, leading to a contradiction. □

Corollary 7.5. *Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{P}$ be two monomial ideals. $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$, if and only if the monomial set*

$$P(\mathcal{I}, \mathcal{J}) := \Big\{x^\mu \in \mathcal{J} \setminus \mathcal{I} \mid \frac{x^\mu}{x_{\mathrm{cls}(x^\mu)}} \notin \mathcal{J}\Big\}$$

is finite. In this case, $P(\mathcal{I}, \mathcal{J})$ is the unique minimal monomial Pommaret basis of $\mathcal{J}$ relative to $\mathcal{I}$.

Proof. Suppose that $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$. One sees easily that the set H defined in (10) is equal to $P(\mathcal{I}, \mathcal{J})$ and thus it was already shown in the proof of Proposition 7.4, that (10) is a finite *weak* Pommaret basis of $\mathcal{J}$ relative to $\mathcal{I}$. There only remains to show that it is in fact a strong basis. Assume that there exist two generators $x^\lambda, x^\mu \in P(\mathcal{I}, \mathcal{J})$ such that $x^\lambda \neq x^\mu$ and $x^\lambda = x^\sigma x^\mu$ where x^σ contains only multiplicative variables for x^μ for the relative Pommaret division. It follows that $\mathrm{cls}(x^\lambda) < \mathrm{cls}(x^\mu)$ and in turn $x^\lambda / x_{\mathrm{cls}(x^\lambda)} \in \mathcal{J}$, leading to a contradiction.

Conversely, suppose that $P(\mathcal{I}, \mathcal{J})$ is finite. Assume that for some monomial $x^\mu \in \mathcal{J} \setminus \mathcal{I}$, for some index $i > \mathrm{cls}(x^\mu)$ and for each exponent s we have $x_i^s x^\mu / x_{\mathrm{cls}(x^\mu)} \notin \mathcal{J}$ so that $\mathcal{J}$ is not quasi-stable for $\mathcal{I}$. Note that $x_i^s x^\mu$ and x^μ have the same class. Thus, by definition of $P(\mathcal{I}, \mathcal{J})$, for each s the monomial $x_i^s x^\mu / x_{\mathrm{cls}(x^\mu)}$ must lie in $P(\mathcal{I}, \mathcal{J})$ contradicting its finiteness. □

In the sequel, we use the degree reverse lexicographical ordering $\prec$ with $x_1 \prec \cdots \prec x_n$. The notion of ideals in quasi-stable position can be defined in the relative setting as follows.

Definition 7.6. Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{P}$ be two polynomial ideals. We say that $\mathcal{J}$ is in *quasi-stable position relative to* $\mathcal{I}$, if $\mathrm{lm}(\mathcal{J})$ is quasi-stable relative to $\mathrm{lm}(\mathcal{I})$.

As a consequence of [26, Theorem 4.3.15] and Lemma 7.3, we get the next result.

Proposition 7.7. *Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{P}$ be two homogeneous polynomial ideals. If K is an infinite field, then a generic linear change of variables transforms $\mathcal{J}$ into quasi-stable position relative to $\mathcal{I}$.*

Thus, given homogeneous ideals $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{P}$, $\mathcal{J}$ need not be in quasi-stable position relative to $\mathcal{I}$, but after a sufficiently general linear change of variables $\Phi : \mathcal{P} \to \mathcal{P}$, the ideal $\Phi(\mathcal{J}) \trianglelefteq \mathcal{P}$ will be in quasi-stable position relative to $\Phi(\mathcal{I})$. Under the assumption that the coefficient field K is large enough, [15, Alg. 2] describes a deterministic algorithm returning for a given homogeneous ideal a sparse linear change of variables such that the transformed ideal is in quasi-stable position. In our situation where two homogeneous ideals $\mathcal{I} \subseteq \mathcal{J} \trianglelefteq \mathcal{P}$ are given, we look for a linear change of variables Φ such that $\Phi(\mathcal{J})$ is in quasi-stable position relative to $\Phi(\mathcal{I})$.[2] For this, we extend the approach of [15] to the relative case. The following definition of an ordering on ordered tuples of leading monomials is identical to the one used in [15].

Definition 7.8. Let $F \subset \mathcal{P}$ be a finite set of polynomials with $\mathrm{lm}(F) = \{t_1, \dots, t_\ell\}$ such that $t_1 >_{revlex} \cdots >_{revlex} t_\ell$ where $>_{revlex}$ refers to the pure reverse lexicographic ordering with $x_1 < \cdots < x_n$. Then we denote the ordered tuple of these leading monomials by $\mathcal{L}(F) = (t_1, \dots, t_\ell)$. If $F, \tilde{F} \subset \mathcal{P}$ are two finite sets of polynomials with $\mathcal{L}(F) = (t_1, \dots, t_\ell)$ and $\mathcal{L}(\tilde{F}) = (\tilde{t}_1, \dots, \tilde{t}_{\tilde{\ell}})$, then we define an ordering on the corresponding tuples of monomials by

$$\mathcal{L}(F) <_{\mathcal{L}} \mathcal{L}(\tilde{F}) \iff \begin{cases} \exists\, j \le \min(\ell, \tilde{\ell})\ \forall\, i < j : t_i = \tilde{t}_i \wedge t_j <_{revlex} \tilde{t}_j & \text{or} \\ \forall\, j \le \min(\ell, \tilde{\ell}) : t_j = \tilde{t}_j \wedge \ell < \tilde{\ell}\,. \end{cases}$$

The definition of quasi-stability relative to a monomial ideal leads immediately to a simple test realised in Algorithm 6. As we are not concerned with efficiency questions here, the test returns in the negative case simply the first obstruction detected.

Algorithm 6: Relative Quasi-Stable Test

Data: A monomial ideal $\mathcal{I}$ and the minimal generating B of the monomial ideal $\mathcal{J}$.
Result: True if $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$ and false otherwise.
begin
  for $x^\mu \in B$ **do**
    for *i from* $\mathrm{cls}(x^\mu) + 1$ *to* n **do**
      if *for each s we have* $x_i^s x^\mu \notin \mathcal{I}$ *and* $x_i^s x^\mu / x_{\mathrm{cls}(x^\mu)} \notin \mathcal{J}$ **then**
        return $(false, x_i, x_{\mathrm{cls}(x^\mu)})$
  return $true$

Based on this test, it is straightforward to design a relative version of the algorithm in [15]. Algorithm 7 is based on the repeated determination of reduced Gröbner bases for our chosen ordering $\prec$ realised in the function `ReducedGrobnerBasis`(F). A key point is the inner **while** loop ensuring that in each iteration of the outer loop some progress is made – see the discussion in [15].

Theorem 7.9. *Algorithm 7 is correct and terminates in finitely many steps for a sufficiently large field K.*

Proof. The main issue with this algorithm is its termination. Indeed, it is easy to see that upon termination the output satisfies the specification. Let $\mathcal{J}$ be an ideal which is not in quasi-stable position relative to $\mathcal{I}$, i. e. there exists a monomial $x^\mu \in \mathcal{J}$ with $x_i^s x^\mu / x_{\mathrm{cls}(x^\mu)} \notin \mathcal{J}$ for some index $i > \mathrm{cls}(x^\mu)$ and for all exponents s and Algorithm 6 will return x_i and $x_{\mathrm{cls}(x^\mu)}$. If we perform

[2]It should be noted that, by Lemma 7.3, it follows that this change may be sparser than the change that we need to transform $\mathcal{J}$ into quasi-stable position.

Algorithm 7: Relative Quasi-Stable Position

Data: A homogeneous Gröbner basis G of $\mathcal{I} := \langle G \rangle \trianglelefteq \mathcal{P}$, finite set of homogeneous polynomials $F \subset \mathcal{P}$ with $\langle F \rangle + \mathcal{I} = \mathcal{J}$ and the monomial ordering $\prec$.
Result: A linear change Φ such that $\Phi(\mathcal{J})$ is in quasi-stable position relative to $\Phi(\mathcal{I})$.
begin
  $\Phi \longleftarrow$ The identity linear change
  $K \longleftarrow$ `ReducedGrobnerBasis`(G)
  $H \longleftarrow$ `ReducedGrobnerBasis`(F)
  $A \longleftarrow$ `RelativeQuasiStableTest`$(\langle \mathrm{lm}(K) \rangle, \mathrm{lm}(H))$
  while $A \neq \mathrm{true}$ **do**
    $\phi \longleftarrow (A[3] \mapsto A[3] + A[2])$; $\Phi \longleftarrow \phi \circ \Phi$
    $\tilde{K} \longleftarrow$ `ReducedGrobnerBasis`$(\phi(K))$
    $\tilde{H} \longleftarrow$ `ReducedGrobnerBasis`$(\phi(H))$
    while $\mathcal{L}(H) \succeq_{\mathcal{L}} \mathcal{L}(\tilde{H})$ **do**
      $\phi \longleftarrow (A[3] \mapsto A[3] + A[2])$; $\Phi \longleftarrow \phi \circ \Phi$
      $\tilde{K} \longleftarrow$ `ReducedGrobnerBasis`$(\phi(\tilde{K}))$
      $\tilde{H} \longleftarrow$ `ReducedGrobnerBasis`$(\phi(\tilde{H}))$
    $G \longleftarrow \tilde{K}$
    $H \longleftarrow \tilde{H}$
    $A \longleftarrow$ `RelativeQuasiStableTest`$(\langle \mathrm{lm}(K) \rangle, \mathrm{lm}(H))$
  return Φ

now a linear change of coordinates ϕ mapping $x_{\mathrm{cls}(x^\mu)} \mapsto x_{\mathrm{cls}(x^\mu)} + a x_i$ with a positive integer a and keeping all other variables unchanged, then, by [15, Prop. 6.9], $\mathcal{L}(H) \prec_{\mathcal{L}} \mathcal{L}(\tilde{H})$ where H is a Gröbner basis of $\mathcal{J}$ and $\tilde{H}$ is a Gröbner basis of $\phi(\mathcal{J})$. Finally, [15, Thm. 6.11] guarantees the termination of the algorithm in any characteristic for a sufficiently large field $\mathbb{K}$. □

Example 7.10. For a better understanding of Algorithm 7, we illustrate its steps with a concrete example. Let $\mathcal{P} = \mathbb{K}[x_1, x_2, x_3]$ and consider $\mathcal{I} = \langle x_1x_2 + x_2^2 \rangle$ and $\mathcal{J} = \langle x_1x_3, x_1x_2 + x_2^2 \rangle$. One sees that $\mathcal{J}$ is not in quasi-stable position relative to $\mathcal{I}$. Set $G = \{x_1x_2 + x_2^2\}$ and $H = \{x_1x_3, x_1x_2 + x_2^2\}$. Since $x_2^2 x_1 x_3 \in \langle \mathrm{lm}(G) \rangle$ and $x_3^s x_3 \notin \langle \mathrm{lm}(H) \rangle$ for any s, the algorithm `RelativeQuasiStableTest` returns $(false, x_3, x_1)$. Now, by performing the linear change $\phi := x_1 \mapsto x_1 + x_3$ on $\mathcal{I}$ and $\mathcal{J}$, we get $\tilde{G} = \{x_1x_2 + x_2^2 + x_2x_3\}$ and $\tilde{H} = \{x_1x_2 + x_2^2 + x_2x_3, x_1x_3 + x_3^2, x_1x_2^2 + x_2^3\}$. Therefore, we have $\mathcal{L}(H) = (x_1x_3, x_2^2) \prec_{\mathcal{L}} \mathcal{L}(\tilde{H}) = (x_3^2, x_3x_2, x_2^3)$. It can be seen that $\phi(\mathcal{J})$ is in quasi-stable position relative to $\phi(\mathcal{I})$ and the algorithm terminates.

As mentioned above, an alternative way to obtain quasi-stable position consists of comparing the Janet and the Pommaret multiplicative variables. We present a relative version of this approach. It is based on the following result (see [25, Prop. 4.3.6, Thm. 4.3.12] for more information).

Lemma 7.11. *Let $\mathcal{J} \trianglelefteq \mathcal{P}$ be a monomial ideal and B a Janet basis for $\mathcal{J}$ which is involutively autoreduced with respect to the Pommaret division. Then, $\mathcal{J}$ is quasi-stable, if and only if for each monomial $x^\mu \in B$ the sets of Janet respectively Pommaret multiplicative variables coincide.*

In the next lemma, we give a variant of this lemma in relative setting.

Lemma 7.12. *Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{P}$ be two monomial ideals and $B \subset \mathcal{J} \setminus \mathcal{I}$ a set of monomials Pommaret autoreduced relative to $\mathcal{I}$ such that $\langle B \rangle + \mathcal{I} = \mathcal{J}$. Then, the following statements hold:*

(1) *For any monomial $x^\mu \in B$, any Pommaret multiplicative variable relative to $\mathcal{I}$ is also Janet multiplicative relative to $\mathcal{I}$.*
(2) *If for all monomials in B the sets of Janet and Pommaret multiplicative variables relative to $\mathcal{I}$ coincide, then $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$.*
(3) *Let $\mathcal{J}$ be quasi-stable relative to $\mathcal{I}$ and B a Janet basis for $\mathcal{J}$ relative to $\mathcal{I}$. Assume that for the monomial $x^\mu \in B$ the variable x_i is Janet multiplicative relative to $\mathcal{I}$ and $x_i^s x^\mu \notin \mathcal{I}$ for any exponent s. Then, x_i is also Pommaret multiplicative relative to $\mathcal{I}$ for x^μ.*

Proof. (1) Assume that x_i is Pommaret multiplicative relative to $\mathcal{I}$ for x^μ. Then two cases may arise. If $x_i x^\mu \in \mathcal{I}$, then, by definition, it is Janet multiplicative relative to $\mathcal{I}$ as well and we are done. Otherwise, x_i is Pommaret multiplicative for x^μ. It is easy to see that B is Pommaret autoreduced. Then, by [12, Prop. 3.10], it follows that x_i is Janet multiplicative with respect to B and this proves the claim.

(2) Suppose that $\mathcal{J}$ is not quasi-stable relative to $\mathcal{I}$. Then there exists a monomial $x^\mu \in \mathcal{J} \setminus \mathcal{I}$ and an index $\operatorname{cls}(x^\mu) < i \leq n$ such that for each $s \geq 0$ we have $x_i^s x^\mu \notin \mathcal{I}$ and $x_i^s x^\mu / x_{\operatorname{cls}(x^\mu)} \notin \mathcal{J}$. If x_i is Janet multiplicative for x^μ, then by assumption it is also Pommaret multiplicative for x^μ, contradicting $\operatorname{cls}(x^\mu) < i$. Otherwise, there exists a monomial $x^\nu \in B$ such that x_i is Janet multiplicative for x^ν, $\mu_{i+1} = \nu_{i+1}, \dots, \mu_n = \nu_n$ and $\mu_i < \nu_i$ where $\mu = (\mu_1, \dots, \mu_n)$ and $\nu = (\nu_1, \dots, \nu_n)$. If $\operatorname{cls}(x^\nu) < i$ then x_i is Pommaret non-multiplicative and in turn it is Janet non-multiplicative which leads to a contradiction. Otherwise, $\operatorname{cls}(x^\nu) = i$ and it follows that $x_i^s x^\mu / x_{\operatorname{cls}(x^\mu)} \in \mathcal{J}$ for some s which leads again to a contradiction,

(3) Arguing by reductio ad absurdum, suppose that x_i is not Pommaret multiplicative relative to $\mathcal{I}$ for x^μ. Since $x_i^s x^\mu \notin \mathcal{I}$ for each s then x_i is Janet but not Pommaret multiplicative for x^μ and it follows that $\operatorname{cls}(x^\mu) < i$. From assumption, there exists s such that $x_i^s x^\mu / x_{\operatorname{cls}(x^\mu)} \in \mathcal{J}$. On the other hand, B is a Janet basis for $\mathcal{J}$ relative to $\mathcal{I}$. Thus, there exists $x^\nu \in B$ such that x^ν divides $x_i^s x^\mu / x_{\operatorname{cls}(x^\mu)}$ using Janet division relative to $\mathcal{I}$. Since $x_i^s x^\mu \notin \mathcal{I}$ we conclude that $x_i^s x^\mu / x_{\operatorname{cls}(x^\mu)} \notin \mathcal{I}$ and in turn x^ν divides $x_i^s x^\mu / x_{\operatorname{cls}(x^\mu)}$ using the ordinary Janet division. It yields that $\mu_{i+1} = \nu_{i+1}, \dots, \mu_n = \nu_n$ and $\mu_i < \nu_i$ where $\mu = (\mu_1, \dots, \mu_n)$ and $\nu = (\nu_1, \dots, \nu_n)$. We obtain a contradiction with the fact that x_i is Janet multiplicative for x^μ, proving the claim. □

Remark 7.13. The converse of the second item in Lemma 7.12 does not hold in general. For example, in the ring $\mathcal{P} = \mathbb{K}[x_1, x_2, x_3]$, let $\mathcal{I} = \langle x_2^3, x_3^3 \rangle$ and $\mathcal{J} = \langle x_1 x_3, x_2^3, x_3^3 \rangle$. One easily sees that $\mathcal{J}$ is quasi-stable relative to $\mathcal{I}$ and that for $x_1 x_3$ – the only generator of $\mathcal{J}$ not in $\mathcal{I}$ – all variables are Janet multiplicative relative to $\mathcal{I}$, whereas only x_1 is also Pommaret multiplicative relative to $\mathcal{I}$.

Algorithm 8 uses this lemma to compare the Pommaret and the Janet multiplicative variables relative to $\mathcal{I}$ for a set which is a Pommaret autoreduced Janet basis relative to $\mathcal{I}$.

Algorithm 8: Relative Janet-Pommaret Test

Data: A monomial ideal $\mathcal{I}$ and a monomial set B which is a Janet basis for the monomial ideal $\mathcal{J}$ relative to the monomial ideal $\mathcal{I}$ and Pommaret autoreduced relative to $\mathcal{I}$.
Result: True if for each $x^\mu \in B$ and each x_i, we have either $x_i^s x^\mu \in \mathcal{I}$ for some s or x_i is Janet and Pommaret multiplicative relative to $\mathcal{I}$ and false otherwise.

```
begin
    for x^μ ∈ B do
        for i from cls(x^μ) + 1 to n do
            if for each s we have x_i^s x^μ ∉ I and x_i is Janet multiplicative for x^μ then
                return (false, x_i, x_cls(x^μ))
    return true
```

Algorithm 9 follows a similar strategy as Algorithm 7: with the help of Algorithm 8 it constructs deterministically a linear change of coordinates such that $\mathcal{J}$ is in quasi-stable position relative to $\mathcal{I}$. However, instead of reduced Gröbner bases is uses Janet bases relative to $\mathcal{I}$. More precisely, the function `RelativeJanetBasis`(G, F) computes a Janet basis for $\langle F \rangle$ relative to $\langle G \rangle$ which is Pommaret autoreduced relative to $\langle G \rangle$. While classically a Pommaret autoreduced Janet basis of an ideal in quasi-stable position is automatically also a Pommaret basis, the situation is slightly more complicated in the relative case and we need the following additional construction.

Definition 7.14. Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{P}$ be two monomial ideals and $B \subset \mathcal{J} \setminus \mathcal{I}$ a set of monomials with $\langle B \rangle + \mathcal{I} = \mathcal{J}$. Then *Pommaret completion* of B relative to $\mathcal{I}$, denoted by $\mathrm{PommComp}(\mathcal{I}, \mathrm{B})$, is the set of all monomials $x_{i_1}^{j_1} \cdots x_{i_k}^{j_k} x^\mu \notin \mathcal{I}$ such that $x^\mu \in B$ and for each ℓ we have $i_\ell > \mathrm{cls}(x^\mu)$ and $x_{i_\ell}^{s_\ell} x^\mu \in \mathcal{I}$ for some s_ℓ.

Corollary 7.15. *Let $\mathcal{I} \subset \mathcal{J} \trianglelefteq \mathcal{P}$ be two monomial ideals and B a Janet basis for $\mathcal{J}$ relative to $\mathcal{I}$ which is Pommaret autoreduced relative to $\mathcal{I}$. Assume that for each monomial $x^\mu \in B$ and each variable x_i we have either $x_i^s x^\mu \in \mathcal{I}$ for some exponent s or x_i is Janet multiplicative relative to $\mathcal{I}$, if and only if it is also Pommaret multiplicative relative to $\mathcal{I}$. Then $B \cup \mathrm{PommComp}(\mathcal{I}, \mathrm{B})$ is a finite weak Pommaret basis for $\mathcal{J}$ relative to $\mathcal{I}$.*

Algorithm 9: Relative Pommaret Basis

Data: A homogeneous Gröbner basis G of $\mathcal{I} := \langle G \rangle \trianglelefteq \mathcal{P}$, finite set of homogeneous polynomials $F \subset \mathcal{P}$ with $F \cap \mathcal{I} = \varnothing$, $\mathrm{NF}_G(F) = F$ and $\langle F \rangle + \mathcal{I} = \mathcal{J}$ and the monomial ordering $<$.

Result: A linear change Φ such that $\Phi(\mathcal{J})$ has a finite Pommaret basis relative to $\Phi(\mathcal{I})$ and such a basis.

begin

- $\Phi \longleftarrow$ The identity linear change
- $K \longleftarrow G$
- $H \longleftarrow$ `RelativeJanetBasis`(K, F)
- $A \longleftarrow$ `RelativeJanetPommaretTest`$(\langle \mathrm{lm}(K) \rangle, \mathrm{lm}(H))$
- **while** $A \neq$ true **do**
  - $\phi \longleftarrow (A[3] \mapsto A[3] + A[2])$; $\Phi \longleftarrow \phi \circ \Phi$
  - $\tilde{K} \longleftarrow$ `ReducedGrobnerBasis`$(\phi(K))$
  - $\tilde{H} \longleftarrow$ `RelativeJanetBasis`$(\tilde{K}, \phi(H))$
  - $\tilde{A} \longleftarrow$ `RelativeJanetPommaretTest`$(\langle \mathrm{lm}(\tilde{K}) \rangle, \mathrm{lm}(\tilde{H}))$
  - **while** $A \neq \tilde{A}$ **do**
    - $\phi \longleftarrow (A[3] \mapsto A[3] + A[2])$; $\Phi \longleftarrow \phi \circ \Phi$
    - $\tilde{K} \longleftarrow$ `ReducedGrobnerBasis`$(\phi(\tilde{K}))$
    - $\tilde{H} \longleftarrow$ `RelativeJanetBasis`$(\tilde{K}, \phi(\tilde{H}))$
    - $\tilde{A} \longleftarrow$ `RelativeJanetPommaretTest`$(\langle \mathrm{lm}(\tilde{K}) \rangle, \mathrm{lm}(\tilde{H}))$
  - $G \longleftarrow \tilde{K}$
  - $H \longleftarrow \tilde{H}$
  - $A \longleftarrow \tilde{A}$
- **return** $(\Phi, H \cup \mathrm{PommComp}(\mathcal{I}, \mathrm{B})(\langle K \rangle, H))$

Theorem 7.16. *Algorithm 9 is correct and terminates in finitely many steps.*

Proof. Assume that we are given a finite generating set F of $\mathcal{J}$ (that is $\langle F\rangle + \mathcal{I} = \mathcal{J}$) and a Gröbner basis G of ideal $\mathcal{I}$. If the algorithm `RelativeJanetPommaretTest` finds an obstruction $(x_i, x_{\mathrm{cls}(x^\mu)})$ for a monomial $x^\mu \in \mathrm{lm}(F)$, then we claim that it remains also an obstruction for some monomial in $\mathrm{lm}(F) \cup \mathrm{lm}(G)$. We know that $x_i^s x^\mu \notin \mathrm{lm}(\mathcal{I})$ for each s and x_i is Janet but not Pommaret multiplicative for $x^\mu \in \mathrm{lm}(F)$. Suppose that x_i is not Janet multiplicative for $x^\mu \in \mathrm{lm}(F) \cup \mathrm{lm}(G)$, but Janet multiplicative for some $x^\nu \in \mathrm{lm}(G)$. Two case may occur. If $\mathrm{cls}(x^\nu) < \mathrm{cls}(x^\mu)$, then x_i is not Pommaret multiplicative for x^ν and therefore $(x_i, x_{\mathrm{cls}(x^\mu)})$ remains an obstruction for $x^\nu \in \mathrm{lm}(F) \cup \mathrm{lm}(G)$. Otherwise, we must have $\mathrm{cls}(x^\nu) = i$ and it follows that $x_i^s x^\mu \in \mathrm{lm}(\mathcal{I})$ for some s and this leads to a contradiction, proving the claim. Thus, based on Corollary 7.15 and the correctness and termination of the algorithm similar to Algorithm 9 in the classical setting (see [26, Theorem 4.3.12]), the correctness and termination of Algorithm 9 is guaranteed. Finally, we note that, $\mathrm{PommComp}(\mathcal{I}, \mathrm{B})(\langle K\rangle, H)$ is a finite set. □

Example 7.17. To illustrate the steps of Algorithm 9, let us consider again the ideals given in Example 7.10. We know that $G = \{x_1x_2 + x_2^2\}$ is the reduced Gröbner basis for $\mathcal{I}$ and $H = \{x_1x_3\}$ is the Janet basis for $\mathcal{J}$ relative to $\mathcal{I}$ which is Pommaret autoreduced relative to $\mathcal{I}$. Since $x_2^2x_1x_3 \in \langle \mathrm{lm}(G)\rangle$, Algorithm `RelativeJanetPommaretTest` returns $(false, x_3, x_1)$. By performing the linear change $\phi := x_1 \mapsto x_1 + x_3$ on $\mathcal{I}$ and $\mathcal{J}$, we get $\tilde{G} = \{x_1x_2 + x_2^2 + x_2x_3\}$ and $\tilde{H} = \{x_1x_3 + x_3^2,\ x_1x_2^2 + x_2^3\}$. One sees that $\{x_1, x_2, x_3\}$ is the set of the Janet multiplicative variables for x_3^2 and x_2^3 relative to $\langle x_2x_3\rangle$. Since $x_3x_2^3 \in \mathrm{lm}(\phi(\mathcal{I}))$, Algorithm `RelativeJanetPommaretTest` returns true and in turn $\tilde{H}$ is the weak Pommaret basis for $\phi(\mathcal{J})$ relative to $\phi(\mathcal{I})$.

Example 7.18. Consider in the polynomial ring $\mathcal{P} = \mathbb{K}[x_1, x_2, x_3]$ the monomial ideals $\mathcal{I} = \langle x_2^3,\ x_3^3\rangle$ and $\mathcal{J} = \langle x_1x_3,\ x_2^3,\ x_3^3\rangle$. Since Algorithm `RelativeJanetPommaretTest` returns true, the set $\{x_1x_3,\ x_1x_2x_3,\ x_1x_2^2x_3,\ x_1x_3^2,\ x_1x_2x_3^2,\ x_1x_2^2x_3^2\}$ is a weak Pommaret basis of $\mathcal{J}$ relative to $\mathcal{I}$.

8. Conclusion

In this paper, we introduced the notions of relative Gröbner bases as well as relative involutive bases. In addition, we established a relative Schreyer Theorem. We developed the required concepts and tools to present algorithms for the constructions of these bases. In particular, we introduced the notion of relative quasi-stable position and applied it to describe an algorithm to compute finite relative Pommaret bases. In future works, we will investigate the applications of the concepts introduced in this paper. Specially, since involutive bases provide effective tools to compute many homological invariants of an ideal like the Castelnuovo-Mumford regularity, a natural question consists of designing similar tools in the relative case.

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