Computing the Resolution Regularity of Bi-Homogeneous Ideals

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Abstract

We present an effective method to compute the resolution regularity (vector) of bi-homogeneous ideals. For this purpose, we first introduce the new notion of an *x-Pommaret basis* and describe an algorithm to compute a linear change of coordinates for a given bi-homogeneous ideal such that the new ideal obtained after performing this change possesses a finite x-Pommaret basis. Then, we show that the x-component of the bi-graded regularity of a bi-homogeneous ideal is equal to the x-degree of its x-Pommaret basis (after performing the mentioned linear change of variables). Finally, we introduce the new notion of an ideal in *x-quasi stable position* and show that a bi-homogeneous ideal has a finite x-Pommaret basis iff it is in x-quasi stable position.

Keywords: Polynomial ideals, Gröbner bases, Involutive divisions, Janet bases, Pommaret bases, Quasi stable position, Castelnuovo-Mumford regularity, Resolution regularity.

1. Introduction

The Castelnuovo-Mumford regularity of a homogeneous ideal or a graded module is a fundamental invariant in commutative algebra and algebraic geometry which, roughly speaking, measures the computational complexity of the ideal or module. Since its introduction by Mumford (Mumford, 2005), it has been studied by many authors including Goto and Eisenbud (Goto and Eisenbud, 1984), Bayer and Stillman (Bayer and Stillman, 1987) and Bayer and Mumford (Bayer and Mumford, 1993). Two different (and inequivalent) approaches have been proposed for an extension to multi-graded ideals and modules. Maclagan and Smith (Maclagan and Smith, 2004) defined a *multi-graded Castelnuovo-Mumford regularity* via the vanishing of multi-graded

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pieces of local cohomology modules (see (Hoffman and Wang, 2004) for the special case of bi-graded modules). We will use in this work the regularity defined for a bi-homogeneous ideal by Aramova et al. (Aramova et al., 2000) via the bi-degrees in a bi-graded minimal free resolution of the ideal. Following Hà (Hà, 2007), we will speak of the *resolution regularity (vector)*. Hà (Hà, 2007) also investigated the relationship between these two approaches.

Aramova et al. (Aramova et al., 2000) also defined the concept of a *bi-generic initial ideal* for a bi-homogeneous ideal and introduced *bi-stable* monomial ideals. They showed that the bi-generic initial ideal of a bi-homogeneous ideal is always bi-stable and that the resolution regularity of a bi-stable ideal can be immediately read off from the minimal generators of its bi-generic initial ideal. This represents natural extensions of analogous results in the simply graded case.

For simplicity, we will consider here the resolution regularity only for bi-homogeneous ideals, as the generalisation of our results to multi-graded modules is straightforward. Thus we assume that $\mathcal{P} = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ is a polynomial ring in $m + n$ variables over an infinite field $\mathbb{K}$ and we consider the grading on $\mathcal{P}$ given by $\deg(x_i) = (1, 0)$ and $\deg(y_j) = (0, 1)$ for all i and j. An ideal is called *bi-homogeneous*, if it can be generated by homogeneous polynomials with respect to this grading. For a given bi-homogeneous ideal, the resolution regularity is the integer pair (p, q) such that the i-th syzygy module in a minimal bi-graded free resolution is generated by elements whose x-degree (resp. y-degree) is at most $p + i$ (resp. $q + i$) for all i.

One of our main techniques will be *involutive bases*. As a special form of (non-reduced) Gröbner bases, involutive bases have interesting combinatorial properties and are a valuable tool in a wide range of applications in commutative algebra and algebraic geometry. They combine ideas of the Janet-Riquier theory of partial differential equations with the theory of Gröbner bases. Gerdt and Blinkov (Gerdt and Blinkov, 1998) developed the general theory of involutive bases; Zharkov and Blinkov (Zharkov and Blinkov, 1996) introduced already earlier the special case of Pommaret bases (see also (Janet, 1924, pages 30–31)). It is well-known that Pommaret bases do not always exist, but the existence of finite Pommaret basis for a given ideal can always be achieved by means of a suitable linear change of variables. In a constructive fashion, one can say that an ideal has a finite Pommaret basis, if and only if it is in quasi stable position. For a general survey on involutive bases with special emphasis on Pommaret bases see e. g. (Seiler, 2009a,b, 2010).

In this paper, we will consider the problem of computing the resolution regularity of a bi-homogeneous ideal. We first introduce the new notion of an *x-Pommaret basis* for a given bi-homogeneous ideal and propose an algorithm to compute a linear change of the x-variables such that the ideal obtained after performing this change possesses a finite x-Pommaret basis. In particular, we will show that the first component of the resolution regularity of a given bi-homogeneous ideal is equal to the x-degree of the x-Pommaret basis of the ideal (after performing the computed linear change of variables). Note that the second component of the resolution regularity of an ideal may be similarly obtained by computing its (analogously defined) y-Pommaret basis. In analogy to the theory of Pommaret bases, we introduce the new notions of bi-Pommaret bases and ideals in *x-quasi stable* and *bi-stable* positions and show that a bi-homogeneous ideal has a finite x-Pommaret basis, if and only if it is in x-quasi stable position. In addition, we show that a monomial ideal is bi-stable, if and only if its minimal basis is a bi-Pommaret basis. All the algorithms presented in this paper have been implemented in Maple and their application is illustrated by some worked-out examples

This work is organised as follows. In the next section, we will review the basic definitions and notations that we will use throughout. In Section 3, we will introduce the new notion of an x-

Pommaret division and show that it is an involutive division which is non-Noetherian, continuous and constructive. Section 4 is devoted to the description of an algorithm to compute a linear change of variables to transform a given bi-homogeneous ideal into an ideal possessing a finite x-Pommaret basis. In Section 5, after defining the notions of bi-Pommaret bases and ideals in bi-stable position, we will study the properties of the ideals in bi-stable position by establishing its connection to the existence of finite bi-Pommaret bases. In the last section, we will investigate the computation of the resolution regularity of a bi-homogeneous ideal.

2. Preliminaries

We assume throughout that $\mathbb{K}$ is an infinite field and $\mathcal{R} = \mathbb{K}[x_1, \dots, x_n]$ is the polynomial ring in n variables $x_1, \dots, x_n$. Let $\mathbb{M}$ be the monoid of all terms $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ with an exponent vector $\alpha \in \mathbb{N}^n$. We denote by $\deg_j(x^\alpha) = \alpha_j$ the x_j-degree of such a term. We will always use the degree reverse lexicographic term ordering with $x_n \prec \cdots \prec x_1$. For a non-zero polynomial $f \in \mathcal{R}$, the *leading term* $\mathrm{LT}(f)$ of f is the greatest term occurring in f. The *leading coefficient* of f is denoted by $\mathrm{LC}(f)$ and the *leading monomial* of f by $\mathrm{LM}(f) = \mathrm{LC}(f)\mathrm{LT}(f)$. The *leading ideal* of $\mathcal{I}$ is given by $\mathrm{LT}(\mathcal{I}) = \langle \mathrm{LT}(f) \mid 0 \neq f \in \mathcal{I} \rangle$. For a finite set $F = \{f_1, \dots, f_k\}$ of polynomials, $\mathrm{LT}(F) = \{\mathrm{LT}(f_1), \dots, \mathrm{LT}(f_k)\}$. A finite set $G = \{g_1, \dots, g_t\}$ of non-zero polynomials is called a *Gröbner basis* of $\mathcal{I}$, if $G \subset \mathcal{I}$ and $\mathrm{LT}(\mathcal{I}) = \langle \mathrm{LT}(g_1), \dots, \mathrm{LT}(g_t) \rangle$. We refer e. g. to (Cox et al., 2015) for more details on Gröbner bases.

We briefly recall some basic facts from the theory of involutive bases of polynomial ideals. We refer to (Gerdt and Blinkov, 1998; Gerdt, 2005; Seiler, 2010) for a general discussion of this theory.

Definition 1. *An* involutive division $\mathcal{L}$ *is defined on* $\mathbb{M}$ *by assigning to any finite set* $U \subset \mathbb{M}$ *and to any term* $u \in U$ *a set of* multiplicative variables $M_{\mathcal{L}}(u, U) \subseteq \{x_1, \dots, x_n\}$ *such that the following conditions are satisfied:*

- *if there exist* $u, v \in U$ *such that* $uL(u, U) \cap vL(v, U) \neq \emptyset$*, then either* $v \in uL(u, U)$ *or* $u \in vL(v, U)$*,*
- *if* $v \in U$ *and* $v \in uL(u, U)$*, then* $L(v, U) \subseteq L(u, U)$*,*
- *if* $V \subset U$ *then* $L(u, U) \subseteq L(u, V)$ *for all* $u \in V$*,*

where $L(u, U)$ *denotes the set of all terms containing only the variables in* $M_{\mathcal{L}}(u, U)$*. The set* $\{x_1, \dots, x_n\} \setminus M_{\mathcal{L}}(u, U)$*, denoted by* $NM_{\mathcal{L}}(u, U)$*, is called the set of* non-multiplicative variables. *If* $v \in uL(u, U)$*, then* u *is called an* $\mathcal{L}$-involutive divisor *of* v *and this property is denoted by* $u \mid_{\mathcal{L}} v$*.*

For most purposes, the following two involutive divisions are the most important ones and they are the only ones used in this work. Note an important difference in their definitions: the Pommaret division is global meaning that the assignment of multiplicative variables to a term u is independent of the choice of a "container" set $U \subset \mathbb{M}$ containing u.

Definition 2. *Let* $U \subset \mathbb{M}$ *be a finite set of terms. For each sequence* $d_1, \dots, d_i$ *of positive integers of length* $i < n$*, we introduce the subset*

$$[d_1, \dots, d_i] = \{v \in U \mid \deg_j(v) = d_j,\ 1 \leq j \leq i\} \subseteq U\,.$$

The Janet division J *assigns multiplicative variables to each term* $u \in U$ *as follows: the variable* x_1 *is Janet multiplicative for* u*, if* u *is of maximal* x_1*-degree in the full set* U*. For* $i > 1$*, the variable* x_i *is Janet multiplicative for* u*, if* u *is of maximal* x_i*-degree in the subset* $[d_1, \dots, d_{i-1}]$.

Definition 3. *The* class *of a term* $u = x^\alpha \in \mathbb{M}$*, denoted by* $\operatorname{cls}(x^\alpha)$*, is the greatest integer* $1 \leq i \leq n$ *such that* $\alpha_i > 0$*. The* Pommaret division*, denoted by* P*, assigns to* u *the multiplicative variables* $x_i, \ldots, x_n$ *where* $i = \operatorname{cls}(u)$*. By convention, if* $u = 1$ *then we set* $\operatorname{cls}(u) = 1$*.*

We can now define the notion of an involutive basis first for monomial ideals and then extend it to polynomial ideals using the natural reduction relation induced by an involutive division.

Definition 4. *Let* $U \subset \mathbb{M}$ *be a finite set and* $\mathcal{L}$ *an involutive division. Let* $u \in U$ *be an arbitrary element. The* involutive cone $\mathcal{C}_\mathcal{L}(u, U)$ *of* u *for* $\mathcal{L}$ *is*

$$\mathcal{C}_\mathcal{L}(u, U) = \{uv \mid v \in L(u, U)\} \subseteq u\mathbb{M} .$$

The involutive span *of the set* U *is the union* $\mathcal{C}_\mathcal{L}(U) = \cup_{u\in U}\mathcal{C}_\mathcal{L}(u, U)$ *of all involutive cones. Obviously, it is generally only a subset of the monoid* $\mathcal{C}(U) = \cup_{u\in U} u\mathbb{M}$ *generated by* U *in* $\mathbb{M}$*. The set* U *is an* involutive basis *of the monomial ideal generated by it, if* $\mathcal{C}_\mathcal{L}(U) = \mathcal{C}(U)$ *and all involutive cones are disjoint (in other words, if for every term* $v \in \mathcal{C}(U)$ *there exists exactly one generator* $u \in U$ *such that* $v \in \mathcal{C}_\mathcal{L}(u, U)$*).*

Definition 5. *Let* $F \subset \mathcal{R}$ *be a finite set of polynomials and* $\mathcal{L}$ *an involutive division.* F *is called* $\mathcal{L}$-autoreduced*, if no polynomial* $h \in F$ *contains a term* x^α *such that* $\operatorname{LT}(f) \mid_\mathcal{L} x^\alpha$ *for some* $f \in F \setminus \{h\}$*. An* $\mathcal{L}$*-autoreduced set* H *is an* involutive basis *of the ideal* $\langle H\rangle \subset \mathcal{R}$ *for* $\mathcal{L}$*, if the leading terms* $\operatorname{LT}(H)$ *form an involutive basis of the leading ideal* $\operatorname{LT}(\langle H\rangle)$ *for* $\mathcal{L}$*.*

It follows immediately from this definition that any involutive basis of an ideal $\mathcal{I} \subset \mathcal{R}$ is trivially also a (generally non-reduced) Gröbner basis. However, an involutive basis always induces a combinatorial decomposition of the form

$$\mathcal{I} = \bigoplus_{h\in H} \mathbb{K}[M_{\mathcal{L},\operatorname{LT}(H)}(\operatorname{LT}(h))] \cdot h \tag{1}$$

which in turn implies that any polynomial $f \in \mathcal{I}$ has a unique *involutive standard representation* whereas in the theory of Gröbner bases standard representations are very rarely unique.

It is rather straightforward to prove that every ideal $\mathcal{I} \subset \mathcal{R}$ possesses a Janet basis (see e.g., (Gerdt and Blinkov, 1998)). By contrast, one can easily produce examples of (even monomial) ideals for which no finite Pommaret basis exists. More precisely, the ideal $\mathcal{I}$ has a Pommaret basis, if and only if $\operatorname{LT}(\mathcal{I})$ is quasi stable in the sense defined below – see (Seiler, 2009b, Proposition 4.4) for more details.

Definition 6. *A monomial ideal* $\mathcal{J} \subset \mathcal{R}$ *is* quasi stable*, if for any term* $m \in \mathcal{J}$ *and any integers* i, j, s *with* $1 \leq j < i \leq n$ *and* $s > 0$ *such that* $x_i^s \mid m$ *there exists an integer* $t \geq 0$ *such that* $x_j^t m/x_i^s \in \mathcal{J}$*. A polynomial ideal* $\mathcal{I} \subset \mathcal{R}$ *is in* quasi stable position*, if* $\operatorname{LT}(\mathcal{I})$ *is quasi stable.*

Example 7. *The ideal* $\mathcal{J} = \langle x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^3\rangle$ *is quasi stable.*

In contrast to Gröbner bases, Pommaret bases reflect many important properties of the ideal they generate, in particular homological properties related to the minimal free resolution of the ideal (see (Seiler, 2010, Chapter 5)). For our purposes, the most relevant result is that the Castelnuovo-Mumford regularity can easily be read off from a Pommaret basis.

Definition 8. *A homogeneous ideal $\mathcal{I} \subset \mathcal{R}$ is m*-regular, *if its minimal graded free resolution*

$$0 \longrightarrow \bigoplus_j \mathcal{R}(-e_{rj}) \longrightarrow \cdots \longrightarrow \bigoplus_j \mathcal{R}(-e_{1j}) \longrightarrow \bigoplus_j \mathcal{R}(-e_{0j}) \longrightarrow \mathcal{I} \longrightarrow 0$$

satisfies $e_{ij} - i \leq m$ for all i, j. The Castelnuovo-Mumford regularity *of $\mathcal{I}$, denoted by* $\operatorname{reg}(\mathcal{I})$, *is the smallest m such that $\mathcal{I}$ is m-regular.*

It is worth noting that $\operatorname{reg}(\mathcal{I})$ is equal to the maximum degree of the elements in the reduced Gröbner basis of $\mathcal{I}$ with respect to the reverse lexicographic ordering when working with generic coordinates, see (Bayer and Stillman, 1987). For more details on this invariant, we refer to (Mumford, 2005; Goto and Eisenbud, 1984). If an ideal $\mathcal{I}$ possesses a Pommaret basis H, then $\operatorname{reg}(\mathcal{I})$ equals the maximal degree of an element of H, (Seiler, 2009b, Theorem 9.2). The main goal of this work is to derive a similar characterisation of the resolution regularity of a bi-homogeneous ideal the definition of which we now recall following (Aramova et al., 2000; Hà, 2007).

Let $\mathcal{P} = \mathbb{K}[x_1, \ldots, x_n, y_1, \ldots, y_m]$ be a polynomial ring in $n + m$ variables. We denote again by $\mathbb{M}$ the set of all terms in $\mathcal{P}$. In the sequel, we write a term $x_1^{\alpha_1} \ldots x_n^{\alpha_n} y_1^{\beta_1} \ldots y_m^{\beta_m}$ briefly as $x^\alpha y^\beta$ where $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{N}^m$. We consider the grading defined by $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. A given polynomial $f \in \mathcal{P}$ is called *bi-homogeneous* of bi-degree (a, b) if for each term $x^\alpha y^\beta$ appearing in f we have $\deg(x^\alpha) = a$ and $\deg(y^\beta) = b$. An ideal $\mathcal{I}$ is bi-homogeneous, if it can be generated by bi-homogeneous polynomials.

Definition 9. *The* resolution regularity *of a bi-homogeneous ideal $\mathcal{I} \subset \mathcal{P}$ is the pair $(p, q) \in \mathbb{N}^2$, if its minimal bi-graded free resolution*

$$0 \longrightarrow \bigoplus_j \mathcal{P}(-r_{tj}, -s_{tj}) \longrightarrow \cdots \longrightarrow \bigoplus_j \mathcal{P}(-r_{1j}, -s_{1j}) \longrightarrow \bigoplus_j \mathcal{P}(-r_{0j}, -s_{0j}) \longrightarrow \mathcal{I} \longrightarrow 0$$

satisfies $r_{ij} - i \leq p$, $s_{ij} - i \leq q$ for all i, j and both p and q are minimal with this property.

3. The x-Pommaret Division and its Properties

We now introduce a new involutive division, called x-Pommaret division, and show that it is algorithmically a good division. If $u = x^\alpha y^\beta$ is a term in $\mathbb{M}$, then x^α is called the *x-part* of u and is denoted by u_x. If U is a finite set of terms, we write $U_x = \{u_x \mid u \in U\}$. Analogous notations are used for the y-part.

Definition 10. *Let $U \subset \mathbb{M}$ be a finite set. The* x-Pommaret division *P_x assigns multiplicative variables as follows: A given variable is x-Pommaret multiplicative for a term $u = x^\alpha y^\beta \in U$, if it is either Pommaret multiplicative for $x^\alpha \in \mathbb{K}[x_1, \ldots, x_n]$ or Janet multiplicative for $y^\beta \in U_y \subset \mathbb{K}[y_1, \ldots, y_m]$.*

Example 11. *Consider $U = \{x_1^2 y_1^2 y_2^2, x_1^2 x_2 y_1^2, x_1 y_2^3\} \subset \mathbb{K}[x_1, x_2, y_1, y_2]$. The following table illustrates the multiplicative variables of the elements of U for different involutive divisions.*

term	*Pommaret* ($u_x \in U_x$)	*Janet* ($u_y \in U_y$)	*x-Pommaret*
$x_1^2 y_1^2 y_2^2$	$\{x_1, x_2\}$	$\{y_1, y_2\}$	$\{x_1, x_2, y_1, y_2\}$
$x_1^2 x_2 y_1^2$	$\{x_2\}$	$\{y_1\}$	$\{x_2, y_1\}$
$x_1 y_2^3$	$\{x_1, x_2\}$	$\{y_2\}$	$\{x_1, x_2, y_2\}$

If u is a term and U is a finite set of terms then, in what follows, by $P_x(u, U)$, we mean the set of all terms containing only the multiplicative variables of $u \in U$ with respect to the x-Pommaret division. Similar notations are also defined for all kind of involutive divisions. For a particular division like the Pommaret division which is global, the container set U is omitted and the set of all multiplicative terms is denoted by $P(u)$.

Proposition 12. *The x-Pommaret division is an involutive division.*

Proof. We must prove that the three conditions contained in Definition 1 are satisfied. For the first condition, suppose that $C_{P_x}(u, U) \cap C_{P_x}(v, U) \neq \emptyset$ for two terms $u \neq v \in U$ and that w lies in this intersection. Let us assume that $u = x_1^{a_1} \cdots x_k^{a_k} y_1^{b_1} \cdots y_m^{b_m}$ and $v = x_1^{r_1} \cdots x_l^{r_l} y_1^{s_1} \cdots y_m^{s_m}$ with $a_k, r_l > 0$. If now $w = x_1^{\alpha_1} \cdots x_n^{\alpha_n} y_1^{\beta_1} \cdots y_m^{\beta_m}$, then there must exist two terms $s \in P_x(u, U)$ and $t \in P_x(v, U)$ such that we can write $w = s \cdot u$ and $w = t \cdot v$. Hence,

$$\begin{aligned} w_x &= s_x \cdot (x_1^{a_1} \cdots x_k^{a_k}) = t_x \cdot (x_1^{r_1} \cdots x_l^{r_l}) && s_x \in P(u_x),\ t_x \in P(v_x) \\ w_y &= s_y \cdot (y_1^{b_1} \cdots y_m^{b_m}) = t_y \cdot (y_1^{s_1} \cdots y_m^{s_m}) && s_y \in J(u_y, U_y),\ t_y \in J(v_y, U_y) \end{aligned}$$

First, we consider the y-parts and show that $u_y = v_y$. We claim first that $b_1 = s_1$. If the degree in y_1 of s_y is zero, then trivially $b_1 \geq s_1$. Otherwise, y_1 is Janet-multiplicative for u_y which requires by the definition of the Janet division that $b_1 \geq s_1$. Swapping the roles of u and v, we can show analogously that also $s_1 \geq b_1$ which entails the assertion. We proceed now to y_2. By the considerations above, we know that $u_y, v_y \in [b_1]$ which allows us to repeat the same argument for y_2. Continuing this way, we eventually find $u_y = v_y$.

Now, we discuss the x-parts. Without loss of generality, assume that $k \leq l$. By the definition of the Pommaret division, we then have $\alpha_i = a_i = r_i$ for all $i < k$. If $a_k \leq r_k$, then $v_x \in C_P(u_x)$. If $r_k < a_k$, then we must distinguish two cases. If $k = l$, then $u_x \in C_P(v_x)$. Otherwise, we have $r_k < a_k \leq \alpha_k$ and thus x_k is Pommaret-multiplicative for v_x which contradicts the assumption $k < l$. Hence, either $v \in C_{P_x}(u, U)$ or $u \in C_{P_x}(v, U)$ which shows that the first condition holds.

Concerning the second condition in Definition 1, assume that there are $u, v \in U$ with $v \in C_{P_x}(u, U)$. Then $v_x \in C_P(u_x)$ implying that $M_P(v_x) \subseteq M_P(u_x)$ and $v_y \in C_J(u_y, U_y)$ which is only possible if $v_y = u_y$. By the definition of x-Pommaret division, thus $M_{P_x}(v, U) \subseteq M_{P_x}(u, U)$.

To prove the third condition, take a subset $V \subseteq U$ and a term $u \in V$ in it. It follows from the definitions of the Janet and Pommaret division, resp., that $P(u_x, U_x) = P(u_x, V_x)$ and $J(u_y, U_y) \subseteq J(u_y, V_y)$ which together implies that $P_x(u, U) \subseteq P_x(u, V)$. □

The usual algorithms for computing an involutive basis require that the division is continuous and constructive. After briefly recalling the corresponding definitions, we show that the x-Pommaret division has both properties.

Definition 13. *Let $\mathcal{L}$ be an involutive division and $U \subset \mathbb{M}$ a finite set of terms. Let also $(u_1, \ldots, u_k)$ be a finite sequence of elements of U such that*

$$(\forall i < k)(\exists z_i \in NM_{\mathcal{L}}(u_i, U))[u_{i+1} \mid_{\mathcal{L},U} u_i \cdot z_i]. \tag{2}$$

Then $\mathcal{L}$ is continuous*, if for any such sequence we have $u_i \neq u_j$ for all $i \neq j$.*

Lemma 14. *Let $(v_1, \ldots, v_s)$ be a sequence of terms in $\mathbb{M}$. If $v_i \mid_P v_{i-1}$ for $2 \leq i \leq s-1$ and $v_s \mid_P v_{s-1} \cdot z$ where z is not Pommaret-multiplicative for v_{s-1}, then z is also not Pommaret-multiplicative for v_1 and we have $v_s \mid_P v_1 \cdot z$.*

Proof. According to our assumption $v_i \mid_P v_{i-1}$ for $2 \leq i \leq s-1$ and by the definition of the Pommaret division, we must have $v_{i-1} \in v_i P(v_i)$. So, by the second condition in Definition 1 of an involutive division, we have $M_P(v_{i-1}) \subseteq M_P(v_i)$ for $2 \leq i \leq s-1$. Thus we deduce that $M_P(v_1) \subseteq M_P(v_{s-1})$ which shows the first claim. From the inclusion $M_P(v_{i-1}) \subseteq M_P(v_i)$ for $2 \leq i \leq s-1$, it follows again that $v_{s-1} \mid_P v_1$. On the other hand, from the assumption $v_s \mid_P v_{s-1} \cdot z$ where z is not Pommaret-multiplicative for v_{s-1}, we conclude that $M_P(v_{s-1}) \subseteq M_P(v_s)$ and these arguments prove that $v_s \mid_P v_1 \cdot z$. □

Lemma 15. *Let $(v_1, \ldots, v_s)$ be a sequence of terms in $\mathbb{M}$. If $v_i \mid_P v_{i-1}$ for $3 \leq i \leq s$ and $v_2 \mid_P v_1 \cdot z$ where z is not Pommaret-multiplicative for v_1, then $v_s \mid_P v_1 \cdot z$.*

Proof. The properties of the Pommaret division imply that $v_s \mid_P v_2$ and $P(v_2) \subseteq P(v_s)$. Thus $v_2 = tv_s$ for a term $t \in P(v_s)$. As $v_2 \mid_P v_1 \cdot z$, we must have $v_1 \cdot z = uv_2$ for a term $u \in P(v_2)$. So $v_1 \cdot z = utv_s$ and $ut \in P(v_s)$. Thus $v_s \mid_P v_1 \cdot z$. □

Proposition 16. *The x-Pommaret division is continuous.*

Proof. Let $U \subset \mathbb{M}$ be a finite set of terms and $(u_1, \ldots, u_k)$ a finite sequence of elements in U satisfying condition (2) for P_x. For a reductio ad absurdum suppose that there are $i \neq j$ such that $u_i = u_j$. Without loss of generality, we may assume that $i = 1$ and $j = k$. We first exclude two trivial cases. If all non-multiplicative variables in condition (2) belong to the set $\{x_1, \ldots, x_n\}$, then the induced sequence $(u_{1x}, \ldots, u_{kx})$ of elements in U_x satisfies

$$(\forall i < k)(\exists z_i \in NM_P(u_{ix}))[u_{i+1x} \mid_P u_{ix} \cdot z_i].$$

Furthermore, by assumption $u_{1x} = u_{kx}$ which contradicts the continuity of the Pommaret division. If all non-multiplicative variables in condition (2) belong to the set $\{y_1, \ldots, y_m\}$, then by repeating the same argument as in the first case, we obtain a contradiction with the continuity of the Janet division.

Now consider the case that some of the non-multiplicative variables in the condition (2) belong to the set $\{x_1, \ldots, x_n\}$ and some of them belong to $\{y_1, \ldots, y_m\}$. By assumption, $u_{1x} = u_{kx}$. Suppose that $(u_{i_1}, \ldots, u_{i_k})$ is the largest subsequence such that $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, k\}$ and there exists a Pommaret non-multiplicative variable $z_{i_{j-1}}$ for $(u_{i_{j-1}})_x$ with $(u_{i_j})_x \mid_P (u_{i_{j-1}})_x \cdot z_{i_{j-1}}$.

Let us first assume that $i_k = k$. If $i_1 = 1$, then we obtain again a contradiction as above by considering the sequence $(u_{1x}, (u_{i_2})_x, \ldots, (u_{i_{k-1}})_x, u_{kx})$. If $i_1 \neq 1$, then $(u_{i_2})_x \mid_P (u_{i_1})_x \cdot z_{i_1}$ where $z_{i_1} \in NM_P((u_{i_1})_x)$. Since $(u_{i_1})_x \mid_P u_{1x}$, by Lemma 14, we have $(u_{i_2})_x \mid_P u_{1x} \cdot z_{i_1}$ and $z_{i_1} \in NM_P(u_{1x})$. Thus we may replace u_{i_1} by u_1 and the sequence $(u_{1x}, (u_{i_2})_x, \ldots, (u_{i_{k-1}})_x, u_{kx})$ leads again to a contradiction.

If $i_k \neq k$, we again suppose first that $i_1 = 1$. We know that $u_{kx} \mid_P (u_{i_k})_x$. According to our assumption, $(u_{i_k})_x \mid_P (u_{i_{k-1}})_x \cdot z_{i_{k-1}}$ with $z_{i_{k-1}} \in NM_P((u_{i_{k-1}})_x)$. By Lemma 15, we find $u_{kx} \mid_P (u_{i_{k-1}})_x \cdot z_{i_{k-1}}$ and thus may replace u_{i_k} by u_k. Then the sequence $((u_{i_1})_x, \ldots, (u_{i_{k-1}})_x, u_{kx})$ provides us again with a contradiction. If $i_1 \neq 1$, then using Lemmas 14 and 15, we may replace u_{i_1} by u_1 and u_{i_k} by u_k, respectively, and the sequence $(u_{1x}, (u_{i_2})_x, \ldots, (u_{i_{k-1}})_x, u_{kx})$ yields the desired contradiction. □

Definition 17. *A continuous involutive division $\mathcal{L}$ is* constructive*, if it satisfies the following property. Let $U \subset \mathbb{M}$ be a finite set and assume that $u \in U$ and $z \in NM_{\mathcal{L}}(u, U)$ are such that*

1. *$u \cdot z \notin C_{\mathcal{L}}(U)$ and*

2. *for each term $v \in U$ and $\tilde{z} \in NM_{\mathcal{L}}(v, U)$ such that $v \cdot \tilde{z} \mid u \cdot z$ and $v \cdot \tilde{z} \neq u \cdot z$ we have that $v \cdot \tilde{z} \in C_{\mathcal{L}}(U)$.*

Then there cannot exist a term $w \in C_{\mathcal{L}}(U)$ such that $u \cdot z \in C_{\mathcal{L}}(w, U \cup \{w\})$

Proposition 18. *The x-Pommaret division is constructive.*

Proof. In the setting of Definition 17 above, we consider first the case that $z \in \{x_1, \dots, x_n\}$. By assumption, $u \cdot z$ has no x-Pommaret divisor in U implying that $z \in NM_P(u_x)$ and $u_x \cdot z \notin C_P(U_x)$. By definition, for any term $w \in C_{P_x}(U)$ there exists a term $v \in U$ such that $w \in C_{P_x}(v, U)$. Since the Pommaret division is global, $C_P(w_x) \subseteq C_P(v_x)$ and consequently adding w_x to U_x does not change the involutive span $C_P(U_x)$. Thus, if $u_x \cdot z \notin C_P(U_x)$, then $u_x \cdot z \notin C_P(U_x \cup \{w_x\})$ and $u \cdot z \notin C_{P_x}(w, U \cup \{w\})$.

Now assume that $z \in \{y_1, \dots, y_m\}$. We follow the idea of the proof of (Gerdt and Blinkov, 1998, Proposition 4.13). From the assumption, we know that $u \cdot z = (u_1 \cdot v) \cdot w_1$ where $u_1 \in U$, $u_1 v \in C_{P_x}(U)$ and $w_1 \in P_x(u_1 v, U \cup \{u_1 v\})$. Let us consider the term ordering $\prec_{lex}$ with $x_n \prec_{lex} \cdots \prec_{lex} x_1 \prec_{lex} y_m \prec_{lex} \cdots \prec_{lex} y_1$. Since z is Janet non-multiplicative for u, we have $u \prec_{lex} u_1$. We claim that w_1 contains a variable in $\{y_1, \dots, y_m\}$. Otherwise – since the Pommaret division is global – $u \cdot z \in C_{P_x}(U)$ which contradicts our assumption and we may assume that y_{i_1} divides w_1. Choose $y_{i_1} \in NM_{P_x}(u_1, U)$. Therefore, using the fact that $u_1 \cdot y_{i_1}$ divides properly $u \cdot z$ and assumption (2) in Definition 17, we can write

$$u \cdot z = (u_1 \cdot y_{i_1}) \frac{v w_1}{y_{i_1}} = (u_2 w_2) \frac{v w_1}{y_{i_1}}$$

where $u_2 \in U$, $u_2 w_2 \in C_{P_x}(U)$. It follows that $u \prec_{lex} u_1 \prec_{lex} u_2$. If $v w_1 / y_{i_1}$ contains a Janet non-multiplicative variable for u_2, then we can continue this process leading to the chain $u \prec_{lex} u_1 \prec_{lex} u_2 \prec_{lex} \cdots$ which must be finite by the continuity of P_x (note that all u_i divide $u \cdot z$). This implies that there exists $u_\ell \in U$ so that $u \cdot z \in C_{P_x}(u_\ell, U)$ what contradicts our assumption $u \cdot z \notin C_{\mathcal{L}}(U)$. $\square$

Remark 19. *It follows trivially from the fact that the Pommaret division is not Noetherian that the x-Pommaret division cannot be Noetherian either.*

4. Computation of x-Pommaret Bases

Above it was shown that the x-Pommaret division is continuous and constructive. Thus in principle the usual completion algorithms (see e.g., (Seiler, 2010, Algorithm 4.1)) can be applied for the construction of an x-Pommaret basis. However, as the x-Pommaret division is not Noetherian, termination is a non-trivial question. We will now show that – as for the Pommaret division – any ideal possesses a finite x-Pommaret basis after a linear coordinate transformation and we will describe a concrete algorithm for the simultaneous construction of such a transformation and the corresponding basis following the ideas developed in (Hashemi et al., 2018).

Suppose that $\mathcal{I} = \langle f_1, \dots, f_k \rangle$ is an ideal generated by the bi-homogeneous polynomials $f_1, \dots, f_k$ in $\mathcal{P} = \mathbb{K}[y_1, \dots, y_m, x_1, \dots, x_n]$ (due to some computational issues, we swap here the order of the x- and the y-variables). Furthermore, let $\prec$ be the degree reverse lexicographic term ordering with $x_n \prec \cdots \prec x_1 \prec y_m \prec \cdots \prec y_1$. We first compute a minimal Janet basis H for $\mathcal{I}$ with respect to this ordering using Gerdt's algorithm (Gerdt, 2005). We then apply the algorithm

TEST to check whether $\mathrm{LT}(f)_x$ has the same Janet and Pommaret multiplicative variables for every polynomial $f \in H$. If this is the case, we may conclude that H is already an x-Pommaret basis for $\mathcal{I}$. Otherwise, we apply an elementary coordinate transformation as in the proof of (Seiler, 2009b, Theorem 2.13) on H and repeat this process, until we find an x-Pommaret basis for $\mathcal{I}$. Since we can always reach in this way a finite Pommaret basis for $\mathcal{I}$ after a finite number of linear changes, the same holds for the x-Pommaret division (alternatively, one can prove this explicitly by adapting the corresponding proofs in (Hashemi et al., 2018)).

The mentioned procedure to compute Pommaret bases uses the following result (Gerdt and Blinkov, 1998, Proposition 3.10): If a finite set $U \subset \mathbb{K}[x_1, \ldots, x_n]$ of terms is Pommaret autoreduced, then for any $u \in U$ we have $M_P(u, U) \subseteq M_J(u, U)$. We now provide an extension of this result to the x-Pommaret division.

Proposition 20. *Suppose that $U \subset \mathbb{M}$ is a finite set of terms which is autoreduced for the x-Pommaret division. Then for any term $u \in U$ we have $M_{P_x}(u, U) \subseteq M_J(u, U)$. Moreover, an ideal $\mathcal{I}$ possesses a finite x-Pommaret basis, if and only if for any polynomial f in a minimal Janet basis H of $\mathcal{I}$ we have $M_{P_x}(\mathrm{LT}(f), \mathrm{LT}(H)) = M_J(\mathrm{LT}(f), \mathrm{LT}(H))$.*

Proof. The assertions are immediate from (Gerdt and Blinkov, 1998, Proposition 3.10) and (Gerdt, 2000, Theorem 1) (see also (Seiler, 2009b, Theorem 2.13)), respectively. □

These considerations lead to Algorithm 1 for the effective construction of an x-Pommaret basis. It is not necessary to perform in it an explicit x-Pommaret autoreduction. Indeed, the output of Gerdt's algorithm is a minimal Janet basis and any minimal Janet basis is automatically x-Pommaret autoreduced. In the sequel, if A is a list, then we denote by $A[i]$ its i-th element.

Algorithm 1 x-PommaretBasis

Require: A finite set $F \subset \mathcal{P}$ of bi-homogeneous polynomials
Ensure: An x-Pommaret basis H for the ideal $\mathcal{I} = \langle F \rangle$

```
  H := GERDT(F, J, ≺)
  chen := ∅
  A := TEST(H, ≺)
  while A ≠ true do
    H := Replace A[3] by A[3] + cA[2] in H where c ∈ 𝕂 is random
    K :=GERDT(H, J, ≺)
    B := TEST(K, ≺)
    if B ≠ A then
      chen := chen ∪ {A[3] = A[3] + cA[2]}
      H := K
      A := B
    end if
  end while
  return (H, chen)
```

The subalgorithm TEST checks whether or not any term in $\mathrm{LT}(F)$ has the same Janet and Pommaret multiplicative variables.

Algorithm 2 TEST

Require: A finite set $F \subset \mathcal{P}$ of bi-homogeneous polynomials

Ensure: True if for each $f \in F$ the x-part of the leading term of f has the same Pommaret and Janet multiplicative variables; and (false, x_i, x_j) otherwise where x_i, x_j are two variables

```
for f ∈ F do
    M := M_{J,≺}(LT(f), LT(F)) \ {y_1, ..., y_m}
    N := M_{P,≺}(LT(f), LT(F)) \ {y_1, ..., y_m}
    if N ≠ M then
        V := M \ N
        return (false, V[1], x_{cls(LT(f)_x)})
    end if
end for
return (true)
```

Theorem 21. *The algorithm x-PommaretBasis terminates in finitely many steps and is correct.*

Proof. If an ideal possesses a finite Pommaret basis with respect to the mentioned ordering of the variables, then that basis is also an x-Pommaret basis for the ideal. On the other hand, Seiler (Seiler, 2009b, Remark 9.11) proved that after a finite number of coordinate transformations on the variables $y_1, \ldots, y_m, x_1, \ldots, x_n$ (including changes of the form used in x-PommaretBasis algorithm) one always can obtain a finite Pommaret basis for $\mathcal{I}$. Note that an x-Pommaret basis may be obtained already in the middle of the Pommaret basis computation when we restrict the coordinate transformations to the variables $x_1, \ldots, x_n$ and this completes the proof of termination. The correctness of the algorithm is a consequence of Proposition 20. □

Example 22. *Consider the set $F = \{x_1y_1 + x_2y_2, x_1^2y_1 + x_1x_2y_2, -3x_1x_2y_1 + x_2^2y_2, x_1^2x_2y_1 + 4x_2^3y_2\}$ and the variable ordering $x_2 \prec_{drl} x_1 \prec_{drl} y_2 \prec_{drl} y_1$. According to the structure of the algorithm x-PommaretBasis[1], we must first compute a Janet basis for $\mathcal{I} = \langle F \rangle$ with respect to $\prec$. Gerdt's algorithm yields the Janet basis $\{4x_2^2y_2, x_1y_1 + x_2y_2\}$. For the term $4x_2^2y_2$ the set of Janet multiplicative variables is $\{x_1, x_2, y_2\}$, whereas only x_2 is Pommaret multiplicative. Consequently, we perform the coordinate transformation $x_2 = x_2 - 2x_1$. Now the output of the algorithm will be as $H = \{16x_1^2y_2 - 16x_1x_2y_2 + 4x_2^2y_2, x_1y_1 - 2x_1y_2 + x_2y_2\}$. We can see easily that H is only an x-Pommaret basis and not a Pommaret basis for $\mathcal{I}$ (see Example 27 below).*

A question naturally arising here is whether or not one can define a *bi-Janet* division and use a bi-Janet basis in the algorithm x-PommaretBasis instead of the usual Janet basis. Below, we define such a division and show that it is not helpful for the construction of x-Pommaret bases.

Definition 23. *Let $U \subset \mathbb{M}$ be a finite set of terms in $\mathcal{P}$. A given variable is called* bi-Janet multiplicative *for $u = x^\alpha y^\beta \in U$, if it is Janet multiplicative for either $x^\alpha \in U_x \subset \mathbb{K}[x_1, \ldots, x_n]$ or $y^\beta \in U_y \subset \mathbb{K}[y_1, \ldots, y_m]$.*

Example 24. *Consider the set $\mathcal{F} = \{x_1y_1y_2, y_1^2y_2\} \subset \mathbb{K}[x_1, y_1, y_2]$ with the variable ordering $x_1 \prec_{drl} y_1 \prec_{drl} y_2$. By applying Gerdt's algorithm, we compute a bi-Janet basis $\mathcal{H} = \{x_1y_1y_2, y_1^2y_2, x_1y_1^2y_2\}$ for $\langle \mathcal{F} \rangle$ with respect to $\prec$. Considering the term $v = y_1^2y_2$, we observe that the set of the Pommaret multiplicative variables for $v_x = 1$ is $\{x_1\}$ however the Janet one is the empty set. Hence we note that there is no analogue for the crucial Proposition 20.*

[1]The Maple code of our implementations are available at `http://amirhashemi.iut.ac.ir/softwares`

We now extend the notion of quasi stability to the bigraded case and show that for a bi-homogeneous ideal having a finite x-Pommaret basis is equivalent to being in x-quasi stable position generalising (Seiler, 2009b, Proposition 4.4).

Definition 25. *A monomial ideal $\mathcal{J} \subset \mathcal{P}$ is called* x-quasi stable*, if for every term $x^\alpha y^\beta \in \mathcal{J}$ and all integers i, j, s with $1 \leq j < i \leq n$, $s > 0$ and $x_i^s \mid x^\alpha y^\beta$ there exists an integer $t \geq 0$ such that $x_j^t x^\alpha y^\beta / x_i^s \in \mathcal{J}$. A bi-homogeneous ideal $\mathcal{I} \subset \mathcal{P}$ is in* x-quasi stable position *if* $\mathrm{LT}(\mathcal{I})$ *is x-quasi stable.*

Proposition 26. *A monomial ideal $\mathcal{J}$ is x-quasi stable, if and only if it possesses a finite x-Pommaret basis.*

Proof. Let us assume that $\mathcal{J}$ is x-quasi stable and that H is a minimal Janet basis of $\mathcal{J}$. We will prove that this Janet basis is simultaneously an x-Pommaret basis for $\mathcal{J}$. For this purpose, it is sufficient to show that for any $u \in H$ we have $M_P(u_x) = M_J(u_x, \mathrm{LT}(H)_x)$. Since H is a minimal Janet basis of $\mathcal{J}$, we have $M_P(u_x) \subseteq M_J(u_x, \mathrm{LT}(H)_x)$ for any $u \in H$. Suppose there exists a term $v = y_1^{\beta_1} \cdots y_m^{\beta_m} x_1^{\alpha_1} \cdots x_k^{\alpha_k} \in H$ with $\mathrm{cls}(v_x) = k$ such that $x_l \in M_J(v_x, \mathrm{LT}(H)_x)$ but $x_l \notin M_P(v_x)$, i. e. $l < k$. Since $\mathcal{J}$ is x-quasi stable, there exists α such that $w = v \cdot x_l^\alpha / x_k^{\alpha_k}$ belongs to $\mathcal{J}$. Now, there is a term $s \in H$ with $w = sr$ where r contains only Janet multiplicative variables of $s \in H$. It follows that r must contain x_l and this contradicts the fact that x_l belongs to $M_J(v_x)$. Thus $M_J(u_x, \mathrm{LT}(H)_x) \subseteq M_P(u_x)$ for all $u \in H$ and we are done.

Conversely, let H be a finite x-Pommaret basis for $\mathcal{J}$ and $u \in \mathcal{J}$ with $\mathrm{cls}(u_x) = k$. Then for $1 \leq i < k$, x_i is a non-multiplicative variable for u_x with respect to Pommaret division. Suppose that t is the maximum power of x_i among the elements of H. As H is an x-Pommaret basis of $\mathcal{J}$ it must contain a term v such that $v_x \mid_P u_x \cdot x_i^{t+1}$. This implies that there exists a term $A \in P(v_x)$ such that $u_x \cdot x_i^{t+1} = v_x \cdot A$. Thus, A must contain x_i and $\mathrm{cls}(v_x) = i$. Therefore, we have $\deg_k(A) = \deg_k(u_x)$. It follows that $u \cdot x_i^{t+1} / x_k^{\alpha_k}$ belongs to $\mathcal{J}$. Based on this argument, one observes that $\mathcal{J}$ is x-quasi stable and this ends the proof. ☐

Example 27. *In Example 22 we saw that the transformation $x_2 = x_2 - 2x_1$ yields the x-Pommaret basis $H = \{16x_1^2y_2 - 16x_1x_2y_2 + 4x_2^2y_2, x_1y_1 - 2x_1y_2 + x_2y_2\}$. By Definition 25, the ideal $\langle \mathrm{LT}(H) \rangle$ is x-quasi stable and it possesses a finite x-Pommaret basis. However it is not quasi stable because $y_1 \mid x_1y_1$ but no pure power of x_1 belongs to $\langle \mathrm{LT}(H) \rangle$. Indeed, after performing the changes $x_2 = x_2 + 4x_1$, $x_1 = x_1 + 5y_2$, $x_2 = x_2 - 4x_1$ and $y_2 = y_2 - 4y_1$ we get the following Pommaret basis*

$$
\begin{aligned}
\Big\{ & -\frac{1024}{15}x_1^2y_1y_2 + \frac{256}{15}x_1^2y_2^2 + \frac{64}{75}x_1^3y_1 + \frac{256}{15}x_1x_2y_1y_2 - \frac{64}{15}x_1x_2y_2^2 - \\
& \frac{32}{75}x_1^2x_2y_1 + \frac{64}{75}x_1^2x_2y_2 + \frac{4}{75}x_1x_2^2y_1 - \frac{16}{75}x_1x_2^2y_2, \\
& -\frac{256}{9}y_1y_2^2 + \frac{64}{9}y_2^3 + \frac{4352}{45}x_1y_1y_2 - \frac{1024}{45}x_1y_2^2 - \frac{256}{225}x_1^2y_1 - \frac{128}{45}x_2y_1y_2 + \\
& \frac{32}{45}x_2y_2^2 + \frac{128}{225}x_1x_2y_1 - \frac{256}{225}x_1x_2y_2 - \frac{16}{225}x_2^2y_1 + \frac{4}{225}x_2^2y_2, \\
& x_1y_1 - 4x_2y_1 + x_2y_2 + 300y_1^2 - 155y_1y_2 + 20y_2^2 \Big\}
\end{aligned}
$$

for the ideal $\mathcal{I}$.

5. Bi-Quasi Stable Ideals

We discuss now another generalisation of quasi stable ideal and provide then an algorithm to transform any bi-homogeneous ideal into the corresponding generic position. Furthermore, we introduce the notion of a bi-Pommaret basis and relate it with bi-quasi stable ideals.

Definition 28. *A monomial ideal $\mathcal{J} \in \mathcal{P}$ is* bi-quasi stable*, if for every term $x^\alpha y^\beta \in \mathcal{J}$ the following two conditions are satisfied.*

1. *for all integers i, j, s with $1 \leq j < i \leq n$, $s > 0$ and $x_i^s \mid x^\alpha y^\beta$ there exists an integer $t \geq 0$ such that $x_j^t x^\alpha y^\beta / x_i^s \in \mathcal{J}$,*
2. *for all integers i, j, s with $1 \leq j < i \leq m$, $s > 0$ and $y_i^s \mid x^\alpha y^\beta$ there exists an integer $t \geq 0$ such that $y_j^t x^\alpha y^\beta / y_i^s \in \mathcal{J}$.*

An ideal $\mathcal{I}$ is in bi-quasi stable position*, if* $\mathrm{LT}(\mathcal{I})$ *is bi-quasi stable.*

Example 29. *Let $\mathcal{J} = \langle x_1^2y_1^2, x_1^2y_1y_2, x_1^2y_2^2, x_1^2y_1y_3, x_1x_2y_1^2, x_1x_2y_1y_2, x_1x_2y_2^2, x_1x_2y_1y_3, x_2^2y_1^2\rangle$ be an ideal in $\mathbb{K}[x_1, x_2, y_1, y_2, y_3]$. One can check easily that $\mathcal{J}$ is bi-quasi stable.*

Based on Definition 28, there exists a simple algorithm verifying whether or not a given monomial ideal is bi-quasi stable.

Algorithm 3 BQS-TEST

Require: A minimal basis G of the monomial ideal $\mathcal{J} \subset \mathcal{P}$
Ensure: Answer to the question: "Is $\mathcal{J}$ bi-quasi stable?"
  $q := \max\{\deg(u_x), \deg(u_y) \mid u \in G\}$
  for $x^\alpha y^\beta \in G$ with $\mathrm{cls}(x^\alpha) = k$ and $\mathrm{cls}(y^\beta) = \ell$ **do**
    for i **from** 1 **to** $k-1$ **do**
      if $x_i^q x^\alpha y^\beta / x_k^{\alpha_k} \notin \mathcal{J}$ **then**
        return$(false, x_k, x_i)$
      end if
    end for
    for i **from** 1 **to** $\ell - 1$ **do**
      if $y_i^q x^\alpha y^\beta / y_\ell^{\beta_k} \notin \mathcal{J}$ **then**
        return$(false, y_\ell, y_i)$
      end if
    end for
  end for
  return (true)

Theorem 30. *The algorithm* BQS-TEST *terminates in finitely many steps and is correct.*

Proof. The termination of the algorithm is trivial. Its correctness is a straight-forward generalisation of the corresponding result for quasi stable ideals – see (Seiler, 2012, Lemma 3.4). □

A method to transform deterministically any ideal into quasi stable position was proposed in (Hashemi et al., 2018), see also (Seiler, 2009b, Remark 9.11). Algorithm 4 adapts this method to the bigraded case. Since transforming a given ideal into bi-quasi stable position is simply a step towards transforming the ideal into quasi stable position, its termination and correctness follow directly from the results in (Hashemi et al., 2018).

Algorithm 4 BQS-TRANSFORM

Require: A bi-homogeneous polynomial ideal $\mathcal{I} = \langle F \rangle \subset \mathcal{P}$
Ensure: A linear change Ψ s.t. $\Psi(\mathcal{I})$ is in bi-quasi stable position

$\Psi := \emptyset$
$G :=$GRÖBNERBASIS$(F, \prec)$
$A :=$BQS-TEST(LT(G))
while $A \neq$ *true* **do**
  $G :=$ Replace $A[2]$ by $A[2] + cA[3]$ in G where $c \in \mathbb{K}$ is a random element
  $H :=$GRÖBNERBASIS$(G, \prec)$
  $B :=$BQS-TEST(H)
  if $B \neq A$ **then**
    $\Psi := \Psi \cup \{A[2] = A[2] + cA[3]\}$
    $G := H$
    $A := B$
  end if
end while
return (Ψ)

Theorem 31. *The algorithm* BQS-TRANSFORM *terminates in finitely many steps and is correct.*

Example 32. *Consider the ideal* $\mathcal{I} = \langle x_1y_1 + x_2y_2, x_1^2y_1 + x_1x_2y_2, -3x_1x_2y_1 + x_2^2y_2, x_1^2x_2y_1 + 4x_2^3y_2\rangle$ *and the variable ordering* $x_2 \prec_{drl} x_1 \prec_{drl} y_2 \prec_{drl} y_1$. *According to the algorithm* BQS-TRANSFORM, *we compute first the Gröbner basis* $G = \{x_1y_1 + x_2y_2, x_2^2y_2\}$ *for* $\mathcal{I}$. *Since* $\dfrac{x_1^t x_2^2 y_2}{x_2^2} \notin \langle \mathrm{LT}(G)\rangle$ *for any t, we perform the coordinate transformation* $\Psi_1 : (x_2 = x_2 + 2x_1)$ *on G which yields the set* $F_1 = \{4x_1^2y_2 + 4x_1x_2y_2 + x_2^2y_2, x_1y_1 + 2x_1y_2 + x_2y_2\}$. *The Gröbner basis of the ideal generated by* F_1 *is* $G_1 = \{x_2^2y_1^2y_2, 4x_1x_2y_2^2 - x_2^2y_1y_2 + 2x_2^2y_2^2, 4x_1^2y_2 + 4x_1x_2y_2 + x_2^2y_2, x_1y_1 + 2x_1y_2 + x_2y_2\}$. *However,* $\mathrm{LT}(\Psi_1(\mathcal{I}))$ *is still not bi-quasi stable as* $\dfrac{y_1^2x_2^2y_1^2y_2}{y_2} \notin \langle \mathrm{LT}(G_1)\rangle$. *Thus we perform a second coordinate transformation of the form* $\Psi_2 : (y_2 = y_2 - y_1)$. *One can easily check that* $\mathrm{LT}(\Psi_2(\Psi_1(\mathcal{I})))$ *is bi-quasi stable.*

We now recall the definition of bi-stable ideals from (Aramova et al., 2000, Definition 1.1). Furthermore, after defining bi-Pommaret bases, we show that a monomial ideal is bi-stable, if and only if its minimal basis is a bi-Pommaret basis.

Remark 33. *Throughout this section, if u is a term in* $\mathbb{M}$, *then we write* $\mathrm{cls}_x(u) = \max\{i \mid \deg_i(u_x) > 0\}$ *and* $\mathrm{m}_x(u) = \min\{i \mid \deg_i(u_x) > 0\}$. *The notations* $\mathrm{cls}_y(u)$ *and* $\mathrm{m}_y(u)$ *are defined analogously.*

Definition 34. *Let* $\mathcal{I} \subset \mathcal{P}$ *be a bi-homogeneous ideal. A finite generating set* $H \subset \mathcal{I}$ *is a* bi-Pommaret basis *of* $\mathcal{I}$, *if for any* $f \in \mathcal{I}$ *there exists* $g \in H$ *such that* $\mathrm{LT}(g) \mid \mathrm{LT}(f)$ *and* $t \in \mathbb{K}[x_{\mathrm{cls}_x(\mathrm{LT}(g))}, \ldots, x_n, y_{\mathrm{cls}_y(\mathrm{LT}(g))}, \ldots, y_m]$ *where* $t = \mathrm{LT}(f)/\mathrm{LT}(g)$.

Definition 35. *Let* $\mathcal{J} \subset \mathcal{P}$ *be a monomial ideal.* $\mathcal{J}$ *is called* bi-stable, *if for any term* $u \in \mathcal{J}$ *the following two conditions hold:*

1. *for every* $i < \mathrm{cls}_x(u)$ *one has* $x_i \cdot u/x_{\mathrm{cls}_x(u)} \in \mathcal{J}$,

2. *for every $j < \mathrm{cls}_y(u)$ one has $y_j \cdot u/y_{\mathrm{cls}_y(u)} \in \mathcal{J}$.*

Proposition 36. *Let $\mathcal{J}$ be a monomial ideal in $\mathcal{P}$. Then $\mathcal{J}$ is* bi-stable*, if and only if its minimal basis H is a bi-Pommaret basis.*

Proof. Let us assume that $\mathcal{J}$ is bi-stable. We then have to show that H is a bi-Pommaret basis for $\mathcal{J}$. We know that for any term $v \in \mathcal{J}$ there exists a decomposition of the form

$$v = uw, \qquad \mathrm{cls}_x(u) \leq \mathrm{m}_x(w), \qquad \mathrm{cls}_y(u) \leq \mathrm{m}_y(w) \tag{3}$$

where $u \in H$, see (Aramova et al., 2000, Lemma 2.1). It follows that H is a bi-Pommaret basis for $\mathcal{J}$. For the converse, assume that the minimal basis H is a bi-Pommaret basis for $\mathcal{J}$. Thus every term in $\mathcal{J}$ has a decomposition of the form (3). Let $v \in \mathcal{J}$ be a term and $i < \mathrm{cls}_x(v)$. As $x_i \cdot v \in \mathcal{J}$, there exists a term $u \in H$ so that $x_i \cdot v = zu$ with $\mathrm{cls}_x(u) \leq \mathrm{m}_x(z)$ and $\mathrm{cls}_y(u) \leq \mathrm{m}_y(z)$. Thus $\mathrm{cls}_x(x_i \cdot v) = \mathrm{cls}_x(v)$. Since $u \in H$ cannot be a multiple of v (because H is minimal), $z \neq 1$ and $x_{\mathrm{cls}_x(v)} \mid z$. Setting $z = z' \cdot x_{\mathrm{cls}_x(v)}$, we have $x_i \cdot v = uz'.x_{\mathrm{cls}_x(v)}$. Therefore $\frac{x_i \cdot v}{x_{\mathrm{cls}_x(v)}} = z' \cdot u \in \mathcal{J}$ and so the first condition of Definition 35 holds. Similarly, we can prove the second condition and we are done. □

The following example shows that the bi-Pommaret division is not involutive.

Example 37. *Let $U_1 = \{x_1^2y_1^2y_2^2, x_1^2x_2y_1^2\}$ be a subset of $\mathbb{K}[y_1, y_2, x_1, x_2]$. Then $x_1^2y_1^2y_2^2 \cdot x_2 = x_1^2x_2y_1^2 \cdot y_2^2$ where $\{x_1, x_2, y_2\}$ and $\{x_2, y_1, y_2\}$ are the sets of the bi-Pommaret multiplicative variables of $x_1^2y_1^2y_2^2$ and $x_1^2x_2y_1^2$, respectively. However $x_1^2y_1^2y_2^2 \nmid x_1^2x_2y_1^2$ and $x_1^2x_2y_1^2 \nmid x_1^2y_1^2y_2^2$.*

We conclude this section with an example which shows that the method described in (Seiler, 2009b) for computing Pommaret bases cannot be applied to compute bi-Pommaret bases.

Example 38. *Let $\mathcal{F} = \{y_1x_1x_2, x_1^2x_2, y_1y_2x_2\} \subset \mathbb{K}[y_1, y_2, x_1, x_2]$ be a finite set and $x_2 \prec_{drl} x_1 \prec_{drl} y_2 \prec_{drl} y_1$. First we compute a Janet basis for $\mathcal{J} = \langle \mathcal{F} \rangle$ with respect to $\prec$. By applying Gerdt's algorithm, one finds that $\mathcal{F}$ is already a Janet basis for $\mathcal{J}$. We consider $u = x_1^2x_2$ and observe that the set of Janet multiplicative variables for u_x is $\{x_1, x_2\}$, however, only x_2 is also Pommaret multiplicative. We perform a coordinate transformation of the form $x_2 = x_2 + 5x_1$ and find that the set $H = \{5x_1^3+x_1^2x_2, 5y_1x_1^2+y_1x_1x_2, 5y_1y_2x_1+y_1y_2x_2\}$ is a Janet basis for $\langle H \rangle$ with respect to $\prec$. Now we consider the polynomial $f = 5y_1x_1^2+y_1x_1x_2$ and we observe that the set of the Pommaret multiplicative variables for $\mathrm{LT}(f)_y = y_1$ is $\{y_1, y_2\}$, however, only y_1 is also Janet multiplicative. Therefore $\mathrm{LT}(f)_y$ possesses more Pommaret than Janet multiplicative variables and one cannot apply the method of (Seiler, 2009b).*

6. Computing the Resolution Regularity

Recall that the resolution regularity of a bi-homogeneous ideal is a pair of integers. In this section, we focus on computing only the first component, which we call the *x-regularity* of the ideal. We show how one can apply the theory of x-Pommaret bases to compute the x-regularity of a bi-homogeneous ideal. For this, we show first that the x-Pommaret division is of *Schreyer type*. For a given integer s, let us consider $\mathcal{P}^s$ as an $\mathcal{P}$-module and represent each of its element by an s-tuple $\mathbf{f} = (f_1, \ldots, f_s)$ with $f_i \in \mathcal{P}$. Denoting the standard basis of $\mathcal{P}^s$ as $\{\mathbf{e}_1, \ldots, \mathbf{e}_s\}$, we can write $\mathbf{f} = f_1\mathbf{e}_1 + \cdots + f_s\mathbf{e}_s$. A (module) term is a vector of the form $\mathbf{t} = t\mathbf{e}_i$ with $t \in \mathbb{M}$ and $1 \leq i \leq s$.

Definition 39. *Let $\mathcal{I} \subset \mathcal{P}$ be a bi-homogeneous ideal. The* x-regularity *of $\mathcal{I}$, denoted by* $\mathrm{reg}_x(\mathcal{I})$, *is the least integer q such that its i-th bi-graded syzygy module is generated by the elements of x-degree (degree with respect to the variables x_i which is denoted by* $\deg_x$*) less than or equal to $q + i$.*

The Castelnuovo-Mumford regularity of an ideal generated by a Pommaret basis H is equal to the maximal degree of an element of H (Seiler, 2009b, Theorem 9.2). On the other hand, by the definition of the x-regularity, one sees that performing a linear change of the x-coordinates only on a given bi-homogeneous ideal does not change its x-regularity. Therefore, one could hope that to determine the x-regularity of a bi-homogeneous ideal, one might compute first a Pommaret basis for the ideal and then take the maximal x-degree of the basis as the x-regularity of ideal. The next example shows that this is not true in general.

Example 40. *Let $F = \{x_1y_1y_2, y_1^2y_2, x_1x_2y_2, x_1x_2^2, x_2^3\} \subset \mathbb{K}[x_1, x_2, y_1, y_2]$ be a finite set of terms and $x_2 \prec_{drl} x_1 \prec_{drl} y_2 \prec_{drl} y_1$. After performing the changes of coordinates $x_2 = x_2 + 2y_1$ and $x_2 = x_2 + 2y_2$ on F, we obtain the Pommaret basis $H = \{\frac{1}{48}x_2^4y_2 + \frac{1}{6}x_2^3y_2^2 + \frac{1}{2}x_2^2y_2^3 + \frac{2}{3}x_2y_2^4 + \frac{1}{3}y_2^5, \frac{1}{3}x_1y_2^4 + \frac{1}{24}x_1x_2^3y_2, -\frac{1}{8}x_2^3y_2 - \frac{3}{4}x_2^2y_2^2 - \frac{3}{2}x_2y_2^3 - y_2^4 - \frac{3}{4}y_2y_1x_2^2 - 3x_2y_1y_2^2 - 3y_1y_2^3, -x_1y_1y_2^2, -x_1y_2^3 + \frac{1}{4}y_2x_1x_2^2, x_2^3 + 6x_2^2y_1 + 6x_2^2y_2 + 12x_2y_1^2 + 24x_2y_1y_2 + 12x_2y_2^2 + 8y_1^3 + 24y_1^2y_2 + 24y_1y_2^2 + 8y_2^3, y_1^2y_2, x_1x_2^2 + 4x_1x_2y_1 + 4x_1x_2y_2 + 4x_1y_1^2 + 8x_1y_1y_2 + 4x_1y_2^2, x_1x_2y_2 + 2x_1y_1y_2 + 2x_1y_2^2, -\frac{1}{2}y_2x_1x_2 - x_1y_2^2\}$ for the transformed ideal. One observes that the maximal x-degree of the elements of H is* 4*. On the other hand, performing the linear change $y_2 = y_2 + 5y_1$ and $x_2 = x_2 - 2x_1$ on F, we obtain the Gröbner basis $G = \{5x_2^3y_1^2 + x_2^3y_1y_2, 10x_1x_2^2y_1 + 2x_1x_2^2y_2 - 5x_2^3y_1 - x_2^3y_2, 5y_1^3 + y_1^2y_2, 5x_1y_1^2 + x_1y_1y_2, 10x_1^2y_1 + 2x_1^2y_2 - 5x_1x_2y_1 - x_1x_2y_2, 4x_1^3 - 3x_1x_2^2 + x_2^3, 4x_1^2x_2 - 4x_1x_2^2 + x_2^3\}$ for the transformed ideal. We notice that $\langle \mathrm{LT}(G) \rangle$ is bi-stable and hence the x-regularity of $\langle F \rangle$ is the maximal x-degree of the elements of G which is equal to* 3*, see (Aramova et al., 2000, Theorem 2.2).*

As usual, a total ordering $<$ on $\mathcal{P}^s$ is called a *module term ordering*, if for any $r \in \mathbb{M}$ and for any two module terms $\mathbf{t}, \mathbf{s} \in \mathbb{M}^s$ the following two conditions hold: (i) $\mathbf{t} \leq r\mathbf{t}$ and (ii) if $\mathbf{s} < \mathbf{t}$ then $r\mathbf{s} < r\mathbf{t}$. This allows for a natural extension of the notions of leading term, leading monomial and leading coefficient to an element $\mathbf{f}$ of $\mathcal{P}^s$ and we denote them by $\mathrm{LT}_<(\mathbf{f})$, $\mathrm{LM}_<(\mathbf{f})$ and $\mathrm{LC}_<(\mathbf{f})$ respectively.

Example 41 (Schreyer ordering)**.** *Suppose that $F = \{f_1, \ldots, f_s\} \subset \mathcal{P}$ and $\prec$ is a term ordering on $\mathcal{P}$. We recall that the* Schreyer module term ordering *$<_F$ with respect to F on $\mathcal{P}^s$ is defined as follows: let $\mathbf{r} = r\mathbf{e}_i$ and $\mathbf{t} = t\mathbf{e}_j$ be two module terms in $\mathbb{M}^s$; we write $\mathbf{r} <_S \mathbf{t}$ if either $\mathrm{LT}_\prec(rf_i) \prec \mathrm{LT}_\prec(tf_j)$ or $\mathrm{LT}_\prec(rf_i) = \mathrm{LT}_\prec(tf_j)$ and $j < i$.*

The following definitions are taken from (Seiler, 2009b). Let H be a finite subset of $\mathcal{P}^s$, $<$ a module term ordering on $\mathbb{M}^s$ and $\mathcal{L}$ an involutive division on $\mathcal{P}$. We divide H into s disjoint subsets $H_i = \{\mathbf{h} \in H \mid \mathrm{LM}_<(\mathbf{h}) = t\mathbf{e}_i,\ t \in \mathbb{M}\}$. This partitioning leads to s sets $B_i = \{t \in \mathbb{M} \mid t\mathbf{e}_i \in \mathrm{LM}_<(H_i)\}$. Now we assign to each $\mathbf{h} \in H_i$ the multiplicative variables $M_{\mathcal{L},H,<}(\mathbf{h}) = \{x_i \in M_{\mathcal{L},B_i}(t) \mid \mathrm{LM}_<(\mathbf{h}) = t\mathbf{e}_i\}$.

Definition 42. *An involutive division $\mathcal{L}$ is of* Schreyer type*, if for any monomial $\mathcal{L}$-involutive set $H \subset \mathbb{M}$ and for any term $h \in H$ the set of non-multiplicative variables $NM_{\mathcal{L},H}(h) \subset \mathbb{M}$ is again $\mathcal{L}$-involutive.*

Lemma 43. *The x-Pommaret division is of Schreyer type.*

Proof. Let $H \subset \mathbb{M}$ be a monomial x-Pommaret basis and h an arbitrary element of H. According to the definition of the x-Pommaret division, the set of non-multiplicative variables of h is of the form $\{y_{i_1}, \ldots, y_{i_\ell}, x_1, \ldots, x_k\}$ for some $i_1, \ldots, i_\ell, k$. Using the fact that both the Pommaret and the Janet division are of Schreyer type (Seiler, 2009b, Lemma 5.9), one can show easily that this set is x-Pommaret involutive. $\square$

Let $H = \{\mathbf{h}_1, \ldots, \mathbf{h}_r\}$ be an x-Pommaret basis of the submodule $M \subseteq \mathcal{P}^s$. Then, we know that for each $\mathbf{h}_\alpha \in H$ and each non-multiplicative variable, say x_k, we can write $x_k \mathbf{h}_\alpha$ as $\sum_{i=1}^r p_i \mathbf{h}_i$ using an involutive normal form algorithm. The corresponding syzygy is written as $S_{\alpha,[k],[]} = x_k \mathbf{e}_\alpha - \sum_{i=1}^r p_i \mathbf{e}_i$. If we consider a non-multiplicative variable y_ℓ, then the corresponding syzygy is denoted by $S_{\alpha,[],[\ell]}$. The set of all such syzygies is denoted by H_{Syz}. Using Lemma 43 and similar to (Seiler, 2009b, Theorem 5.10), we can show that H_{Syz} is an x-Pommaret basis for the syzygy module of $\mathbf{h}_1, \ldots, \mathbf{h}_r$ for the Schreyer ordering. We note that if we start with a set of bi-homogeneous elements, then the syzygy modules are all bi-graded. Based on these observations, we claim that for a given x-Pommaret basis H we can write a minimal bi-graded free resolution of length $m + n$ of the form:

$$0 \longrightarrow \mathcal{P}^{r_{m+n}} \longrightarrow \cdots \longrightarrow \mathcal{P}^{r_1} \longrightarrow \mathcal{P}^{r_0} \longrightarrow \langle H \rangle \longrightarrow 0\,. \tag{4}$$

Remark 44. *Let $F = \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_t\} \subset \mathcal{P}^s$ be a set of bi-graded elements. Thus the syzygy module $Syz(F)$ may be considered as a bi-graded submodule of $\mathcal{P}^t$ where the grading on the module $\mathcal{P}^t$ is defined as* $\deg_x(x^\alpha y^\beta \boldsymbol{e}_i) = |\alpha| + \deg_x(\boldsymbol{f}_i)$. *Furthermore,* $\deg_x(F) = \max\{\deg_x(\boldsymbol{f}_1), \ldots, \deg_x(\boldsymbol{f}_t)\}$. *Analogous notation* $\deg_y$ *is defined for an element in $\mathcal{P}^t$ as well.*

Theorem 45. *Let $\mathcal{I}$ be a bi-homogeneous ideal and Ψ a linear change of coordinates such that $\Psi(\mathcal{I})$ has a finite x-Pommaret basis H. Then* $\mathrm{reg}_x(\mathcal{I}) = \deg_x(H)$.

Proof. The proof is inspired by (Seiler, 2009b, Theorem 9.2). First we note that the x-regularity of a bi-homogeneous ideal is stable under any linear change of the x-coordinates. This shows that $\mathrm{reg}_x(\mathcal{I}) = \mathrm{reg}_x(\langle H \rangle)$. Therefore, without loss of generality, we may assume that $\mathcal{I}$ has a finite x-Pommaret basis $H = \{h_1, \ldots, h_t\}$ and $q = \deg_x(H)$. We show that $\mathrm{reg}_x(\mathcal{I}) = q$. From the above observation, we know that the i-th syzygy module of the resolution (4) induced by the basis H is generated by elements of x-degree at most $q + i$. Thus by Definition 39, we have $\mathrm{reg}_x(\mathcal{I}) \leq q$. Hence it remains to show that $\mathrm{reg}_x(\mathcal{I}) = q$. Let h_γ be an element of H with $\deg_x(h_\gamma) = q$ so that $\mathrm{cls}_x(h_\gamma) = \max\{\mathrm{cls}_x(h_\alpha) \mid h_\alpha \in H, \deg_x(h_\alpha) = q\}$. If $\mathrm{cls}_x(h_\gamma) = 1$, then it is of the form $\mathrm{LT}(h_\gamma) = x_1^q y_1^{\gamma_1} \cdots y_m^{\gamma_m}$. As the x-Pommaret basis H is minimal, the leading term of any other generator h_α with $\mathrm{cls}_x(h_\alpha) = 1$ cannot divide $\mathrm{LT}(h_\gamma)$. Thus h_γ cannot be eliminated from H. Hence we have $\mathrm{reg}_x(\mathcal{I}) = q$. Now, assume that $\mathrm{cls}_x(h_\gamma) = 1 + i$ for some $i > 0$ and the rest of the proof is essentially the same as (Seiler, 2009b, Theorem 9.2). $\square$

Example 46. *Let $\mathcal{F} = \{x_1x_2y_1, x_1x_3y_2, x_1x_2x_3y_2, x_2x_3y_1y_2, x_1x_3y_1\}$ be a subset of the polynomial ring $\mathcal{P} = \mathbb{K}[x_1, x_2, y_1, y_2]$, $\mathcal{I} = \langle F \rangle$ and $x_2 \prec_{drl} x_1 \prec_{drl} y_2 \prec_{drl} y_1$. By applying the algorithm* x-PommaretBasis, *we obtain after the change of coordinates $x_3 = x_3 - x_1$ and $x_3 = x_3 - x_2$ on F the x-Pommaret basis $H = \{x_1x_2y_1, -x_1^2y_2 - x_1x_2y_2 + x_1x_3y_2, -x_1^2y_1 - x_1x_2y_1 + x_1x_3y_1, -x_2^2y_1y_2 + x_2x_3y_1y_2, x_1x_2y_1y_2, -x_1^2y_1y_2 - x_1x_2y_1y_2 + x_1x_3y_1y_2\}$ for the transformed ideal. We have* $\deg_x(H) = 2$ *and thus $reg_x(\mathcal{I}) = 2$. In addition, by applying Buchberger's algorithm after performing the linear changes $x_3 = x_3 + x_2$, $y_2 = y_2 + 4y_1$, $x_2 = x_2 - 4x_1$ and $x_3 = x_3 + 5x_2$ on F, we obtain the Gröbner basis $G = \{16x_1x_3y_1^2 + 4x_1x_3y_1y_2 + 120x_2^2y_1^2 + 30x_2^2y_1y_2 + 20x_2x_3y_1^2 + 5x_2x_3y_1y_2, 20x_1^2y_1 + x_1x_3y_1, 4x_1^2y_2 - 6x_1x_2y_2 - x_1x_3y_2, 5x_1x_2y_1 + x_1x_3y_1\}$ for the transformed ideal. We can check easily that $\langle \mathrm{LT}(G) \rangle$ is bi-stable. Therefore we have $reg_x(\mathcal{I}) = \max(\{\deg_x(\mathrm{LT}(u)) \mid u \in G\}) = 2$.*

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