Singularities of Implicit Differential Equations and Static Bifurcations

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Abstract. We discuss geometric singularities of implicit ordinary differential equations from the point of view of Vessiot theory. We show that quasi-linear systems admit a special treatment leading to phenomena not present in the general case. These results are then applied to study static bifurcations of parametric ordinary differential equations.

1 Introduction

The Vessiot theory [17] provides a powerful framework for analysing differential equations geometrically. It uses vector fields in contrast to the differential forms of the more familiar Cartan-Kähler theory [2]. Fackerell [3] applied it in the context of symmetry analysis of partial differential equations. In [5, 6], we gave a rigorous proof that Vessiot's construction of solutions works, if and only if one is dealing with an involutive equation.

Vessiot's intention was to provide an alternative proof of the basic existence and uniqueness theorem for general systems of partial differential equations, the Cartan-Kähler Theorem. In our opinion, there are better approaches for this (see e. g. the proof in [14] which mainly follows [9]). However, his results are very useful for the further geometric analysis of involutive equations. In particular, one can easily see that Arnold's treatment of singularities of implicit ordinary differential equations in [1] is in fact based on what we call the Vessiot distribution. This identification provides us with a starting point for extending singularity theory to more general systems. For example, Tuomela [15, 16] used it for differential algebraic equations and we considered in [7] partial differential equations of finite type.

In this work, we first review the theory of geometric singularities of general nonlinear systems of ordinary differential equations which are not underdetermined. Then we discuss the particularities appearing in quasi-linear systems. In contrast to the general case, the Vessiot distribution is here projectable to the base manifold which leads to types of singular behaviour which are not admitted by fully non-linear equations. In particular, solutions can be extended to certain points where strictly speaking the differential equation is not even defined. We recover and clarify here results by Rabier [10–12] with alternative proofs.

Finally, we apply the obtained results to the analysis of static bifurcations in parametric autonomous ordinary differential equations. This is possible due to the simple observation that differentiation of the equilibrium condition yields a quasi-linear equation. The projected Vessiot distribution contains now much information about bifurcations: turning points (also called saddle-node bifurcations) are characterised by a vertical distribution, whereas at pitchfork or transcritical bifurcations the distribution vanishes. Furthermore, the bifurcation diagram consists of integral manifolds of the Vessiot distribution.

2 Geometric Theory of Differential Equations

The geometric modelling of differential equations is based on jet bundles [8, 9, 13, 14]. Let $\pi : \mathcal{E} \to \mathcal{T}$ be a fibred manifold. For ordinary differential equations, we may assume that $\dim \mathcal{T} = 1$. For simplicity, we will work in local coordinates, although we will use throughout a "global notation". As coordinate on the base space $\mathcal{T}$ we use t and fibre coordinates in the total space $\mathcal{E}$ will be $\boldsymbol{u} = (u^1, \dots, u^m)$. The first derivative of u^α will be denoted by $\dot{u}^\alpha$; higher derivatives are written in the form $u^\alpha_k = d^k u^\alpha / dt^k$. Adding all derivatives u^α_k with $k \leq q$ (collectively denoted by $\boldsymbol{u}^{(q)}$) defines a coordinate system for the q-th order jet bundle $J_q\pi$. There are natural fibrations $\pi^q_r : J_q\pi \to J_r\pi$ for $r < q$ and $\pi^q : J_q\pi \to \mathcal{T}$. Sections $\sigma : \mathcal{T} \to \mathcal{E}$ of the fibration π correspond to functions $\boldsymbol{u} = \boldsymbol{s}(t)$, as locally they can always be written in the form of a graph $\sigma(t) = \big(t, \boldsymbol{s}(t)\big)$. To such a section σ, we associate its *prolongation* $j_q\sigma : \mathcal{T} \to J_q\pi$, a section of the fibration π^q given by $j_q\sigma(t) = \big(t, \boldsymbol{s}(t), \dot{\boldsymbol{s}}(t), \ddot{\boldsymbol{s}}(t), \dots\big)$.

The geometry of $J_q\pi$ is to a large extent determined by its *contact structure* describing intrinsically the relationship between the different types of coordinates. The *contact distribution* is the smallest distribution $\mathcal{C}_q \subset T(J_q\pi)$ that contains the tangent spaces $T(\operatorname{im} j_q\sigma)$ of all prolonged sections and any field in it is a *contact vector field*. In local coordinates, $\mathcal{C}_q$ is generated by one transversal and m vertical fields:

$$C^{(q)}_{\text{trans}} = \partial_t + \sum_{j=0}^{q-1} u^\alpha_{j+1} \partial_{u^\alpha_j} \;, \tag{1a}$$

$$C^{(q)}_\alpha = \partial_{u^\alpha_q} \;, \qquad 1 \leq \alpha \leq m \;. \tag{1b}$$

Proposition 1. *A section $\gamma : \mathcal{T} \to J_q\pi$ is of the form $\gamma = j_q\sigma$ with $\sigma : \mathcal{T} \to \mathcal{E}$, if and only if $T_{\gamma(t)}(\operatorname{im}\gamma) \subseteq \mathcal{C}_q|_{\gamma(t)}$ for all points $t \in \mathcal{T}$ where γ is defined.*

The following intrinsic geometric definition of a differential equation generalises the usual one, as it allows for certain types of singular behaviour. It imposes considerably weaker conditions on the restricted projection $\hat{\pi}^q$ which in the standard definition is expected to be a surjective submersion. Note that we do not distinguish between scalar equations and systems. Indeed, when we speak of a differential equation in the sequel, we will always mean a system, if not explicitly stated otherwise.

Definition 2. *An* (ordinary) differential equation *of order q is a submanifold $\mathcal{R}_q \subseteq J_q\pi$ such that the restriction $\hat{\pi}^q$ of the projection $\pi^q : J_q\pi \to \mathcal{T}$ to $\mathcal{R}_q$*

has a dense image. A (strong) solution *is a (local) section* $\sigma : \mathcal{T} \to \mathcal{E}$ *such that* $\operatorname{im} j_q\sigma \subseteq \mathcal{R}_q$.

Locally, a differential equation $\mathcal{R}_q \subseteq J_q\pi$ can be described as the zero set of some smooth functions $\Phi : J_q\pi \to \mathbb{R}$. Differentiating every function yields the *prolonged equation* $\mathcal{R}_{q+1} \subseteq J_{q+1}\pi$ defined by all equations $\Phi(t, \boldsymbol{u}^{(q)}) = 0$ and $D_t\Phi(t, \boldsymbol{u}^{(q+1)}) = 0$ with the formal derivative

$$D_t\Phi = C^{(q)}_{\text{trans}}(\Phi) + \sum_{\alpha=1}^{m} u^\alpha_{q+1} C^{(q)}_\alpha(\Phi) \; . \tag{2}$$

Iteration of this process gives the higher prolongations $\mathcal{R}_{q+r} \subseteq J_{q+r}\pi$. A subsequent projection leads to the differential equation $\mathcal{R}^{(1)}_q = \pi^{q+1}_q(\mathcal{R}_{q+1}) \subseteq \mathcal{R}_q$ which will be a proper submanifold, if integrability conditions are hidden. $\mathcal{R}_q$ is *formally integrable*, if at any prolongation order $r > 0$ the equality $\mathcal{R}^{(1)}_{q+r} = \mathcal{R}_{q+r}$ holds (see [14] for more details). It is easy to show that (under some regularity assumptions) every consistent ordinary differential equation $\mathcal{R}_q$ leads after a finite number of projection and prolongation cycles to a formally integrable equation $\mathcal{R}^{(s)}_q \subseteq \mathcal{R}_q$. Therefore, without loss of generality, we will always assume in the sequel that we are already dealing with a formally integrable equation.

More precisely, we will study in this work only *square* first-order equations with a local representation

$$\mathcal{R}_1 \,:\, \left\{ \; \boldsymbol{\Phi}(t, \boldsymbol{u}^{(1)}) = 0 \right. \tag{3}$$

where $\boldsymbol{\Phi} : J_1\pi \to \mathbb{R}^m$ (thus we have as many equations as unknown functions) and where we furthermore assume that the symbol matrix, i. e. the Jacobian $\partial\boldsymbol{\Phi}/\partial\dot{\boldsymbol{u}}$, is almost everywhere non-singular. These assumptions are less restrictive as they may appear. Whenever a first-order equation is not underdetermined, its symbol matrix has almost everywhere rank m. Thus locally we may always assume that a general first-order equation splits into an equation of the form considered here plus some purely algebraic equations. Solving the latter ones, we can eliminate some of the dependent variables u^α and obtain then a smaller equation of the desired form. At least theoretically this is always possible.

By a classical result on jet bundles, the fibration $\pi^1_0 : J_1\pi \to \mathcal{E}$ is an affine bundle. A first-order differential equation $\mathcal{R}_1 \subset J_1\pi$ is *quasi-linear*, if it defines an affine subbundle. For such equations, we assume in the sequel a local representation of the form

$$\mathcal{R}_1 \,:\, \left\{ \; A(t, \boldsymbol{u})\dot{\boldsymbol{u}} = \boldsymbol{r}(t, \boldsymbol{u}) \right. \tag{4}$$

where the $m \times m$ matrix function A is almost everywhere non-singular.

Assumption 3. *In the sequel,* $\mathcal{R}_1 \subseteq J_1\pi$ *will always be a formally integrable square first-order ordinary differential equation with a local representation of the form (3) or (4), respectively, which is not underdetermined.*

3 The Vessiot Distribution

A key insight of Cartan was to study *infinitesimal solutions* of a differential equation $\mathcal{R}_1 \subseteq J_1\pi$, i. e. to consider at any point $\rho \in \mathcal{R}_1$ those linear subspaces $\mathcal{U}_\rho \subseteq T_\rho\mathcal{R}_1$ which are potentially part of the tangent space of a prolonged solution. We will follow here an approach pioneered by Vessiot [17] which is based on vector fields and dual to the more familiar Cartan-Kähler theory of exterior differential systems (see [4, 6, 14] for modern presentations in the context of the geometric theory). By Proposition 1, the tangent spaces $T_\rho(\operatorname{im} j_1\sigma)$ of prolonged sections at points $\rho \in J_1\pi$ are always subspaces of the contact distribution $\mathcal{C}_1|_\rho$. If the section σ is a solution of $\mathcal{R}_1$, it furthermore satisfies $\operatorname{im} j_1\sigma \subseteq \mathcal{R}_1$ by Definition 2 and hence $T(\operatorname{im} j_1\sigma) \subseteq T\mathcal{R}_1$. These considerations motivate the following construction.

Definition 4. *The* Vessiot distribution *of a first-order ordinary differential equation $\mathcal{R}_1 \subseteq J_1\pi$ is the distribution $\mathcal{V}[\mathcal{R}_1] \subseteq T\mathcal{R}_1$ defined by*

$$\mathcal{V}[\mathcal{R}_1] = T\mathcal{R}_1 \cap \mathcal{C}_1|_{\mathcal{R}_1} \ . \tag{5}$$

Computing the Vessiot distribution is straightforward and requires only linear algebra. It follows from Definition 4 that any vector field X contained in $\mathcal{V}[\mathcal{R}_1]$ is a contact field and thus can be written as a linear combination of the basic contact fields (1): $X = aC^{(1)}_{\mathrm{trans}} + \sum_\alpha b^\alpha C^{(1)}_\alpha$. On the other hand, X must be tangent to the manifold $\mathcal{R}_1$. Hence, if $\mathcal{R}_1$ is described by the local system (3), then the field X must satisfy the equations $d\boldsymbol{\Phi}(X) = X(\boldsymbol{\Phi}) = 0$. Evaluation of this condition yields a linear system of equations for the coefficients a, b^α:

$$C^{(1)}_{\mathrm{trans}}(\Phi^\mu)a + \sum_{\alpha=1}^{m} C^{(1)}_\alpha(\Phi^\mu)b^\alpha = 0 \ , \qquad \mu = 1, \dots, m \ . \tag{6}$$

Note that X is vertical with respect to π^1, if and only if the coefficient a vanishes. Concerning our Assumption 3 on the differential equation $\mathcal{R}_1$, we remark that equations of lower order are irrelevant for the Vessiot distribution provided that the equation is indeed formally integrable [14, Prop. 9.5.10].

Determining the Vessiot distribution of $\mathcal{R}_1$ requires essentially the same computations as prolonging it. Indeed, the prolongation $\mathcal{R}_2 \subseteq J_2\pi$ is locally described by the original equations $\Phi^\mu = 0$ together with their prolongations

$$C^{(1)}_{\mathrm{trans}}(\Phi^\mu) + \sum_{\alpha=1}^{m} C^{(1)}_\alpha(\Phi^\mu)\ddot{u}^\alpha = 0 \ , \quad \mu = 1, \dots, m \ . \tag{7}$$

These coincide with (6), if we set $a = 1$ and $b^\alpha = \ddot{u}^\alpha$, i. e. for transversal solutions of (6). One may say that (6) is a “projective” version of (7).

It should be stressed that we allow that the rank of a distribution varies from point to point. In fact, this will be important for certain types of singularities. The following, fairly elementary result is the basis of Vessiot’s approach to the existence theory of differential equations. It relates solutions with certain subdistributions of the Vessiot distribution. We formulate it here only for first-order ordinary differential equations; for the general case see [14, Prop. 9.5.7].

Lemma 5. *If the section $\sigma : \mathcal{T} \to \mathcal{E}$ is a solution of the first-order ordinary differential equation $\mathcal{R}_1 \subseteq J_1\pi$, then the tangent bundle $T(\operatorname{im} j_1\sigma)$ is a one-dimensional subdistribution of $\mathcal{V}[\mathcal{R}_1]|_{\operatorname{im} j_1\sigma}$ transversal to the fibration π^1. Conversely, if the subdistribution $\mathcal{U} \subseteq \mathcal{V}[\mathcal{R}_1]$ is one-dimensional and transversal, then any integral curve of $\mathcal{U}$ has locally the form $\operatorname{im} j_1\sigma$ for a solution σ of $\mathcal{R}_1$.*

Definition 6. *A* generalised solution *of the first-order ordinary differential equation $\mathcal{R}_1 \subseteq J_1\pi$ is an integral curve $\mathcal{N} \subseteq \mathcal{R}_1$ of the Vessiot distribution $\mathcal{V}[\mathcal{R}_1]$. The projection $\pi^1_0(\mathcal{N}) \subseteq \mathcal{E}$ is a* geometric solution*.*

Note that generalised solutions live in the jet bundle $J_1\pi$ and not in the base manifold $\mathcal{E}$. If the section $\sigma : \mathcal{T} \to \mathcal{E}$ is a classical solution, then the image of its prolongation $j_1\sigma : \mathcal{T} \to J_1\pi$ is a generalised solution. However, not every generalised solution $\mathcal{N}$ projects on a classical one: this will be the case, if and only if $\mathcal{N}$ is everywhere transversal to the fibration $\pi^1 : J_1\pi \to \mathcal{T}$.

4 Geometric Singularities

A *geometric singularity* of a differential equation $\mathcal{R}_1$ is a critical point $\rho \in \mathcal{R}_1$ of the restricted projection $\hat{\pi}^1_0 : \mathcal{R}_1 \to \mathcal{E}$, i. e. a point where the tangent map $T_\rho\hat{\pi}^1_0$ is not surjective. Following the terminology of Arnold [1] in the scalar case, we distinguish three types of points on $\mathcal{R}_1$. Note that this taxonomy makes only sense for formally integrable equations which are not underdetermined.

Definition 7. *Let $\mathcal{R}_1 \subseteq J_1\pi$ be a first-order ordinary differential equation satisfying Assumption 3. A point $\rho \in \mathcal{R}_1$ is* regular*, if $\mathcal{V}_\rho[\mathcal{R}_1]$ is one-dimensional and transversal to π^1, and* regular singular*, if $\mathcal{V}_\rho[\mathcal{R}_1]$ is one-dimensional and vertical. If $\dim \mathcal{V}_\rho[\mathcal{R}_1] > 1$, then ρ is an* irregular singularity*.*

For an equation of the form (3), we define the $m \times m$ matrix $A(t, \boldsymbol{u}^{(1)})$ and the m-dimensional vector $\boldsymbol{r}(t, \boldsymbol{u}^{(1)})$ by

$$A = \frac{\partial \boldsymbol{\Phi}}{\partial \dot{\boldsymbol{u}}} , \qquad \boldsymbol{d} = \frac{\partial \boldsymbol{\Phi}}{\partial t} + \frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{u}} \cdot \dot{\boldsymbol{u}} . \tag{8}$$

In addition to the (symbol) matrix $A(t, \boldsymbol{u}^{(1)})$, we introduce its determinant $\delta(t, \boldsymbol{u}^{(1)}) = \det A(t, \boldsymbol{u}^{(1)})$ and its adjugate $C(t, \boldsymbol{u}^{(1)}) = \operatorname{adj} A(t, \boldsymbol{u}^{(1)})$. Because of the well-known identity $\delta \mathbb{1}_m = AC = CA$, we find $\operatorname{im} C(\rho) \subseteq \ker A(\rho)$ and $\operatorname{im} A(\rho) \subseteq \ker C(\rho)$ for any point $\rho \in J_1\pi$ where $\delta(\rho) = 0$. For later use, we note that if additionally $\dim \ker A(\rho) = 1$ at such a point, then $C(\rho) \neq 0$, as $A(\rho)$ must possess at least one non-vanishing minor. Hence in this case, we even find $\operatorname{im} C(\rho) = \ker A(\rho)$ and $\operatorname{im} A(\rho) = \ker C(\rho)$.

The following considerations recover results by Rabier [10] from the point of view of Vessiot's theory and provide simpler alternative proofs. They show in particular that geometric singularities are characterised by the vanishing of δ. We therefore call the subset $\mathcal{S}_1 = \{\rho \in \mathcal{R}_1 \mid \delta(\rho) = 0\}$ the *singular locus* of the differential equation $\mathcal{R}_1$.

Theorem 8. *Let $\mathcal{R}_1 \subseteq J_1\pi$ be a first-order ordinary differential equation satisfying Assumption 3. A point $\rho \in \mathcal{R}_1$ is regular, if and only if $\operatorname{rk} A(\rho) = m$ and regular singular, if and only if $\operatorname{rk} A(\rho) = m - 1$ and $\boldsymbol{d}(\rho) \notin \operatorname{im} A(\rho)$. A regular point ρ has an open simply connected neighbourhood $\mathcal{U} \subseteq \mathcal{R}_1$ without any geometric singularity and there exists locally a unique strong solution σ such that $\rho \in \operatorname{im} j_1\sigma \subseteq \mathcal{U}$. A regular singular point ρ has an open simply connected neighbourhood $\mathcal{U} \subseteq \mathcal{R}_1$ without any irregular singularity and there exists locally a unique generalised solution $\mathcal{N}$ such that $\rho \in \mathcal{N} \subseteq \mathcal{U}$. If the neighbourhood $\mathcal{U}$ is chosen sufficiently small, then in both cases the Vessiot distribution is generated in $\mathcal{U}$ by the vector field*

$$X = \delta C^{(1)}_{\text{trans}} - (C\boldsymbol{d})^t \boldsymbol{C}^{(q)} \ . \tag{9}$$

Proof. The first two assertions follow from the fact that the matrix of the linear system (6) evaluated at a point $\rho \in \mathcal{R}_1$ is $\big(\boldsymbol{d}(\rho) \mid A(\rho)\big)$ and that the Vessiot distribution is one-dimensional, if and only if its rank is m. Since rank is an upper semicontinuous function, the Vessiot distribution will remain one-dimensional in a whole neighbourhood $\mathcal{U}$ of such a point. Hence any such neighbourhood cannot contain an irregular singularity where the rank of the matrix $\big(\boldsymbol{d}(\rho) \mid A(\rho)\big)$ must be less than m. If the point ρ is regular, we can apply the same argument to the matrix $A(\rho)$ and find that $\mathcal{U}$ cannot even contain a regular singularity.

If the neighbourhood $\mathcal{U}$ is chosen simply connected, then the one-dimensional distribution $\mathcal{V}[\mathcal{R}_1]$ can be generated in it by a single vector field without any zero. The explicit generator X of (9) is now obtained by simply multiplying (6) with the adjugate C. Note that the field X does indeed vanish nowhere, as even at a regular singularity where $\delta(\rho) = 0$ we find by the considerations above that $\ker C(\rho) = \operatorname{im} A(\rho)$ and thus $C(\rho)\boldsymbol{d}(\rho) \neq 0$ since $\boldsymbol{d}(\rho) \notin \operatorname{im} A(\rho)$.

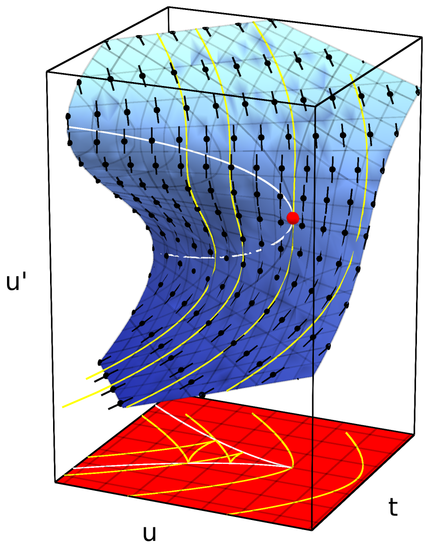
The existence of a unique local integral curve $\mathcal{N} \subseteq \mathcal{U}$ of X and thus of a unique generalised solution through ρ follows now by the usual existence and uniqueness theorems for vector fields. If ρ is a regular point, then X (and thus also $\mathcal{N}$) is everywhere on $\mathcal{U}$ transversal to π^1_0 and we can write $\mathcal{N} = \operatorname{im} \gamma$ for some section $\gamma : \mathcal{T} \to J_1\pi$. By Proposition 1, $\gamma = j_1\sigma$ for a section $\sigma : \mathcal{T} \to \mathcal{E}$. Thus in this case ρ even lies on a unique (prolonged) strong solution. □

A different formulation of the existence and uniqeness part of Theorem 8 can be found in [7, Thm. 4.1]. There the standard existence and uniqueness result for ordinary differential equations solved for the derivatives is generalised to arbitrary formally integrable equations without irregular singularities. In particular, it is shown that any strong solution can be extended until its prolongation hits either the boundary of $\mathcal{R}_1$ or a regular singularity.

Note that at a regular singularity $\delta = 0$ and hence the vector field X defined by (9) is indeed vertical. By an extension of the above argument, one can prove the following statement about irregular singularities [7, Thm. 4.2]. Any irregular singularity ρ lies on the boundary of an open simply connected neighbourhood $\mathcal{U} \subseteq \mathcal{R}_1$ without further irregular singularities. By Theorem 8, $\mathcal{V}[\mathcal{R}_1]$ is generated on $\mathcal{U}$ by a vector field X. Then any extension of X to ρ will vanish. The dynamics around an irregular singular point ρ is now to a large extent determined by the eigenvalues of the Jacobian of X at ρ. Usually, there are infinitely many strong (prolonged) solutions beginning or ending at such point.

Example 9. From a geometric point of view, it is straightforward to understand what happens at a regular singularity ρ (cf. also [7]). There are two possibilities.[1] If the sign of the ∂_t-component of X along $\mathcal{N}$ does change at ρ (the generic behaviour), then there exist precisely *two* strong solutions which both either end or start at $\pi^1_0(\rho)$ and which arise through the "folding" of the generalised solution $\mathcal{N}$ during the projection. No prolonged strong solution can go through ρ in this case, as the projection of $\mathcal{N}$ is not a graph at $\pi^1_0(\rho)$. Otherwise, the projection of $\mathcal{N}$ is a strong solution which, however, is only of class $\mathcal{C}^1$, as its second derivative at ρ cannot exist by the above comparison of prolongation and determination of the Vessiot distribution.

As a concrete example, we consider the equation $\dot{u}^3 - u\dot{u} - t = 0$ whose surface $\mathcal{R}_1 \subseteq J_1\pi$ corresponds to the elementary catastrophe known as gather or Whitney pleat. We call this equation the *elliptic gather*. The blue surface in the picture on the right is $\mathcal{R}_1$; the short black lines indicate the direction defined by the Vessiot distribution $\mathcal{V}[\mathcal{R}_1]$ at some points $\rho \in \mathcal{R}_1$. The white curve shows the singular locus. All points on it are regular singular points. The yellow curves depict some generalised solutions determined by numerically integrating the vector field X of Theorem 8 (which has also been computed numerically). Their projection to $\mathcal{E}$ is shown on the red plane.



Whenever a generalised solution crosses the singular locus, the projected curve changes its direction and thus ceases to be the graph of a function. An exception is the one generalised solution that goes through the "tip" of the singular locus, as here the Vessiot distribution is tangential to the singular locus. Its projection is still the graph of a function which, however, is at $t = 0$ only once differentiable; the second derivative blows up at this point.

As the following result shows, generically the Vessiot distribution is transversal to the singular locus at regular singularities and the singular locus is almost

[1] We ignore here the degenerate case that the generalised solution $\mathcal{N}$ through ρ is completely vertical, i. e. $\mathcal{N} \subseteq (\pi^1_0)^{-1}\bigl(\pi^1_0(\rho)\bigr)$; see, however, the next section.

everywhere a smooth manifold. We omit a proof of this proposition, as we can simply use the one given by Rabier [10]. He does not interpret the non-degeneracy condition in terms of the Vessiot distribution. But given the expression (9) for the generator X of $\mathcal{V}[\mathcal{R}_1]$, it is easy to see that his results imply the transversality of $\mathcal{V}[\mathcal{R}_1]$ to $\mathcal{S}_1$.

Proposition 10. *Let $\rho \in \mathcal{R}_1$ be a regular singular point. If*

$$\boldsymbol{v}^t \frac{\partial A}{\partial \dot{\boldsymbol{u}}}(\rho)\boldsymbol{v} \notin \operatorname{im} A(\rho) \tag{10}$$

for all non-vanishing vectors $\boldsymbol{v} \in \ker A(\rho)$, then the Vessiot distribution $\mathcal{V}_\rho[\mathcal{R}_1]$ is transversal to the singular locus $\mathcal{S}_1$ and the singular locus $\mathcal{S}_1$ is a smooth manifold in a neighbourhood of ρ.

Remark 11. One should note that we do *not* get here an equivalence between the condition (10) and transversality of the Vessiot distribution—or the vector field X—to the singular locus. The converse may become invalid when the determinant δ is of the form $\delta = \zeta^k$ for some function ζ and an exponent $k > 1$. In this case, transversality cannot be decided by using the differential $d\delta$, as it vanishes everywhere on the singular locus $\mathcal{S}_1$. However, Rabier's proof of Proposition 10 uses $d\delta$ and thus cannot be inverted in such situations.

5 Quasi-Linear Equations

We specialise now the results of the previous section to quasi-linear equations of the form (4). As we already indicated with our notations, the matrix A is then indeed the Jacobian with respect to the derivatives. Thus here the only difference is the fact that in the quasi-linear case A does not depend on the derivatives and we will continue to denote its determinant by δ and its adjugate by C. However, the vectors $\boldsymbol{r}$ and $\boldsymbol{d}$ are not related.

Lemma 12. *Let $\mathcal{R}_1 \subseteq J_1\pi$ be a quasi-linear equation satisfying Assumption 3. According to Theorem 8, on a simply conncected open subset $\mathcal{U} \subseteq \mathcal{R}_1$ without any irregular singularity the Vessiot distribution $\mathcal{V}[\mathcal{R}_1]$ is generated by the vector field X given by (9). This field is projectable to a vector field $Y = (\pi^1_0)_* X$ defined on $\pi^1_0(\mathcal{U}) \subseteq \mathcal{R}^{(1)}_0 \subseteq \mathcal{E}$ by*

$$Y = \delta\partial_t + (C\boldsymbol{r})^t\partial_{\boldsymbol{u}}\,. \tag{11}$$

Proof. Multiplication of (4) with the adjugate C yields the equation $\delta\dot{\boldsymbol{u}} = C\boldsymbol{r}$. Thus on $\mathcal{R}_1$ we may write $X = Y + (C\boldsymbol{d})^t\partial_{\dot{\boldsymbol{u}}}$. As all coefficients of Y depend only on t and $\boldsymbol{u}$, the field X is projectable to $\mathcal{E}$ and $(\pi^1_0)_* X = Y$. □

The coordinate form (11) shows that the field Y can be locally continued to points outside of the projection $\mathcal{R}^{(1)}_0$. More precisely, it follows from the definition of the adjugate that Y can be defined at any point $\xi \in \mathcal{E}$ where both A and $\boldsymbol{r}$ are defined. This observation allows us to study the singularities of $\mathcal{R}_1$ on $\mathcal{E}$ using the vector field Y instead of working in $J_1\pi$ with the Vessiot distribution $\mathcal{V}[\mathcal{R}_1]$.

Definition 13. *Let $\mathcal{R}_1 \subseteq J_1\pi$ be a quasi-linear differential equation with local representation (4) and $\mathcal{D} \subseteq \mathcal{E}$ the subset where both A and $\boldsymbol{r}$ are defined. $\xi \in \mathcal{D}$ is a* regular point *for $\mathcal{R}_1$, if Y_ξ is transversal to the fibration $\pi : \mathcal{E} \to \mathcal{T}$, and an* impasse point *otherwise. An impasse point is* irregular*, if $Y_\xi = 0$, and* regular *otherwise. The set of all impasse points is the* impasse hypersurface *$\mathcal{S}_0 \subseteq \mathcal{E}$. A* geometric solution *of the differential equation $\mathcal{R}_1$ is an integral curve $\mathcal{N} \subseteq \mathcal{E}$ of the vector field Y.*

In analogy to Theorem 8, one obtains the following characterisation of the different cases. We stress again that for quasi-linear equation all conditions live in $\mathcal{E}$ and not in $J_1\pi$.

Theorem 14. *Let $\mathcal{R}_1 \subseteq J_1\pi$ be a quasi-linear differential equation satisfying Assumption 3. A point $\xi \in \mathcal{D}$ is regular for $\mathcal{R}_1$, if and only if $\operatorname{rk} A(\xi) = m$. It is a regular impasse point, if and only if $\operatorname{rk} A(\xi) = m - 1$ and $\xi \notin \mathcal{R}_0^{(1)}$.*

Proof. The case of a regular point is obvious. If $\operatorname{rk} A(\xi) < m-1$, then all minors of $A(\xi)$ of size $m-1$ and thus also the adjugate $C(\xi)$ vanish. This fact implies then trivially that $Y_\xi = 0$. Hence there only remains the case $\operatorname{rk} A(\xi) = m - 1$. Recall from above that then $\operatorname{im} A(\xi) = \ker C(\xi)$. Since $\xi \notin \mathcal{R}_0^{(1)}$ is equivalent to $\boldsymbol{r}(\xi) \notin \operatorname{im} A(\xi)$, we thus find in this case that $Y_\xi \neq 0$, if and only if $\xi \notin \mathcal{R}_0^{(1)}$. □

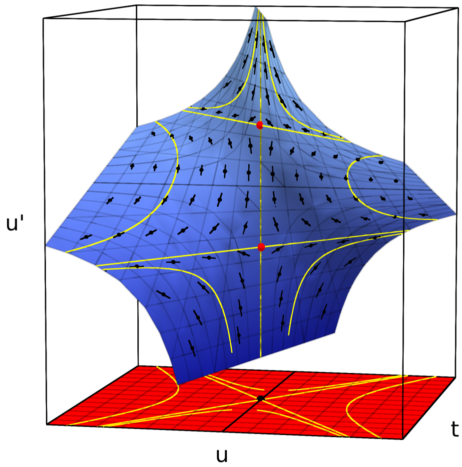
We compare now these notions with the corresponding ones for general equations introduced in Definition 7. Let $\rho \in \mathcal{R}_1$ be a point on our given differential equation and $\xi = \pi_0^1(\rho)$ its projection to $\mathcal{E}$. If ρ is a regular point, then trivially ξ is regular, too. Indeed, it follows immediately from (9) that in this case X_ρ is transversal to π^1 and hence its projection Y_ξ to π. As discussed above, it follows from [7, Thm. 4.2] that X_ρ vanishes, if ρ is an irregular singularity. Hence in this case ξ is an irregular impasse point.

Implicit in the above proof is the observation that regular singular points of a quasi-linear equation $\mathcal{R}_1$ *always* show a degenerate behaviour. Indeed, if ρ is a regular singularity, then the fibre $\mathcal{N} = (\pi_0^1)^{-1}(\xi) \cap \mathcal{R}_1$ is one-dimensional and consists entirely of regular singularities. Furthermore, in this degenerate case $\mathcal{N}$ is the unique generalised solution through ρ and it does not project onto a curve in $\mathcal{E}$ but the single point ξ.

The behaviour normally associated with regular singularities appears for quasi-linear equations at regular impasse points. One possibility is that two strong solutions can be extended so that they either start or end at ξ and together define the geometric solution through ξ. This will happen, if the sign of the ∂_t-component of Y changes at ξ. If the sign remains the same, then we find a unique "strong" solution through ξ which, however, is only $\mathcal{C}^0$ in ξ and thus strictly speaking cannot be considered as a solution. Concrete examples for both cases will be given in the next section.

Again a degenerate situation may arise. Let $\mathcal{R}_1$ be a *semi-linear* equation where the symbol matrix A depends only on the independent variable t. If now $\operatorname{rk} A(t_0) = m - 1$ for some point $t_0 \in \mathcal{T}$, then at all points $\xi \in \pi^{-1}(t_0) \subset \mathcal{E}$ the vector Y_ξ is either vertical or vanishes. Assuming that the latter happens only on some lower-dimensional subset, we find geometric solutions $\mathcal{N}$ lying completely in the fibre $\pi^{-1}(t_0)$. As these project on the single point $t_0 \in \mathcal{T}$, they cannot be interpreted as strong solutions with some singularities.

Example 15. Consider the quasi-linear equation $2u\dot{u} - t = 0$. Its singular locus $\mathcal{S}_1$ is the vertical line $t = u = 0$ which is simultaneously a generalised solution. Two points on it, $(0, 0, \pm 1)$, are irregular singularities; all other points are regular singularities. The impasse manifold $\mathcal{S}_0$ given by $u = 0$ contains one irregular impasse point at the origin. An explicit integration of this equation is easily possible and the solutions are of the implicit form $u^2 - t^2/2 = c$ for a constant $c \in \mathbb{R}$. For $c < 0$, the two branches of the square root always meet on $\mathcal{S}_0$ where the solution is not differentiable. Note that on one branch $\dot{u} \to \infty$ whereas on the other branch $\dot{u} \to -\infty$. For $c = 0$ one obtains the two lines intersecting at the origin. For $c > 0$ each branch of the square root yields one strong solution.



There also exists a special version of Proposition 10 for autonomous quasi-linear equations. In this case the condition (10) must be replaced by

$$\boldsymbol{v}^t \frac{\partial A}{\partial \boldsymbol{u}}(\xi)\boldsymbol{v} \notin \operatorname{im} A(\xi) \tag{12}$$

for all non-vanishing vectors $\boldsymbol{v} \in \ker A(\xi)$ and it ensures that the field Y is transversal to the impasse surface $\mathcal{S}_0$ at the impasse point ξ and that $\mathcal{S}_0$ is a smooth manifold in a neighbourhood of ξ. As already discussed in Remark 11, this condition is only sufficient but not necessary. The quasi-linear equation associated with a hysteresis point (see Example 18 below) represents a concrete counterexample where condition (12) fails at the hysteresis point but nevertheless the vector field Y is there transversal to the smooth manifold $\mathcal{S}_0$.

6 Static Bifurcations

We apply the results of the last section to the analysis of static bifurcations of a parametrised autonomous ordinary differential equation of the standard form $\boldsymbol{u}' = \boldsymbol{\phi}(t, \boldsymbol{u})$. Opposed to the conventions used so far in this work, t represents now the parameter and we denote the (not explicitly appearing) independent variable by x and hence derivatives with respect to it by $\boldsymbol{u}'$. We continue to consider t and $\boldsymbol{u}$ as coordinates on a fibred manifold $\pi : \mathcal{E} \to \mathcal{T}$.

For static bifurcations one analyses the dependence of solutions of the algebraic system $\boldsymbol{\phi}(t,\boldsymbol{u}) = 0$ on the parameter t, i.e. how the equilibria change as t varies. The solution set may be considered as a bifurcation diagram. Note that this represents a purely algebraic problem in $\mathcal{E}$. At certain bifurcation values of the parameter t the number of equilibria changes.

Definition 16. *The point $\xi = (t,\boldsymbol{u}) \in \mathcal{E}$ is a* turning point, *if*

$$\boldsymbol{\phi}(\xi) = 0 \quad \wedge \quad \dim\ker \frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{u}}(\xi) = 1 \quad \wedge \quad \frac{\partial\boldsymbol{\phi}}{\partial t}(\xi) \notin \operatorname{im}\frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{u}}(\xi)\,. \tag{13}$$

At a bifurcation point $\xi \in \mathcal{E}$, *the third condition is replaced by its converse:*

$$\boldsymbol{\phi}(\xi) = 0 \quad \wedge \quad \dim\ker \frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{u}}(\xi) = 1 \quad \wedge \quad \frac{\partial\boldsymbol{\phi}}{\partial t}(\xi) \in \operatorname{im}\frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{u}}(\xi)\,. \tag{14}$$

The rationale behind the above distinction is that at a turning point ξ all solutions of $\boldsymbol{\phi} = 0$ still lie on one smooth curve and the number of solutions only changes because this curve "turns" at ξ. At a bifurcation point several solution curves meet. In the bifurcation literature, much emphasis is put on distinguishing *simple* turning or bifurcation points from higher ones. In particular, the numerical analysis differs for non-simple points. It will turn out that in our approach such a distinction is irrelevant. We try to write the solutions as a function $\boldsymbol{u}(t)$. Differentiating the given algebraic system with respect to the parameter t yields then a square quasi-linear differential equation for this function:

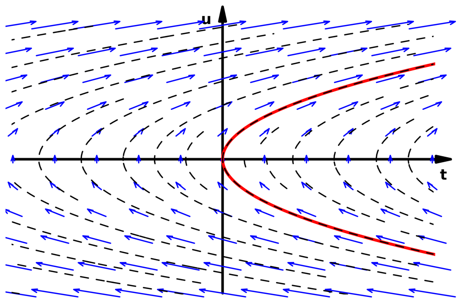
$$\frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{u}}(t,\boldsymbol{u})\dot{\boldsymbol{u}} + \frac{\partial\boldsymbol{\phi}}{\partial t}(t,\boldsymbol{u}) = 0\,. \tag{15}$$

Thus we set $A(t,\boldsymbol{u}) = \frac{\partial\boldsymbol{\phi}}{\partial\boldsymbol{u}}(t,\boldsymbol{u})$ and $\boldsymbol{d}(t,\boldsymbol{u}) = \frac{\partial\boldsymbol{\phi}}{\partial t}(t,\boldsymbol{u})$. The bifurcation diagram consists now of those geometric solutions of (15) on which $\boldsymbol{\phi}$ vanishes.

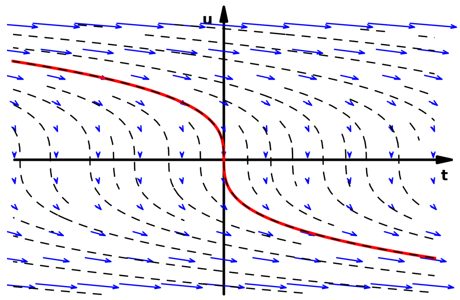
Theorem 17. *$\xi \in \mathcal{E}$ is a turning point, if and only if $\boldsymbol{\phi}(\xi) = 0$ and ξ is a regular impasse point of (15). $\xi \in \mathcal{E}$ is a bifurcation point, if and only if $\boldsymbol{\phi}(\xi) = 0$ and ξ is an irregular impasse point of (15) where* $\operatorname{rk} A(\xi) = m - 1$. *In this case each branch of the bifurcation diagram is tangent to an eigenvector of the Jacobian of the vector field Y at ξ for an eigenvalue with non-vanishing real part.*

Proof. The first two assertions follow immediately from comparing Definition 16 with Theorem 14. For the last assertion we note that an irregular impasse point is, by definition, an equilibrium of the vector field Y and hence it follows from basic dynamical systems theory. $\square$

Example 18. We compare a *simple turning point* or (*saddle node bifurcation*) with a *hysteresis point* (a degenerate turning point). As all our geometric considerations remain invariant under coordinate transformation, we take for simplicity the well-known normal forms of such points: $\phi_1(t,u) = t - u^2$ and $\phi_2 = t - u^3$. The corresponding quasi-linear equations are $2u\dot{u} = 1$ and $3u^2\dot{u} = 1$. Thus in both cases the impasse manifold is given by the equation $u = 0$ and consists entirely of regular impasse points. A straightforward calculation yields for the vector field defined by (11) $Y_1 = 2u\partial_t + \partial_u$ and $Y_2 = 3u^2\partial_t + \partial_u$, respectively. In both cases the origin is the sole turning point in the sense of Definition 16.



Simple Turning Point

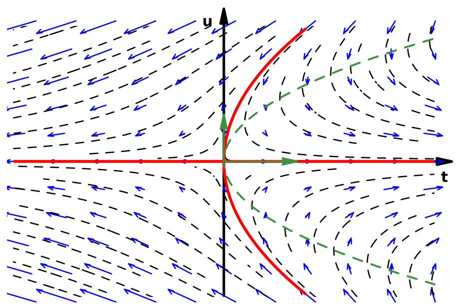


Hysteresis Point

The above pictures show the fields Y_1 and Y_2 and their streamlines.[2] The red curve is the bifurcation diagram. One clearly sees that even in the degenerate case there are no particular numerical problems. The only difference between the two cases is the behaviour of the ∂_t-component of Y: on the left it changes sign when going through the impasse point, on the right it does not. As explained in the previous section, this observation entails that on the left we have two strong solutions starting arbitrarily close to the origin and extendable to the origin, whereas on the right we find one solution going through the origin which is, however, not differentiable there.

Example 19. The normal form of a *pitchfork bifurcation* is $\phi(t,u) = tu - u^3$. The associated quasi-linear equation is then $(t - 3u^2)\dot{u} + u = 0$ with singular locus $\mathcal{S}_1 \subseteq \mathcal{R}_1$ given by the points satisfying in addition $3u^2 = t$. The impasse surface $\mathcal{S}_0 \subseteq \mathcal{E}$ is also described by this equation. The origin is an irregular impasse point; all other points on $\mathcal{S}_0$ are regular. The vector field of (11) is given by $Y = (t - 3u^2)\partial_t - u\partial_u$ and vanishes at the origin as required for an irregular impasse point.

The picture on the right exhibits the vector field Y, its streamlines and the eigenvectors of the Jacobian of Y at the origin $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Obviously, they are tangent to the bifurcation diagram consisting of the invariant manifolds of Y. The picture also nicely shows how the pitchfork bifurcation arises when two turning points moving on the dashed green line collide.



[2] Only one of the streamlines satisfies the algebraic equation $\boldsymbol{\phi}(\xi) = 0$ and thus represents the bifurcation diagram we want. The other streamlines can be interpreted as bifurcation diagrams of the perturbed problems $\boldsymbol{\phi}(\xi) = \epsilon$ with a constant $\epsilon \in \mathbb{R}$. Indeed, such a constant disappears when we differentiate in order to obtain our quasi-linear equation. Thus strictly speaking, we simultaneously analyse a whole family of bifurcation problems in our approach.

We remark that from this geometric point of view a pitchfork and a *transcritical bifurcation* are very similar. There arise only minor differences how the invariant manifolds lie relative to the eigenspaces. We omit therefore the details for a transcritical bifurcation.

7 Conclusions

In this work we used Vessiot's vector field based approach to differential equations for an analysis of geometric singularities of ordinary differential equations satisfying certain basic assumptions. We clarified the special role of quasi-linear equations where the geometric analysis can be performed on the basic fibred manifold $\mathcal{E}$ instead of the jet bundle $J_1\pi$. This observation is not surprising, as one can see similar effects also at other places in the theory of differential equations. In [14, Addendum Sect. 9.5] the method of characteristics is reviewed from a geometric point of view. There it also turns out that one may consider the characteristics for linear equations even on $\mathcal{T}$ and for quasi-linear ones also on $\mathcal{E}$. Only for fully non-linear equations one must use the jet bundle $J_1\pi$.

As an application we studied the theory of static bifurcations of autonomous ordinary differential equations. We associated a quasi-linear differential equation and thus a certain vector field Y on the base manifold $\mathcal{E}$ with such a bifurcation problem. Then we showed that the distinction between turning and bifurcation points corresponds to the distinction between regular and irregular impasse points of the associated quasi-linear equation. For equations admitting only static bifurcation points, these results may lead to a simpler way to determine bifurcation diagrams. Instead of using continuation methods one can simply integrate the vector field Y. Furthermore, around a bifurcation point ξ there is no need to search where branches may head, as they can only emerge in the direction of eigenvectors of the Jacobian of Y at ξ.

We did not consider the question of recognising simple turning or bifurcation points. As already noted by Rabier and Rheinboldt [11], simple turning points are characterised by the fact that there also the condition (10) is satisfied. We discussed in Remark 11 that from a geometric point of view this condition and consequently the concept of a simple turning point is not fully satisfactory (or of a more technical nature), since (10) is not equivalent to the transversality of the vector field X to the singular locus $\mathcal{S}_1$ (or the transversality of Y to the impasse hypersurface $\mathcal{S}_0$, respectively). The condition (10) always fails, if the determinant δ provides a degenerate description of $\mathcal{S}_1$ (or $\mathcal{S}_0$).

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