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Spencer Cohomology, Differential Equations, and Pommaret Bases

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1 Introduction

A key notion in the theory of general (i. e. including under- or overdetermined) systems of differential equations is involution. As we will see it may be understood as a simultaneous abstraction and generalisation of Gröbner bases for polynomial ideals to differential equations (without any restriction to linear or polynomial systems). Without the concept of involution (or some variation of it like passivity in Janet–Riquier theory [27, 48] or differential Gröbner bases [37]), one cannot prove general existence and uniqueness theorems like the Cartan–Kähler theorem.

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The terminology "involutive" appeared probably first in the 19th century in the analysis of overdetermined systems of first-order linear differential equations in one unknown function. Nowadays these works are subsumed by the Frobenius theorem which is usually treated in differential geometry (where one still has the notion of an involutive distribution) and no longer in differential equations theory. The first complete theories of arbitrary differential equations were reached in the early 20th century with the Janet–Riquier and the Cartan–Kähler theory [6, 9, 24, 29], the latter one formulated in the language of exterior differential systems (Cartan also provided a differential equations version for linear first-order systems in [8]).

The Janet–Riquier theory is completely based on local coordinate computations and requires the introduction of a ranking in analogy to the term orders used to define Gröbner bases. By contrast, the Cartan–Kähler theory is in principle intrinsic, but in the classical approach the decision whether or not a given exterior differential system is involutive requires for the so-called Cartan test at least the introduction of a local basis on the tangent bundle which is often done via coordinates.

Only much later it was realised that Cartan's test is actually of a homological nature. The homological approach to involution was mainly pioneered by Spencer [57] and collaborators [18, 19, 46]; later discussions can be found e. g. in [6, 10, 12, 28, 32, 30, 34, 36, 38, 42]. However, one should mention that the Spencer cohomology appeared first not in the context of differential equations but in deformation theory [56].

This contribution is largely a review; most results are well-known to specialists. However, these specialists are divided into two classes: many experts in the formal theory of differential equations are familiar with Spencer cohomology but much less with commutative algebra; conversely, few experts in commutative algebra know the formal theory. This clear division into two communities is the main reason why even elementary facts like that the degree of involution and the Castelnuovo–Mumford regularity coincide have remained unnoticed for a long time. It is our hope that this article may help to bridge this gap.

Some novel aspects are contained in the use of Pommaret bases; this concerns in particular Chapter 5 (parts of this material is also contained in [22]). While we do not discuss here any algorithmic aspects (this is done in [22]), it should be mentioned that by relating concepts like involution or the Castelnuovo–Mumford regularity to Pommaret bases, we make them immediately accessible for effective computations.

The article is organised as follows. The next chapter introduces axiomatically the polynomial de Rham complex and its dualisation, the Koszul complex. The Spencer cohomology and the Koszul homology of a (co)module arise then by (co)tensoring with the (co)module. Since the symmetric algebra is Noetherian by Hilbert's basis theorem, it is straightforward to prove a number of finiteness statements for the Spencer cohomology via dualisation to the Koszul side which are otherwise quite hard to obtain. It seems that the duality between the Spencer cohomology and the Koszul homology was first noted by Singer and Sternberg [55] (but see also [46, Lemma 5.5]) who attributed it to private discussions with Grothendieck and Mumford. An independent proof was later given by Ruiz [49]. The chapter closes by defining involution as the vanishing of the Spencer cohomology (or the dual Koszul homology, resp.).

Chapter 2 is concerned with the Cartan test for deciding whether a symbolic system is involutive. It represents a homological reformulation of the classical Cartan test in the theory of exterior differential systems and is due to Matsushima [39, 40]. We then discuss the dual version of the Cartan test developed by Serre in a letter appended to [21]. While the notion of involution is intrinsically defined, any form of Cartan's test requires the introduction of coordinates and it turns out that in certain "bad" coordinate systems the test fails. This problem is known under the name δ- or quasi-regularity and appears in all versions of the Cartan test.

Chapter 4 recalls briefly the notion of an involutive basis for a polynomial ideal with particular emphasis on Pommaret bases. Involutive bases represent a special kind of Gröbner bases with additional combinatorial properties; they were introduced by Gerdt and Blinkov [15] combining ideas from the Janet–Riquier theory of differential equations with the classical theory of Gröbner bases. It is shown that the Pommaret basis with respect to the degree reverse lexicographic order contains many structural information. This chapter essentially summarises some of the results of [51].

Most invariants that can be read off from a Pommaret basis are of a homological nature. Therefore we study in Chapter 5 the relation between the Pommaret basis (for the degree reverse lexicographic order) of an ideal $\mathcal{I}$ and the Koszul homology of the factor algebra $\mathcal{P}/\mathcal{I}$ in more detail. The presented results only scratch at the surface of this question. It is a conjecture of us that for Pommaret bases the Schreyer Theorem can be significantly generalised so that it yields explicit bases for the whole Koszul homology and not only for the degree-1-part. This entails that in contrast to general Gröbner bases this special kind of bases is to a large extent determined by the structure of the ideal.

The last two chapters demonstrate how the algebraic theory developed in the previous chapters can be applied to general differential equations. For this purpose, a differential equation is defined geometrically as submanifold of a jet bundle. The fundamental identification leads to a natural polynomial structure in the hierarchy of jet bundles. It allows us to associate with each differential equation a symbolic system (or dually a polynomial module) so that involution can be effectively decided with any form of Cartan's test. We also discuss why 2-acyclicity implies formal integrability; going to the Koszul side this becomes an elementary statements about syzygies.

Finally, some conclusions are given. Two small appendices fix the used notations concerning multi indices and term orders, respectively. A slightly larger appendix gives an introduction to coalgebras and comodules.

2 Spencer Cohomology and Koszul Homology

Let $\mathcal{V}$ be an n-dimensional vector space over a field $\mathbb{k}$;[1] over $\mathcal{V}$ one has the symmetric algebra $S\mathcal{V}$ and the exterior algebra $\Lambda\mathcal{V}$. We introduce two natural complexes based on the product spaces $S_q\mathcal{V} \otimes \Lambda_p\mathcal{V}$. Any element of such a space may be written as a $\mathbb{k}$-linear sum of separable elements, i. e. elements of the form $w_1 \cdots w_q \otimes v_1 \wedge \cdots \wedge v_p$ with $w_i, v_j \in \mathcal{V}$. By convention, we set $S_j\mathcal{V} = 0$ for $j < 0$.

[1] For simplicity, we assume throughout that $\operatorname{char} \mathbb{k} = 0$.

Definition 2.1 For any integer $r \geq 0$ the complex

$$\begin{array}{c} 0 \longrightarrow S_r\mathcal{V} \xrightarrow{\delta} S_{r-1}\mathcal{V} \otimes \mathcal{V} \xrightarrow{\delta} S_{r-2}\mathcal{V} \otimes \Lambda_2\mathcal{V} \xrightarrow{\delta} \cdots \\ \cdots \xrightarrow{\delta} S_{r-n}\mathcal{V} \otimes \Lambda_n\mathcal{V} \longrightarrow 0 \end{array} \tag{2.1}$$

where the differential δ is defined by[2]

$$\delta(w_1 \cdots w_q \otimes v_1 \wedge \cdots \wedge v_p) = \sum_{i=1}^{q} w_1 \cdots \widehat{w_i} \cdots w_q \otimes w_i \wedge v_1 \wedge \cdots \wedge v_p \tag{2.2}$$

is called the *polynomial de Rham complex* $R_r(S\mathcal{V})$ at degree r over the vector space $\mathcal{V}$. The *Koszul complex* $K_r(S\mathcal{V})$ at degree r over $\mathcal{V}$ is given by

$$\begin{array}{c} 0 \longrightarrow S_{r-n}\mathcal{V} \otimes \Lambda_n\mathcal{V} \xrightarrow{\partial} S_{r-n+1}\mathcal{V} \otimes \Lambda_{n-1}\mathcal{V} \xrightarrow{\partial} \cdots \\ \cdots \xrightarrow{\partial} S_r\mathcal{V} \longrightarrow 0 \end{array} \tag{2.3}$$

where now the differential ∂ is defined as

$$\partial(w_1 \cdots w_q \otimes v_1 \wedge \cdots \wedge v_p) = \sum_{i=1}^{p} (-1)^{i+1} w_1 \cdots w_q v_i \otimes v_1 \wedge \cdots \wedge \widehat{v_i} \wedge \cdots \wedge v_p\ . \tag{2.4}$$

It is trivial to verify that, due to the skew-symmetry of the wedge product, the differentials satisfy $\delta^2 = 0$ and $\partial^2 = 0$, so that we are indeed dealing with complexes.
Let $\{x^1, \dots, x^n\}$ be a basis of $\mathcal{V}$. Then a basis of the vector space $S_q\mathcal{V}$ is given by all terms x^μ with μ a multi index[3] of length q. For a basis of the vector space $\Lambda_p\mathcal{V}$ we use the following convention: let I be a *sorted* repeated index of length p, i. e. $I = (i_1, \dots, i_p)$ with $1 \leq i_1 < i_2 < \cdots < i_p \leq n$; then we write x^I for $x^{i_1} \wedge \cdots \wedge x^{i_p}$ and the set of all such "terms" provides a basis of $\Lambda_p\mathcal{V}$. With respect to these bases, we obtain the following expressions for the above differentials:

$$\delta(x^\mu \otimes x^I) = \sum_{i=1}^{n} \operatorname{sgn}\bigl(\{i\} \cup I\bigr) \mu_i x^{\mu - 1_i} \otimes x^{\{i\} \cup I} \tag{2.5}$$

and

$$\partial(x^\mu \otimes x^I) = \sum_{j=1}^{p} (-1)^{j+1} x^{\mu + 1_{i_j}} \otimes x^{I \setminus \{i_j\}}\ . \tag{2.6}$$

Formally, (2.5) looks like the exterior derivative applied to a differential p-form with polynomial coefficients. This observation explains the name "polynomial de Rham complex" for (2.1) and in principle one should use the usual symbol d for the differential but the notation δ has become standard.

[2] The hat signals that the corresponding factor is omitted.
[3] See Appendix A for the used conventions on multi indices.

Remark 2.2 While the de Rham differential δ indeed depends on the algebra structure of the exterior algebra $\Lambda\mathcal{V}$, it exploits only the vector space structure of the symmetric algebra $S\mathcal{V}$. Thus we may substitute the symmetric algebra $S\mathcal{V}$ by the symmetric *coalgebra*[4] $\mathfrak{S}\mathcal{V}$ and define δ on the components of the free $\mathfrak{S}\mathcal{V}$-comodule $\mathfrak{S}\mathcal{V} \otimes \Lambda\mathcal{V}$, since both are identical as vector spaces. It is not difficult to verify that with this interpretation the differential δ is a comodule morphism. In fact, we will see later that in our context this comodule interpretation is even more natural. It is somewhat surprising that this point of view was introduced only very recently in [33]. For the Koszul differential ∂ we have the opposite situation: we need the algebra $S\mathcal{V}$ but only the vector space $\Lambda\mathcal{V}$. Thus one could similarly $\Lambda\mathcal{V}$ replace by the exterior coalgebra, however, this will not become relevant for us.

Lemma 2.3 *We have* $\bigl(\delta \circ \partial + \partial \circ \delta\bigr)(\omega) = (p+q)\omega$ *for all* $\omega \in S_q\mathcal{V} \otimes \Lambda_p\mathcal{V}$.

Proof. For $\omega = w_1 \cdots w_q \otimes v_1 \wedge \cdots \wedge v_p$ one readily computes that

$$(\partial \circ \delta)(\omega) = q\omega + \sum_{i=1}^{q}\sum_{j=1}^{p}(-1)^j w_1 \cdots \widehat{w_i} \cdots w_q v_j \otimes w_i \wedge v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_p \quad (2.7)$$

and similarly

$$(\delta \circ \partial)(\omega) = p\omega + \sum_{j=1}^{p}\sum_{i=1}^{q}(-1)^{j+1} w_1 \cdots \widehat{w_i} \cdots w_q v_j \otimes w_i \wedge v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_p \quad (2.8)$$

which immediately implies our claim. ∎

Proposition 2.4 *The complexes* $R_q(S\mathcal{V})$ *and* $K_q(S\mathcal{V})$ *are exact for all values* $q > 0$. *For* $q = 0$ *both complexes are of the form* $0 \to \Bbbk \to 0$.

Proof. This is an immediate consequence of Lemma 2.3. It implies that for $q > 0$ the map ∂ induces a contracting homotopy for $R_q(S\mathcal{V})$ and conversely δ for $K_q(S\mathcal{V})$ connecting the respective identity and zero maps. It is well-known that the existence of such a map entails exactness. ∎

For the polynomial de Rham complex, this result is also known as the *formal Poincaré Lemma*, as one may interpret it as a special case of the Poincaré Lemma for general differential forms. We consider the complexes $R_q(S\mathcal{V})$ and $K_q(S\mathcal{V})$ as homogeneous components of complexes $R(S\mathcal{V})$ and $K(S\mathcal{V})$ over the $S\mathcal{V}$-modules $S\mathcal{V} \otimes \Lambda_i\mathcal{V}$. Since $S_0\mathcal{V} = \Bbbk$, we find that the Koszul complex $K(S\mathcal{V})$ defines a free resolution of the ground field $\Bbbk$. Similarly, the polynomial de Rham complex $R(S\mathcal{V})$ may be considered as a free coresolution of $\Bbbk$.
The polynomial de Rham and the Koszul complex are related by duality [46, 49, 55]. Recall that we may introduce for any complex of $\mathcal{R}$-modules its dual complex obtained by applying the functor $\mathrm{Hom}_{\mathcal{R}}(\cdot, \mathcal{R})$. In the case of finite-dimensional vector spaces, it is well-known that the homology of the dual complex is the dual space of the cohomology of the original complex.

[4] See Appendix C for some information about coalgebras and comodules.

Remark 2.5 There exists a canonical isomorphism $S_q(\mathcal{V}^*) \cong (S_q\mathcal{V})^*$: any separable element $\phi_1 \cdots \phi_q \in S_q(\mathcal{V}^*)$ is interpreted as the linear map on $S_q\mathcal{V}$ obtained by setting

$$(\phi_1 \cdots \phi_q)\bigl(v_1 \cdots v_q\bigr) = \sum_{\pi \in \mathcal{S}_q} \prod_{i=1}^{q} \phi_i(v_{\pi(i)}) \tag{2.9}$$

where $\mathcal{S}_q$ denotes the symmetric group of all permutations of $1, \dots, q$. The same construction can be applied to exterior products and thus we can extend to a canonical isomorphism $S_q(\mathcal{V}^*) \otimes \Lambda_p(\mathcal{V}^*) \cong (S_q\mathcal{V} \otimes \Lambda_p\mathcal{V})^*$.
At the level of bases, this isomorphism takes the following form. We denote again by $\{x^1, \dots, x^n\}$ a basis of $\mathcal{V}$ and by $\{y_1, \dots, y_n\}$ the corresponding dual basis of $\mathcal{V}^*$. Then the monomials x^μ with $|\mu| = q$ form a basis of $S_q\mathcal{V}$ and similarly the monomials $y_\mu = y_1^{\mu_1} \cdots y_n^{\mu_n}$ with $|\mu| = q$ form a basis of $S_q(\mathcal{V}^*)$. However, these two bases are *not* dual to each other, since according to (2.9) $y_\mu(x^\nu) = \mu!\delta_\mu^\nu$. Thus the dual basis consists of the *divided powers* $\frac{y_\mu}{\mu!}$. For the exterior algebra no such combinatorial factor arises, as the evaluation of the expression corresponding to the right hand side of (2.9) on basis vectors yields only one non-vanishing summand.
Another way to see that the dualisation leads to the divided powers is based on the coalgebra approach of Remark 2.2. If we substitute in the definition of the polynomial de Rham complex the symmetric algebra $S\mathcal{V}$ by the symmetric coalgebra $\mathfrak{S}\mathcal{V}$, then the dual algebra is $S(\mathcal{V}^*)$ and evaluation of the convolution product (C.3) leads to (2.9).

Proposition 2.6 *$\bigl(R(S\mathcal{V})^*, \delta^*\bigr)$ is isomorphic to $\bigl(K\bigl(S(\mathcal{V}^*)\bigr), \partial\bigr)$.*

Proof. There only remains to show that ∂ is indeed the pull-back of δ. Choosing the above described dual bases, this is a straightforward computation. By definition of the pull-back,

$$\delta^*\Bigl(\frac{y^\mu}{\mu!} \otimes y^I\Bigr)(x^\nu \otimes x^J) = \begin{cases} v_j \operatorname{sgn}\bigl(\{j\} \cup J\bigr) & \text{if } \exists j : \left\{\begin{array}{l} \mu = \nu - 1_j \\ I = \{j\} \cup J \end{array}\right\}, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.10}$$

Note that $\nu_j = \frac{\nu!}{\mu!}$ if $\mu = \nu - 1_j$; hence we find that

$$\delta^*(y^\mu \otimes y^I) = \sum_{j=1}^{p} (-1)^{j+1} y^{\mu + 1_{i_j}} \otimes y^{I \setminus \{i_j\}}\,. \tag{2.11}$$

Comparison with (2.6) yields the desired result. □

For reasons that will become apparent in Chapter 7 when we apply the here developed algebraic theory to differential equations, we prefer to consider the Koszul complex over the vector space $\mathcal{V}$ and the polynomial de Rham complex over its dual space $\mathcal{V}^*$. Thus we will always use $R\bigl(S(\mathcal{V}^*)\bigr)$ and $K(S\mathcal{V})$. If $\mathcal{U}$ is a further finite-dimensional vector space over $\Bbbk$ with $\dim \mathcal{U} = m$, then we may extend to the tensor product complex $R\bigl(S(\mathcal{V}^*) \otimes \mathcal{U}\bigr) = R\bigl(S(\mathcal{V}^*)\bigr) \otimes \mathcal{U}$ and dually to $K(S\mathcal{V} \otimes \mathcal{U}^*) = K(S\mathcal{V}) \otimes \mathcal{U}^*$.

Everything we have done so far remains valid with trivial modifications, as the differentials of the complexes are essentially unaffected by this operation. Basically, one must only add a factor $u \in \mathcal{U}$ (or $\nu \in \mathcal{U}^*$, respectively) to each equation and consider all our computations above as componentwise.

Definition 2.7 Let $\mathcal{N}_q \subseteq S_q(\mathcal{V}^*) \otimes \mathcal{U}$ be an arbitrary vector subspace. Its *(first) prolongation* is the subspace

$$\mathcal{N}_{q,1} = \left\{ f \in S_{q+1}(\mathcal{V}^*) \otimes \mathcal{U} \mid \delta(f) \in \mathcal{N}_q \otimes \mathcal{V}^* \right\} . \tag{2.12}$$

A sequence of vector subspaces $\left(\mathcal{N}_q \subseteq S_q(\mathcal{V}^*) \otimes \mathcal{U}\right)_{q\in\mathbb{N}_0}$ is called a *symbolic system* over $\mathcal{V}^*$, if $\mathcal{N}_{q+1} \subseteq \mathcal{N}_{q,1}$ for all $q \in \mathbb{N}_0$.

We may equivalently introduce the prolongation as

$$\mathcal{N}_{q,1} = (\mathcal{V} \otimes \mathcal{N}_q) \cap \left(S_{q+1}(\mathcal{V}^*) \otimes \mathcal{U}\right) \tag{2.13}$$

with the intersection understood to take place in $\mathcal{V}^* \otimes \left(S_q(\mathcal{V}^*) \otimes \mathcal{U}\right)$. This follows immediately from the definition of the differential δ. The extension to higher prolongations $\mathcal{N}_{q,r} \subseteq S_{q+r}(\mathcal{V}^*) \otimes \mathcal{U}$ proceeds either by induction, $\mathcal{N}_{q,r+1} = (\mathcal{N}_{q,r})_{,1}$ for all $r \in \mathbb{N}$, or alternatively by generalising (2.13) to $\mathcal{N}_{q,r} = \left(\bigotimes_{i=1}^r \mathcal{V}^* \otimes \mathcal{N}_q\right) \cap \left(S_{q+r}(\mathcal{V}^*) \otimes \mathcal{U}\right)$ where the intersection is now understood to take place in $\bigotimes_{i=1}^r \mathcal{V}^* \otimes \left(S_q(\mathcal{V}^*) \otimes \mathcal{U}\right)$. The notion of a symbolic system is fairly classical in the formal theory of differential equations (see Proposition 7.6). The next result shows, however, that if we take the coalgebra point of view of the polynomial de Rham complex mentioned in Remark 2.2, then a symbolic system is equivalent to a simple algebraic structure.

Lemma 2.8 *Let $(\mathcal{N}_q)_{q\in\mathbb{N}_0}$ be a symbolic system. Then $\mathcal{N} = \bigoplus_{q=0}^\infty \mathcal{N}_q$ is a graded (right) subcomodule of the free $\mathfrak{S}(\mathcal{V}^*)$-comodule $\mathfrak{S}(\mathcal{V}^*) \otimes \mathcal{U}$. Conversely, the sequence $(\mathcal{N}_q)_{q\in\mathbb{N}_0}$ of the components of any graded (right) subcomodule $\mathcal{N} \subseteq \mathfrak{S}(\mathcal{V}^*) \otimes \mathcal{U}$ defines a symbolic system.*

Proof. Let $(\mathcal{N}_q)_{q\in\mathbb{N}_0}$ be a symbolic system and $f \in \mathcal{N}_q$. Then $\delta f \in \mathcal{N}_{q-1} \otimes \mathcal{V}$ and hence $\partial f / \partial x^i \in \mathcal{N}_{q-1}$ for all $1 \leq i \leq n$, since our differential δ is just the exterior derivative. Using induction we thus find that $\partial^{|\mu|} f / \partial x^\mu \in \mathcal{N}_{q-r}$ for all μ with $|\mu| = r$. By the definition of the polynomial coproduct, this is equivalent to $\Delta(f) \in \mathcal{N} \otimes \mathfrak{S}(\mathcal{V}^*)$ and hence $\mathcal{N}$ is a subcomodule. For the converse, we simply revert every step of this argument to find that $\Delta(f) \in \mathcal{N} \otimes \mathfrak{S}(\mathcal{V}^*)$ implies that $\mathcal{N}_q \subseteq \mathcal{N}_{q-1,1}$ for all $q > 0$. ◻

Example 2.9 Let $\mathcal{V}$ be a two-dimensional space. The subspaces $\mathcal{N}_0 = \mathbb{k}$, $\mathcal{N}_1 = \mathcal{V}^*$ and $\mathcal{N}_q = \langle y_1^q \rangle \subset S_q(\mathcal{V}^*)$ for $q \geq 2$ define a symbolic system where $\mathcal{N}_{q,1} = \mathcal{N}_{q+1}$ for all $q \geq 2$. Indeed, if $k + \ell = q$, then $\delta\left(y_1^k y_2^\ell\right) = y_1^{k-1} y_2^\ell \otimes y_1 + y_1^k y_2^{\ell-1} \otimes y_2$ so that the result lies in $\mathcal{N}_{q-1} \otimes \mathcal{V}^*$ only for $\ell = 0$. We will see later that this symbolic system is associated with partial differential equations of the form $u_{22} = F(\mathbf{x}, u^{(1)})$, $u_{12} = G(\mathbf{x}, u^{(1)})$ where the shorthand $u^{(q)}$ denotes the unknown function u depending on $\mathbf{x} = (x^1, x^2)$ and all its derivatives up to order q.

Another simple symbolic system over the same dual space $\mathcal{V}^*$ is given by $\mathcal{N}_0 = \Bbbk$, $\mathcal{N}_1 = \mathcal{V}^*$, $\mathcal{N}_2 = S_2(\mathcal{V}^*)$, $\mathcal{N}_3 = \langle y_1^2 y_2, y_1 y_2^2\rangle$, $\mathcal{N}_4 = \langle y_1^2 y_2^2\rangle$ and $\mathcal{N}_q = 0$ for all $q \geq 5$. This system is related to partial differential equations of the form $u_{222} = F(\mathbf{x}, u^{(2)})$, $u_{111} = G(\mathbf{x}, u^{(2)})$. One can show that any such equation has a finite-dimensional solution space and this fact is reflected by the vanishing of the associated symbolic system beyond a certain degree.

From now on, we will not distinguish between a symbolic system $(\mathcal{N}_q)_{q\in\mathbb{N}_0}$ and the corresponding subcomodule $\mathcal{N} \subseteq \mathfrak{S}(\mathcal{V}^*) \otimes \mathcal{U}$. We are particularly interested in subcomodules $\mathcal{N}$ where almost all components $\mathcal{N}_q$ are different from zero (i. e. as a vector space $\mathcal{N}$ is infinite-dimensional). Recall that it follows immediately from the definition of the polynomial coproduct that cogeneration in $\mathfrak{S}(\mathcal{V}^*)$ always leads to elements of at most the same degree as the cogenerator; hence a finitely cogenerated comodule is necessarily finite-dimensional as vector space. However, the duality between $\mathfrak{S}(\mathcal{V}^*)$ and $S\mathcal{V}$ yields easily the following result.

Corollary 2.10 *Let $(\mathcal{N}_q)_{q\in\mathbb{N}_0}$ be an arbitrary symbolic system. There exists an integer $r_0 \geq 0$ such that $\mathcal{N}_{r+1} = \mathcal{N}_{r,1}$ for all $r \geq r_0$.*

Proof. It is well-known that the annihilator $\mathcal{N}^0 \subseteq S\mathcal{V} \otimes \mathcal{U}^*$ is an $S\mathcal{V}$-submodule. Now $\mathcal{N}_{r+1} \subsetneq \mathcal{N}_{r,1}$ implies that any minimal basis of $\mathcal{N}^0$ contains at least one generator of degree r. Since, by Hilbert's Basis Theorem, any polynomial ring in a finite number of variables and hence also the symmetric algebra $S\mathcal{V}$ is Noetherian, an upper bound r_0 for such values r exists. ∎

By this corollary, we may consider symbolic systems as a kind of finitely cogenerated "differential comodules": since the truncated comodule $\mathcal{N}_{\leq r_0}$ is a finite-dimensional vector space, it is obviously finitely cogenerated and by repeated prolongations of the component $\mathcal{N}_{r_0}$ we obtain the remainder of the comodule $\mathcal{N}$. Thus we conclude that every symbolic system is uniquely determined by a finite number of elements.

Definition 2.11 Let $\mathcal{N}$ be a graded comodule over the coalgebra $\mathcal{C} = \mathfrak{S}(\mathcal{V}^*)$. Its *Spencer complex* $\bigl(R(\mathcal{N}), \delta\bigr)$ is the cotensor product[5] complex $\mathcal{N} \boxtimes_{\mathcal{C}} R\bigl(\mathfrak{S}(\mathcal{V}^*)\bigr)$. The *Spencer cohomology* of $\mathcal{N}$ is the corresponding bigraded cohomology; the cohomology group at $\mathcal{N}_q \otimes \Lambda_p(\mathcal{V}^*)$ in $\bigl(R_{q+p}(\mathcal{N}), \delta\bigr)$ is denoted by $H^{q,p}(\mathcal{N})$.

Since $\mathcal{N} \boxtimes_{\mathcal{C}} \mathcal{C} \cong \mathcal{N}$ for any $\mathcal{C}$-comodule $\mathcal{N}$, the components of the cotensored complex $\mathcal{N} \boxtimes_{\mathcal{C}} R\bigl(\mathfrak{S}(\mathcal{V}^*)\bigr)$ are indeed just the vector spaces $\mathcal{N}_q \otimes \Lambda_p(\mathcal{V}^*)$. We are mainly interested in the special case that $\mathcal{N}$ is a subcomodule of a free comodule $\mathcal{C} \otimes \mathcal{U}$ and then the differential in the Spencer complex $R(\mathcal{N})$ is simply given by the restriction of the differential δ in the polynomial de Rham complex $R\bigl(\mathfrak{S}(\mathcal{V}^*)\bigr)$ to the subspaces $\mathcal{N}_q \otimes \Lambda_p(\mathcal{V}^*) \subseteq \mathfrak{S}_q(\mathcal{V}^*) \otimes \Lambda_p(\mathcal{V}^*) \otimes \mathcal{U}$; this observation explains why we keep the notation δ for the differential. One can also verify by direct computation that this restriction makes sense whenever $(\mathcal{N}_q)_{q\in\mathbb{N}_0}$ defines a symbolic system (this is basically the same computation as the one showing the equivalence of the two definitions (2.12)

[5] The definition of the cotensor product $\boxtimes_{\mathcal{C}}$ over a coalgebra $\mathcal{C}$ is dual to the one of the usual tensor product; it was introduced by Eilenberg and Moore [13].

and (2.13) of the prolongation); in fact, this restriction is the classical approach to define the Spencer complex.

Remark 2.12 It is important to note here that the Spencer cohomology is bigraded. If we ignore the polynomial degree and consider only the form degree, we obtain the modules $H^p(\mathcal{N}) = \bigoplus_{q=0}^{\infty} H^{q,p}(\mathcal{N})$. For these, another point of view is possible. Since any free comodule is injective, we have exactly the situation of the definition of *cotorsion*: we are given an injective coresolution (of $\Bbbk$) and cotensor it with a comodule. Thus we may consider the Spencer cohomology as the right derived functor of $\mathcal{N} \boxtimes_{\mathcal{C}} \cdot$ and identify $H^p(\mathcal{N}) = \operatorname{Cotor}^p_{\mathcal{C}}(\mathcal{N}, \Bbbk)$.
As for arbitrary derived functors, the definition of $\operatorname{Cotor}^p_{\mathcal{C}}(\mathcal{N}, \Bbbk)$ is independent of the coresolution used for its computation or, more precisely, the results obtained with different coresolutions are isomorphic. However, given some other way to explicitly determine $\operatorname{Cotor}^p_{\mathcal{C}}(\mathcal{N}, \Bbbk)$, say via a coresolution of $\mathcal{N}$, it may be a non-trivial task to recover the bigrading of the Spencer cohomology.

Lemma 2.13 *Let $\mathcal{N} \subseteq \mathfrak{S}(\mathcal{V}^*) \otimes \mathcal{U}$ be a symbolic system. Then $H^{q,0}(\mathcal{N}) = 0$ and $\dim H^{q-1,1}(\mathcal{N}) = \dim \big(\mathcal{N}_{q-1,1}/\mathcal{N}_q\big)$ for all $q > 0$.*

Proof. The first claim follows immediately from the formal Poincaré Lemma (Proposition 2.4). For the second claim consider a non-vanishing element $f \in \mathcal{N}_{q-1,1} \setminus \mathcal{N}_q$. Then $g = \delta f \in \ker \delta\big|_{\mathcal{N}_{q-1} \otimes \mathcal{V}^*}$ and, because of the formal Poincaré Lemma, $g \neq 0$. However, by construction, $g \notin \operatorname{im} \delta\big|_{\mathcal{N}_q}$ and hence we find $0 \neq [g] \in H^{q-1,1}(\mathcal{N})$. This implies immediately the inequality $\dim H^{q-1,1}(\mathcal{N}) \geq \dim \big(\mathcal{N}_{q-1,1}/\mathcal{N}_q\big)$. Conversely, consider an arbitrary non-vanishing cohomology class $[g] \in H^{q-1,1}(\mathcal{N})$. Again by the formal Poincaré Lemma, an element $f \in \mathfrak{S}_q(\mathcal{V}^*) \otimes \mathcal{U}$ exists such that $g = \delta f$ and, by definition of the prolongation, $f \in \mathcal{N}_{q-1,1} \setminus \mathcal{N}_q$. Thus we also have the opposite inequality $\dim H^{q-1,1}(\mathcal{N}) \leq \dim \big(\mathcal{N}_{q-1,1}/\mathcal{N}_q\big)$. ∎

Note that Corollary 2.10 implies that $H^{q,1}(\mathcal{N}) = 0$ for a sufficiently high degree q. Dualisation of Definition 2.11 leads to the following classical construction in commutative algebra with a polynomial module.

Definition 2.14 Let $\mathcal{M}$ be a graded module over the symmetric algebra $\mathcal{P} = S\mathcal{V}$. Its *Koszul complex* $\big(K(\mathcal{M}), \partial\big)$ is the tensor product complex $\mathcal{M} \otimes_{\mathcal{P}} K(S\mathcal{V})$. The *Koszul homology* of $\mathcal{M}$ is the corresponding bigraded homology; the homology group at $\mathcal{M}_q \otimes \Lambda_p \mathcal{V}$ is denoted by $H_{q,p}(\mathcal{M})$.

Remark 2.15 We observed already above that the Koszul complex defines a free resolution of the field $\Bbbk$. Hence, as for the Spencer cohomology, we may take another point of view and consider the Koszul homology as the right derived functor of $\mathcal{M} \otimes_{\mathcal{P}} \cdot$. According to the definition of the torsion modules, this leads to the identification $H_p(\mathcal{M}) = \bigoplus_{q=0}^{\infty} H_{q,p}(\mathcal{M}) = \operatorname{Tor}^{\mathcal{P}}_p(\mathcal{M}, \Bbbk)$ where we consider $\Bbbk$ as a $\mathcal{P}$-module. But again this interpretation ignores the natural bigrading of the Koszul complex $K(\mathcal{M})$.
An alternative way to compute $\operatorname{Tor}^{\mathcal{P}}_p(\mathcal{M}, \Bbbk)$ consists of using a free resolution of the module $\mathcal{M}$. If $\mathcal{F} \to \mathcal{M} \to 0$ is such a resolution, then the Koszul homology $H_\bullet(\mathcal{M})$ is

isomorphic to the homology of the tensor product complex $\mathcal{F}\otimes_{\mathcal{P}}\Bbbk$. Each component in $\mathcal{F}$ is of the form $\mathcal{P}^m$ and therefore $\mathcal{P}^m\otimes_{\mathcal{P}}\Bbbk=\Bbbk^m$. Now assume that we actually have a *minimal* resolution. In this case all differentials in $\mathcal{F}$ possess a positive degree and it follows from the $\mathcal{P}$-action on $\Bbbk$ that the induced differential on the complex $\mathcal{F}\otimes_{\mathcal{P}}\Bbbk$ is the zero map. Hence we find that $H_\bullet(\mathcal{M})\cong\mathcal{F}\otimes_{\mathcal{P}}\Bbbk$ and $\dim H_p(\mathcal{M})$ is just the pth Betti number of $\mathcal{M}$. In this sense we may say that the Koszul homology corresponds to a minimal free resolution.

Lemma 2.16 *Let $\mathcal{M}$ be a graded $\mathcal{P}$-module. Then $H_{q,0}(\mathcal{M})=\mathcal{M}_q/\mathcal{V}\mathcal{M}_{q-1}$ and thus $\dim H_{q,0}(\mathcal{M})$ gives the numbers of generators of degree q in any minimal basis of $\mathcal{M}$. Furthermore,*

$$H_{q,n}(\mathcal{M})\cong\left\{m\in\mathcal{M}_q\mid\operatorname{Ann}(m)=S_+\mathcal{V}\right\}\,. \tag{2.14}$$

Proof. The first assertion follows trivially from the definition of the Koszul homology. Elements of $H_{q,n}(\mathcal{M})$ are represented by cycles in $\mathcal{M}_q\otimes\Lambda_n\mathcal{V}$. If $\{x^1,\dots,x^n\}$ is a basis of $\mathcal{V}$, these are forms $\omega=m\otimes x^1\wedge\cdots\wedge x^n$ and the condition $\partial\omega=0$ is equivalent to $x^i m=0$ for $1\le i\le n$. □

Lemma 2.17 *Let $\mathcal{M}$ be a graded $\mathcal{P}$-module. Multiplication by an arbitrary element of $S_+\mathcal{V}$ induces the zero map on the Koszul homology $H_\bullet(\mathcal{M})$.*

Proof. We first observe that if $\omega\in\mathcal{M}_q\otimes\Lambda_p\mathcal{V}$ is a cycle, i. e. $\partial\omega=0$, then for any $v\in\mathcal{V}$ the form $v\omega$ is a boundary, i. e. $v\omega\in\operatorname{im}\partial$. Indeed,

$$\partial\bigl(v\wedge\omega\bigr)=-v\wedge(\partial\omega)+v\omega=v\omega\,. \tag{2.15}$$

Since ∂ is $S\mathcal{V}$-linear, this observation remains true, if we take for v an arbitrary element of $S_+\mathcal{V}$, i. e. a polynomial without constant term. □

Each subcomodule $\mathcal{N}\subseteq\mathfrak{S}(\mathcal{V}^*)\otimes\mathcal{U}$ induces a submodule $\mathcal{M}=\mathcal{N}^0\subseteq S\mathcal{V}\otimes\mathcal{U}^*$, its annihilator. Conversely, the annihilator of any submodule $\mathcal{M}\subseteq S\mathcal{V}\otimes\mathcal{U}^*$ defines a subcomodule $\mathcal{N}=\mathcal{M}^0\in\mathfrak{S}(\mathcal{V}^*)\otimes\mathcal{U}$. In view of the duality between the polynomial de Rham and the Koszul complex, we expect a simple relation between the Spencer cohomology $H^\bullet(\mathcal{N})$ of the comodule $\mathcal{N}$ and the Koszul homology $H_\bullet(\mathcal{N}^0)$ of its annihilator $\mathcal{N}^0$.

Such a relation is easily obtained with the help of the $S\mathcal{V}$-module $\mathcal{N}^*$ dual to $\mathcal{N}$. If we take the dual $\pi^*:\bigl((\mathfrak{S}(\mathcal{V}^*)\otimes\mathcal{U})/\mathcal{N}\bigr)^*\to\bigl(\mathfrak{S}(\mathcal{V}^*)\otimes\mathcal{U}\bigr)^*=S\mathcal{V}\otimes\mathcal{U}^*$ of the canonical projection $\pi:\mathfrak{S}(\mathcal{V}^*)\otimes\mathcal{U}\to\bigl(\mathfrak{S}(\mathcal{V}^*)\otimes\mathcal{U}\bigr)/\mathcal{N}$, then $\operatorname{im}\pi^*=\mathcal{N}^0$. Like for any map, we have for π the canonical isomorphism $\operatorname{coker}(\pi^*)\cong(\ker\pi)^*=\mathcal{N}^*$ and hence may identify $\mathcal{N}^*$ with the factor module $(S\mathcal{V}\otimes\mathcal{U}^*)/\mathcal{N}^0$.

Proposition 2.18 *Let $\mathcal{N}\subseteq\mathfrak{S}(\mathcal{V}^*)\otimes\mathcal{U}$ be a symbolic system. Then for all $q\ge0$ and $1\le p\le n$*

$$\bigl(H^{q,p}(\mathcal{N})\bigr)^*\cong H_{q,p}(\mathcal{N}^*)\cong H_{q+1,p-1}(\mathcal{N}^0) \tag{2.16}$$

where the second isomorphism is induced by the Koszul differential ∂.

Proof. The first isomorphism follows from Proposition 2.6. For the second one we note that the considerations above lead to the short exact sequence

$$0 \longrightarrow \mathcal{N}^0 \overset{\iota}{\hookrightarrow} S\mathcal{V} \otimes \mathcal{U}^* \xrightarrow{\pi} \mathcal{N}^* \longrightarrow 0 \tag{2.17}$$

where the first map is the natural inclusion and the second one the canonical projection. Tensoring with the vector space $\Lambda_p\mathcal{V}$ is a flat functor and hence does not affect the exactness so that we obtain a short exact sequence of Koszul complexes:

$$0 \longrightarrow K(\mathcal{N}^0) \hookrightarrow K(S\mathcal{V} \otimes \mathcal{U}^*) \longrightarrow K(\mathcal{N}^*) \longrightarrow 0\,. \tag{2.18}$$

Since $K(S\mathcal{V} \otimes \mathcal{U}^*)$ is exact in positive exterior degree, the long exact homological sequence for (2.18) yields an isomorphism $H_p(\mathcal{N}^*) \to H_{p-1}(\mathcal{N}^0)$. Furthermore, as the maps in the exact sequence (2.17) are so simple, it follows straightforwardly from the construction of the connecting homomorphism that this isomorphism is induced by the Koszul differential ∂. Hence, taking the bigrading into account, we obtain an isomorphism $H_{q,p}(\mathcal{N}^*) \to H_{q+1,p-1}(\mathcal{N}^0)$. □

Remark 2.19 From a computational point of view, it is often more convenient to work with the annihilator $\mathcal{N}^0$ instead of the dual module $\mathcal{N}^*$. The way we proved the lemma gave the isomorphism only in one direction. However, Lemma 2.3 allows us to derive easily an explicit expression for the inverse.

Let $\omega \in \mathcal{N}^*_q \otimes \Lambda_p\mathcal{V}$ be a cycle and $\tilde{\omega} \in S_q\mathcal{V} \otimes \Lambda_p\mathcal{V} \otimes \mathcal{U}^*$ an arbitrary form such that $\pi(\tilde{\omega}) = \omega$. Then $\partial\omega = 0$ implies that $\bar{\omega} = \partial\tilde{\omega} \in \mathcal{N}^0_{q+1} \otimes \Lambda_{p-1}\mathcal{V}$. Now the isomorphism used in the proof above simply maps $[\omega] \mapsto [\bar{\omega}]$. For the inverse we note that, by Lemma 2.3, $\delta\bar{\omega} = (p+q)\tilde{\omega} - \partial(\delta\tilde{\omega})$ and hence $\left[\frac{1}{p+q}\delta\bar{\omega}\right] = [\tilde{\omega}]$. But this implies that the inverse of our isomorphism is given by the map $[\bar{\omega}] \mapsto \left[\frac{1}{p+q}\pi(\delta\bar{\omega})\right]$.

For our purposes, the most important property of the Spencer cohomology is the following finiteness result obviously requiring the bigrading. A direct proof would probably be not easy, but the duality to the Koszul homology (Proposition 2.18) allows us to restrict to the dual situation where the finiteness is a trivial corollary to Lemma 2.17.

Theorem 2.20 *Let $\mathcal{N} \subseteq \mathfrak{S}(\mathcal{V}^*) \otimes \mathcal{U}$ be a symbolic system. Then there exists an integer $q_0 \geq 0$ such that $H^{q,p}(\mathcal{N}) = 0$ for all $q \geq q_0$ and $0 \leq p \leq n$. Dually, let $\mathcal{M}$ be a finitely generated graded polynomial module. Then there exists an integer $q_0 \geq 0$ such that $H_{q,p}(\mathcal{M}) = 0$ for all $q \geq q_0$ and $0 \leq p \leq n$.*

Proof. As mentioned above, it suffices to consider the case of a polynomial module $\mathcal{M}$. The cycles in $\mathcal{M} \otimes \Lambda_p\mathcal{V}$ form a finitely generated $S\mathcal{V}$-module. Thus there exists an integer $q_0 \geq 0$ such that the polynomial degree of all elements in a finite generating set of it is less than q_0. All cycles of higher polynomial degree are then linear combinations of these generators with polynomial coefficients without constant terms. By Lemma 2.17, they are therefore boundaries. Hence $H_{q,p}(\mathcal{M}) = 0$ for all $q \geq q_0$. □

Definition 2.21 The *degree of involution* of the $\mathfrak{S}(\mathcal{V}^*)$-comodule $\mathcal{N}$ is the smallest value q_0 such that $H^{q,p}(\mathcal{N}) = 0$ for all $q \geq q_0$ and $0 \leq p \leq n = \dim \mathcal{V}$. More generally, we say that $\mathcal{N}$ is s-*acyclic* at degree q_0 for an integer $0 \leq s \leq n$, if $H^{q,p}(\mathcal{N}) = 0$ for all $q \geq q_0$ and $0 \leq p \leq s$. A comodule that is n-acyclic at degree q_0 is called *involutive* at degree q_0. Dually, we call an $S\mathcal{V}$-module $\mathcal{M}$ *involutive* at degree q_0, if its Koszul homology vanishes beyond degree q_0: $H_{q,p}(\mathcal{M}) = 0$ for all $q \geq q_0$ and $0 \leq p \leq n$.

With this terminology we may formulate Lemma 2.13 as follows: if the symbolic system $\mathcal{N}$ is such that its annihilator $\mathcal{N}^0$ is generated in degree less than or equal to r_0, then $\mathcal{N}$ is 1-acyclic at degree r_0, and if conversely r_0 is the smallest degree at which $\mathcal{N}$ is 1-acyclic, then any generating set of $\mathcal{N}^0$ contains an element of degree r_0 or higher. We will see later in Theorem 7.15 that 2-acyclicity is very important for checking formal integrability. It follows trivially from the definition that if $\mathcal{N}$ is involutive at some degree q_0, then it is also involutive at any higher degree $q \geq q_0$.

For complexity considerations, it is of great interest to bound for a given comodule $\mathcal{N}$ or module $\mathcal{M}$, respectively, its degree of involution. In our applications to differential equations we will be mainly concerned with the special case that $\mathcal{M}$ is a submodule of a free $S\mathcal{V}$-module of rank m generated by homogeneous elements of degree q. Sweeney [61, Corollary 7.7] derived for this situation a bound $\bar{q}$ depending only on the values of n, m and q. It may be expressed as a nested recursion relation:

$$\begin{aligned}
\bar{q}(n,m,q) &= \bar{q}\Bigl(n, m\binom{q+n-1}{n}, 1\Bigr) ,\\
\bar{q}(n,m,1) &= m\binom{\bar{q}(n-1,m,1)+n}{n-1} + \bar{q}(n-1,m,1) + 1 ,\\
\bar{q}(0,m,1) &= 0 .
\end{aligned} \tag{2.19}$$

Table 2.1 shows $\bar{q}(n,m,1)$ for different values of m and n. One sees that the values rapidly explode, if n increases. The situation is still worse for modules generated in higher order. It seems to be an open question whether this bound is sharp, i. e. whether for some modules the degree of involution is really that high. Fortunately, $\bar{q}(n,m,q)$ yields usually a coarse over-estimate of the actual degree of involution.

$n\backslash m$	1	2	3	4
1	2	3	4	5
2	7	14	23	34
3	53	287	999	2.699
4	29.314	8.129.858	503.006.503	13.151.182.504

Table 2.1 $\bar{q}(n,m,1)$ for different values of m and n.

Example 2.22 Let us consider the homogeneous ideal $\mathcal{I}$ (i. e. $m = 1$) generated by the two monomials $(x^1)^q$ and $(x^2)^q$ for some value $q > 0$ in the polynomial ring $\Bbbk[x^1, x^2]$ in $n = 2$ variables. For the value $q = 3$ this ideal is just the annihilator $\mathcal{N}^0$ of the second symbolic system $\mathcal{N}$ considered in Example 2.9. A trivial computation yields that the only non-vanishing Koszul homology modules are $H_{q,0}(\mathcal{I}) = \langle[(x^1)^q], [(x^2)^q]\rangle$ and $H_{2q-1,1}(\mathcal{I}) = \langle[(x^1)^q(x^2)^{q-1} \otimes x^2 - (x^1)^{q-1}(x^2)^q \otimes x^1]\rangle$. Hence the degree of involution of $\mathcal{I}$ is $2q - 1$. By contrast, evaluation of Sweeney's bound (2.19) yields

$$\bar{q}(2, 1, q) = \frac{1}{4}q^4 + \frac{1}{2}q^3 + \frac{9}{4}q^2 + 2q + 2\,, \tag{2.20}$$

i. e. a polynomial in q of degree 4.

3 Cartan's Test

We study now some explicit criteria for a (co)module to be involutive. We start with a symbolic system $\mathcal{N} \subseteq \mathfrak{S}(\mathcal{V}^*) \otimes \mathcal{U}$. As before, let $\{x^1, \dots, x^n\}$ be an ordered basis of $\mathcal{V}$ and $\{y_1, \dots, y_n\}$ the corresponding dual basis of $\mathcal{V}^*$. Then we introduce for any $0 \leq k \leq n$ the following subspaces of the homogeneous component $\mathcal{N}_q$:

$$\begin{aligned}\mathcal{N}_q^{(k)} = &\Big\{f \in \mathcal{N}_q \mid f(x^i, v_1, \dots, v_{q-1}) = 0\,,\\ &\qquad\qquad \forall 1 \leq i \leq k\,,\ \forall v_1, \dots, v_{q-1} \in \mathcal{V}\Big\}\\ = &\Big\{f \in \mathcal{N}_q \mid \frac{\partial f}{\partial y_i} = 0\ \forall 1 \leq i \leq k\Big\}\,.\end{aligned} \tag{3.1}$$

In the first line we interpreted elements of $\mathcal{N}_q$ as multilinear maps on $\mathcal{V}$ and in the last line we considered them as polynomials in the "variables" $y_1, \dots, y_n$.
Obviously, these subspaces define a filtration

$$0 = \mathcal{N}_q^{(n)} \subseteq \mathcal{N}_q^{(n-1)} \subseteq \cdots \subseteq \mathcal{N}_q^{(1)} \subseteq \mathcal{N}_q^{(0)} = \mathcal{N}_q\,. \tag{3.2}$$

It is clear that this filtration (and in particular the dimensions of the involved subspaces) depend on the chosen basis for $\mathcal{V}^*$. Thus it distinguishes certain bases. This effect is known as the problem of δ-regularity. In the next chapter we will see it reappear in a different form for Pommaret bases.

Definition 3.1 Let $\mathcal{N} \subseteq \mathfrak{S}(\mathcal{V}^*) \otimes \mathcal{U}$ be a symbolic system. With respect to a given basis $\{y_1, \dots, y_n\}$ of $\mathcal{V}^*$, we define the *Cartan characters* of the component $\mathcal{N}_q$ as

$$\alpha_q^{(k)} = \dim \mathcal{N}_q^{(k-1)} - \dim \mathcal{N}_q^{(k)}\,, \qquad 1 \leq k \leq n\,. \tag{3.3}$$

A basis $\{y_1, \dots, y_n\}$ of $\mathcal{V}^*$ is *δ-regular* for the component $\mathcal{N}_q$, if the sum $\sum_{k=1}^n k\alpha_q^{(k)}$ attains a minimal value for it.

One can show that generic bases are always δ-regular. Hence conceptually trivial solutions of the problem of δ-regularity are to use either a random basis (which is δ-regular with probability 1) or to work with a general (i. e. parametrised) basis. However, from a computational point of view both approaches are extremely expensive and useless in larger calculations. In the context of Pommaret bases much more efficient solutions have been developed (see Remark 4.6 below for references).

We know from the proof of Lemma 2.8 that differentiation with respect to a variable y_k maps $\mathcal{N}_{q+1}$ into $\mathcal{N}_q$. It follows trivially from the definition of the subspaces $\mathcal{N}_q^{(k)}$ that we may consider the restrictions $\partial_{y_k} : \mathcal{N}_{q+1}^{(k-1)} \to \mathcal{N}_q^{(k-1)}$.

Proposition 3.2 *Let $\mathcal{N} \subseteq \mathfrak{S}(\mathcal{V}^*) \otimes \mathcal{U}$ be a symbolic system and $\{y_1, \dots, y_n\}$ a basis of $\mathcal{V}^*$. Then we have for any $q \geq 0$ the inequality*

$$\dim \mathcal{N}_{q+1} \leq \sum_{k=0}^{n-1} \dim \mathcal{N}_q^{(k)} = \sum_{k=1}^{n} k\alpha_q^{(k)} \ . \tag{3.4}$$

Equality holds, if and only if the restricted maps $\partial_{y_k} : \mathcal{N}_{q+1}^{(k-1)} \to \mathcal{N}_q^{(k-1)}$ are surjective for all $1 \leq k \leq n$.

Proof. By definition of the subspaces $\mathcal{N}_q^{(k)}$, we have the exact sequences

$$0 \longrightarrow \mathcal{N}_{q+1}^{(k)} \hookrightarrow \mathcal{N}_{q+1}^{(k-1)} \xrightarrow{\partial_{y_k}} \mathcal{N}_q^{(k-1)} \tag{3.5}$$

implying the inequalities $\dim \mathcal{N}_{q+1}^{(k-1)} - \dim \mathcal{N}_{q+1}^{(k)} \leq \dim \mathcal{N}_q^{(k-1)}$. Summing over all $0 \leq k \leq n$ yields immediately the inequality (3.4). Equality in (3.4) is obtained, if and only if in all these dimension relations equality holds. But this is the case, if and only if all the maps ∂_{y_k} are surjective. $\square$

Proposition 3.3 *The symbolic system $\mathcal{N} \subseteq \mathfrak{S}(\mathcal{V}^*) \otimes \mathcal{U}$ is involutive at degree q_0, if and only if a basis $\{y_1, \dots, y_n\}$ of $\mathcal{V}^*$ exists such that the maps $\partial_{y_k} : \mathcal{N}_{q+1}^{(k-1)} \to \mathcal{N}_q^{(k-1)}$ are surjective for all degrees $q \geq q_0$ and all values $1 \leq k \leq n$.*

Proof. We prove only one direction; the converse will follow from our subsequent considerations for the dual Koszul homology of $\mathcal{N}^*$ (see Remark 3.13). Let us take an arbitrary cycle $\omega \in \mathcal{N}_q \otimes \Lambda_p(\mathcal{V}^*)$ with $1 \leq p \leq n$ and $q \geq q_0$; we will show that, if all maps ∂_{y_k} are surjective, then a form $\eta \in \mathcal{N}_{q+1} \otimes \Lambda_{p-1}(\mathcal{V}^*)$ exists with $\omega = \delta\eta$. This implies that $H^{q,p}(\mathcal{N}) = 0$.

We do this demonstration in an iterative process, assuming first that the exterior part of ω depends only on $y_k, y_{k+1}, \dots, y_n$. Then we may decompose $\omega = \omega_1 + y_k \wedge \omega_2$ where the exterior parts of both ω_1 and ω_2 depend only on $y_{k+1}, \dots, y_n$. Since ω is a cycle, we have $\delta\omega = \delta\omega_1 - y_k \wedge \delta\omega_2 = 0$. Consider now in this equation those terms where the exterior part is of the form $y_\ell \wedge y_k \wedge \cdots$ with $\ell \leq k$. Such terms occur only in the second summand and hence we must have $\partial\omega_2/\partial y_l = 0$ for all $1 \leq \ell < k$. This implies $\omega_2 \in \mathcal{N}_q^{(k-1)} \otimes \Lambda_{p-1}(\mathcal{V}^*)$.

By assumption, the map $\partial_{y_k} : \mathcal{N}_{q+1}^{(k-1)} \to \mathcal{N}_q^{(k-1)}$ is surjective so that there exists a form $\eta^{(k)} \in \mathcal{N}_{q+1}^{(k-1)} \otimes \Lambda_{p-1}(\mathcal{V}^*)$ such that $\partial_{y_k}\eta^{(k)} = \omega_2$. Hence the exterior part of the form $\omega^{(k)} = \omega - \delta\eta^{(k)}$ depends only on $y_{k+1}, \dots, y_n$ and we can iterate. Thus starting with $k = 1$ we finally obtain $\omega = \delta\big(\eta^{(1)} + \cdots + \eta^{(n-1)}\big)$. ∎

While Proposition 3.3 is nice from a theoretical point of view, it is not very useful computationally, as we must check infinitely many conditions, namely one for each degree $q \geq q_0$. Under modest assumptions it suffices to consider only the one degree q_0 and then we obtain an effective criterion for involution representing an algebraic reformulation of the classical Cartan test in the theory of exterior differential systems. It uses only linear algebra with the two finite-dimensional components $\mathcal{N}_q$ and $\mathcal{N}_{q+1}$ (note, however, that the test can only be applied in δ-regular bases). In particular, it is not necessary to determine explicitly any Spencer cohomology module. In the context of differential equations, this observation will later translate into the fact that it is easier to check involution than formal integrability.

Theorem 3.4 (Cartan Test) *Let $\mathcal{N} \subseteq \mathfrak{S}(\mathcal{V}^*) \otimes \mathcal{U}$ be a symbolic system such that $\mathcal{N}_{q,1} = \mathcal{N}_{q+1}$ for all $q \geq q_0$. Then $\mathcal{N}$ is involutive at degree q_0, if and only if a basis $\{y_1, \dots, y_n\}$ of $\mathcal{V}^*$ exists such that we have equality in (3.4) for $q = q_0$.*

Implicitly, a proof of this result was already given by Janet [26]. Later the theorem was explicitly demonstrated by Matsushima [39, 40]. We do not give here a proof, as it will follow automatically from later results on Pommaret bases (see Remark 4.15); in spirit this corresponds to the proof of Janet.

Example 3.5 Let us consider over a three-dimensional vector space $\mathcal{V}$ the symbolic system $\mathcal{N} \subset \mathfrak{S}(\mathcal{V}^*)$ defined by $\mathcal{N}_0 = \mathbb{k}$, $\mathcal{N}_1 = \mathcal{V}^*$, $\mathcal{N}_2 = \langle y_1^2, y_1y_2, y_1y_3, y_2^2\rangle$ and $\mathcal{N}_q = \mathcal{N}_{q-1,1}$ for $q \geq 3$. One easily verifies that here $\mathcal{N}_2^{(1)} = \langle y_2^2\rangle$ and $\mathcal{N}_2^{(2)} = \mathcal{N}_2^{(3)} = 0$ and therefore the only non-vanishing Cartan characters of $\mathcal{N}$ are $\alpha_2^{(1)} = 3$ and $\alpha_2^{(2)} = 1$. Furthermore, $\mathcal{N}_3 = \langle y_1^3, y_1^2y_2, y_1^2y_3, y_1y_2^2, y_2^3\rangle$. Since $\alpha_2^{(1)} + 2\alpha_2^{(2)} = 5 = \dim \mathcal{N}_3$, the symbolic system $\mathcal{N}$ passes the Cartan test and is involutive at degree $q = 2$. One also immediately sees that the map $\partial_{y_1} : \mathcal{N}_3 \to \mathcal{N}_2$ is indeed surjective and that the map $\partial_{y_2} : \mathcal{N}_3^{(1)} = \langle y_2^3\rangle \to \mathcal{N}_2^{(1)}$ is even bijective (there is no need to consider also ∂_{y_3}, since both $\mathcal{N}_2^{(2)}$ and $\mathcal{N}_3^{(2)}$ vanish).

Example 3.6 For an instance where the Cartan test is not passed, we return to the second symbolic system in Example 2.9. Since $\mathcal{N}$ vanishes beyond degree 5, it is trivially involutive at degree 5. We verify now that it is not involutive at a lower degree. It is clear that $\partial_{y_1} : \mathcal{N}_5 \to \mathcal{N}_4$ cannot be surjective and also $\alpha_4^{(1)} = 1 > \dim \mathcal{N}_5 = 0$. Hence $\mathcal{N}$ is not involutive at degree 4.

Given the duality between the polynomial de Rham and the Koszul complex, we expect that a similar criterion for involution exists for polynomial modules. The essence of the proof of Proposition 2.6 is that differentiation with respect to y_k is dual to multiplication with x^k. Hence when we now study, following the letter of Serre appended to [21], the dualisation of the considerations above, it is not surprising that the multiplication with elements $v \in \mathcal{V}$ is central.

Lemma 3.7 *Let $\mathcal{M}$ be a finitely generated graded $S\mathcal{V}$-module and $q > 0$ an integer. Then the following statements are equivalent.*

(i) $H_{r,n}(\mathcal{M}) = 0$ *for all* $r \geq q$*.*

(ii) *If* $\operatorname{Ann}(m) = S_+\mathcal{V}$ *for an* $m \in \mathcal{M}$*, then* $m \in \mathcal{M}_{<q}$*.*

(iii) *The existence of an element* $v \in \mathcal{V}$ *with* $v \cdot m = 0$ *entails* $m \in \mathcal{M}_{<q}$*.*

(iv) *For all* $v \in \mathcal{V}$ *except the elements of a finite number of proper subspaces of* $\mathcal{V}$ *the equation* $v \cdot m = 0$ *entails* $m \in \mathcal{M}_{<q}$*.*

Proof. The equivalence of (i) and (ii) follows immediately from Lemma 2.16. Furthermore, it is trivial that (iv) implies (iii) implies (ii). Hence there only remains to show that (iv) is a consequence of (ii).
Assume that (ii) holds and let $\mathcal{A} = \{m \in \mathcal{M}_{<q} \mid \operatorname{Ann}(m) = S_+\mathcal{V}\}$. We choose a complement $\mathcal{K}$ such that $\mathcal{M}_{<q} = \mathcal{A} \oplus \mathcal{K}$ and set $\overline{\mathcal{M}} = \mathcal{K} \oplus \bigoplus_{r \geq q} \mathcal{M}_r$. Because of (ii) no element of $\overline{\mathcal{M}} \setminus \{0\}$ is annihilated by $S_+\mathcal{V}$ and hence $S_+\mathcal{V}$ is not an associated prime ideal of the module $\overline{\mathcal{M}}$. By a standard result in commutative algebra, the set $\operatorname{Ass}\overline{\mathcal{M}}$ of all associated prime ideals of $\overline{\mathcal{M}}$ contains only finitely many elements. The intersection of any of these with $\mathcal{V}$ is a proper subspace. If we choose $v \in \mathcal{V}$ such that it is not contained in any of these subspaces, then $v \cdot m = 0$ entails $m \in \mathcal{M}_{<q}$. □

The property of v in Part (iii) will become so important in the sequel that we provide a special name for it. It is closely related to the classical notion of a regular sequence in commutative algebra except that for the latter it is not permitted that the multiplication with v has a non-trivial kernel whereas here we only restrict the degree of the kernel.

Definition 3.8 A vector $v \in \mathcal{V}$ is called *quasi-regular* at degree q for the module $\mathcal{M}$, if $v \cdot m = 0$ entails $m \in \mathcal{M}_{<q}$. A finite sequence $(v_1, \dots, v_k)$ of elements of $\mathcal{V}$ is *quasi-regular* at degree q for the module $\mathcal{M}$, if each v_i is quasi-regular at degree q for the factor module $\mathcal{M}/\langle v_1, \dots, v_{i-1}\rangle\mathcal{M}$.

Obviously, if a vector v is quasi-regular at degree q, it is also quasi-regular at any degree $r > q$. Furthermore, the vectors in a quasi-regular sequence are linearly independent. Thus such a sequence of length $n = \dim \mathcal{V}$ defines a basis of the vector space $\mathcal{V}$.

Lemma 3.9 *Let $v \in \mathcal{V}$ be quasi-regular at degree q. For each $r \geq q$ and $1 \leq p \leq n$ there is a short exact sequence*

$$0 \longrightarrow H_{r,p}(\mathcal{M}) \xrightarrow{\alpha} H_{r,p}(\mathcal{M}/v\mathcal{M}) \xrightarrow{\beta} H_{r,p-1}(\mathcal{M}) \longrightarrow 0 \tag{3.6}$$

and the multiplication with v is injective on $\mathcal{M}_{\geq q}$.

Proof. As above we decompose $\mathcal{M} = \mathcal{A} \oplus \overline{\mathcal{M}}$. Since $\mathcal{A} \subseteq \mathcal{M}_{<q}$, we have the equality $H_{r,p}(\mathcal{M}) = H_{r,p}(\overline{\mathcal{M}})$ for all $r \geq q$ and, because of $v\mathcal{A} = 0$, similarly $H_{r,p}(\mathcal{M}/v\mathcal{M}) = H_{r,p}(\overline{\mathcal{M}}/v\overline{\mathcal{M}})$ for all $r \geq q$.
It follows trivially from the definition of quasi-regularity that multiplication with v is injective on $\mathcal{M}_{\geq q}$. In fact, it is injective on $\overline{\mathcal{M}}$. Indeed, suppose that $v \cdot m = 0$ for some homogeneous element $m \in \mathcal{M}$. Let us assume first that $m \in \mathcal{M}_{q-1}$. Then $v \cdot (w \cdot m) = 0$ for all $w \in \mathcal{V}$ and since $w \cdot m \in \mathcal{M}_q$, this is only possible, if $w \cdot m = 0$ and thus $\operatorname{Ann}(m) = S_+\mathcal{V}$ implying $m \in \mathcal{A}$. Iterating this argument, we conclude that m cannot be contained in $\overline{\mathcal{M}}_{q-2}$ either and so on. Hence $m \in \mathcal{A}$.
Because of the injectivity, the sequence

$$0 \longrightarrow \mathcal{M} \xrightarrow{v} \mathcal{M} \xrightarrow{\pi} \mathcal{M}/v\mathcal{M} \longrightarrow 0 \tag{3.7}$$

of graded modules is exact at all degrees $r \geq q$. Tensoring with the vector space $\Lambda\mathcal{V}$ yields a similar sequence for the corresponding Koszul complexes $K(\mathcal{M})$ and $K(\mathcal{M}/v\mathcal{M})$, respectively, with the same exactness properties. Now we consider the associated long exact homological sequence

$$\begin{aligned} \cdots \longrightarrow H_{r-1,p}(\mathcal{M}) &\xrightarrow{H(v)} H_{r,p}(\mathcal{M}) \xrightarrow{H(\pi)} \\ \longrightarrow H_{r,p}(\mathcal{M}/v\mathcal{M}) &\xrightarrow{\beta} H_{r,p-1}(\mathcal{M}) \xrightarrow{H(v)} \cdots . \end{aligned} \tag{3.8}$$

Since, by Lemma 2.17, $H(v)$ is the zero map, it decomposes into the desired short exact sequences with $\alpha = H(\pi)$ and $\beta\big([\omega]\big) = \big[\frac{1}{v} \circ \partial(\omega)\big]$. ∎

Proposition 3.10 *Let $\mathcal{M}$ be a finitely generated graded $\mathcal{P}$-module and the sequence $(v_1, \dots, v_k)$ quasi-regular at degree q. Then $H_{r,p}(\mathcal{M}) = 0$ for all values $r \geq q$ and $n - k < p \leq n$. If we set $\mathcal{M}^{(i)} = \mathcal{M}/\langle v_1, \dots, v_i\rangle\mathcal{M}$, then*

$$H_{r,n-k}(\mathcal{M}) \cong H_{r,n-k+1}\big(\mathcal{M}^{(1)}\big) \cong \cdots \cong H_{r,n}\big(\mathcal{M}^{(k)}\big) \tag{3.9}$$

for all $r \geq q$.

Proof. We proceed by induction over the length k of the quasi-regular sequence. For $k = 1$, it follows from Lemma 3.7 that $H_{r,n}(\mathcal{M}) = 0$ for all $r \geq q$. Entering this result into the short exact sequence (3.6) of Lemma 3.9 gives immediately an isomorphism $H_{r,n-1}(\mathcal{M}) \cong H_{r,n}(\mathcal{M}^{(1)})$.
Assume now that the proposition holds for any quasi-regular sequence of length less than k. Then we know already that $H_{r,p}(\mathcal{M}) = 0$ for all $r \geq q$ and $n - k + 1 < p \leq n$ and that $H_{r,n-k+1}(\mathcal{M}) \cong H_{r,n}(\mathcal{M}^{(k-1)})$. Since v_k is quasi-regular at degree q for $\mathcal{M}^{(k-1)}$, the latter homology group vanishes by Lemma 3.7 proving the first assertion. Applying the induction hypothesis to the module $\mathcal{M}^{(i-1)}$ and the quasi-regular sequence $(v_i, \dots, v_k)$ shows that $H_{r,n-k+i}(\mathcal{M}^{(i-1)}) = 0$. Now we may use again the exact sequence of Lemma 3.9 to conclude that $H_{r,n-k+i}(\mathcal{M}^{(i)}) \cong H_{r,n-k+i-1}(\mathcal{M}^{(i-1)})$. This proves the second assertion. ∎

Proposition 3.11 *Let $\mathcal{M}$ be a graded $S\mathcal{V}$-module finitely generated in degree less than $q > 0$. The module $\mathcal{M}$ is involutive at degree q, if and only if a basis $\{x^1, \dots, x^n\}$ of $\mathcal{V}$ exists such that the maps*

$$\mu_k : \mathcal{M}_r/\langle x^1, \dots, x^{k-1}\rangle\mathcal{M}_{r-1} \longrightarrow \mathcal{M}_{r+1}/\langle x^1, \dots, x^{k-1}\rangle\mathcal{M}_r \tag{3.10}$$

induced by the multiplication with x^k are injective for all $r \geq q$ and $1 \leq k \leq n$.

Proof. We first note that the statement that $\mathcal{M}$ is generated in degree less than q is equivalent to $H_{r,0}(\mathcal{M}) = 0$ for all $r \geq q$ by Lemma 2.16.
If $\mathcal{M}$ is involutive at degree q, then $H_{r,n}(\mathcal{M}) = 0$ for all $r \geq q$ and Lemma 3.7 implies that a generic vector $x^1 \in \mathcal{V}$ is quasi-regular. Now we proceed by iteration. Setting $\mathcal{M}^{(k)} = \mathcal{M}/\langle x^1, \cdots, x^k\rangle\mathcal{M}$, we find that $H_{r,n}(\mathcal{M}^{(k)}) = H_{r,n-k}(\mathcal{M}) = 0$ for all $r \geq q$ by Lemma 3.9. Thus we may again apply Lemma 3.7 in order to show that for any $1 \leq k < n$ the quasi-regular sequence $(x^1, \dots, x^k)$ can be extended by a generic vector $x^{k+1} \in \mathcal{V}$. As already remarked above, an quasi-regular sequence of length n defines a basis of $\mathcal{V}$. Now the injectivity of the maps μ_k follows from Lemma 3.9.
Conversely, if all maps μ_k are injective, then obviously $(x^1, \dots, x^n)$ defines an quasi-regular sequence of length n. Now the vanishing of all homology groups $H_{r,p}(\mathcal{M})$ with $r \geq q$ and $1 \leq p \leq n$ follows from Proposition 3.10 and $\mathcal{M}$ is involutive. □

Again we face the problem that this proposition requires an infinite number of checks and thus cannot be applied effectively. Quillen [46, App., Prop. 8] was the first to show that for a certain class of modules, it suffices to consider only the components $\mathcal{M}_q$ and $\mathcal{M}_{q+1}$. This leads to the following dual formulation of the Cartan test (Theorem 3.4); again we refer to Remark 4.15 for a proof (or alternatively to [35]).

Theorem 3.12 (Dual Cartan Test) *Let $\mathcal{N}^0 \subseteq S\mathcal{V} \otimes \mathcal{U}^*$ be a homogeneous submodule of the free $S\mathcal{V}$-module $S\mathcal{V} \otimes \mathcal{U}^*$ finitely generated in degree less than $q > 0$. Then the factor module $\mathcal{M} = (S\mathcal{V} \otimes \mathcal{U}^*)/\mathcal{N}^0$ is involutive at degree q, if and only if a basis $\{x^1, \dots, x^n\}$ of $\mathcal{V}$ exists such that the maps*

$$\mu_k : \mathcal{M}_q/\langle x^1, \dots, x^{k-1}\rangle\mathcal{M}_{q-1} \longrightarrow \mathcal{M}_{q+1}/\langle x^1, \dots, x^{k-1}\rangle\mathcal{M}_q \tag{3.11}$$

induced by the multiplication with x^k are injective for all $1 \leq k \leq n$.

Remark 3.13 Let $\mathcal{N} \subseteq \mathfrak{S}(\mathcal{V}^*) \otimes \mathcal{U}$ be a symbolic system and consider the dual $S\mathcal{V}$-module $\mathcal{M} = \mathcal{N}^* \cong \left(S\mathcal{V} \otimes \mathcal{U}^*\right)/\mathcal{N}^0$. Let furthermore $\{x^1, \dots, x^n\}$ be a basis of $\mathcal{V}$ and $\{y_1, \dots, y_n\}$ the dual basis of $\mathcal{V}^*$. Then we find that $\mu_1^* = \partial_{y_1}$ and hence that $\left(\mathcal{N}^{(1)}\right)^* = (\ker \partial_{y_1})^* \cong \operatorname{coker} \mu_1 = \mathcal{M}^{(1)}$. Iteration of this argument yields $\left(\mathcal{N}^{(k)}\right)^* \cong \mathcal{M}^{(k)}$ for all $1 \leq k \leq n$ (considering always ∂_{y_k} as a map on $\mathcal{N}^{(k-1)}$ so that $\mathcal{N}^{(k)} = \ker \partial_{y_k}$ and μ_k as a map on $\mathcal{M}^{(k-1)}$ so that $\mathcal{M}^{(k)} = \operatorname{coker} \mu_k$). We also have $\mu_k = \partial_{y_k}^*$ and hence obtain the isomorphisms $(\ker \mu_k)^* \cong \operatorname{coker} \partial_{y_k}^*$ (again considering the maps on the appropriate domains of definition). Thus injectivity of all the maps μ_k is equivalent to surjectivity of all the maps ∂_{y_k}. Hence applying Proposition 3.11 to $\mathcal{M}$ proves dually Proposition 3.3 for $\mathcal{N}$ and similarly for the Theorems 3.4 and 3.12. Furthermore, it is obvious that if the basis $\{x^1, \dots, x^n\}$ is quasi-regular at degree q, then the dual basis

$\{y_1, \dots, y_n\}$ is δ-regular for $\mathcal{N}_q$. The converse does not necessarily hold, as δ-regularity is a much weaker condition than quasi-regularity (the latter implies involution via the dual Cartan test; the former is only a necessary condition for applying the Cartan test).

Example 3.14 Recall the symbolic system $\mathcal{N}$ of Example 3.5. Its annihilator $\mathcal{N}^0$ is the ideal $\mathcal{I} \subset \mathcal{P} = S\mathcal{V}$ generated by the monomials x^2x^3 and $(x^3)^2$. We apply now the dual Cartan test to the the factor module $\mathcal{M} = \mathcal{P}/\mathcal{I}$. For the two relevant module components we obtain after a trivial computation that $\mathcal{M}_2 \cong \langle (x^1)^2, x^1x^2, x^1x^3, (x^2)^2 \rangle \cong \mathcal{N}_2$ and $\mathcal{M}_3 \cong \langle (x^1)^3, (x^1)^2x^2, (x^1)^2x^3, x^1(x^2)^2, (x^2)^3 \rangle \cong \mathcal{N}_3$. Similarly, we find that the non-vanishing factor modules required for the dual Cartan test are given by $\mathcal{M}_2^{(1)} \cong \langle (x^2)^2 \rangle \cong \mathcal{N}_2^{(1)}$ and $\mathcal{M}_3^{(1)} \cong \langle (x^2)^3 \rangle \cong \mathcal{N}_3^{(1)}$. It is now trivial to see that the map $\mu_1 : \mathcal{M}_2 \to \mathcal{M}_3$ induced by the multiplication with x^1 is injective and the map $\mu_2 : \mathcal{M}_2^{(1)} \to \mathcal{M}_3^{(1)}$ induced by the multiplication with x^2 is even bijective. Hence by the dual Cartan test the module $\mathcal{M}$ is involutive at degree 2.

Remark 3.15 Another way to formulate the assumptions of Theorem 3.12 is to require that $\mathcal{M}$ is a finitely generated graded $S\mathcal{V}$-module such that $H_{r,0}(\mathcal{M}) = H_{r,1}(\mathcal{M}) = 0$ for all $r \geq q$. Indeed, any such module can be finitely presented and thus is isomorphic to a factor module $(S\mathcal{V} \otimes \mathcal{U}^*)/\mathcal{N}^0$ for an appropriately chosen $\mathcal{U}$. By the same argument as in the proof of Proposition 2.18, $H_{r,1}(\mathcal{M}) \cong H_{r+1,0}(\mathcal{N}^0)$. Since the latter homology is determined by the minimal generators of the submodule $\mathcal{N}^0$, the two sets of assumptions are equivalent.

Consider the monomial ideal $\mathcal{I} = \langle (x^1)^3, (x^2)^3 \rangle \subset \Bbbk[x^1, x^2]$ generated in degree 3; it is the annihilator of the second symbolic system $\mathcal{N}$ in Example 2.9. It is trivial that here $\mu_1 : \mathcal{I}_4 \to \mathcal{I}_5$ is injective. For the map μ_2 we note that $\mathcal{I}_4/x^1\mathcal{I}_3 \cong \langle (x^1)^3x^2, (x^2)^4 \rangle$ and thus it is again easy to see that μ_2 is injective.

If we consider the map $\mu_2 : \mathcal{I}_5/x^1\mathcal{I}_4 \to \mathcal{I}_6/x^1\mathcal{I}_5$, then we find (using the identification $\mathcal{I}_5/x^1\mathcal{I}_4 \cong \langle (x^1)^3(x^2)^2, (x^2)^5 \rangle$) that $\mu_2\bigl([(x^1)^3(x^2)^2]\bigr) = [(x^1)^3(x^2)^3] = 0$ so that μ_2 is not injective and the Theorem 3.12 is not valid here. The observation that at some lower degree the maps μ_1 and μ_2 are injective may be understood from the syzygies. $\mathrm{Syz}(\mathcal{I}) \cong H_1(\mathcal{I})$ is generated by the single element $(x^2)^3\mathbf{e}_1 - (x^1)^3\mathbf{e}_2$ of degree 6. As its coefficients are of degree 3, nothing happens with the maps μ_i before we encounter $\mathcal{I}_6$ and then the equation $\mu_2\bigl([(x^1)^3(x^2)^2]\bigr) = 0$ is a trivial consequence of this syzygy.

4 Involutive Bases

Involutive bases are a special form of Gröbner bases with additional combinatorial properties. They were introduced by Gerdt and Blinkov [15, 16] generalising earlier ideas by Janet [25] in the theory of partial differential equations (a special case was slightly earlier discovered by Wu [63]); an introduction into their basic theory can be found in [50]. We assume in the sequel that the reader is familiar with the basic concepts in the theory of Gröbner bases; classical introductory texts are [1, 4, 11]. A

Gröbner basis is defined with respect to a term order; for an involutive basis we need one further ingredient, namely a so-called involutive division.

While the precise definition of an involutive division is somewhat technical, the underlying idea is simple. We consider first the monomial case. Let $\mathcal{T} = \{t_1, \dots, t_s\}$ be a set of terms in the ring $\mathcal{P} = \Bbbk[X]$ where $X = \{x^1, \dots, x^n\}$. Then the ideal $\mathcal{I} \subseteq \mathcal{P}$ generated by $\mathcal{T}$ consists of all polynomials $f = \sum_{i=1}^{s} P_i t_i$ where the coefficients P_i are arbitrary polynomials, i. e. $\mathcal{I}$ is the linear span $\langle \mathcal{T} \rangle$. An involutive division L assigns to each generator t_i a set of multiplicative variables $X_{L,\mathcal{T}}(t_i) \subseteq X$ (the remaining variables are denoted by $\overline{X}_{L,\mathcal{T}}(t_i)$). The involutive span $\langle \mathcal{T} \rangle_L$ consists of all linear combinations $f = \sum_{i=1}^{s} P_i t_i$ where now the coefficients must satisfy $P_i \in \Bbbk[X_{L,\mathcal{T}}(t_i)]$. Thus in general it contains only a subset of the ideal $\mathcal{I}$.

We call the set $\mathcal{T}$ a *weak involutive basis* of $\mathcal{I}$ for the involutive division L, if $\langle \mathcal{T} \rangle_L = \mathcal{I}$. For a term $x^\mu \in \mathcal{I}$, we call any generator $t_i \in \mathcal{T}$ such that $x^\mu \in \Bbbk[X_{L,\mathcal{T}}(t_i)] \cdot t_i$ an *involutive divisor*. Thus $\mathcal{T}$ is a weak involutive basis, if every term in $\mathcal{I}$ has at least one involutive divisor. For a *(strong) involutive basis* we require additionally that this involutive divisor is unique (in other words, for any two generators $t_i \neq t_j \in \mathcal{T}$ we have $\Bbbk[X_{L,\mathcal{T}}(t_i)] \cdot t_i \cap \Bbbk[X_{L,\mathcal{T}}(t_j)] \cdot t_j = \{0\}$).

The assignment of the multiplicative variables by the involutive division cannot be arbitrary but must satisfy certain conditions which we omit here (they can be found in the above references), as we are here only interested in one particularly simple division, namely the *Pommaret*[1] *division* P. While for general involutive divisions the assignment of multiplicative variables depends on the set $\mathcal{T}$ (i. e. the same term t may be assigned different variables if considered as element of different sets $\mathcal{T}$), the Pommaret division is a so-called global division where the assignment is independent of $\mathcal{T}$. If the term t is of the form $t = x^\mu$ with a multi index of class $\operatorname{cls}\mu = k$, then we simply assign as multiplicative variables $X_P(t) = \{x^1, \dots, x^k\}$.

Example 4.1 Consider again the second symbolic system $\mathcal{N}$ in Example 2.9. Its annihilator is the ideal $\mathcal{I}$ generated by $\mathcal{T} = \{(x^1)^3, (x^2)^3\}$. Since the first generator is of class 1, x^2 is not multiplicative for it with respect to the Pommaret division. As a consequence the monomial $(x^1)^3 x^2 \in \mathcal{I}$ is not contained in $\langle \mathcal{T} \rangle_P$ and thus the minimal basis $\mathcal{T}$ of $\mathcal{I}$ is not an involutive basis. An involutive basis is obtained, if we add the monomials $(x^1)^3 x^2$ and $(x^1)^3 (x^2)^2$, as one easily verifies.

The extension to general ideals is now straightforward. Let $\mathcal{F} = \{f_1, \dots, f_s\}$ be an arbitrary set of polynomials, $\prec$ a term order and L an involutive division. $\mathcal{F}$ is a *weak involutive basis* of $\mathcal{I} = \langle \mathcal{F} \rangle$ for $\prec$ and L, if the monomial set $\operatorname{lt} \mathcal{F} = \{\operatorname{lt} f_1, \dots, \operatorname{lt} f_s\}$ is a weak involutive basis of the leading ideal $\operatorname{lt} \mathcal{I}$. Note that this definition trivially implies that any weak involutive basis is a Gröbner basis, too. We call $\mathcal{F}$ a *(strong) involutive basis*, if no two elements of $\mathcal{F}$ have the same leading term and $\operatorname{lt} \mathcal{F}$ is a strong involutive basis of $\operatorname{lt} \mathcal{I}$. An involutive basis with respect to the Pommaret division is briefly called *Pommaret basis*. We also introduce multiplicative variables for generators $f \in \mathcal{F}$ by setting $X_{L,\mathcal{F},\prec}(f) = X_{L,\operatorname{lt}\mathcal{F}}(\operatorname{lt} f)$.

[1] Historically the terminology "Pommaret division" is a misnomer, as this division was already introduced by Janet. But the name has become generally accepted and therefore we stick to it.

Remark 4.2 One easily shows that by a simple elimination process any weak involutive basis can be reduced to a strong one and thus we will exclusively work with strong bases (this is no longer possible in more general situations with e. g. local term orders or coefficient rings) [50]. A particular property of the Pommaret division (in fact of any global division) is that the Pommaret basis of any monomial ideal is unique.

The above definition does not provide us with an effective criterion for recognising an involutive basis. For arbitrary involutive divisions no such criterion has been discovered so far. However, the Pommaret division belongs to the so-called *continuous* divisions for which the situation is more favourable. A finite set $\mathcal{F} \subset \mathcal{P}$ is *locally involutive* for the division L, if for every polynomial $f \in \mathcal{F}$ and for every non-multiplicative variable $x^j \in \overline{X}_{L,\mathcal{F},\prec}(f)$ the product $x^j f$ can be involutively reduced[2] to zero with respect to $\mathcal{F}$. Obviously, this property can be checked effectively.

Theorem 4.3 *If the finite set $\mathcal{F} \subset \mathcal{P}$ is involutively (head) autoreduced and locally involutive for a continuous division L, then $\mathcal{F}$ is an involutive basis of $\langle \mathcal{F} \rangle$ for L.*

As the proof of this theorem is rather technical and requires some concepts not introduced here, we refer to [15, 50]. We show here only a simpler special case. However, it will turn out later (Remarks 4.5 and 4.15) that this special case entails the Cartan test (in fact, this approach is almost identical with Janet's proof [26]).

Proposition 4.4 *Let $\mathcal{H}_q \subset \mathcal{P}_q$ be a finite triangular set of homogeneous polynomials of degree q which is locally involutive for the Pommaret division and a term order $\prec$. Then the set*

$$\mathcal{H}_{q+1} = \left\{ x^i h \mid h \in \mathcal{H}_q,\ x^i \in X_P(h) \right\} \subset \mathcal{P}_{q+1} \tag{4.1}$$

is also triangular and locally involutive (by induction this implies that $\mathcal{H}_q$ is involutive).

Proof. It is trivial to see that $\mathcal{H}_{q+1}$ is again triangular (all leading terms are different). For showing that it is also locally involutive, we consider an element $x^i h \in \mathcal{H}_{q+1}$. By construction, $\operatorname{cls}(x^i h) = i \leq \operatorname{cls} h$. We must show that for any non-multiplicative index $i < j \leq n$ the polynomial $x^j(x^i h)$ is expressible as a linear combination of polynomials $x^k \bar{h}$ where $\bar{h} \in \mathcal{H}_{q+1}$ and $x^k \in X_{P,\prec}(\bar{h})$. In the case that $j \leq \operatorname{cls} h$, this is trivial, as we may choose $\bar{h} = x^j h$ and $k = i$.

Otherwise x^j is non-multiplicative for h and since $\mathcal{H}_q$ is assumed to be locally involutive, the polynomial $x^j h$ can be written as a $\Bbbk$-linear combination of elements of $\mathcal{H}_{q+1}$. For exactly one summand $\bar{h}$ in this linear combination we have $\operatorname{lt} \bar{h} = \operatorname{lt}(x^j h)$ and hence $x^i \in X_{P,\prec}(\bar{h})$. If x^i is also multiplicative for all other summands, we are done. If the variable x^i is non-multiplicative for some summand $\bar{h}' \in \mathcal{H}_{q+1}$, then we analyse the product $x^i \bar{h}'$ in the same manner writing $\bar{h}' = x^k h'$ for some $h' \in \mathcal{H}_q$. Since $\operatorname{lt} \bar{h}' \prec \operatorname{lt}(x^j h)$, this process terminates after a finite number of steps leading to an involutive standard representation of $x^j(x^i h)$. ∎

[2] Involutive reducibility is defined as in the standard Gröbner theory; the sole difference is that a reduction is permitted only, if the reducing element $f \in \mathcal{F}$ is multiplied with a polynomial in $\Bbbk[X_{L,\mathcal{F},\prec}(f)]$.

Remark 4.5 Proposition 4.4 may be considered as an involutive basis version of the Cartan test. Let $\mathcal{I} \subseteq \mathcal{P}$ be a homogeneous ideal and $\mathcal{H}_q$ a triangular vector space basis of $\mathcal{I}_q$ for some degree q. The set $\mathcal{H}_q$ is locally involutive (and thus a Pommaret basis of $\mathcal{I}_{\geq q}$), if and only if the set $\mathcal{H}_{q+1}$ defined by (4.1) is a vector space basis of $\mathcal{I}_{q+1}$. Denoting by $\beta_q^{(k)}$ the number of elements of $\mathcal{F}$ where the leading term is of class k, these considerations lead to the inequality

$$\dim \mathcal{I}_{q+1} \geq \sum_{k=1}^{n} k\beta_q^{(k)} \tag{4.2}$$

and equality holds, if and only if $\mathcal{F}$ is a Pommaret basis. Remark 4.15 below shows that (4.2) does not only formally looks like (3.4) but that it is actually equivalent.

While it is almost trivial to prove that any ideal $\mathcal{I} \subseteq \mathcal{P}$ possesses a Gröbner basis for any term order, the existence of involutive bases is a more difficult question and depends on the precise form of the chosen division. For some divisions the existence is always guaranteed; one speaks of *Noetherian* divisions.

For Pommaret bases the situation is more complicated. It is easy to find ideals without a Pommaret basis—consider for example $\langle x^1x^2\rangle \subset \Bbbk[x^1, x^2]$ where a Pommaret basis would have to include all terms $x^1(x^2)^k$ with $k \geq 1$. A closer look reveals that this is actually only a problem of the chosen coordinates. If we begin as above with the symmetric algebra $S\mathcal{V}$ and consider $\mathcal{I}$ as an ideal in it, then for a generic basis of $\mathcal{V}$ the corresponding polynomial ideal $\mathcal{I}$ has a Pommaret basis. We call a basis (or coordinates) $\{x^1, \dots, x^n\}$ such that $\mathcal{I}$ possesses a Pommaret basis *δ-regular* for the ideal $\mathcal{I}$.[3] The use of the same terminology as in the Cartan test is no coincidence, as we will show in the next chapter. In the example above the transformation $x^1 \to x^1 + x^2$ leads to the ideal $\langle (x^2)^2 + x^1x^2\rangle$ which has a Pommaret basis for any term order where $x^2 \succ x^1$ (if $x^2 \prec x^1$, we can use the transformation $x^2 \to x^1 + x^2$).

Remark 4.6 For general information about the algorithmic determination of involutive bases we refer to [15, 16, 17]. Effective criteria for recognising δ-singular and effective methods for the construction of δ-regular coordinates for a given ideal $\mathcal{I}$ are discussed in detail in [22]. From a strictly algorithmic point of view, it is unpleasant that the Pommaret division is not Noetherian. But we will see in the remainder of this chapter that this seeming disadvantage has a number of benefits, as for many applications in algebraic geometry it is of considerable interest to know "good" coordinates.

We turn now to properties of involutive bases, in particular to those not shared by ordinary Gröbner bases. For simplicity, we always assume that we are dealing with a homogeneous ideal $\mathcal{I}$ and that also all considered bases of $\mathcal{I}$ are homogeneous.

If $\mathcal{H} = \{h_1, \dots, h_s\}$ is a Gröbner basis of the ideal $\mathcal{I}$, then it is well-known that any polynomial $f \in \mathcal{I}$ possesses a standard representation $f = \sum_{i=1}^{s} P_i h_i$ where the coefficients $P_i \in \mathcal{P}$ satisfy $\operatorname{lt}(P_i h_i) \preceq \operatorname{lt} f$ whenever $P_i \neq 0$. However, even with this constraint this representation is in general not unique. This changes, if we assume

[3] Of course the used term order is here of great importance: it follows immediately from the definition of an involutive basis that the δ-regularity of a coordinate system is completely determined by $\operatorname{lt}\mathcal{I}$.

that $\mathcal{H}$ is an involutive basis for an involutive division L. Imposing now the additional constraint that $P_i \in \Bbbk\bigl[X_{L,\operatorname{lt}\mathcal{H}}(\operatorname{lt} h_i)\bigr]$, we obtain the unique *involutive standard representation* of f with respect to $\mathcal{H}$. This uniqueness is the key to most applications of involutive bases.[4]

Another way to express the uniqueness of the involutive standard representation is to say that $\mathcal{H}$ induces a *Stanley decomposition* of $\mathcal{I}$. Because of the assumed homogeneity, $\mathcal{I}$ may be considered as a graded vector space with respect to the natural grading given by the total degree. A Stanley decomposition is then an isomorphism of graded vector spaces $\mathcal{I} \cong \bigoplus_{t\in\mathcal{T}} \Bbbk[X_t]\cdot t$ where $\mathcal{T}$ is some finite set of generators and $X_t \subseteq X$ is some subset of variables. In our case, we obtain the decomposition

$$\mathcal{I} \cong \bigoplus_{i=1}^{s} \Bbbk\bigl[X_{L,\operatorname{lt}\mathcal{H}}(\operatorname{lt} h_i)\bigr]\cdot h_i\ . \tag{4.3}$$

Pommaret bases lead to a special kind of Stanley decompositions, so-called *Rees decompositions* [47], where the subsets X_t are always of the form $\{x^1,\dots,x^{k_t}\}$ for some value $0 \le k_t \le n$.

A simple application of a Stanley decomposition (in fact, the one which motivated its introduction by Stanley [58][5]) is that one can trivially read off the *Hilbert series* of $\mathcal{I}$:

$$\mathcal{H}_{\mathcal{I}}(\lambda) = \sum_{t\in\mathcal{T}} \frac{\lambda^{q_t}}{(1-\lambda)^{k_t}} \tag{4.4}$$

where we introduced $q_t = \deg t$ and $k_t = |X_t|$. In particular, the *(Krull) dimension* of the ideal $\mathcal{I}$ is given by $D = \max_{t\in\mathcal{T}} k_t$ and the *multiplicity* (or *degree*) by the number of generators $t \in \mathcal{T}$ with $k_t = D$.

For most purposes, it is of greater interest to obtain a *complementary decomposition*, i. e. a Stanley decomposition of the factor algebra $\mathcal{A} = \mathcal{P}/\mathcal{I}$. Sturmfels and White [60] presented a recursive algorithm for computing such a decomposition given a Gröbner basis of $\mathcal{I}$. Somewhat surprising, the knowledge of an arbitrary involutive basis does not seem to give an advantage here. The situation changes, if one considers special involutive divisions. In the context of determining formally well-posed initial conditions for overdetermined systems of partial differential equations, Janet [27, §15] presented an algorithmic solution to this problem already in the 1920s. For Pommaret bases the solution is almost trivial; in fact, one only needs the degree of the basis (the reason will become evident below when we discuss the Castelnuovo–Mumford regularity).

Proposition 4.7 *The homogeneous ideal $\mathcal{I} \subseteq \mathcal{P}$ possesses a Pommaret basis $\mathcal{H}$ with $\deg\mathcal{H} = q$, if and only if the two sets $\bar{\mathcal{T}}_0 = \{x^\mu \in \mathbb{T}\setminus\operatorname{lt}\mathcal{I} \mid \deg x^\mu < q\}$ and $\bar{\mathcal{T}}_1 = \{x^\mu \in \mathbb{T}\setminus\operatorname{lt}\mathcal{I} \mid \deg x^\mu = q\}$ yield the complementary decomposition*

$$\mathcal{A} \cong \bigoplus_{t\in\bar{\mathcal{T}}_0} \Bbbk\cdot t \oplus \bigoplus_{t\in\bar{\mathcal{T}}_1} \Bbbk[X_P(t)]\cdot t\ . \tag{4.5}$$

[4] One easily shows that with respect to a *weak* involutive basis also every ideal member has an involutive standard representation. However, it will be unique, if and only if one is dealing with a strong involutive basis. For this reason, for most advanced applications of involutive bases only the strong ones are of real interest.

[5] One should note that in the context of partial differential equations Janet [27] derived already much earlier a similar expression for the Hilbert function.

Remark 4.8 Stanley decompositions are not unique. The complementary decomposition (4.5) is generally rather redundant. One can show that any Pommaret basis is simultaneously a Janet basis. Applying Janet's algorithm to it almost always yields a more compact decomposition with less generators. However, from a theoretical point of view, Proposition 4.7 is very useful, as it provides a closed formula and not only an algorithm. Note that (4.5) is again a Rees decomposition.

Proposition 4.9 *Let $\mathcal{H}$ be a homogeneous Pommaret basis of the homogeneous ideal $\mathcal{I} \subseteq \mathcal{P}$ with $\deg \mathcal{H} = q$. Then $D = \dim \mathcal{A}$ is given by*

$$D = \min \left\{ i \mid \langle \mathcal{H}, x^1, \dots, x^i \rangle_q = \mathcal{P}_q \right\} . \tag{4.6}$$

Remark 4.10 As a corollary to this result, one can easily show that $\{x^1, \dots, x^D\}$ is a maximal strongly independent set modulo $\mathcal{I}$ (see [20, 31] for the notion of an independent set modulo an ideal and its relation to the dimension). Here we see for the first time that the knowledge of δ-regular coordinates is of some interest, as generally no maximal independent set of this particularly simple form exists.
In fact, combining this observation with Proposition 4.7 yields that the restriction of the canonical projection $\pi : \mathcal{P} \to \mathcal{A} = \mathcal{P}/\mathcal{I}$ to the subring $\Bbbk[x^1, \dots, x^D]$ is a *Noether normalisation* of $\mathcal{A}$. Thus computing δ-regular coordinates determines automatically a Noether normalisation. One can show that δ-regularity is equivalent to simultaneous Noether normalisations of $\operatorname{lt} \mathcal{I}$ and all its primary components [5, 51].

Another measure for the size of $\mathcal{A}$ is its *depth*. It can also be immediately read off from a Pommaret basis. The proof of this fact provided by [51] relies on a direct verification that the given sequence is regular. Later in this article (Theorem 5.6) we will provide a homological proof of the following statement about the depth.

Proposition 4.11 *Let $\mathcal{H}$ be a homogeneous Pommaret basis of the homogeneous ideal $\mathcal{I} \subseteq \mathcal{P}$ for a class respecting term order $\prec$ and $d = \min_{h \in \mathcal{H}} \operatorname{cls} h$. Then $(x^1, \dots, x^{d-1})$ is a maximal regular sequence for $\mathcal{A}$ and hence $\operatorname{depth} \mathcal{A} = d - 1$.*

Remark 4.12 Combining Propositions 4.9 and 4.11 leads immediately to the so-called *Hironaka criterion* for Cohen–Macaulay algebras: the factor algebra $\mathcal{A} = P/\mathcal{I}$ is Cohen–Macaulay, if and only if it possesses a Rees decomposition where all generators are of the same class.

Definition 4.13 A homogeneous ideal $\mathcal{I} \subseteq \mathcal{P}$ is called *q-regular*, if its ith syzygy module can be generated by elements of degree less than or equal to $q+i$; the *Castelnuovo–Mumford regularity* $\operatorname{reg} \mathcal{I}$ is the least value q for which $\mathcal{I}$ is q-regular.

Among other applications, $\operatorname{reg} \mathcal{I}$ represents an important measure for the complexity of Gröbner basis computations [2]. According to Bayer and Stillman [3], *generically* the reduced Gröbner basis with respect to the degree reverse lexicographic order has the degree $\operatorname{reg} \mathcal{I}$ and no other term order yields a lower degree. However, one rarely knows whether or not one is in the generic case so that this result is only of limited use for concrete computations. For Pommaret bases we rediscover here again simply the question of δ-regularity.

Theorem 4.14 *Let $\mathcal{I} \subseteq \mathcal{P}$ be a homogeneous ideal. Then $\operatorname{reg}\mathcal{I} = q$, if and only if $\mathcal{I}$ has in some coordinates a homogeneous Pommaret basis $\mathcal{H}$ with respect to the degree reverse lexicographic order such that $\deg\mathcal{H} = q$.*

This result implies that in δ-regular coordinates the equality $\operatorname{reg}\mathcal{I} = \operatorname{reg}(\operatorname{lt}\mathcal{I})$ holds whereas in general we only have the inequality $\operatorname{reg}\mathcal{I} \leq \operatorname{reg}(\operatorname{lt}\mathcal{I})$. Another remarkable implication is that in arbitrary coordinates $x^1, \dots, x^n$ the ideal $\mathcal{I}$ either does not possess a finite Pommaret basis or the basis is of the fixed degree $\operatorname{reg}\mathcal{I}$.

Remark 4.15 According to Remark 2.15, the Koszul homology of a module $\mathcal{M}$ is equivalent to its minimal free resolution. Thus if $\operatorname{reg}\mathcal{I} = q$, then all homology modules $H_{r,p}(\mathcal{I})$ with $r > q$ vanish. Taking into account the degree shift in (2.16), for the factor module $\mathcal{M} = \mathcal{P}/\mathcal{I}$ thus all homology modules $H_{r,p}(\mathcal{M})$ with $r \geq q$ vanish. Hence the Castelnuovo–Mumford regularity of $\mathcal{I}$ is the same as the degree of involution of $\mathcal{M}$. It is very surprising that this elementary fact remained unobserved until very recently; it is implicitly contained in [53] and explicitly mentioned by Malgrange [35].
Combining this observation with Theorem 4.14 and Remark 4.5, we finally see that Proposition 4.4 may indeed be considered as an involutive basis version of the Cartan test. Let—as in Remark 4.5—the set $\mathcal{H}_q$ be a basis of the vector space $\mathcal{I}_q$ where all generators have different leading terms. Then we may choose as representatives of a basis of $\mathcal{M}_q$ polynomials which have as leading terms exactly those terms which do not appear in $\operatorname{lt}\mathcal{H}_q$. Elementary combinatorics shows that if $\mathcal{H}_q$ contains $\beta_q^{(k)}$ elements with a leading term of class k, then our basis of $\mathcal{M}_q$ contains

$$\alpha_q^{(k)} = m\binom{q+n-k-1}{q-1} - \beta_q^{(k)} \tag{4.7}$$

representatives with a leading term of class k.
It is no coincidence that we use here the same notation as for the Cartan characters. As a vector space the symbolic system $\mathcal{N} = \mathcal{I}^0$ is isomorphic to $\mathcal{P}/\mathcal{I}$; a concrete isomorphism is given by replacing in the above representatives x^i by y_i. If we use a class respecting term order, then it follows from Lemma B.1 that $\dim\mathcal{N}_q^{(k)} = \sum_{j=k+1}^{n} \alpha_q^{(j)}$ so that the numbers $\alpha_q^{(k)}$ are indeed the Cartan characters. A well-known identity for binomial coefficients proves now that (3.4) and (4.2) are equivalent inequalities.

Theorem 4.14 represents probably the simplest method for computing $\operatorname{reg}\mathcal{I}$. It requires the knowledge of δ-regular coordinates, but as already mentioned in Remark 4.6 these can be constructed effectively. In recent years, a number of methods for the determination of $\operatorname{reg}\mathcal{I}$ have been developed [3, 5, 62]. However, they all also require the use of generic coordinates (in [51] their relation to Pommaret bases is studied in detail).

Example 4.16 Consider the homogeneous ideal

$$\mathcal{I} = \langle z^8 - wxy^6,\ y^7 - x^6z,\ yz^7 - wx^7\rangle \subset \mathbb{Q}[w,x,y,z]\ . \tag{4.8}$$

The given basis of degree 8 is already a Gröbner basis for the degree reverse lexicographic term order. If we perform a permutation of the variables and consider $\mathcal{I}$ as an ideal in $\mathbb{Q}[w, y, x, z]$, then we obtain for the degree reverse lexicographic term order (in the new variables!) the following Gröbner basis of degree 50:

$$\begin{aligned}\bigl\{y^7 - x^6 z,\ yz^7 - wx^7,\ z^8 - wxy^6,\ y^8 z^6 - wx^{13},\\ y^{15}z^5 - wx^{19},\ y^{22}z^4 - wx^{25},\ y^{29}z^3 - wx^{31},\\ y^{36}z^2 - wx^{37},\ y^{43}z - wx^{43},\ y^{50} - wx^{49}\bigr\}\ .\end{aligned} \tag{4.9}$$

Unfortunately, neither coordinate system is generic: as $\operatorname{reg}\mathcal{I} = 13$, one yields a basis of too low degree and the other one one of too high degree.
With a Pommaret basis it is no problem to determine the Castelnuovo-Mumford regularity, as the first coordinate system is δ-regular. A Pommaret basis of $\mathcal{I}$ for the degree reverse lexicographic term order is obtained by adding the polynomials $z^k(y^7 - x^6 z)$ for $1 \leq k \leq 6$ and thus the degree of the basis is indeed 13.

Remark 4.17 In order to obtain their above mentioned result, Bayer and Stillman first proved the following characterisation of a q-regular ideal (which may be considered as a variant of δ-regularity): if $\mathcal{I}$ is a homogeneous ideal which can be generated by elements of degree less than or equal to q, then it is q-regular, if and only if for some value $0 \leq d \leq n$ linear forms $y_1, \dots, y_d \in \mathcal{P}_1$ exist such that

$$\bigl(\langle \mathcal{I}, y_1, \dots, y_{j-1}\rangle : y_j\bigr)_q = \langle \mathcal{I}, y_1, \dots, y_{j-1}\rangle_q\ , \quad 1 \leq j \leq d\ , \tag{4.10a}$$

$$\langle \mathcal{I}, y_1, \dots, y_d\rangle_q = \mathcal{P}_q\ . \tag{4.10b}$$

We will discuss later in Remark 5.5 that this characterisation of q-regularity is equivalent to the dual Cartan test.

5 Pommaret Bases and Homology

Now we study the relationship between Pommaret bases and the homological constructions introduced in Chapters 2 and 3. We assume throughout that a fixed basis $\{x^1, \dots, x^n\}$ of $\mathcal{V}$ has been chosen so that we may identify $S\mathcal{V} = \Bbbk[x^1, \dots, x^n] = \mathcal{P}$. For simplicity, we restrict to homogeneous ideals $\mathcal{I} \subseteq \mathcal{P}$. We only consider Pommaret bases for the degree reverse lexicographic order $\prec_{\text{degrevlex}}$, as for any other term order the corresponding Pommaret basis (if it exists) cannot be of lower degree by the inequality $\operatorname{reg}\mathcal{I} \leq \operatorname{reg}(\operatorname{lt}\mathcal{I})$ and Theorem 4.14.
It turns out that this relationship takes its simplest form, if we compare the Pommaret basis of the ideal $\mathcal{I}$ and the Koszul homology of its factor algebra $\mathcal{P}/\mathcal{I}$ which we consider here as a $\mathcal{P}$-module in order to be consistent with the terminology introduced in Chapters 2 and 3. Like for general Gröbner bases, essentially everything relevant for involutive bases can be read off the leading ideal. Therefore, we show first that at least for our chosen term order quasi-regularity is also already decided by the leading ideal.

Lemma 5.1 *Let $\mathcal{I} \subseteq \mathcal{P}$ be a homogeneous ideal and $\prec$ the degree reverse lexicographic order. The sequence $(x^1, \dots, x^n)$ is quasi-regular at degree q for the module $\mathcal{M} = \mathcal{P}/\mathcal{I}$, if and only if it is quasi-regular at degree q for $\mathcal{M}' = \mathcal{P}/\operatorname{lt}\mathcal{I}$.*

Proof. Let $\mathcal{G}$ be a Gröbner basis of $\mathcal{I}$ for $\prec$. Then the normal form with respect to the basis $\mathcal{G}$ defines an isomorphism between the vector spaces $\mathcal{M}$ and $\mathcal{M}'$. One direction is now trivial, as an obvious necessary condition for $m = [f] \in \mathcal{M}$ to satisfy $x^1 \cdot m = 0$ is that $x^1 \cdot [\operatorname{lt} f] = 0$ in $\mathcal{M}'$. Hence quasi-regularity of x^1 for $\mathcal{M}'$ implies quasi-regularity of x^1 for $\mathcal{M}$ and by iteration the same holds true for the whole sequence (note that here we could have used any term order).
For the converse let $r \geq q$ be an arbitrary degree. Because of the mentioned isomorphism, we may choose for the vector space $\mathcal{M}_r$ a basis where each member is represented by a monomial, i. e. the representatives simultaneously induce a basis of $\mathcal{M}'_r$. Let x^μ be one of these monomials. As we assume that x^1 is quasi-regular for $\mathcal{M}$, we must have $x^1 \cdot [x^\mu] \neq 0$ in $\mathcal{M}$. Suppose now that $x^1 \cdot [x^\mu] = 0$ in $\mathcal{M}'$ so that x^1 is not quasi-regular for $\mathcal{M}'$.
Thus $x^{\mu+1_1} \in \operatorname{lt}\mathcal{I}$. Since $\operatorname{lt}\mathcal{I} = \langle \operatorname{lt}\mathcal{G} \rangle$ by the definition of a Gröbner basis, $\mathcal{G}$ must contain a polynomial g with $\operatorname{lt} g \mid x^{\mu+1_1}$. Because of the assumption $x^\mu \notin \operatorname{lt}\mathcal{I}$, we must have $\operatorname{cls}(\operatorname{lt} g) = 1$. By Lemma B.1, this implies that every term in g is of class 1. Iteration of this argument shows that the normal form of $x^{\mu+1_1}$ with respect to $\mathcal{G}$ is divisible by x^1, i. e. it can be written as $x^1 f$ with $f \in \mathcal{P}_r$ and $\operatorname{lt} f \prec x^\mu$. Consider now the polynomial $\bar{f} = x^\mu - f \in \mathcal{P}_r \setminus \{0\}$. As it consists entirely of terms not contained in $\operatorname{lt}\mathcal{I}$, we have $[\bar{f}] \neq 0$ in $\mathcal{M}_r$. However, by construction $x^1 \cdot [\bar{f}] = 0$ contradicting the injectivity of multiplication by x^1 on $\mathcal{M}_r$.
For the remaining elements of the sequence $(x^1, \dots, x^n)$ we note the isomorphism $\mathcal{M}^{(k)} = \mathcal{M}/\langle x^1, \dots, x^k \rangle \mathcal{M} \cong \mathcal{P}^{(k)}/\mathcal{I}^{(k)}$ for each $1 \leq k < n$ where we introduced the abbreviations $\mathcal{P}^{(k)} = \Bbbk[x^{k+1}, \dots, x^n]$ and $\mathcal{I}^{(k)} = \mathcal{I} \cap \mathcal{P}^{(k)}$. It implies that we may iterate the arguments above so that indeed quasi-regularity of $(x^1, \dots, x^n)$ for $\mathcal{M}'$ is equivalent to quasi-regularity of the sequence for $\mathcal{M}'$. ∎

Note that restriction to the degree reverse lexicographic order is here essential, as in general we have only the inequality $\operatorname{reg}\mathcal{M} \leq \operatorname{reg}(\operatorname{lt}\mathcal{M})$ and if it is strict, then a sequence may be quasi-regular for $\mathcal{M}$ at any degree $\operatorname{reg}\mathcal{M} \leq q < \operatorname{reg}(\operatorname{lt}\mathcal{M})$, but it cannot be quasi-regular for $\mathcal{M}'$ at such a degree by the results below.

Theorem 5.2 *The basis $\{x^1, \dots, x^n\}$ is δ-regular for the homogeneous ideal $\mathcal{I} \subseteq \mathcal{P}$ in the sense that $\mathcal{I}$ possesses a Pommaret basis $\mathcal{H}$ for the degree reverse lexicographic term order with $\deg\mathcal{H} = q$, if and only if the sequence $(x^1, \dots, x^n)$ is quasi-regular for the factor algebra $\mathcal{P}/\mathcal{I}$ at degree q but not at any lower degree.*

Proof. It suffices to consider monomial ideals $\mathcal{I}$: for Pommaret bases it is obvious from their definition that a basis is δ-regular for $\mathcal{I}$, if and only if it is so for $\operatorname{lt}\mathcal{I}$; a similar statement holds for quasi-regularity by Lemma 5.1.
Let us first assume that the basis is $\{x^1, \dots, x^n\}$ is δ-regular in the described sense. By Proposition 4.7, the leading terms $\operatorname{lt}\mathcal{H}$ induce a complementary decomposition of the form (4.5) of $\mathcal{M} = \mathcal{P}/\mathcal{I}$ where all generators are of degree $q = \deg\mathcal{H}$ or less. Thus, if $\mathcal{M}_q \neq 0$ (otherwise there is nothing to show), then we can choose a vector space basis

of it as part of the complementary decomposition and the variable x^1 is multiplicative for all its members. But this observation immediately implies that multiplication with x^1 is injective from degree q on, so that x^1 is quasi-regular for $\mathcal{M}$ at degree q.

For the remaining elements of $\{x^1, \dots, x^n\}$ we proceed as in the proof of Lemma 5.1 and use the isomorphism $\mathcal{M}^{(k)} \cong \mathcal{P}^{(k)}/\mathcal{I}^{(k)}$. One easily verifies that a Pommaret basis of $\mathcal{I}^{(k)}$ is obtained by setting $x^1 = \cdots = x^k = 0$ in the partial basis $\mathcal{H}^{(k)} = \{h \in \mathcal{H} \mid \operatorname{cls} h > k\}$. Thus we can again iterate for each $1 < k \leq n$ the argument above so that indeed $(x^1, \dots, x^n)$ is a quasi-regular sequence for $\mathcal{M}$ at degree q.

For the converse, we first show that quasi-regularity of the sequence $(x^1, \dots, x^n)$ implies the existence of a Rees decomposition for $\mathcal{P}/\mathcal{I}$. Exploiting again the isomorphism $\mathcal{M}^{(k)} \cong \mathcal{P}^{(k)}/\mathcal{I}^{(k)}$, one easily sees that a vector space basis of $\mathcal{M}_q^{(k)}$ is induced by all terms $x^\mu \notin \mathcal{I}$ with $|\mu| = q$ and $\operatorname{cls} \mu \geq k$. By the definition of quasi-regularity, multiplication with x^k is injective on $\mathcal{M}^{(k)}$, hence we take $\{x^1, \dots, x^k\}$ as multiplicative variables for such a term (which is exactly the assignment used in the Rees decomposition induced by a Pommaret basis according to Proposition 4.7).

We claim now that this assignment yields a Rees decomposition of $\mathcal{M}_{\geq q}$ (and hence induces one of $\mathcal{P}/\mathcal{I}$, since we only have to add all terms $x^\mu \notin \mathcal{I}$ such that $|\mu| < q$ without any multiplicative variables). The only thing to prove is that our decomposition indeed covers all of $(\mathcal{P}/\mathcal{I})_{\geq q}$. But this is trivial. If $x^\mu \notin \mathcal{I}$ is an arbitrary term with $|\mu| = q + 1$ and $\operatorname{cls} \mu = k$, then we can write $x^\mu = x^k \cdot x^{\mu - 1_k}$. Obviously, $x^\mu \notin \mathcal{I}$ implies $x^{\mu - 1_k} \notin \mathcal{I}$ and $\operatorname{cls}(\mu - 1_k) \geq k$ so that x^k is multiplicative for it. Hence all of $\mathcal{M}_{q+1}$ is covered and an easy induction shows that we have a decomposition of $\mathcal{M}_{\geq q}$.

Proposition 4.7 entails now that $\mathcal{I}$ possesses a *weak* Pommaret basis of degree q. Since the reduction to a strong basis as mentioned in Remark 4.2 can only decrease the degree, we conclude that $\mathcal{I}$ has a strong Pommaret basis of degree at most q. However, if the degree of the basis actually decreased, then, by the converse statement already proven, $(x^1, \dots, x^n)$ would be a quasi-regular sequence for $\mathcal{M}$ at a lower degree than q contradicting our assumptions.

The same "reverse" argument shows that if $\mathcal{I}$ has a Pommaret basis of degree q, then the sequence $(x^1, \dots, x^n)$ cannot be quasi-regular for $\mathcal{M}$ at any degree less than q, as otherwise a Pommaret basis of lower degree would exist which is not possible by the discussion following Theorem 4.14. ∎

For *monomial* ideals $\mathcal{I} \subseteq \mathcal{P}$ a much stronger statement is possible. Using again the isomorphism $\mathcal{M}^{(k)} \cong \mathcal{P}^{(k)}/\mathcal{I}^{(k)}$, we may identify elements of $\mathcal{M}^{(k)}$ with linear combinations of the terms $x^\nu \notin \mathcal{I}$ satisfying $\operatorname{cls} x^\nu > k$. Finally, if we denote as before by $\mu_k : \mathcal{M}^{(k-1)} \to \mathcal{M}^{(k-1)}$ the map induced by multiplication with x^k, then we obtain a simple relationship between the (unique!) Pommaret basis of the monomial ideal $\mathcal{I}$ and the kernels of the maps μ_k.

Proposition 5.3 *Let the basis $\{x^1, \dots, x^n\}$ of $\mathcal{V}$ be δ-regular for the monomial ideal $\mathcal{I} \subseteq \mathcal{P}$. Furthermore, let $\mathcal{H}$ be the Pommaret basis of $\mathcal{I}$ and set $\mathcal{H}_k = \{x^\nu \in \mathcal{H} \mid \operatorname{cls} \nu = k\}$ for any $1 \leq k \leq n$. Then the set $\{x^{\nu - 1_k} \mid x^\nu \in \mathcal{H}_k\}$ is a basis of $\ker \mu_k$.*

Proof. Assume that $x^\nu \in \mathcal{H}_k$. Then $x^{\nu-1_k} \notin \mathcal{I}$, as otherwise the Pommaret basis $\mathcal{H}$ was not involutively autoreduced, and hence we find $x^{\nu-1_k} \in \ker \mu_k$.

Conversely, suppose that $x^\nu \in \ker \mu_k$. Obviously, this implies $x^{\nu+1_k} \in \mathcal{I}$ and the Pommaret basis $\mathcal{H}$ must contain an involutive divisor of $x^{\nu+1_k}$. If this divisor was not $x^{\nu+1_k}$ itself, the term x^ν would have to be an element of $\mathcal{I}$ which is obviously not possible. Since $x^\nu \in \ker \mu_k$ entails $\operatorname{cls}(\nu + 1_k) = k$, we thus find $x^{\nu+1_k} \in \mathcal{H}_k$. ☐

We noted already in Remark 4.15 that the degree of involution is nothing but the Castelnuovo–Mumford regularity. There we used the equivalence of the Koszul homology to the minimal free resolution. With the help of Theorem 5.2, we can also give a simple direct proof.

Corollary 5.4 *Let $\mathcal{I} \subseteq \mathcal{P}$ be a homogeneous ideal. Then the factor module $\mathcal{M} = \mathcal{P}/\mathcal{I}$ is involutive at degree q but not at any lower degree, if and only if the Castelnuovo–Mumford regularity takes the value $\operatorname{reg}\mathcal{I} = q$.*

Proof. By Theorem 4.14, $\operatorname{reg}\mathcal{I} = q$, if and only if $\mathcal{I}$ possesses in suitable variables $x^1, \dots, x^n$ a Pommaret basis $\mathcal{H}$ with $\deg \mathcal{H} = q$. According to Theorem 5.2, the sequence $(x^1, \dots, x^n)$ is then quasi-regular for $\mathcal{M}$ at degree q but not any lower degree, so that by the dual Cartan test (Theorem 3.12) the module $\mathcal{M}$ is involutive at degree q but not any lower degree. ☐

Remark 5.5 Given this result, it is not so surprising to see that the characterisation of the Castelnuovo–Mumford regularity mentioned in Remark 4.17 and the dual Cartan test in Theorem 3.12 are equivalent. Consider a homogeneous ideal $\mathcal{I} \subseteq \mathcal{P}$ for which the basis $\{x^1, \dots, x^n\}$ of $\mathcal{V}$ is δ-regular and assume that for some degree $q \geq 0$ the condition (4.10a) is violated for some $1 \leq j \leq D = \dim(\mathcal{P}/\mathcal{I})$. Thus there exists a polynomial $f \in \mathcal{P}_{q-1}$ such that $f \notin \langle \mathcal{I}, x^1, \dots, x^{j-1} \rangle$ but $x^j f$ is contained in this ideal. If we set $\mathcal{M}_j = \mathcal{P}/\langle \mathcal{I}, x^1, \dots, x^j \rangle$, then obviously the equivalence class $[f]$ lies in the kernel of the map $\mu_j : \mathcal{M}_{j-1} \to \mathcal{M}_{j-1}$ induced by multiplication with x^j. Since trivially for $\mathcal{M} = \mathcal{P}/\mathcal{I}$ the module $\mathcal{M}^{(j)} = \mathcal{M}/\langle x^1, \dots, x^j \rangle \mathcal{M}$ is isomorphic to $\mathcal{M}_j$, the conditions of Theorem 3.12 are not satisfied for $\mathcal{M}$ either. Conversely, any representative of a non-trivial element of $\ker \mu_j$ of degree q provides us at once with such a polynomial f. There is no need to consider a value $j > D$, since we know from Proposition 4.9 that $(\mathcal{M}_D)_{\geq \operatorname{reg}\mathcal{I}} = 0$.

As an application we consider the following theorem providing a classical characterisation of the depth via Koszul homology which in fact is often even used as definition of $\operatorname{depth}\mathcal{M}$ (see e. g. [54, Sect. IV.A.4]). Note that, taking into account the relation between the minimal free resolution of a module and its Koszul homology discussed in Remark 2.15, it also trivially implies the Auslander–Buchsbaum formula relating depth and projective dimension.

Theorem 5.6 *Let $\mathcal{M}$ be a $\mathcal{P}$-module. Then $\operatorname{depth}\mathcal{M} = d$, if and only if $H_{n-d}(\mathcal{M}) \neq 0$ and $H_{n-d+1}(\mathcal{M}) = \cdots = H_n(\mathcal{M}) = 0$.*

Proof. For simplicity, we give the proof only for the case of an ideal $\mathcal{I} \subseteq \mathcal{P}$. The extension to modules is straightforward. Let $\mathcal{H}$ be a Pommaret basis of $\mathcal{I}$ with respect to the degree reverse lexicographic order, $d = \min_{h\in\mathcal{H}} \operatorname{cls} h$ (and thus $\operatorname{depth} \mathcal{I} = d$) and $\mathcal{H}_d = \{h \in \mathcal{H} \mid \operatorname{cls} h = d\}$. We choose a polynomial $\bar{h} \in \mathcal{H}_d$ of maximal degree and show now that it induces a non-zero element of $H_{n-d}(\mathcal{I})$.

By Lemma B.1, $\bar{h} \in \langle x^1, \dots, x^d \rangle$ and thus it possesses a unique representation $\bar{h} = x^1\bar{h}^{(1)} + \cdots + x^d\bar{h}^{(d)}$ with $\bar{h}^{(i)} \in \Bbbk[x^i, \dots, x^n]$. The polynomial $\bar{h}^{(d)}$ cannot lie in $\mathcal{I}$, as otherwise there would exist an $h \in \mathcal{H}$ with $\operatorname{lt} h \,|_P\, \operatorname{lt} \bar{h}^{(d)} \,|_P\, \operatorname{lt} \bar{h}$ contradicting the fact that any Pommaret basis is involutively head autoreduced. We claim now that for any $d < k \leq n$ polynomials $P_h \in \langle x^1, \dots, x^d \rangle$ exist such that $x^k\bar{h} = \sum_{h\in\mathcal{H}} P_h h$.

Obviously, the variable x^k is non-multiplicative for $\bar{h}$. By definition of a Pommaret basis, for each generator $h \in \mathcal{H}$ a polynomial $P_h \in \Bbbk[x^1, \dots, x^{\operatorname{cls} h}]$ exists such that $x^k\bar{h} = \sum_{h\in\mathcal{H}} P_h h$. No polynomial h with $\operatorname{cls} h > d$ lies in $\langle x^1, \dots, x^d \rangle$ (obviously $\operatorname{lt} h \notin \langle x^1, \dots, x^d \rangle$). As the leading terms cannot cancel in the sum, this implies already that $P_h \in \langle x^1, \dots, x^d \rangle$ for all $h \in \mathcal{H} \setminus \mathcal{H}_d$. For all $h \in \mathcal{H}_d$ we know that $P_h \in \Bbbk[x^1, \dots, x^d]$ and thus the only possibility for $P_h \notin \langle x^1, \dots, x^d \rangle$ is that P_h contains a constant term. However, as $\mathcal{I}$ is a homogeneous ideal and as the degree of $\bar{h}$ is maximal in $\mathcal{H}_d$, this is not possible for degree reason. As above, each of the coefficients may thus be uniquely decomposed $P_h = x^1P_h^{(1)} + \cdots + x^dP_h^{(d)}$ with $P_h^{(i)} \in \Bbbk[x^i, \dots, x^n]$. Because of the uniqueness of these decompositions we find that $x^k\bar{h}^{(i)} = \sum_{h\in\mathcal{H}} P_h^{(i)} h$ and therefore we conclude that $x^k\bar{h}^{(i)} \in \mathcal{I}$ for any $d < k \leq n$.

Let $I = (i_1, \dots, i_{d-1})$ be a repeated index with $i_1 < i_2 < \cdots, i_{d-1}$. Then its complement $\bar{I} = \{1, \dots, n\} \setminus I$ is a repeated index of length $n - d + 1$ and we may represent any element $\bar{\omega} \in \mathcal{P} \otimes \Lambda_{n-d+1}\mathcal{V}$ in the form $\bar{\omega} = \sum_{|I|=d-1} \bar{f}_I \mathrm{d}x^{\bar{I}}$. We consider now in particular all repeated indices with $i_{d-1} \leq d$. For each of them a unique value $i \in \{1, \dots, d\}$ exists such that $i \notin I$ and we set $\bar{f}_I = (-1)^{d-i}\bar{h}^{(i)}$. For all remaining coefficients we only assume that $\bar{f}_I \in \mathcal{I}$. Then, by our considerations above, the so constructed form $\bar{\omega}$ is *not* contained in $\mathcal{I} \otimes \Lambda_{n-d+1}\mathcal{V}$.

We claim that $\omega = \partial\bar{\omega} \in \mathcal{I} \otimes \Lambda_{n-d}\mathcal{V}$. If we write $\omega = \sum_{|I|=d} f_I \mathrm{d}x^{\bar{I}}$, then by definition of the Koszul differential $f_I = \sum_{j=1}^{d} (-1)^j x^{i_j} \bar{f}_{I\setminus\{i_j\}}$. Let us first assume that $i_d > d$. Then it follows from our choice of $\bar{\omega}$ that $f_{I\setminus\{i_j\}} \in \mathcal{I}$ for all $j < d$ and that always $x^{i_d} f_{I\setminus\{i_d\}} \in \mathcal{I}$ implying trivially that $f_I \in \mathcal{I}$. If $i_d = d$, then one easily verifies that we have chosen $\bar{\omega}$ precisely such that $f_I = \bar{h} \in \mathcal{I}$. Hence our claim is proven.

If we can now show that it is not possible to choose a form $\tilde{\omega} \in \mathcal{P} \otimes \Lambda_{n-d+2}\mathcal{V}$ such that $\bar{\omega} + \partial\tilde{\omega} \in \mathcal{I} \otimes \Lambda_{n-d+1}\mathcal{V}$, then we have constructed a non-zero element $[\omega] \in H_{n-d}(\mathcal{I})$. But this is easy to achieve by considering in particular the coefficient $\bar{f}_{(1,2,\dots,d-1)} = \bar{h}^{(d)} \notin \mathcal{I}$. The corresponding coefficient of the form $\partial\tilde{\omega}$ is given by $\sum_{j=1}^{d-1} (-1)^j x^j \tilde{f}_{(1,2,\dots,d-1)\setminus\{j\}} \in \langle x^1, \dots, x^{d-1} \rangle$. As noted above, we have $\bar{h}^{(d)} \in \Bbbk[x^d, \dots, x^n]$ so that it is not possible to eliminate it in this manner and hence no form $\bar{\omega} + \partial\tilde{\omega}$ can be contained in $\mathcal{I} \otimes \Lambda_{n-d+1}\mathcal{V}$.

There remains to show that $H_{n-d+1}(\mathcal{I}) = \cdots = H_n(\mathcal{I}) = 0$ under our assumptions. $H_n(\mathcal{I}) = 0$ follows immediately from Lemma 2.16. Consider now a cycle $\omega \in \mathcal{I} \otimes \Lambda_{n-k}\mathcal{V}$ with $0 < k < d$. Since the Koszul complex $K(\mathcal{P})$ is exact, a form $\bar{\omega} \in \mathcal{P} \otimes \Lambda_{n-k+1}\mathcal{V}$ exists with $\partial\bar{\omega} = \omega$. For all I we have by assumption

$f_I = \sum_{j=1}^{d}(-1)^j x^{i_j}\bar{f}_{I\setminus\{i_j\}} \in \mathcal{I}$; our goal is to show that (modulo $\operatorname{im}\partial$) we can always choose $\bar{\omega}$ such that all coefficients $\bar{f}_J \in \mathcal{I}$, too.
Without loss of generality, we may assume that all coefficients $\bar{f}_J$ are in normal form with respect to the Gröbner basis $\mathcal{H}$, as the difference is trivially contained in $\mathcal{I}$. In addition, we may assume that $\operatorname{lt} f_I = \operatorname{lt}(x^{i_j}\bar{f}_{I\setminus\{i_j\}})$ for some value j. Indeed, it is easy to see that cancellations between such leading terms can always be eliminated by subtracting a suitable form $\partial\tilde{\omega}$ from $\bar{\omega}$.
We begin with those repeated indices $I = (i_1,\dots,i_k)$ for which all indices satisfy $i_j < d = \min_{h\in\mathcal{H}} \operatorname{cls} h$. In this case $\operatorname{lt} f_I \in \langle \operatorname{lt}\mathcal{H}\rangle_P = \operatorname{lt}\mathcal{I}$ implies that already $\operatorname{lt}\bar{f}_{I\setminus\{i_j\}} \in \operatorname{lt}\mathcal{I}$ for the above j. But unless $\bar{f}_{I\setminus\{i_j\}} = 0$, this observation contradicts our assumption that all f_J are in normal form and thus do not contain any terms from $\operatorname{lt}\mathcal{I}$. Therefore all $\bar{f}_J$ where all entries of J are less than d must vanish.
We continue with those repeated indices $I = (i_1,\dots,i_k)$ where only one index $i_\ell > d$. Then, by our considerations above, $\bar{f}_{I\setminus\{i_\ell\}} = 0$ and hence $\operatorname{lt} f_I = \operatorname{lt}(x^{i_j}\bar{f}_{I\setminus\{i_j\}})$ for some value $j \neq \ell$. Thus $i_j < d$ and the same argument as above implies that all such $\bar{f}_{I\setminus\{i_j\}} = 0$. A trivial induction proves now that in fact all $\bar{f}_J = 0$ and therefore we find $\bar{\omega} \in \mathcal{I}\otimes\Lambda_{n-k+1}\mathcal{V}$. □

6 Formal Geometry of Differential Equations

In the next chapter we will demonstrate how the algebraic and homological theory presented so far naturally appears in the analysis of differential equations. Perhaps somewhat paradoxically, the key for applying algebraic methods lies in first providing a differential geometric framework. For this purpose, we must briefly recall some basic notions from the formal geometry of differential equations [30, 32, 42, 53].
Let $\pi : \mathcal{E} \to \mathcal{X}$ be a fibred manifold with an n-dimensional base space $\mathcal{X}$ and an $(m+n)$-dimensional total space $\mathcal{E}$ (in the simplest case $\mathcal{X} = \mathbb{R}^n$ and $\mathcal{E} = \mathbb{R}^{n+m}$ with π being the projection on the first n components). Local coordinates on $\mathcal{X}$ are $\mathbf{x} = (x^1,\dots,x^n)$ and fibre coordinates on $\mathcal{E}$ are $\mathbf{u} = (u^1,\dots,u^m)$. A section is then a map[1] $\sigma : \mathcal{X} \to \mathcal{E}$ satisfying $\pi\circ\sigma = \mathrm{id}_{\mathcal{X}}$. In local coordinates, such a section σ corresponds to a smooth function $\mathbf{u} = \mathbf{s}(\mathbf{x})$, as $\sigma(\mathbf{x}) = \big(\mathbf{x}, \mathbf{s}(\mathbf{x})\big)$.
A q-*jet* is an equivalence class $[\sigma]^{(q)}_{\mathbf{x}_0}$ of sections where two sections σ_1, σ_2 are considered as equivalent, if their graphs have at the point $\sigma_i(\mathbf{x}_0)$ a contact of order q, in other words if their Taylor expansions at $\mathbf{x}_0$ coincides up to order q (thus we may consider a q-jet as a truncated Taylor series). The *qth order jet bundle* $J_q\pi$ is then defined to be the set of all such q-jets. One easily verifies that $J_q\pi$ is an $(n+m\binom{n+q}{q})$-dimensional manifold. Projection on the expansion point $\mathbf{x}_0$ defines a fibration $\pi^q : J_q\pi \to \mathcal{X}$. As fibre coordinates for the point $[\sigma]^{(q)}_{\mathbf{x}_0}$ we may use $\mathbf{u}^{(q)} = (u^\alpha_\mu)$ with $1 \leq \alpha \leq n$ and a multi index μ where $0 \leq |\mu| \leq q$ and the interpretation that u^α_μ is the value of $\partial^{|\mu|}s^\alpha/\partial x^\mu$ at the expansion point $\mathbf{x}_0 \in \mathcal{X}$.

[1] For notational simplicity, we do not explicitly mention local charts and use a global notation. Nevertheless all construction are to be understood purely locally.

A *differential equation* is now a fibred submanifold $\mathcal{R}_q \subseteq J_q\pi$. We will always assume that it may be locally described as the zero set of a function $\mathbf{\Phi} : J_q\pi \to \mathbb{R}^t$; thus we recover the usual picture of a differential equation $\mathbf{\Phi}(\mathbf{x}, \mathbf{u}^{(q)}) = 0$ (note that we do not distinguish between a scalar equation and a system). The *prolongation* of a section $\sigma : \mathcal{X} \to \mathcal{E}$ is the section $j_q\sigma : \mathcal{X} \to J_q\pi$ locally defined by $u^\alpha_\mu = \partial^{|\mu|}s^\alpha/\partial x^\mu(\mathbf{x})$. We call σ a *solution* of the differential equation $\mathcal{R}_q$, if $\operatorname{im}(j_q\sigma) \subseteq \mathcal{R}_q$. Expressed in coordinates, this is equivalent to the usual definition.

If $q > r$, then we also have the canonical fibrations $\pi^q_r : J_q\pi \to J_r\pi$ defined by simply "forgetting" the higher-order derivatives. This leads to two natural operations with a differential equation $\mathcal{R}_q$. The first one is the *projection* to lower order: given a differential equation $\mathcal{R}_q \subseteq J_q\pi$ its r-fold projection is $\mathcal{R}^{(r)}_{q-r} = \pi^q_{q-r}(\mathcal{R}_q) \subseteq J_{q-r}\pi$. While the projection is easy to describe geometrically, it is, in particular for non-linear equations, hard to perform effectively, as it requires the elimination of variables.

In the second basic operation, the *prolongation* to higher order, we encounter the opposite situation: while it is easy to perform effectively in local coordinates, it is somewhat cumbersome to provide a rigorous intrinsic definition. Given a differential equation $\mathcal{R}_q \subseteq J_q\pi$, we may consider the restriction $\hat{\pi}^q : \mathcal{R}_q \to \mathcal{X}$ of the projection π^q which provides $\mathcal{R}_q$ with the structure of a fibred manifold over which we may again construct jet bundles. If we consider now both $J_r\hat{\pi}^q$ and $J_{q+r}\pi$ as submanifolds of $J_r\pi^q$ (which is possible with certain straightforward identifications), then the r-fold prolongation of $\mathcal{R}_q$ is the differential equation $\mathcal{R}_{q+r} = J_r\hat{\pi}^q \cap J_{q+r}\pi \subseteq J_{q+r}\pi$.

In local coordinates, prolongation requires only the *formal derivative*. If Φ is an arbitrary smooth function $J_q\pi \to \mathbb{R}$, then its formal derivative $D_i\Phi$ with respect to the variable x^i is a smooth function $J_{q+1}\pi \to \mathbb{R}$ given by the chain rule:

$$D_i\Phi = \frac{\partial\Phi}{\partial x^i} + \sum_{\alpha=1}^{m} \sum_{0\leq|\mu|\leq q} \frac{\partial\Phi}{\partial u^\alpha_\mu} u^\alpha_{\mu+1_i} \,. \tag{6.1}$$

If now the differential equation $\mathcal{R}_q$ is locally described as the zero set of the functions $\Phi^\tau : J_q\pi \to \mathbb{R}$ for $1 \leq \tau \leq t$, then its first prolongation $\mathcal{R}_{q+1}$ is the common zero set of the functions Φ^τ and their formal derivatives $D_i\Phi^\tau$ for $1 \leq i \leq n$ and $1 \leq \tau \leq t$. Higher prolongations are obtained by iteration.

It should be noted that in general neither a projection $\mathcal{R}^{(r)}_{q-r}$ nor a prolongation $\mathcal{R}_{q+r}$ is again a manifold, as we must expect that singularities appear. For simplicity, we will ignore this problem and always assume that we are dealing with a *regular* differential equation where all operations yield manifolds.

One could think that prolongation and projection are some kind of "inverse" operations: if one first prolongs an equation $\mathcal{R}_q \subseteq J_q\pi$ to $\mathcal{R}_{q+r} \subseteq J_{q+r}\pi$ for some $r > 0$ and subsequently projects back to $J_q\pi$ with π^{q+r}_q, one might naively expect that the obtained equation $\mathcal{R}^{(r)}_q$ coincides with the original equation $\mathcal{R}_q$. However, this is in general not correct, as *integrability conditions* may arise: we only get that always $\mathcal{R}^{(r)}_q \subseteq \mathcal{R}_q$.

Example 6.1 From a computational point of view, one may distinguish two different mechanisms for the generation of integrability conditions during prolongations and projections (for ordinary differential equations only the first one occurs): (i) the local

representation of $\mathcal{R}_q$ comprises equations of different orders and formal differentiation of the lower-order equations leads to new (i. e. *algebraically* independent equations), (ii) generalised cross-derivatives.
As a concrete example for the first mechanism consider the trivial ordinary differential equation $\mathcal{R}_1$ in two dependent variables u^1, u^2 and one independent variable x defined by $(u^1)' = u^2$ and $u^1 = x$. The local representation of $\mathcal{R}_2$ contains in addition the equations $(u^1)'' = (u^2)'$ and $(u^1)' = 1$. As the second one is of first order, it survives a subsequent projection back to second order and (after an obvious simplification) the projected system $\mathcal{R}_2^{(1)}$ is given by $(u^1)' = 1$, $u^1 = x$ and $u^2 = 1$.
For demonstrating of the second mechanism, we use a classical example due to Janet, namely the partial differential equation $\mathcal{R}_2$ in one dependent variable u and three independent variables x^1, x^2, x^3 locally described by $u_{33}+x^2u_{11} = 0$ and $u_{22} = 0$. Among others, the second prolongation $\mathcal{R}_4$ contains the equations $u_{2233} + x^2u_{1122} + u_{112} = 0$, $u_{1122} = 0$ and $u_{2233} = 0$. An obvious linear combination of these equations yields the integrability condition $u_{112} = 0$ and hence $\mathcal{R}_3^{(1)}$ is a proper subset of $\mathcal{R}_3$. Note that here the integrability condition is of higher order than the original system; this is a typical phenomenon for partial differential equations.

It is important to note that integrability conditions are not additional restrictions on the solution space of the considered equation $\mathcal{R}_q$; any solution of $\mathcal{R}_q$ *automatically* satisfies them. They represent conditions implicitly contained or hidden in $\mathcal{R}_q$ and which can be made visible by performing a suitable sequence of prolongations and projections. They may be considered as obstructions for the order by order construction of formal power series solutions. In practice, it often considerable simplifies the integration of the equation, if at least some integrability conditions are added.
These considerations motivate the following definition where the term "integrable" is used in its most basic meaning: existence of solutions. As we discuss here only formal solutions, we speak of formal integrability.[2] One should not confuse this concept with other notions like complete integrability where properties like the existence of first integrals or symmetries are considered.

Definition 6.2 The differential equation $\mathcal{R}_q \subseteq J_q\pi$ is called *formally integrable*, if for all $r \geq 0$ the equality $\mathcal{R}_{q+r}^{(1)} = \mathcal{R}_{q+r}$ holds.

While this geometric definition of formal integrability is very natural, it has an obvious and serious drawback: it requires the satisfaction of an infinite number of conditions (surjectivity of the projections $\hat{\pi}^{q+r+1}_{q+r} : \mathcal{R}_{q+r+1} \to \mathcal{R}_{q+r}$ for all $r \geq 0$). Thus in the given form formal integrability cannot be verified effectively. We will see in the next chapter that algebraic and homological methods lead to a finite criterion for formal integrability. The key for the application of these methods lies in a natural polynomial structure hidden in the jet bundle hierarchy, the so-called fundamental identification. As this topic is often ignored in the literature, we discuss it here in some detail. It is based the following crucial observation.

[2] In some applications like Lie symmetry theory *local solvability* is very important [41]. A differential equation $\mathcal{R}_q$ is locally solvable, if for every point $\rho \in \mathcal{R}_q$ a solution σ exists such that $\rho \in \operatorname{im} j_q\sigma$. Again in the sense of existence of the formal solutions, formal integrability trivially implies local solvability.

Proposition 6.3 *The jet bundle $J_q\pi$ of order q is affine over the jet bundle $J_{q-1}\pi$ of order $q-1$.*

Proof. The simplest approach to proving this proposition consists of studying the effect of fibred changes of coordinates $\bar{\mathbf{x}} = \bar{\mathbf{x}}(\mathbf{x})$ and $\bar{\mathbf{u}} = \bar{\mathbf{u}}(\mathbf{x}, \mathbf{u})$ in the total space $\mathcal{E}$ on the derivatives which are fibre coordinates in $J_q\pi$. Using the chain rule one easily computes that in repeated index notation the result for the highest-order derivatives is

$$\bar{u}^\alpha_{j_1\cdots j_q} = \left(\frac{\partial \bar{u}^\alpha}{\partial u^\beta}\frac{\partial x^{i_1}}{\partial \bar{x}^{j_1}}\cdots\frac{\partial x^{i_q}}{\partial \bar{x}^{j_q}}\right) u^\beta_{i_1\cdots i_q} + \cdots \tag{6.2}$$

where the dots represent a complicated expression in the derivatives of lower order and where $\partial\mathbf{x}/\partial\bar{\mathbf{x}}$ represents the inverse of the Jacobian $\partial\bar{\mathbf{x}}/\partial\mathbf{x}$. But this implies that (6.2) is indeed affine in the derivatives of order q as claimed. ∎

An affine space is always modelled on a vector space: the difference between two points may be interpreted as a vector. In our case it is easy to identify this vector space. Let $[\sigma]^{(q)}_{\mathbf{x}}$ and $[\sigma']^{(q)}_{\mathbf{x}}$ be two points in $J_q\pi$ such that $[\sigma]^{(q-1)}_{\mathbf{x}} = [\sigma']^{(q-1)}_{\mathbf{x}}$, i. e. the two points belong to the same fibre with respect to the fibration π^q_{q-1}. Thus $[\sigma]^{(q)}_{\mathbf{x}}$ and $[\sigma']^{(q)}_{\mathbf{x}}$ correspond to two Taylor series truncated at degree q which coincide up to degree $q-1$. Obviously, this observation implies that their difference consists of one homogeneous polynomial of degree q for each dependent variable u^α.

In a more intrinsic language, we may formulate this result as follows. Let $\rho = [\sigma]^{(q)}_{\mathbf{x}}$ be a point in $J_q\pi$ and $\bar{\rho} = [\sigma]^{(q-1)}_{\mathbf{x}} = \pi^q_{q-1}(\rho)$ its projection to $J_{q-1}\pi$; we furthermore set $\xi = \sigma(\mathbf{x}) = \pi^q_0(\rho) \in \mathcal{E}$. Then according to Proposition 6.3, the fibre $(\pi^q_{q-1})^{-1}(\bar{\rho})$ is an affine space modelled on the vector space $S_q(T^*_{\mathbf{x}}\mathcal{X}) \otimes V_\xi\pi$ where S_q denotes again the q-fold symmetric product and $V_\xi\pi \subset T_\xi\mathcal{E}$ is the *vertical bundle* defined as the kernel of the tangent map $T_\xi\pi$. Indeed, this follows immediately from our discussion so far: the symmetric algebra $S(T^*_{\mathbf{x}}\mathcal{X})$ is a coordinate-free form of the polynomial ring and one easily verifies that the homogeneous part of (6.2) obtained by dropping the terms represented by the dots describes the transformation behaviour of vectors in $S_q(T^*_{\mathbf{x}}\mathcal{X}) \otimes V_\xi\mathcal{E}$ (note that we must use the *co*tangent space $T^*_{\mathbf{x}}\mathcal{X}$, as tangent vectors would transform with the inverse matrix).

By Proposition 6.3, the jet bundle $J_q\pi$ is an affine bundle over $J_{q-1}\pi$. This fact implies that the tangent space to the affine space $(\pi^q_{q-1})^{-1}(\bar{\rho})$ at the point $\rho \in J_q\pi$ is canonically isomorphic to the corresponding vector space, i. e. to $S_q(T^*_{\mathbf{x}}\mathcal{X}) \otimes V_\xi\mathcal{E}$. This isomorphism is called the *fundamental identification*. We derive now a local coordinate expression for it. On one side we have the tangent space to the fibre $(\pi^q_{q-1})^{-1}(\bar{\rho})$ at the point ρ, i. e. the vertical space $V_\rho\pi^q_{q-1}$ defined as the kernel of the tangent map $T_\rho\pi^q_{q-1}$. Obviously, it is spanned by all the vectors $\partial_{u^\alpha_\mu}$ with $|\mu| = q$. Let us consider one of these vectors; it is tangent to the curve $\gamma : t \mapsto \rho(t)$ where $\rho(0) = \rho$ and all coordinates of a point $\rho(t)$ coincide with those of ρ except for the one coordinate u^α_μ corresponding to the chosen vector which is increased by t.

On the other side, we may compute the difference quotient $(\rho(t) - \rho)/t$ interpreting the points as above as truncated Taylor series. The u^α-component of the result is obviously the polynomial $(u^\alpha_\mu(t)-u^\alpha_\mu)x^\mu/\mu!$. Hence the fundamental identification is just the map $\epsilon_q : V_\rho\pi^q_{q-1} \to S_q(T^*_{\mathbf{x}}\mathcal{X}) \otimes V_\xi\mathcal{E}$ given by

$$\epsilon_q(\partial_{u^\alpha_\mu}) = \frac{1}{\mu!}\mathrm{d}x^\mu \otimes \partial_{u^\alpha}\ . \tag{6.3}$$

Note the combinatorial factor $\frac{1}{\mu!}$ having its origin in Taylor's formula!

7 Algebraic Analysis of Differential Equations

Definition 7.1 Let $\mathcal{R}_q \subseteq J_q\pi$ be a differential equation. The *(geometric) symbol* $\mathcal{N}_q$ of $\mathcal{R}_q$ is a family of vector spaces over $\mathcal{R}_q$ where the value at $\rho \in \mathcal{R}_q$ is given by

$$(\mathcal{N}_q)_\rho = T_\rho\mathcal{R}_q \cap V_\rho\pi^q_{q-1} = V_\rho\big(\pi^q_{q-1}|_{\mathcal{R}_q}\big)\ . \tag{7.1}$$

Thus the symbol is the vertical part of the tangent space of the submanifold $\mathcal{R}_q$ with respect to the fibration π^q_{q-1}. If $\mathcal{R}_q$ is globally described by a map $\Phi : J_q\pi \to \mathcal{E}'$ with a vector bundle $\pi' : \mathcal{E}' \to \mathcal{X}$, then we introduce the *symbol map* $\sigma : V\pi^q_{q-1} \to T\mathcal{E}'$ given by $\sigma = T\Phi|_{V\pi^q_{q-1}}$ and define $\mathcal{N}_q = \ker\sigma$. Locally, this leads to the following picture. Let $(\mathbf{x}, \mathbf{u}^{(q)})$ be coordinates on $J_q\pi$ in a neighbourhood of ρ. We first determine $T_\rho\mathcal{R}_q$ as a subspace of $T_\rho(J_q\pi)$. Let $(\mathbf{x}, \mathbf{u}^{(q)}; \dot{\mathbf{x}}, \dot{\mathbf{u}}^{(q)})$ be the induced coordinates on $T_\rho(J_q\pi)$; every vector $X \in T_\rho(J_q\pi)$ has the form $X = \dot{x}^i\partial_{x^i} + \dot{u}^\alpha_\mu\partial_{u^\alpha_\mu}$. Assuming that $\mathcal{R}_q$ is locally defined by $\Phi^\tau(\mathbf{x}, \mathbf{u}^{(q)}) = 0$ with $\tau = 1, \dots, t$, its tangent space $T_\rho\mathcal{R}_q$ consists of all vectors X such that $\mathrm{d}\Phi^\tau(X) = X\Phi^\tau = 0$. The symbol $\mathcal{N}_q$ is by definition the vertical part of this tangent space. Hence we are only interested in those solutions of the above conditions where $\dot{\mathbf{x}} = \dot{u}^{(q-1)} = 0$ and locally $\mathcal{N}_q$ can be described as the solution space of the following system of linear equations:

$$(\mathcal{N}_q)_\rho\ :\ \left\{\ \sum_{\substack{1\leq\alpha\leq m\\ |\mu|=q}} \frac{\partial\Phi^\tau}{\partial u^\alpha_\mu}(\rho)\,\dot{u}^\alpha_\mu = 0\ , \qquad \tau = 1, \dots, t\ .\right. \tag{7.2}$$

This is a system with real coefficients, as the derivatives $\partial\Phi^\tau/\partial u^\alpha_\mu$ are evaluated at the point $\rho \in \mathcal{R}_q$. We call its matrix the *symbol matrix* and denote it by $M_q(\rho)$. It is also the matrix of the symbol map σ in local coordinates.
The symbol is most easily understood for linear differential equations. Loosely speaking, the geometric symbol is then simply the highest-order or principal part of the system (considered as algebraic equations). For non-linear systems we perform a brute force linearisation at the point ρ in order to obtain $(\mathcal{N}_q)_\rho$. Obviously, $\dim(\mathcal{N}_q)_\rho$ might vary with ρ. For this reason, we speak in Definition 7.1 only of a family of vector spaces and not of a vector bundle. Only if the dimension remains constant over $\mathcal{R}_q$, the symbol $\mathcal{N}_q$ is a vector subbundle of $V\pi^q_{q-1}$. For simplicity, we will assume that all considered symbols are vector bundles.

Example 7.2 For Janet's partial differential equation $\mathcal{R}_2$ considered in Example 6.1 the symbol equations are $\dot{u}_{33}+x^2\dot{u}_{11}=0$ and $\dot{u}_{22}=0$ (as $\mathcal{R}_2$ is a linear system without lower-order terms, the symbol equations look formally like the differential equation itself; however, the symbol equations are *algebraic* and not differential equations). Hence the symbol $\mathcal{N}_2$ is here the one-dimensional distribution spanned by the vector field $\partial_{u_{11}} - x^2\partial_{u_{33}}$.

Of course, not only the original equation $\mathcal{R}_q$ has a symbol $\mathcal{N}_q$, but also every prolongation $\mathcal{R}_{q+r} \subseteq J_{q+r}\pi$ of it possesses a symbol $\mathcal{N}_{q+r} \subseteq T(J_{q+r}\pi)$. It follows easily from the coordinate expression (6.1) of the formal derivative that for obtaining a local representation of the prolonged symbol $\mathcal{N}_{q+r}$, there is no need to explicitly compute a local representation of the prolonged differential equation $\mathcal{R}_{q+r}$. We can directly derive it from a local representation of $\mathcal{N}_q$, as we need only the partial derivatives $\partial D_i\Phi^\tau/\partial u^\alpha_\nu$ with $1 \leq i \leq n$ and $|\nu| = q+1$, i. e. the highest-order part of the formal derivative $D_i\Phi^\tau$, for determining the symbol $\mathcal{N}_{q+1}$. It is given by $\partial D_i\Phi^\tau/\partial u^\alpha_\nu = \partial\Phi^\tau/\partial u^\alpha_{\nu-1_i}$ (if $\nu_i = 0$, the derivative vanishes) and thus a local representation of $\mathcal{N}_{q+1}$ is

$$(\mathcal{N}_{q+1})_\rho \,:\, \begin{cases} \displaystyle\sum_{\substack{1\leq\alpha\leq m\\ |\mu|=q}} \frac{\partial\Phi^\tau}{\partial u^\alpha_\mu}\,\dot{u}^\alpha_{\mu+1_i} = 0\;, & \begin{array}{l}\tau = 1,\dots,p\;,\\ i = 1,\dots,n\;.\end{array}\end{cases} \tag{7.3}$$

In our geometric approach to integrability conditions, their existence is signalled by a dimension inequality: $\dim\mathcal{R}^{(1)}_q < \dim\mathcal{R}_q$. By the following result, which follows from a straightforward analysis of the Jacobians of the involved differential equations, the dimension of $\mathcal{R}^{(1)}_q$ is related to $\dim\mathcal{N}_{q+1}$, i. e. analysing the prolonged symbol matrix M_{q+1} gives information about possible integrability conditions.

Proposition 7.3 *If $\mathcal{N}_{q+1}$ is a vector bundle, then* $\dim\mathcal{R}^{(1)}_q = \dim\mathcal{R}_{q+1} - \dim\mathcal{N}_{q+1}$.

In the classical theory of partial differential equations a different notion of symbol is used which should not be confused with the geometric symbol introduced above: the classical symbol is *not* an intrinsic object. Our notion of symbol is closely related to what is traditionally called the principal symbol which is intrinsically defined.

Assume we are given a one-form $\chi \in T^*\mathcal{X}$. It induces for every $q > 0$ a map $\iota_{\chi,q} : V\pi \to V\pi^q_{q-1}$ defined by $\iota_{\chi,q}(v) = \epsilon_q(\chi^q \otimes v)$ where ϵ_q is the fundamental identification and χ^q denotes the q-fold symmetric product of χ. In local coordinates, we write $\chi = \chi_i\mathrm{d}x^i$ and obtain $\iota_{\chi,q} : v^\alpha\partial_{u^\alpha} \mapsto \chi_\mu v^\alpha\partial_{u^\alpha_\mu}$ where μ runs over all multi indices of length q and $\chi_\mu = \chi_1^{\mu_1}\cdots\chi_n^{\mu_n}$.

Let σ be the symbol map of the differential equation $\mathcal{R}_q$ globally described by the map $\Phi : J_q\pi \to \mathcal{E}'$. Then the *principal symbol* of $\mathcal{R}_q$ is the linear map $\tau_\chi : V\pi \to T\mathcal{E}'$ defined by $\tau_\chi = \sigma\circ\iota_{\chi,q}$. Locally, we can associate a matrix $T[\chi]$ with τ_χ:

$$T^\tau_\alpha[\chi] = \sum_{|\mu|=q}\frac{\partial\Phi^\tau}{\partial u^\alpha_\mu}\chi^\mu\;. \tag{7.4}$$

If $\dim \mathcal{E} = m$ and $\dim \mathcal{E}' = p$, it has p rows and m columns. Its entries are homogeneous polynomials of degree q in the coefficients of χ. We may think of $T[\chi]$ as a kind of contraction of the symbol matrix M_q. Both matrices have the same number of rows. The column with index α of $T[\chi]$ is a linear combination of all columns in M_q corresponding to a variable $\dot{u}^\alpha_\mu$ with the coefficients given by χ_μ.

Remark 7.4 Using the matrix $T[\chi]$ of the principal symbol, we may relate the construction of integrability conditions with syzygy computations. Assume that the functions Φ^τ with $1 \leq \tau \leq t$ locally representing the differential equation $\mathcal{R}_q$ lie in a differential field $\mathbb{F}$. Then the entries of $T[\chi]$ are polynomials in $\mathcal{P} = \mathbb{F}[\chi_1, \dots, \chi_n]$. The rows of $T[\chi]$ may be considered as elements of $\mathcal{P}^m$ and generate a submodule $\mathcal{M} \subseteq \mathcal{P}^m$. Let $\mathbf{S} \in \mathrm{Syz}(\mathcal{M}) \subseteq \mathcal{P}^t$ be a syzygy of the rows of $T[\chi]$. The substitution $\chi_i \to D_i$ transforms each component S_τ of $\mathbf{S}$ into a differential operator $\hat{S}_\tau$. By construction, $\Psi = \sum_{\tau=1}^t \hat{S}_\tau \Phi^\tau$ is a linear combination of differential consequences of $\mathcal{R}_q$ in which the highest-order terms cancel. In fact, this represents nothing but the rigorous mathematical formulation of "taking a cross-derivative."

Example 7.5 For Janet's differential equation the module generated by the rows of $T[\chi]$ is the ideal $\mathcal{I}_1 = \langle \chi_3^2 + x^2\chi_1^2, \chi_2^2 \rangle$. Obviously, its syzygy module is spanned by $\mathbf{S}_1 = \chi_2^2 \mathbf{e}_1 - (\chi_3^2 + x^2\chi_1^2)\mathbf{e}_2$ and applying the corresponding differential operator to Janet's equation yields the above mentioned integrability condition $u_{112} = 0$.

The fundamental identification ϵ_q allows us to identify the symbol $(\mathcal{N}_q)_\rho$ with a subspace of $S_q(T^*_x\mathcal{X}) \otimes V_\xi\pi$ where $\xi = \pi^q_0(\rho)$ and $x = \pi(\xi)$. In local coordinates, ϵ_q is given by (6.3); hence its main effect is the introduction of some combinatorial factors which can be absorbed in the choice of an appropriate basis. More precisely, we recover here the discussion in Remark 2.5. If we take $\{\partial_{x^1}, \dots, \partial_{x^n}\}$ as basis of the tangent space $T_x\mathcal{X}$ and the dual basis $\{\mathrm{d}x^1, \dots, \mathrm{d}x^n\}$ for the cotangent space $T^*_x\mathcal{X}$, then the "terms" $\partial_{x^\mu} = \partial^{\mu_1}_{x^1} \cdots \partial^{\mu_n}_{x^n}$ with $|\mu| = q$ form a basis of $S_q(T_x\mathcal{X})$ whereas the dual basis of $S_q(T^*_x\mathcal{X})$ is given by the "divided powers" $\frac{1}{\mu!}\mathrm{d}x^\mu$. If we express an element $f \in S_q(T^*_x\mathcal{X}) \otimes V_\xi\pi$ in this basis as $f = \frac{1}{\mu!} f^\alpha_\mu \mathrm{d}x^\mu \otimes \partial_{u^\alpha}$ where μ runs over all multi indices with $|\mu| = q$, then the symbol $\mathcal{N}_q$ consists of all such f satisfying the linear system of equations $\sum_{1\leq\alpha\leq m, |\mu|=q} \frac{\partial\Phi^\tau}{\partial u^\alpha_\mu} f^\alpha_\mu = 0$ with $\tau = 1, \dots, t$. Obviously, this is the same linear system as (7.2) defining the symbol as a subspace of $V_\rho\pi^q_{q-1}$.

Proposition 7.6 *Let $\mathcal{R}_q \subseteq J_q\pi$ be a differential equation and $\bigl(\rho_r \in \mathcal{R}_r\bigr)_{r\geq q}$ be a sequence of points such that $\pi^r_q(\rho_r) = \rho_q$ and set $\xi = \pi^q_0(\rho_q)$ and $x = \pi^q(\rho_q)$. If we set $\mathcal{N}_r = \mathfrak{S}_r(T^*_x\mathcal{X}) \otimes V_\xi\pi$ for $0 \leq r < q$, then the sequence $\bigl((\mathcal{N}_r)_{\rho_r}\bigr)_{r\in\mathbb{N}_0}$ defines a symbolic system in $\mathfrak{S}(T^*_x\mathcal{X}) \otimes V_\xi\pi$ which satisfies $\mathcal{N}_{r+1} = \mathcal{N}_{r,1}$ for all $r \geq q$.*

Proof. For notational simplicity, we consider only $r = q$. Let $f = \frac{1}{\nu!} f^\alpha_\nu \mathrm{d}x^\nu \otimes \partial_{u^\alpha}$ where ν runs over all multi indices with $|\nu| = q+1$ be an element of $\mathcal{N}_{q,1}$. By definition of the prolongation, this is equivalent to $\delta(f) \in \mathcal{N}_q \otimes T^*_x\mathcal{X}$ and hence we find for every $1 \leq i \leq n$ that $\frac{\nu_i}{\nu!} f^\alpha_\nu \mathrm{d}x^{\nu-1_i} \otimes \partial_{u^\alpha} \in \mathcal{N}_q$. In other words, the coefficients f^α_ν must satisfy the linear system of equations $\sum_{1\leq\alpha\leq m, |\nu|=q+1, \nu_i>0} \frac{\partial\Phi^\tau}{\partial u^\alpha_{\nu-1_i}} f^\alpha_\nu = 0$

with $\tau = 1, \dots, t$ and $i = 1, \dots, n$. A comparison with (7.3) shows that this system describes the prolonged symbol $\mathcal{N}_{q+1}$. Hence we have $\mathcal{N}_{q+1} = \mathcal{N}_{q,1}$ as claimed. $\square$

In this proposition we used a sequence of points $\rho_r \in \mathcal{R}_r$ with $\pi^r_q(\rho_r) = \rho_q$ in order to consider the symbols $(\mathcal{N}_r)_{\rho_r}$. Obviously, such a sequence does not necessarily exists, unless we are dealing with a formally integrable equation. However, by the final assertion, the obtained symbolic system is independent of the choice of these points, as we may simply set $\mathcal{N}_{r+1} = \mathcal{N}_{r,1}$ for all $r \geq q$. Hence at each point $\rho \in \mathcal{R}_q$ the symbol $(\mathcal{N}_q)_\rho$ induces a symbolic system which, according to Lemma 2.8, we may alternatively consider as a subcomodule $\mathcal{N}[\rho] \subseteq \mathfrak{S}(T^*_x\mathcal{X}) \otimes V_\xi\pi$; we then speak of the *symbol comodule* of $\mathcal{R}_q$ at the point ρ. One can now easily verify that the symbolic systems given in Example 2.9 are associated to the their mentioned differential equations.

Remark 7.7 In Proposition 7.6 and in the definition of the symbol comodule $\mathcal{N}$ we simply set the lower-order components $\mathcal{N}_r$ for $0 \leq r < q$ to the full symmetric product $\mathfrak{S}_r(T^*_x\mathcal{X}) \otimes V_\xi\pi$. In principle, one could use a more precise approach by considering instead the symbols of the projected equations $\mathcal{R}^{(q-r)}_r$. However, for the subsequent involution analysis it only matters what happens in degree q and beyond which is not affected by such changes in lower order. Hence we stick to this simpler approach.

Remark 7.8 The comodules $\mathcal{N}$ arising as symbols are of a special form: their annihilators $\mathcal{N}^0$ possess bases where all generators are homogeneous of the same degree q, namely the order of the underlying differential equation $\mathcal{R}_q$. It follows now immediately from the identification of the degree of involution of $\mathcal{N}$ and the Castelnuovo–Mumford regularity of $\mathcal{N}^0$ that the minimal free resolution of $\mathcal{N}^0$ is linear, i. e. the syzygy modules of any order can be generated by syzygies of degree 1. This was already noted as a “curiosité” by Serre in his letter appended to [21]. As later shown by Eisenbud and Goto [14], this represents in fact a characteristic property of q-regular modules: if $\mathcal{M}$ is q-regular, then the truncation $\mathcal{M}_{\geq q}$ possesses a linear resolution.

Definition 7.9 The symbol $\mathcal{N}_q$ of the differential equation $\mathcal{R}_q \subseteq J_q\pi$ of order q is *involutive* at the point $\rho \in \mathcal{R}_q$, if the symbol comodule $\mathcal{N}[\rho]$ is involutive at degree q.

Choosing local coordinates $(\mathbf{x}, \mathbf{u}^{(q)})$ in a neighbourhood of a given point $\rho \in \mathcal{R}_q$, we can apply Cartan’s test (Theorem 3.4) for deciding involution. Recall that it requires only linear algebra computations with the two symbols $\mathcal{N}_q$ and $\mathcal{N}_{q+1}$ and thus is easily performed effectively. In practice, one uses a dual approach exploiting that the annihilator $\mathcal{N}^0 \subseteq S(T_x\mathcal{X}) \otimes V_\xi\pi$ is an $S(T_x\mathcal{X})$-submodule, the *symbol module*. In our chosen coordinates and bases the submodule $\mathcal{N}^0$ is generated by the “polynomials”

$$\sum_{\substack{1\leq\alpha\leq m\\ |\mu|=q}} \frac{\partial\Phi^\tau}{\partial u^\alpha_\mu}\, \partial^\mu_x \otimes \partial_{u^\alpha}\ , \qquad \tau = 1, \dots, t\ , \tag{7.5}$$

corresponding to the left hand sides in (7.2). Identifying $S(T_x\mathcal{X})$ with the polynomial ring $\mathcal{P} = \mathbb{R}[\partial_{x^1}, \dots, \partial_{x^n}]$, one readily recognises in $\mathcal{N}^0$ the polynomial module generated by the rows of the matrix $T[\chi]$ of the principal symbol which already appeared in

Remark 7.4. We may now apply the theory of Pommaret bases to the submodule $\mathcal{N}^0$. Then the following result follows immediately from Theorem 5.2.

Proposition 7.10 *The symbol $\mathcal{N}_q$ of the differential equation $\mathcal{R}_q \subseteq J_q\pi$ is involutive at the point $\rho \in \mathcal{R}_q$, if and only if in suitable local coordinates an involutive head autoreduction transforms the generators (7.5) into a Pommaret basis of the symbol module $\mathcal{N}^0$ for a class respecting term order.*

In principle, at some points $\rho \in \mathcal{R}_q$ the symbol could be involutive, whereas at other points on the differential equation this is not the case. For notational simplicity, we will assume throughout this work that all points on $\mathcal{R}_q$ behave uniformly and therefore drop from now on the explicit reference to the point $\rho \in \mathcal{R}_q$.
Proposition 7.10 transforms the Cartan test into an easily applicable effective criterion for an involutive symbol. In order to recover some results in the literature, we express it in a less algebraic language. Recall that the columns of M_q correspond to the unknowns $\dot{u}^\alpha_\mu$; we sort them according to a class respecting term order (it suffices, if we take care that a column corresponding to an unknown $\dot{u}^\alpha_\mu$ is always to the left of a column corresponding to the unknown $\dot{u}^\beta_\nu$, if $\operatorname{cls}\mu > \operatorname{cls}\nu$). Now an involutive head autoreduction is equivalent to determining a row echelon form $M_q^\triangle$ of M_q using only row operations. The unknown $\dot{u}^\alpha_\mu$ corresponding to the column where the first non-vanishing entry of a row sits is called the *leader* of this row. If $\beta_q^{(k)}$ is the number of leaders that are of class k, then we call these numbers the *indices* of the symbol $\mathcal{N}_q$.
The problem of δ-regularity concerns this notion. The class of a derivative is not invariant under coordinate transformations. In different coordinate systems we may thus obtain different values for the indices. δ-regular coordinates are distinguished by the fact that the sum $\sum_{k=1}^n k\beta_q^{(k)}$ takes its maximal value. It is not difficult to see that actually we are here only reformulating Remark 4.5. Hence (4.2) immediately implies the following result.

Proposition 7.11 *The symbol $\mathcal{N}_q$ with the indices $\beta_q^{(k)}$ is involutive, if and only if the matrix M_{q+1} of the prolonged symbol $\mathcal{N}_{q+1}$ satisfies* $\operatorname{rank} M_{q+1} = \sum_{k=1}^n k\beta_q^{(k)}$.

Remark 7.12 A special situation arises, if there is only one dependent variable, as then *any* first-order symbol $\mathcal{N}_1$ is involutive. The symbol module $\mathcal{N}^0$ is now an ideal in $\mathcal{P}$ generated by linear polynomials. Using some linear algebra, we may always assume that all generators have different leading terms (with respect to the degree reverse lexicographic order). Because of the linearity, this implies that all leading terms are relatively prime. It is straightforward to show (in fact, this is nothing but Buchberger's first criterion) that all S-polynomials reduce to zero and hence our generating set is a Gröbner basis. As one easily verifies that the leading terms involutively generate the leading ideal, we have a Pommaret basis of $\mathcal{N}^0$ or equivalently $\mathcal{N}_1$ is involutive.
This observation is the deeper reason for a classification of partial differential equations suggested by Drach (see [59, Chapt. 5]). Using a simple trick due to him, we may transform any differential equation $\mathcal{R}_q$ into one with only one dependent variable. If we first rewrite $\mathcal{R}_q$ as a first-order equation, then the transformed equation will be

of second order. Only in special circumstances one can derive a first-order equation in one dependent variables. Thus from a theoretical point of view we may distinguish two basic classes of differential equations: first-order and second-order equations, respectively, in one dependent variable. The first class is much simpler, as its symbol is always involutive (like for ordinary differential equations).

Definition 7.13 The differential equation $\mathcal{R}_q$ is called *involutive*, if it is formally integrable and if its symbol $\mathcal{N}_q$ is involutive.

The term “involution” is often used in a rather imprecise manner. In particular, involution is sometimes taken as a synonym for formal integrability. While Definition 7.13 obviously implies that an involutive equation is also formally integrable, the converse is generally not true: involution is a stronger concept than formal integrability.

Theorem 7.14 *$\mathcal{R}_q$ is an involutive differential equation, if and only if its symbol $\mathcal{N}_q$ is involutive and $\mathcal{R}_q^{(1)} = \mathcal{R}_q$.*

We omit a proof of this theorem, as it follows immediately from Theorem 7.15 below and the finiteness of the Spencer cohomology (Theorem 2.20). Checking whether or not the symbol $\mathcal{N}_q$ is involutive via the Cartan test (Theorem 3.4 or its alternative formulation Proposition 7.11) requires only computations in order q and $q+1$. Obviously, the same is true for verifying the equality $\mathcal{R}_q^{(1)} = \mathcal{R}_q$. Hence Theorem 7.14 represents indeed a finite criterion for involution. A closer look at the above developed homological theory yields a finite criterion for formal integrability independent of involution.

Theorem 7.15 *The differential equation $\mathcal{R}_q$ is formally integrable, if and only if an integer $r \geq 0$ exists such that the symbolic system $\mathcal{N}$ defined by the symbol $\mathcal{N}_q$ and all its prolongations is 2-acyclic at degree $q+r$ and the equality $\mathcal{R}_{q+r'}^{(1)} = \mathcal{R}_{q+r'}$ holds for all values $0 \leq r' \leq r$.*

Proof. One direction is trivial. For a formally integrable equation $\mathcal{R}_q$ we even have $\mathcal{R}_{q+r'}^{(1)} = \mathcal{R}_{q+r'}$ for all $r' \geq 0$ and by Theorem 2.20 the symbolic system $\mathcal{N}$ must become 2-acyclic at some degree $q+r$. For the converse, we first note that, because of Lemma 2.13, the symbolic system $\mathcal{N}$ is trivially 1-acyclic at degree q. Our assumption says that in addition the Spencer cohomology modules $H^{q+s,2}(\mathcal{N})$ vanish for all $s \geq r$. According to Proposition 2.18, this implies dually that the Koszul homology modules $H_{q+s,1}(\mathcal{N}^0)$ of the symbol module $\mathcal{N}^0$ vanish for all $s > r$.
Recall from Remark 2.15 that the Koszul homology corresponds to a minimal free resolution of $\mathcal{N}^0$ and hence our assumption tells us that the maximal degree of a minimal generator in the first syzygy module $\mathrm{Syz}(\mathcal{N}^0)$ is $q+r$. In Remark 7.4 we have seen that the syzygies of $\mathcal{N}^0$ are related to those integrability conditions arising from generalised cross-derivatives between the highest-order equations. If now the equality $\mathcal{R}_{q+r'}^{(1)} = \mathcal{R}_{q+r'}$ holds for all $0 \leq r' \leq r$, then none of these cross-derivatives can produce an integrability condition. Furthermore, no integrability conditions can arise from lower-order equations. Hence $\mathcal{R}_q$ is formally integrable. ∎

An abstract proof of this result was given by Goldschmidt [19]. The proof above is interesting from a computational point of view as it demonstrates that (a generating set of) the Koszul homology module $H_1(\mathcal{N}^0)$ shows us exactly which generalised cross-derivatives may produce integrability conditions (Kruglikov and Lychagin developed recently an alternative approach for the construction of these conditions based on multi-brackets, see [32] and references therein). Of course, we cannot decide solely on the basis of the symbol $\mathcal{N}_q$ whether or not these integrability conditions vanish modulo the equations describing $\mathcal{R}_q$, as this depends on the lower-order terms. Therefore, we must check a finite number of projections $\hat{\pi}^{q+r+1}_{q+r} : \mathcal{R}_{q+r+1} \to \mathcal{R}_{q+r}$ for surjectivity.

Example 7.16 We continue with Janet's partial differential equation $\mathcal{R}_2$ defined by $u_{33} + x^2 u_{11} = u_{22} = 0$. In Example 7.5 above we constructed via the syzygy $\mathbf{S}_1 = \chi_2^2 \mathbf{e}_1 - (\chi_3^2 + x^2\chi_1^2)\mathbf{e}_2$ the integrability condition $u_{112} = 0$. Geometrically, we have arrived then at the equation $\mathcal{R}_3^{(1)}$ defined by this condition, the original equations and their formal derivatives. The rows of the principal symbol of $\mathcal{R}_3^{(1)}$ generate the ideal $\mathcal{I}_2 = \langle \chi_3^2 + x^2\chi_1^2, \chi_2^2, \chi_1^2\chi_2\rangle$. Its syzygy module is spanned by $\mathbf{S}_1$, $\mathbf{S}_2 = \chi_1^2\mathbf{e}_2 - \chi_2\mathbf{e}_3$ and $\mathbf{S}_3 = \chi_1^1\chi_2\mathbf{e}_1 - x^2\chi_2^2\mathbf{e}_3$. Applying the differential operator corresponding to $\mathbf{S}_2$ yields zero, whereas $\mathbf{S}_3$ leads to a further integrability condition: $u_{1111} = 0$.

Geometrically, we are now dealing with the differential equation $\mathcal{R}_4^{(2)}$ described by the two integrability conditions, the original equations and all prolongations up to order 4. The rows of the principal symbol $T[\chi]$ define now the ideal $\mathcal{I}_3 = \mathcal{I}_2 + \langle \chi_1^4\rangle$ and for its syzygy module we need two further generators, namely $\mathbf{S}_4 = \chi_1^2\mathbf{e}_3 - \chi_2\mathbf{e}_4$ and $\mathbf{S}_5 = (\chi_3^2 + x^2\chi_1^2)\mathbf{e}_4 - \chi_1^4\mathbf{e}_1$. One easily checks than none of them leads to a new integrability condition so that $\mathcal{R}_4^{(2)}$ is a formally integrable equation.

However, $\mathcal{R}_4^{(2)}$ is not involutive. One way to see this consisting of noting that in the syzygy $\mathbf{S}_5$ the coefficient of $\mathbf{e}_4$ is of degree 2. Since $\mathbf{e}_4$ represents a differential equation of order 4, the corresponding cross-derivative takes place in order 6. According to Remark 7.8, we can always obtain a linear resolution for an involutive symbol. Indeed, we must prolong here once: $\mathcal{R}_5^{(2)}$ is an involutive equation with vanishing symbol.

Another way to prove this goes as follows. Consider a point $\rho \in \mathcal{R}_4^{(2)}$ where $x^2 = a$ for some constant $a \in \mathbb{R}$ and the ideal $\mathcal{I} = \langle (x^3)^2 + a(x^1)^2, (x^2)^2, (x^1)^2x^2, (x^1)^4\rangle$. Then one easily verifies that the truncated ideal $\mathcal{I}_{\geq 4}$ is the annihilator of the symbol comodule $\mathcal{N}$ at the point ρ. For a Pommaret basis of $\mathcal{I}$ we must add the generators $(x^2)^2x^3$, $(x^1)^2x^2x^3$ and $(x^1)^4x^3$. Since the last generator is of degree 5, we find that $\operatorname{reg}\mathcal{I} = 5$ and hence according to Remark 4.15 the degree of involution of $\mathcal{N}$ is 5.

8 Conclusions

A central question for any differential equation is the existence of solutions. For formal solutions the existence is equivalent to the formal integrability of the equation. The

Cartan–Kähler theorem (see [42, 53] and references therein) provides us with an existence and uniqueness theorem for analytic solutions of involutive analytic equations generalising the classical Cauchy–Kovalevskaya theorem. Compared with alternative approaches like Riquier's existence theorem, the proof of the Cartan–Kähler theorem does not require a convergence analysis of power series. This allows us sometimes to extend it to larger function spaces (which is very important for applications), if additional information about the equation is given; a concrete example can be found in [52] where smooth solutions of hyperbolic systems with elliptic constraints are treated.

For deciding the mere existence of solutions, formal integrability is sufficient. If one is interested in the size of the solution space (or equivalently in the number and form of conditions leading to a unique solution), then one needs more information. The simplest approach (implicitly already exploited by Janet) consists of using a complementary decomposition of the symbol module for deriving a (formally) well-posed initial value problem and is trivial for a not only formally integrable but even involutive system. Generally speaking, the main difference between formal integrability and involution is the same as the one between a Gröbner and a Pommaret basis: the former one is concerned only with the first syzygy module (i. e. $H_1(\mathcal{M})$), the latter one with the full syzygy resolution (i. e. the full Koszul homology $H_\bullet(\mathcal{M})$).

In order to apply such results, it is important that one deals with an involutive differential equation. The *Cartan–Kuranishi theorem* (see again [42, 53] and references therein) asserts that any differential equation $\mathcal{R}_q$ satisfying some modest regularity assumptions is either inconsistent or can be completed to an equivalent involutive equation of the form $\mathcal{R}^{(s)}_{q+r}$; a concrete instance of such a completion process was given in Example 7.16 for Janet's equation. The key for proving this result is the observation that any symbol becomes involutive, if it is sufficiently often prolonged, in other words the finiteness of the Spencer cohomology (Theorem 2.20). The power of the homological approach to involution becomes evident in its trivial proof.

For concrete computations, a direct application of the Cartan–Kuranishi procedure becomes quickly cumbersome, as it requires an explicit local representation of every appearing differential equation. In the (small!) Janet example the final involutive system $\mathcal{R}^{(2)}_5$ is locally described by 44 equations. However, all relevant information can be extracted from just 7 equations corresponding to the Pommaret basis of the symbol module. In the language of [23] these equations comprise the skeleton of $\mathcal{R}^{(2)}_5$. On the basis of this notion, [23] presents a hybrid completion algorithm that combines the algebraic efficiency of Pommaret bases with the intrinsic geometry of the Cartan–Kuranishi procedure.

For lack of space we could not discuss applications of involutive differential equations in this contribution. Pommaret [43, 44, 45] presents in his books many applications, in particular in mathematical physics and control theory. Some applications in numerical analysis can be found in [53] and references therein. Generally speaking, wherever under- or overdetermined systems of differential equations appear, the theory of involution will make any subsequent analysis significantly easier; in many cases such an analysis will even be impossible without the concept of involution (or at least formal integrability).

A Multi Indices

As there exist different kinds of multi indices but apparently no standard names for them, we must introduce our own terminology. Let $x^1, \dots, x^n$ be n variables. For various constructions with them, we distinguish in this article between multi indices and repeated indices. A *multi index* is an element $\mu = [\mu_1, \dots, \mu_n] \in \mathbb{N}_0^n$; the value $|\mu| = \mu_1 + \cdots + \mu_n$ is its length. A typical use of a multi index is $x^\mu = (x^1)^{\mu_1} \cdots (x^n)^{\mu_n}$ or for a function $u = u(x^1, \dots, x^n)$

$$\frac{\partial^{|\mu|} u}{\partial x^\mu} = \frac{\partial^{|\mu|} u}{\partial (x^1)^{\mu_1} \cdots \partial (x^n)^{\mu_n}} \, . \tag{A.1}$$

We furthermore define $\mu! = \mu_1! \cdots \mu_n!$. If k is the smallest value such that $\mu_k \neq 0$, we call it the *class* of the multi index μ and write $\operatorname{cls} \mu = k$.

By convention, we introduce for the special multi index $0 = [0, \dots, 0]$ that $x^0 = 1$ and $\partial^{|0|} u / \partial x^0 = u$. Obviously, $0! = 1$ and $|0| = 0$. In principle, $\operatorname{cls} 0$ is undefined, but in many situations it is convenient to set $\operatorname{cls} 0 = n$. Other special multi indices appearing occasionally are

$$\ell_i = [0, \dots, 0, \ell, 0, \dots, 0] \tag{A.2}$$

where $\ell \in \mathbb{N}$ is the ith entry and all other entries vanish. The addition of multi indices is defined componentwise, i. e. $\mu + \nu = [\mu_1 + \nu_1, \dots, \mu_n + \nu_n]$. If we want to increase the ith entry of a multi index μ by one, we can thus simply write $\mu + 1_i$ using (A.2).

A *repeated index* of length q is an ordered sequence $I = (i_1, \dots, i_q)$ where each entry i_k is an element of $\{1, \dots, n\}$. Now x^I is a shorthand for the product $x^{i_1} x^{i_2} \cdots x^{i_q}$ and correspondingly for partial derivatives. Obviously, here the ordering of the entries does not matter. However, our main use of repeated indices is for exterior forms where the ordering determines the sign. In fact, there we only consider indices $I = (i_1, \dots, i_q)$ with $i_1 < i_2 < \cdots < i_q$, i. e. all entries are different and sorted in ascending order. If I, J are two such repeated indices, then $I \cup J$ denotes the index obtained by first concatenating I and J and then sorting the entries. Obviously, this only yields a valid result, if I and J have no entries in common. We set $\operatorname{sgn}(I \cup J) = \pm 1$ depending on whether an even or odd number of transpositions is required for the sorting. If I and J have entries in common, we set $\operatorname{sgn}(I \cup J) = 0$; this convention is useful to avoid case distinctions in some sums.

B Term Orders

Term orders are crucial for the definition of Gröbner bases and thus of involutive bases. As some of our conventions are inverse to those usually used in commutative algebra, we collect them in this short appendix.

Let $\mathcal{P} = \Bbbk[x^1, \dots, x^n]$ and define the set of *terms* $\mathbb{T} = \{x^\mu \mid \mu \in \mathbb{N}_0^n\}$. Recall that a *term order* is a total order $\prec$ on $\mathbb{T}$ satisfying (i) $1 \preceq t$ and (ii) $r \prec s \Rightarrow rt \prec st$ for all

$r, s, t \in \mathbb{T}$. A term order is *degree compatible*, if $\deg s < \deg t$ implies $s \prec t$. Finally, we say that a term order *respects classes*, if $\deg s = \deg t$ and $\operatorname{cls} s < \operatorname{cls} t$ implies $s \prec t$.

We define now the *lexicographic* order by $x^\mu \prec_{\text{lex}} x^\nu$, if the last non-vanishing entry of $\mu - \nu$ is negative. With respect to the *reverse lexicographic* order, $x^\mu \prec_{\text{revlex}} x^\nu$, if the first non-vanishing entry of $\mu - \nu$ is positive. The latter one is not a term order, as 1 is not the smallest term, but its degree compatible version is a term order: $x^\mu \prec_{\text{degrevlex}} x^\nu$, if $|\mu| < |\nu|$ or if $|\mu| = |\nu|$ and $x^\mu \prec_{\text{revlex}} x^\nu$.

Here we defined the orders inverse to the usual convention in most texts on Gröbner bases: the classical forms arise, if one inverts the order of the variables: $x^1, \dots, x^n \mapsto x^n, \dots, x^1$. Our version fits better to the conventions used in differential equations theory, in particular to our definition of the class of a multi index.

Lemma B.1 *Let $\prec$ be degree compatible and the condition $\operatorname{lt} f \in \langle x^1, \dots, x^k\rangle$ be equivalent to $f \in \langle x^1, \dots, x^k\rangle$ for any homogeneous polynomial $f \in \mathcal{P}$, then $\prec$ is the degree reverse lexicographic order $\prec_{\text{degrevlex}}$.*

The proof of this well-known characterisation lemma is left as an easy exercise to the reader. We note the following simple consequence of it: on terms of the same degree any class respecting term order coincides with the degree reverse lexicographic order.

C Coalgebras and Comodules

Since the notion of a coalgebra and a comodule is still unfamiliar to many mathematicians, we collect here the basic definitions and properties; for an in depth treatment we refer to [7]. Roughly, the idea behind coalgebras is the inversion of certain arrows in diagrams encoding properties of the multiplication in an algebra. Thus, if $\mathcal{A}$ is an algebra over a field $\Bbbk$, then the product is a homomorphism $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ and the unit may be interpreted as a linear map $\Bbbk \to \mathcal{A}$. Correspondingly, a *coalgebra* $\mathcal{C}$ over a field $\Bbbk$ is a vector space equipped with a *coproduct*, a homomorphism $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$, and a *counit*, a linear map $\epsilon : \mathcal{C} \to \mathcal{R}$. The associativity of the product in an algebra and the defining property of the unit dualise to the requirement that the diagrams

$$\begin{tikzcd} \mathcal{C}\otimes\mathcal{C}\otimes\mathcal{C} & \mathcal{C}\otimes\mathcal{C} \arrow[l, "\Delta\otimes\mathrm{id}_{\mathcal{C}}"'] \\ \mathcal{C}\otimes\mathcal{C} \arrow[u, "\mathrm{id}_{\mathcal{C}}\otimes\Delta"] & \mathcal{C} \arrow[l, "\Delta"'] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{C}\otimes\mathcal{C} \arrow[dl, "\epsilon\otimes\mathrm{id}_{\mathcal{C}}"'] \arrow[dr, "\mathrm{id}_{\mathcal{C}}\otimes\epsilon"] & \\ \Bbbk\otimes\mathcal{C} & & \mathcal{C}\otimes\Bbbk \\ & \mathcal{C} \arrow[uu, "\Delta"] \arrow[ul, "\gamma"] \arrow[ur, "\gamma"'] & \end{tikzcd} \tag{C.1}$$

(where γ maps $c \in \mathcal{C}$ to $1 \otimes c$ or $c \otimes 1$, respectively) commute.

Analogously, $\mathcal{C}$-comodules arise from dualising $\mathcal{A}$-modules: a (right) *comodule* is a vector space $\mathcal{N}$ with a *coaction* $\rho : \mathcal{N} \to \mathcal{N} \otimes \mathcal{C}$ such that the two diagrams

$$\begin{tikzcd} \mathcal{N} \arrow[r, "\rho"] \arrow[d, "\rho"'] & \mathcal{N}\otimes\mathcal{C} \arrow[d, "\mathrm{id}_{\mathcal{N}}\otimes\Delta"] \\ \mathcal{N}\otimes\mathcal{C} \arrow[r, "\rho\otimes\mathrm{id}_{\mathcal{C}}"'] & \mathcal{N}\otimes\mathcal{C}\otimes\mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{N} \arrow[r, "\rho"] \arrow[dr, "\mathrm{id}_{\mathcal{N}}"'] & \mathcal{N}\otimes\mathcal{C} \arrow[d, "\mathrm{id}_{\mathcal{N}}\otimes\epsilon"] \\ & \mathcal{N} \end{tikzcd} \tag{C.2}$$

commute. A special case is a *coideal* $\mathcal{N} \subseteq \mathcal{C}$ where the coaction is the coproduct Δ. The subcomodule $\mathcal{L} \subset \mathcal{N}$ *cogenerated* by a set $\mathcal{G} \subseteq \mathcal{N}$ is by definition the intersection of all subcomodules of $\mathcal{N}$ containing $\mathcal{G}$.

The linear dual $\mathcal{C}^*$ of a coalgebra $\mathcal{C}$ has a natural algebra structure via the *convolution product* $\star$. It is defined for arbitrary elements $\phi, \psi \in \mathcal{C}^*$ by requiring that the relation

$$\langle \phi \star \psi, c\rangle = \big\langle \phi \otimes \psi, \Delta(c)\big\rangle \tag{C.3}$$

holds for all $c \in \mathcal{C}$. The unit element of $\mathcal{C}^*$ is simply the counit ϵ. If $\mathcal{N}$ is a $\mathcal{C}$-comodule with coaction ρ, then its dual space $\mathcal{N}^*$ is naturally a right $\mathcal{C}^*$-module with the action $\rho^* : \mathcal{N}^* \otimes \mathcal{C}^* \to \mathcal{N}^*$ defined in similar manner by requiring that the relation

$$\big\langle \rho^*(\nu, \psi), n\big\rangle = \big\langle \nu \otimes \psi, \rho(n)\big\rangle \tag{C.4}$$

holds for all $\nu \in \mathcal{N}^*$, $\psi \in \mathcal{C}^*$ and $n \in \mathcal{N}$. For arbitrary subsets $\mathcal{L} \subseteq \mathcal{N}$ we define in the usual manner the *annihilator* $\mathcal{L}^0 = \{\nu \in \mathcal{N}^* \mid \nu(\ell) = 0\,\forall \ell \in \mathcal{L}\} \subseteq \mathcal{N}^*$. Similarly, for any subset $\mathcal{L}^* \subseteq \mathcal{N}^*$ the annihilator is $(\mathcal{L}^*)^0 = \{n \in \mathcal{N} \mid \lambda(n) = 0\,\forall \lambda \in \mathcal{L}^*\} \subseteq \mathcal{N}$. One can show that if $\mathcal{L} \subseteq \mathcal{N}$ is a subcomodule, then $\mathcal{L}^0 \subseteq \mathcal{N}^*$ is a submodule, and conversely if $\mathcal{L}^* \subseteq \mathcal{N}^*$ is a submodule, then $(\mathcal{L}^*)^0 \subseteq \mathcal{N}$ is a subcomodule.

If $\mathcal{V}$ is a finite-dimensional vector space, then the tensor algebra $T\mathcal{V}$ can be given the structure of a coalgebra with the coproduct

$$\Delta(v_1 \otimes \cdots \otimes v_q) = \sum_{i=0}^{q} (v_1 \otimes \cdots \otimes v_i) \otimes (v_{i+1} \otimes \cdots \otimes v_q)\,. \tag{C.5}$$

and the counit $\epsilon : T\mathcal{V} \to \Bbbk$ which is the identity on $T_0\mathcal{V}$ and zero everywhere else. This coalgebra structure is inherited by the symmetric algebra $S\mathcal{V}$ defined in the usual way as a factor algebra of $T\mathcal{V}$. We denote the symmetric coalgebra by $\mathfrak{S}\mathcal{V}$.

If $\{x^1, \dots, x^n\}$ is a basis of $\mathcal{V}$, then we may use as basis of the symmetric coalgebra $\mathfrak{S}\mathcal{V}$ all monomials x^μ with a multi index $\mu \in \mathbb{N}_0^n$ providing the well-known isomorphy of $S\mathcal{V}$ with the polynomial algebra $\Bbbk[x^1, \dots, x^n]$. In this basis the coproduct of $\mathfrak{S}\mathcal{V}$ is given by "Taylor expansion"

$$\Delta(f) = \sum_{\mu \in \mathbb{N}_0^n} \frac{1}{\mu!} \frac{\partial^{|\mu|} f}{\partial x^\mu} \otimes x^\mu \tag{C.6}$$

for any polynomial $f \in \Bbbk[x^1, \dots, x^n]$

By definition, a subset $\mathcal{J} \subset \mathfrak{S}\mathcal{V}$ is a coideal, if and only if $\Delta(\mathcal{J}) \subseteq \mathcal{J} \otimes \mathcal{C}$ which, by (C.6), is equivalent to the condition $\partial^{|\mu|} f / \partial x^\mu \in \mathcal{J}$ for all $f \in \mathcal{J}$. Similarly, a subset $\mathcal{N} \subseteq (\mathfrak{S}\mathcal{V})^m$ is a subcomodule, if and only if this condition holds in each component. Let $\mathcal{F} \subset \mathfrak{S}_q\mathcal{V}$ be a finite set of homogeneous polynomials of degree q. We are interested in the homogeneous coideal $\mathcal{J}$ cogenerated by $\mathcal{F}$. Obviously, we must take for $\mathcal{J}_q$ the $\mathbb{k}$-linear span of $\mathcal{F}$. In a given basis $\{x^1, \dots, x^n\}$ of $\mathcal{V}$, we set for $0 < r \leq q$

$$\mathcal{J}_{q-r} = \left\{ \frac{\partial^{|\mu|} f}{\partial x^\mu} \mid f \in \mathcal{J}_q,\ \mu \in \mathbb{N}_0^n,\ |\mu| = r \right\}. \tag{C.7}$$

It is easy to see that $\mathcal{J} = \bigoplus_{r=0}^{q} \mathcal{J}_r$ satisfies $\Delta(\mathcal{J}) \subseteq \mathcal{J} \otimes \mathcal{C}$ and that it is the smallest subset of $\mathfrak{S}\mathcal{V}$ containing $\mathcal{F}$ with this property. Note that, in contrast to the algebra case, we obtain components of lower degree and $\mathcal{J}$ is finite-dimensional as vector space.

Bibliography

[1] W.W. Adams and P. Loustaunau, *An Introduction to Gröbner Bases*, Graduate Studies in Mathematics 3, American Mathematical Society, Providence, 1994.

[2] D. Bayer and D. Mumford, *What can be Computed in Algebraic Geometry?*, Computational Algebraic Geometry and Commutative Algebra (D. Eisenbud and L. Robbiano, eds.), Symposia Mathematica 34, Cambridge University Press, Cambridge, 1993, pp. 1–48.

[3] D. Bayer and M. Stillman, *A Criterion for Detecting m-Regularity*, Invent. Math. 87 (1987), pp. 1–11.

[4] Th. Becker and V. Weispfenning, *Gröbner Bases*, Graduate Texts in Mathematics 141, Springer-Verlag, New York, 1993.

[5] I. Bermejo and P. Gimenez, *Saturation and Castelnuovo-Mumford Regularity*, J. Alg. 303 (2006), pp. 592–617.

[6] R.L. Bryant, S.S. Chern, R.B. Gardner, H.L. Goldschmidt, and P.A. Griffiths, *Exterior Differential Systems*, Mathematical Sciences Research Institute Publications 18, Springer-Verlag, New York, 1991.

[7] T. Brzezinski and R. Wisbauer, *Corings and Comodules*, London Mathematical Society Lecture Notes Series 309, Cambridge University Press, Cambridge, 2003.

[8] E. Cartan, *Sur la Théorie des Systèmes en Involution et ses Applications à la Relativité*, Bull. Soc. Math. France 59 (1931), pp. 88–118, (also: Œuvres Complètes d'Élie Cartan, Partie II, Vol. 2, pp. 1199–1230).

[9] ______, *Les Systèmes Différentielles Extérieurs et leurs Applications Géométriques*, Hermann, Paris, 1945.

[10] J. Chrastina, *The Formal Theory of Differential Equations*, Folia Mathematica 6, Masaryk University, Brno, 1998.

[11] D. Cox, J. Little, and D. O'Shea, *Ideals, Varieties, and Algorithms*, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1992.

[12] M. Dubois-Violette, *The Theory of Overdetermined Linear Systems and its Applications to Non-Linear Field Equations*, J. Geom. Phys. 1 (1984), pp. 139–172.

[13] S. Eilenberg and J.C. Moore, *Homology and Fibrations I: Coalgebras, Cotensor Product and its Derived Functors*, Comment. Math. Helv. 40 (1966), pp. 199–236.

[14] D. Eisenbud and S. Goto, *Linear Free Resolutions and Minimal Multiplicity*, J. Alg. 88 (1984), pp. 89–133.

[15] V.P. Gerdt and Yu.A. Blinkov, *Involutive Bases of Polynomial Ideals*, Math. Comp. Simul. 45 (1998), pp. 519–542.

[16] ______, *Minimal Involutive Bases*, Math. Comp. Simul. 45 (1998), pp. 543–560.

[17] V.P. Gerdt, Yu.A. Blinkov, and D.A. Yanovich, *Construction of Janet Bases II: Polynomial Bases*, Computer Algebra in Scientific Computing — CASC 2001 (V.G. Ghanza, E.W. Mayr, and E.V. Vorozhtsov, eds.), Springer-Verlag, Berlin, 2001, pp. 249–263.

[18] H.L. Goldschmidt, *Existence Theorems for Analytic Linear Partial Differential Equations*, Ann. Math. 82 (1965), pp. 246–270.

[19] ______, *Integrability Criteria for Systems of Non-Linear Partial Differential Equations*, J. Diff. Geom. 1 (1969), pp. 269–307.

[20] W. Gröbner, *Moderne Algebraische Geometrie*, Springer-Verlag, Wien, 1949.

[21] V.W. Guillemin and S. Sternberg, *An Algebraic Model of Transitive Differential Geometry*, Bull. Amer. Math. Soc. 70 (1964), pp. 16–47.

[22] M. Hausdorf, M. Sahbi, and W.M. Seiler, *δ- and Quasi-Regularity for Polynomial Ideals*, Global Integrability of Field Theories (J. Calmet, W.M. Seiler, and R.W. Tucker, eds.), Universitätsverlag Karlsruhe, Karlsruhe, 2006, pp. 179–200.

[23] M. Hausdorf and W.M. Seiler, *An Efficient Algebraic Algorithm for the Geometric Completion to Involution*, Appl. Alg. Eng. Comm. Comp. 13 (2002), pp. 163–207.

[24] T.A. Ivey and J.M. Landsberg, *Cartan for Beginners: Differential Geometry via Moving Frames and Exterior Differential Systems*, Graduate Studies in Mathematics 61, American Mathematical Society, Providence, 2003.

[25] M. Janet, *Sur les Systèmes d'Équations aux Dérivées Partielles*, J. Math. Pure Appl. 3 (1920), pp. 65–151.

[26] ______, *Les Modules de Formes Algébriques et la Théorie Générale des Systèmes Différentiels*, Ann. École Norm. Sup. 41 (1924), pp. 27–65.

[27] ______, *Leçons sur les Systèmes d'Équations aux Dérivées Partielles*, Cahiers Scientifiques, Fascicule IV, Gauthier-Villars, Paris, 1929.

[28] J. Johnson, *On Spencer's Cohomology Theory for Linear Partial Differential Equations*, Trans. Amer. Math. Soc. 154 (1971), pp. 137–149.

[29] E. Kähler, *Einführung in die Theorie der Systeme von Differentialgleichungen*, Teubner, Leipzig, 1934.

[30] I.S. Krasilshchik, V.V. Lychagin, and A.M. Vinogradov, *Geometry of Jet Spaces and Nonlinear Partial Differential Equations*, Gordon & Breach, New York, 1986.

[31] H. Kredel and V. Weispfenning, *Computing Dimension and Independent Sets for Polynomial Ideals*, J. Symb. Comp. 6 (1988), pp. 231–247.

[32] B.S. Kruglikov and V.V. Lychagin, *Geometry of Differential Equations*, Preprint IHES/M/07/04, Institut des Hautes Études Scientifiques, Bures-sur-Yvette, 2007.

[33] L.A. Lambe and W.M. Seiler, *Differential Equations, Spencer Cohomology, and Computing Resolutions*, Georg. Math. J. 9 (2002), pp. 723–772.

[34] B. Malgrange, *Cohomologie de Spencer (d'après Quillen)*, Publications du Séminaire Mathématique, Orsay, 1966.

[35] ———, *Cartan Involutiveness = Mumford Regularity*, Contemp. Math. 331 (2003), pp. 193–205.

[36] ———, *Systèmes Différentiels Involutif*, Panoramas et Synthèses 19, Societé Mathématique de France, Paris, 2005.

[37] E.L. Mansfield, *Differential Gröbner Bases*, Ph.D. thesis, Macquarie University, Sydney, 1991.

[38] ———, *A Simple Criterion for Involutivity*, J. London Math. Soc. 54 (1996), pp. 323–345.

[39] Y. Matsushima, *On a Theorem Concerning the Prolongation of a Differential System*, Nagoya Math. J. 6 (1953), pp. 1–16.

[40] ———, *Sur les Algèbres de Lie Linéaires Semi-Involutives*, Colloque de Topologie de Strasbourg, Université de Strasbourg, 1954–55, p. 17.

[41] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics 107, Springer-Verlag, New York, 1986.

[42] J.F. Pommaret, *Systems of Partial Differential Equations and Lie Pseudogroups*, Gordon & Breach, London, 1978.

[43] ———, *Lie Pseudogroups and Mechanics*, Gordon & Breach, London, 1988.

[44] ———, *Partial Differential Control Theory I: Mathematical Tools*, Mathematics and Its Applications 530, Kluwer, Dordrecht, 2001.

[45] ———, *Partial Differential Control Theory II: Control Systems*, Mathematics and Its Applications 530, Kluwer, Dordrecht, 2001.

[46] D.G. Quillen, *Formal Properties of Over-Determined Systems of Linear Partial Differential Equations*, Ph.D. thesis, Harvard University, Cambridge, 1964.

[47] D. Rees, *A Basis Theorem for Polynomial Modules*, Proc. Cambridge Phil. Soc. 52 (1956), pp. 12–16.

[48] C. Riquier, *Les Systèmes d'Équations aux Derivées Partielles*, Gauthier-Villars, Paris, 1910.

[49] C. Ruiz, *Propriétés de Dualité du Prolongement Formel des Systèmes Différentiels Extérieur*, C.R. Acad. Sci. Ser. A 280 (1975), pp. 1625–1627.

[50] W.M. Seiler, *A Combinatorial Approach to Involution and δ-Regularity I: Involutive Bases in Polynomial Algebras of Solvable Type*, Preprint Universität Mannheim, 2002.

[51] ———, *A Combinatorial Approach to Involution and δ-Regularity II: Structure Analysis of Polynomial Modules with Pommaret Bases*, Preprint Universität Mannheim, 2002.

[52] ———, *Completion to Involution and Semi-Discretisations*, Appl. Num. Math. 42 (2002), pp. 437–451.

[53] ———, *Involution — The Formal Theory of Differential Equations and its Applications in Computer Algebra and Numerical Analysis*, Habilitation thesis, Dept. of Mathematics and Computer Science, Universität Mannheim, 2002.

[54] J.-P. Serre, *Local Algebra*, Springer-Verlag, Berlin, 2000.

[55] I.M. Singer and S. Sternberg, *The Infinite Groups of Lie and Cartan I: The Transitive Groups*, J. Anal. Math. 15 (1965), pp. 1–114.

[56] D.C. Spencer, *Deformation of Structures on Manifolds Defined by Transitive, Continuous Pseudogroups: I–II*, Ann. Math. 76 (1962), pp. 306–445.

[57] ______, *Overdetermined Systems of Linear Partial Differential Equations*, Bull. Amer. Math. Soc. 75 (1969), pp. 179–239.

[58] R.P. Stanley, *Hilbert Functions of Graded Algebras*, Adv. Math. 28 (1978), pp. 57–83.

[59] O. Stormark, *Lie's Structural Approach to PDE Systems*, Encyclopedia of Mathematics and its Applications 80, Cambridge University Press, Cambridge, 2000.

[60] B. Sturmfels and N. White, *Computing Combinatorial Decompositions of Rings*, Combinatorica 11 (1991), pp. 275–293.

[61] J.W. Sweeny, *The D-Neumann Problem*, Acta Math. 120 (1968), pp. 223–277.

[62] N.G. Trung, *Evaluations of Initial Ideals and Castelnuovo–Mumford Regularity*, Proc. Amer. Math. Soc. 130 (2002), pp. 1265–1274.

[63] W.T. Wu, *On the Construction of Gröbner Basis of a Polynomial Ideal Based on Riquier-Janet Theory*, Syst. Sci. Math. Sci. 4 (1991), pp. 194–207.

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