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# Taylor and Lyubeznik Resolutions via Gröbner Bases

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**Abstract**

Taylor presented an explicit resolution for arbitrary monomial ideals. Later, Lyubeznik found that already a subcomplex defines a resolution. We show that the Taylor resolution may be obtained by repeated application of the Schreyer Theorem from the theory of Gröbner bases, whereas the Lyubeznik resolution is a consequence of Buchberger's chain criterion. Finally, we relate Fröberg's contracting homotopy for the Taylor complex to normal forms with respect to our Gröbner bases and use it to derive a splitting homotopy that leads to the Lyubeznik complex.

## 1. The Taylor and the Lyubeznik Resolution

Let $\mathcal{M} = \{m_1, \dots, m_r\} \subset \mathcal{P} = \Bbbk[x_1, \dots, x_n]$ be a finite set of monomials. Taylor [1960] constructed in her Ph.D. thesis an explicit free resolution of the monomial ideal $\mathcal{J} = \langle \mathcal{M} \rangle$. The associated complex consists essentially of an exterior algebra and a differential defined via the least common multiples of subsets of $\mathcal{M}$.

Let $\mathcal{V}$ be some $r$-dimensional $\Bbbk$-vector space with the basis $\{v_1, \dots, v_r\}$. If $\mathbf{k} = (k_1, \dots, k_q)$ is a sequence of integers with $1 \leq k_1 < k_2 < \cdots < k_q \leq r$, we set $m_{\mathbf{k}} = \operatorname{lcm}(m_{k_1}, \dots, m_{k_q})$. The $\mathcal{P}$-module $\mathcal{T}_q = \mathcal{P} \otimes \Lambda^q \mathcal{V}$ is then freely generated by all wedge products $v_{\mathbf{k}} = v_{k_1} \wedge \cdots \wedge v_{k_q}$. Finally, we introduce on the algebra $\mathcal{T} = \mathcal{P} \otimes \Lambda \mathcal{V}$ the following $\mathcal{P}$-linear differential $\delta$:

$$\delta v_{\mathbf{k}} = \sum_{\ell=1}^{q} (-1)^{\ell-1} \frac{m_{\mathbf{k}}}{m_{\mathbf{k}_\ell}} v_{\mathbf{k}_\ell} , \tag{1}$$

where $\mathbf{k}_\ell$ denotes the sequence $\mathbf{k}$ with the entry $k_\ell$ removed. Obviously, the differential $\delta$ respects the grading of $\mathcal{T}$ by the form degree, as it maps the component $\mathcal{T}_q$ into $\mathcal{T}_{q-1}$ (however, in general $\delta$ does not respect the natural *bi*grading of $\mathcal{T}$ given by $\mathcal{T}_{rq} = \mathcal{P}_r \otimes \Lambda^q \mathcal{V}$).

One can show that $(\mathcal{T}, \delta)$ is a complex representing a free resolution of the ideal $\mathcal{J}_0 = \langle \mathcal{M} \rangle$. Obviously, $\delta v_i = m_i$ and the length of the resolution is given by the number $r$ of monomials. This implies immediately that the resolution is rarely minimal.* Note that the ordering of the monomials $m_i$ in the set $\mathcal{M}$ has no real influence on the result: the arising resolutions are trivially isomorphic.

Lyubeznik [1988] proved later in his Ph.D. thesis that in fact already a subcomplex $\mathcal{L} \subseteq \mathcal{T}$ defines a free resolution of $\mathcal{J}_0$. Let $\mathbf{k}$ again be an integer sequence; we denote for $1 \leq i < r$ by $\mathbf{k}_{>i}$ the subsequence of all entries $k_j > i$. If we eliminate from the basis of the Taylor complex $\mathcal{T}$ all generators $v_{\mathbf{k}}$ where for at least one $1 \leq i < r$ the monomial $m_i$ divides $m_{\mathbf{k}_{>i}}$, then the remaining part $\mathcal{L}$ is still a complex defining a resolution of $\mathcal{J}$.

Here the ordering of the monomials $m_i$ is crucial; in general, for different orderings different eliminations will be possible. As the Taylor complex is essentially independent of the orderings, one also obtains a free subresolution of $\mathcal{T}$ via a "reverse" form of the Lyubeznik approach. Namely, we define $\mathbf{k}_{<i}$ as the subsequence of all entries $k_j < i$ and then eliminate all generators $v_{\mathbf{k}}$ where for at least one $1 \leq i < r$ the monomial $m_i$ divides $m_{\mathbf{k}_{<i}}$.

Both resolutions are of considerable interest in homological algebra [Johansson et al., 2002, Lambe and Seiler, 2002]. Fröberg [1979] constructed an explicit *contracting homotopy* for the Taylor complex that also restricts to the Lyubeznik subcomplex, i. e. a $\Bbbk$-linear map $\psi : \mathcal{T}_q \to \mathcal{T}_{q+1}$ such that $\delta\psi + \psi\delta = 1$. Given a term $x^\mu v_{\mathbf{k}} \in T_q$, let $\iota = \iota(x^\mu v_{\mathbf{k}})$ be the minimal value for $i$ such that $m_i \mid x^\mu m_{\mathbf{k}}$. Then we define

$$\psi(x^\mu v_{\mathbf{k}}) = [\iota < k_1] \frac{x^\mu m_{\mathbf{k}}}{m_{(\iota,\mathbf{k})}} v_{(\iota,\mathbf{k})} \tag{2}$$

where $(\iota, \mathbf{k})$ denotes the sequence $(\iota, k_1, \ldots, k_q)$ and $[\cdot]$ is the Kronecker-Iverson symbol [Graham et al., 1989] which is 1, if the contained condition is true and 0 otherwise.

## 2. Gröbner Bases and Syzygies

Gröbner bases [Adams and Loustaunau, 1994, Becker and Weispfenning, 1993, Cox et al., 1992] are an important tool in computational algebra and have been introduced in the Ph.D. thesis of Buchberger [1965]. If $\mathcal{J}$ is an ideal in the polynomial ring $\mathcal{P} = \Bbbk[x_1, \ldots, x_n]$, then a finite set $\mathcal{G} \subset \mathcal{J}$ is a *Gröbner basis* of the ideal $\mathcal{J}$ for a term order $\prec$, if the leading term $\mathrm{lt}_\prec f$ of any polynomial $f \in \mathcal{J}$ is divisible by the leading term $\mathrm{lt}_\prec g$ of a generator $g \in \mathcal{G}$. If we write $\mathrm{lt}_\prec \mathcal{J} = \langle \{\mathrm{lt}_\prec f \mid f \in \mathcal{J}\} \rangle$ for the monomial ideal generated by the leading terms of all the elements of $\mathcal{J}$, then we may express this defining condition concisely as $\langle \mathrm{lt}_\prec \mathcal{G} \rangle = \mathrm{lt}_\prec \mathcal{J}$.

If (and only if) the set $\mathcal{G}$ is a Gröbner basis of the ideal $\mathcal{J}$ for the term

*Several characterisations of the case that $\mathcal{T}$ defines a minimal resolution have been given by Fröberg [1979].

order $\prec$, then every polynomial $f \in \mathcal{J}$ possesses a so-called *standard representation* $f = \sum_{g\in\mathcal{G}} P_g g$ where all the polynomials $P_g \in \mathcal{P}$ satisfy $\mathrm{lt}_\prec(P_g g) \preceq \mathrm{lt}_\prec f$. Of course, this representation is not unique, as one may add arbitrary syzygies. More generally, every polynomial $f \in \mathcal{P}$ may be written (with the help of the so-called *division algorithm*) in the form

$$f = \sum_{g\in\mathcal{G}} P_g g + \hat{f} \tag{3}$$

where no term of $\hat{f}$ is contained in $\mathrm{lt}_\prec \mathcal{J}$. The polynomial $\hat{f}$ is called the *normal form* of $f$ with respect to $\mathcal{G}$. One can show that $\hat{f}$ is uniquely defined, if and only if $\mathcal{G}$ is a Gröbner basis.

A central concept in the theory of Gröbner bases is that of an *$S$-polynomial* (the $S$ stands for syzygy). Given two polynomials $f_1, f_2 \in \mathcal{P}$, we define their $S$-polynomial $S_\prec(f_1, f_2)$ as follows. Let $m_i = \mathrm{lt}_\prec f_i$ be the leading term of $f_i$ and set $m_{12} = \mathrm{lcm}(m_1, m_2)$. Then

$$S_\prec(f_1, f_2) = \frac{m_{12}}{\mathrm{lc}_\prec(f_1)m_1} f_1 - \frac{m_{12}}{\mathrm{lc}_\prec(f_2)m_2} f_2 \tag{4}$$

where $\mathrm{lc}_\prec(f_i)$ denotes the leading coefficient of $f_i$. An important criterion for a set $\mathcal{G}$ to be a Gröbner basis of the ideal $\langle\mathcal{G}\rangle$ is that for all $g_1, g_2 \in \mathcal{G}$ the normal form of the $S$-polynomial $S_\prec(g_1, g_2)$ vanishes.

All these notions generalise trivially to submodules of free $\mathcal{P}$-modules. We consider elements of a free $\mathcal{P}$-module of rank $m$ as $m$-dimensional vectors with polynomial entries. The $S$-polynomial is then defined to be zero, if the two leading terms $\mathrm{lt}_\prec \mathbf{f}_1$ and $\mathrm{lt}_\prec \mathbf{f}_2$ belong to different components.

Schreyer [1980] proved in his diploma thesis that the standard representations of the $S$-polynomials lead not only to a generating set of the first syzygy module but in fact again to a Gröbner basis with respect to a special term order $\prec_\mathcal{G}$ induced by the basis $\mathcal{G} = \{g_1, \dots, g_r\}$: let $\{\mathbf{e}_1, \dots, \mathbf{e}_r\}$ denote the standard basis of a free $\mathcal{P}$-module whose rank is $|\mathcal{G}| = r$; then we define for arbitrary monomials $s, t \in \mathcal{P}$ that $s\mathbf{e}_\alpha \prec_\mathcal{G} t\mathbf{e}_\beta$, if either $\mathrm{lt}_\prec(sg_\alpha) \prec \mathrm{lt}_\prec(tg_\beta)$ or both $\mathrm{lt}_\prec(sg_\alpha) = \mathrm{lt}_\prec(tg_\beta)$ and $\beta < \alpha$.

Let $\mathcal{G} = \{g_1, \dots, g_r\}$ and $m_i = \mathrm{lt}_\prec g_i$. Assume that the standard representation of the $S$-polynomial $S_\prec(g_i, g_j)$ is given by $S_\prec(g_i, g_j) = \sum_{k=1}^r P_{ijk} g_k$. Then we set again $m_{ij} = \mathrm{lcm}(m_i, m_j)$ and define the syzygy

$$\mathbf{S}_{ij} = \frac{m_{ij}}{\mathrm{lc}_\prec(g_i)m_i}\mathbf{e}_i - \frac{m_{ij}}{\mathrm{lc}_\prec(g_j)m_j}\mathbf{e}_j - \sum_{k=1}^r P_{ijk}\mathbf{e}_k \in \mathcal{P}^r\,. \tag{5}$$

By the *Schreyer theorem*, the set $\Sigma_\mathcal{G} = \{\mathbf{S}_{ij} \mid 1 \leq i < j \leq r\}$ is a Gröbner basis of the syzygy module $\mathrm{Syz}(\mathcal{G})$ with respect to the term order $\prec_\mathcal{G}$. We also introduce the syzygy $\tilde{\mathbf{S}}_{ij} = \frac{m_{ij}}{m_i}\mathbf{e}_i - \frac{m_{ij}}{m_j}\mathbf{e}_j$ of the leading terms of $g_i$ and $g_j$. As any monomial set is trivially a Gröbner basis of the ideal it generates, we conclude that the set $\tilde{\Sigma}_{\mathcal{G}} = \{\tilde{\mathbf{S}}_{ij} \mid 1 \leq i < j \leq r\}$ is a Gröbner basis of the syzygy module $\mathrm{Syz}(\mathrm{lt}_{\prec}\mathcal{G})$ with respect to the term order $\prec_{\mathcal{G}}$.

*Buchberger's chain criterion* [Buchberger, 1979] asserts that certain $S$-polynomials may be ignored in the above mentioned criterion for a Gröbner basis. It is based on the following observation which will turn out to be crucial for the Lyubeznik resolution. Let $\mathcal{S} \subseteq \Sigma_{\mathcal{G}}$ and assume that (i) the set $\tilde{\mathcal{S}} = \{\tilde{\mathbf{S}}_{ij} \mid \mathbf{S}_{ij} \in \mathcal{S}\}$ generates the syzygy module $\mathrm{Syz}(\mathrm{lt}_{\prec}\mathcal{G})$, (ii) $\tilde{\mathcal{S}}$ contains the three syzygies $\tilde{\mathbf{S}}_{ij}$, $\tilde{\mathbf{S}}_{ik}$, $\tilde{\mathbf{S}}_{jk}$, and (ii) the monomial $m_i$ divides $m_{jk}$. Then the set $\tilde{\mathcal{S}} \setminus \{\tilde{\mathbf{S}}_{jk}\}$ still generates $\mathrm{Syz}(\mathrm{lt}_{\prec}\mathcal{G})$ and the set $\mathcal{S} \setminus \{\mathbf{S}_{jk}\}$ still generates $\mathrm{Syz}(\mathcal{G})$.

## 3. Taylor Resolution via Schreyer Theorem

Our goal is now to show that the Taylor resolution can be constructed via repeated application of the Schreyer theorem. The decisive point will be to define an appropriate ordering of the generators in each Gröbner basis.

As a first step, we introduce on the free polynomial module $\mathcal{T}_q$ defined in Sect. 1 two term orders $\prec_q$ and $\prec_q^r$ as follows. We define on the space of all ascending integer sequences a "lexicographic" order: we set $\mathbf{k} < \mathbf{l}$, if for $j = \min\{i \mid k_i \neq \ell_i\}$ the inequality $k_j < \ell_j$ holds. For $\prec_0 = \prec_0^r$ we choose an arbitrary term order on $\mathcal{P}$; it will turn out that everything we do in the sequel is independent of this choice. Then we define recursively for two sequences $\mathbf{k}$ and $\mathbf{l}$ of length $q+1$ and two monomials $s, t \in \mathcal{P}$ that $su_{\mathbf{k}} \prec_{q+1} tu_{\mathbf{l}}$, if either $\mathrm{lt}_{\prec_q}(s\delta u_{\mathbf{k}}) \prec_q \mathrm{lt}_{\prec_q}(t\delta u_{\mathbf{l}})$ or both $\mathrm{lt}_{\prec_q}(s\delta u_{\mathbf{k}}) = \mathrm{lt}_{\prec_q}(t\delta u_{\mathbf{l}})$ and $\mathbf{k} < \mathbf{l}$. For the "reverse" term order $\prec_{q+1}^r$ the last condition is replaced by $\mathbf{k} > \mathbf{l}$

LEMMA 3.1: *If $\mathbf{k}$ is a sequence of length $q$ with $1 \leq q \leq r$, then*

$$\frac{m_{\mathbf{k}}}{m_{\mathbf{k}_q}} v_{\mathbf{k}_q} \prec_{q-1} \cdots \prec_{q-1} \frac{m_{\mathbf{k}}}{m_{\mathbf{k}_1}} v_{\mathbf{k}_1} \tag{6}$$

*and hence the leading term of $\delta v_{\mathbf{k}}$ is given by*

$$\mathrm{lt}_{\prec_{q-1}}(\delta v_{\mathbf{k}}) = \frac{m_{\mathbf{k}}}{m_{\mathbf{k}_1}} v_{\mathbf{k}_1}\ . \tag{7}$$

*For the reverse order $\prec_{q-1}^r$ we obtain $\frac{m_{\mathbf{k}}}{m_{\mathbf{k}_1}} v_{\mathbf{k}_1} \prec_{q-1}^r \cdots \prec_{q-1}^r \frac{m_{\mathbf{k}}}{m_{\mathbf{k}_q}} v_{\mathbf{k}_q}$ and thus $\mathrm{lt}_{\prec_{q-1}^r}(\delta v_{\mathbf{k}}) = \frac{m_{\mathbf{k}}}{m_{\mathbf{k}_q}} v_{\mathbf{k}_q}$.*

*Proof:* We proceed by induction. For $q = 1$ the assertion is trivial. For $q = 2$ we must compare the two terms $\frac{m_{\mathbf{k}}}{m_{\mathbf{k}_1}} v_{\mathbf{k}_1} = \frac{m_{\mathbf{k}}}{m_{k_2}} v_{k_2}$ and $\frac{m_{\mathbf{k}}}{m_{\mathbf{k}_2}} v_{\mathbf{k}_2} = \frac{m_{\mathbf{k}}}{m_{k_1}} v_{k_1}$. As $\delta v_i = m_i$, we find that $\frac{m_{\mathbf{k}}}{m_{k_1}} \delta v_{k_1} = \frac{m_{\mathbf{k}}}{m_{k_2}} \delta v_{k_2} = m_{\mathbf{k}}$. By the definition of the term order $\prec_1$ this implies that $\frac{m_{\mathbf{k}}}{m_{k_2}} v_{k_2} \prec_1 \frac{m_{\mathbf{k}}}{m_{k_1}} v_{k_1}$ as $k_1 < k_2$ (independent of the choice of the order $\prec_0$ as claimed above).

Now assume that the lemma holds for all sequences of length less than $q$. We must compare the terms $\frac{m_{\mathbf{k}}}{m_{\mathbf{k}_\ell}} v_{\mathbf{k}_\ell}$ with $1 \leq \ell \leq q$. This requires to determine the leading term of $\frac{m_{\mathbf{k}}}{m_{\mathbf{k}_\ell}}\delta v_{\mathbf{k}_\ell}$ with respect to $\prec_{q-2}$. By our induction hypothesis, we obtain for $\ell = 1$ the term $\frac{m_{\mathbf{k}}}{m_{\mathbf{k}_{1,2}}} v_{\mathbf{k}_{1,2}}$ and for $\ell > 1$ the term $\frac{m_{\mathbf{k}}}{m_{\mathbf{k}_{1,\ell}}} v_{\mathbf{k}_{1,\ell}}$. Thus the values $\ell = 1$ and $\ell = 2$ yield both the same term and, as obviously $\mathbf{k}_2 < \mathbf{k}_1$, we find $\frac{m_{\mathbf{k}}}{m_{\mathbf{k}_2}} v_{\mathbf{k}_2} \prec_{q-1} \frac{m_{\mathbf{k}}}{m_{\mathbf{k}_1}} v_{\mathbf{k}_1}$.

In order to compare the terms for the other possible values of $\ell$, we must descend recursively. At the next lower form degree we find that $\ell = 2$ and $\ell = 3$ yield the same term, so that by the definition of our term order, the one for $\ell = 2$ is the greater one. Continuing until degree one we obtain (6).

The proof for the reverse term order $\prec^r_{q-1}$ proceeds completely analogously. This time at form degree $q'$ the terms for $\ell = q'$ and $\ell = q' - 1$ coincide and by definition of the order the one for $\ell = q'$ is greater. □

Now that we know the leading terms of the elements $\delta v_{\mathbf{k}}$, the next step is to show that with respect to both introduced term orders the set

$$\Delta_q = \{\delta v_{\mathbf{k}} \mid \mathbf{k} = (k_1, \dots, k_{q+1})\} \tag{8}$$

is a Gröbner basis of the submodule $\mathcal{J}_q$ it generates in $\mathcal{T}_q$. Note that we have $\mathcal{J}_q = \delta(\mathcal{T}_{q+1})$ because of the $\mathcal{P}$-linearity of $\delta$.

Lemma 3.2: *Let $\mathbf{k}$ be an ascending integer sequence of length $q$ with $0 \leq q < r$ and $i, j$ two further integers with $1 \leq i < j < k_1$. Then the syzygy induced by the $S$-polynomial $\mathbf{S}_{\prec_q}(\delta v_{(i,\mathbf{k})}, \delta v_{(j,\mathbf{k})})$ is*

$$\mathbf{S}_{(j,\mathbf{k}),(i,\mathbf{k})} = \delta v_{(i,j,\mathbf{k})} \ . \tag{9}$$

*Similarly, if $i, j$ are two integers with $k_q < i < j \leq r$, then the syzygy induced by the $S$-polynomial $\mathbf{S}_{\prec^r_q}(\delta v_{(\mathbf{k},i)}, \delta v_{(\mathbf{k},j)})$ is*

$$\mathbf{S}_{(\mathbf{k},i),(\mathbf{k},j)} = (-1)^{q+1}\delta v_{(\mathbf{k},i,j)} \ . \tag{10}$$

*Proof:* We prove the assertion only for $\prec_q$, as the proof for $\prec^r_q$ proceeds again completely analogously. The sign in (10) stems from the fact that there the sequence is manipulated at its end and the sign of the last summand in (1) depends on the length of the sequence.

In order to simplify the notation, we write $\mathbf{i} = (i, \mathbf{k})$, $\mathbf{j} = (j, \mathbf{k})$ and, finally, $\bar{\mathbf{k}} = (i, j, \mathbf{k})$. As $(\mathcal{T}, \delta)$ is a complex, obviously $\delta^2 v_{\bar{\mathbf{k}}} = 0$ and, by the $\mathcal{P}$-linearity of $\delta$, we find that

$$\sum_{\ell=1}^{q+2} (-1)^{\ell-1} \frac{m_{\bar{\mathbf{k}}}}{m_{\bar{\mathbf{k}}_\ell}} \delta v_{\bar{\mathbf{k}}_\ell} = 0 \ . \tag{11}$$

Thus $\delta v_{\bar{\mathbf{k}}} \in \operatorname{Syz}(\Delta_q)$. On the other hand, by definition of an $S$-polynomial and Lemma 3.1, $\mathbf{S}_{\prec_q}(\delta v_{\mathbf{j}}, \delta v_{\mathbf{i}}) = \frac{m_{\bar{\mathbf{k}}}}{m_{\mathbf{j}}}\delta v_{\mathbf{j}} - \frac{m_{\bar{\mathbf{k}}}}{m_{\mathbf{i}}}\delta v_{\mathbf{i}}$. Comparing with (11), we see that these are just the first two summands. Thus we are done, if we can show that the remaining terms of (11) define a standard representation of $\mathbf{S}_{\prec_q}(\delta v_{\mathbf{j}}, \delta v_{\mathbf{i}})$.

$S$-polynomials are defined such that the leading terms of the two summands cancel in the subtraction. Hence in order to find the leading term of our $S$-polynomial we must compare the second largest term in each summand. A straightforward computation shows that applying $\delta$ yields in each case the same leading term. Hence we obtain

$$\mathrm{lt}_{\prec_q}\bigl(\mathbf{S}_{\prec_q}(\delta v_{\mathbf{i}}, \delta v_{\mathbf{j}})\bigr) = \frac{m_{\bar{\mathbf{k}}}}{m_{(j,\mathbf{k}_1)}} v_{(j,\mathbf{k}_1)} \tag{12}$$

and there only remains to show that for all values $3 \leq \ell \leq q+2$ the relation $\mathrm{lt}_{\prec_q}\bigl(\frac{m_{\bar{\mathbf{k}}}}{m_{\bar{\mathbf{k}}_\ell}} \delta v_{\bar{\mathbf{k}}_\ell}\bigr) \preceq_q \frac{m_{\bar{\mathbf{k}}}}{m_{(j,\mathbf{k}_1)}} v_{(j,\mathbf{k}_1)}$ holds. This is straightforward, as Lemma 3.1 implies $\mathrm{lt}_{\prec_q}\bigl(\frac{m_{\bar{\mathbf{k}}}}{m_{\bar{\mathbf{k}}_\ell}} \delta v_{\bar{\mathbf{k}}_\ell}\bigr) = \frac{m_{\bar{\mathbf{k}}}}{m_{(j,\mathbf{k}_\ell)}} v_{(j,\mathbf{k}_\ell)}$. Applying $\delta$ and taking the leading term gives immediately the desired result. Now the lemma follows from the definition of the syzygy $\mathbf{S}_{\mathbf{i},\mathbf{j}}$. □

Proposition 3.1: *For every $q$ with $0 \leq q \leq r$, the set $\Delta_q$ is a Gröbner basis both for $\prec_q$ and for $\prec_q^r$ of the submodule $\mathcal{J}_q$.*

*Proof:* This proposition is a corollary to the above mentioned Schreyer theorem. We proceed again by induction. For $q = 0$, the assertion is trivial, as $\Delta_0 = \mathcal{M}$ and any monomial set is a Gröbner basis of the ideal generated by it for every term order.

Let us assume that the proposition holds for $q - 1$. In order to invoke the Schreyer theorem we must define an ordering of the elements $\delta v_{\mathbf{k}} \in \Delta_q$. Two fairly natural choices are to order them either ascending or descending by the index $\mathbf{k}$ (using the above defined lexicographic ordering of integer sequences). It is easy to see that the term order $\prec_{\Delta_{q-1}}$ used in the Schreyer theorem is in the first case $\prec_q^r$ and in the second case $\prec_q$. The assertion is now a trivial consequence of Schreyer's theorem and the Lemmata 3.1 and 3.2. □

As an immediate corollary we arrive finally at the following theorem.

Theorem 3.1: *The Taylor resolution can be obtained by repeatedly applying the Schreyer theorem, if at each step the generators are ordered in either of the two ways described in the proof above.*

## 4. Lyubeznik Resolution and Buchberger Chain Criterion

The Gröbner basis of the syzygy module obtained via the Schreyer theorem is in general not reduced. A number of generators may be eliminated by applying the Buchberger chain criterion. We show now that this allows us to derive the Lyubeznik subcomplex $\mathcal{L} \subseteq \mathcal{T}$.

Proposition 4.1: *Assume that for some $1 \leq i < r$ and an integer sequence $\mathbf{k}$ of length $q$ the monomial $m_i$ divides $m_{\mathbf{k}_{>i}}$. Then the set $\Delta_q \setminus \{\delta v_{\mathbf{k}}\}$ is still a Gröbner basis of the submodule $\mathcal{J}_q$ for the term order $\prec_q$.*

*Proof:* Suppose $m_i$ divides $m_{\mathbf{k}_{>i}}$. We first assume that $\mathbf{k} = \mathbf{k}_{>i}$, i. e. $i < k_1$. The case $q = 1$ is trivial: the Buchberger chain criterion implies that the syzygy $\mathbf{S}_{k_1,k_2}$ corresponding to the $S$-polynomial of $m_{k_1}$ and $m_{k_2}$ is not needed to generate the syzygy module $\operatorname{Syz}(\mathcal{M})$. By Lemma 3.2, $\mathbf{S}_{k_1,k_2} = \delta v_{\mathbf{k}}$ and thus $\Delta_1 \setminus \{\delta v_{\mathbf{k}}\}$ is still a Gröbner basis.

For $q > 2$ we consider the three elements $\delta v_{\mathbf{k}_1}, \delta v_{\mathbf{k}_2}, \delta v_{(i,\mathbf{k}_{1,2})} \in \Delta_{q-1}$. According to Lemma 3.1, all three leading terms are multiples of $v_{\mathbf{k}_{1,2}}$; the coefficients are $\frac{m_{\mathbf{k}_1}}{m_{\mathbf{k}_{1,2}}}$, $\frac{m_{\mathbf{k}_2}}{m_{\mathbf{k}_{1,2}}}$ and $\frac{m_{(i,\mathbf{k}_{1,2})}}{m_{\mathbf{k}_{1,2}}}$, respectively. Our assumption $m_i \mid m_{\mathbf{k}}$ implies that the least common multiple of the first two is divisible by the third. Thus we can invoke the Buchberger chain criterion and obtain our assertion, as $\mathbf{S}_{\mathbf{k}_1,\mathbf{k}_2} = \delta v_{\mathbf{k}}$ according to Lemma 3.2.

If $\mathbf{k} \neq \mathbf{k}_{>i}$, then we may eliminate at an earlier stage the generator $\delta v_{\mathbf{k}_{>i}}$. The generator $\delta v_{\mathbf{k}}$ arises in our construction of the Taylor resolution by repeated application of the Schreyer theorem only as the result of chains of syzygies starting with one induced by an $S$-polynomial involving $\delta v_{\mathbf{k}_{>i}}$. As the latter one may be eliminated, $\delta v_{\mathbf{k}}$ is not needed either. □

PROPOSITION 4.2: *Assume that for some $1 \leq i < r$ and an integer sequence $\mathbf{k}$ of length $q$ the monomial $m_i$ divides $m_{\mathbf{k}_{<i}}$. Then the set $\Delta_q \setminus \{\delta v_{\mathbf{k}}\}$ is still a Gröbner basis of the submodule $\mathcal{J}_q$ for the term order $\prec_q^r$.*

*Proof:* Completely analogous to the previous proposition. □

THEOREM 4.1: *The (reverse) Lyubeznik resolution arises from the Taylor resolution by repeated application of the Buchberger chain criterion.*

Note that in general we do not invoke all possible instances of the Buchberger chain criterion for deriving the Lyubeznik resolution. This follows already from the trivial fact that the chain criterion is independent of the ordering of the monomials in the original set $\mathcal{M}$ whereas the Lyubeznik criterion for the elimination of generators of $\mathcal{T}$ is not.

## 5. Contracting Homotopies and Normal Forms

The contracting homotopy $\psi$ defined by (2) also possesses an interesting interpretation in terms of our Gröbner bases $\Delta_q$.

LEMMA 5.1: *We have $\psi(x^\mu v_{\mathbf{k}}) = 0$, if and only if $x^\mu v_{\mathbf{k}} \notin \operatorname{lt}_{\prec_q} \mathcal{J}_q$. If, however, $x^\mu v_{\mathbf{k}} \in \operatorname{lt}_{\prec_q} \mathcal{J}_q$, then $\psi(x^\mu v_{\mathbf{k}}) = x^\nu v_{(\iota,\mathbf{k})}$ where $\iota$ is the minimal value of $i$ such that $\operatorname{lt}_{\prec_q} \delta v_{(i,\mathbf{k})} \mid x^\mu v_{\mathbf{k}}$ and $\nu$ is chosen such that $\operatorname{lt}_{\prec_q}(x^\nu \delta v_{(\iota,\mathbf{k})}) = x^\alpha v_{\mathbf{k}}$.*

*Proof:* Let us first assume that in Fröberg's definition $\iota(x^\mu v_{\mathbf{k}}) \geq k_1$ so that $\psi(x^\mu v_{\mathbf{k}}) = 0$. According to Lemma 3.1, the leading term of $\delta v_{\mathbf{l}}$ with respect to the term order $\prec_q$ only lies in the component generated by $v_{\mathbf{k}}$, if $\mathbf{l} = (i, \mathbf{k})$ for some $1 \leq i < k_1$. Thus we have $x^\mu v_{\mathbf{k}} \in \mathrm{lt}_{\prec_q} \mathcal{J}_q$, if and only if an exponent vector $\nu$ exists such that $x^\nu m_{(i,\mathbf{k})} = x^\alpha m_{\mathbf{k}}$. But by the definition of the function $\iota$, the term $m_{(i,\mathbf{k})}$ does not divide $x^\alpha m_{\mathbf{k}}$ for any $i < k_1$.

Now let us assume that $\iota = \iota(x^\mu v_{\mathbf{k}}) < k_1$. Then it follows again from Lemma 3.1 that $\mathrm{lt}_{\prec_q} \delta v_{(\iota,\mathbf{k})} = \frac{m_{(\iota,\mathbf{k})}}{m_{\mathbf{k}}} v_{\mathbf{k}}$. By the definition of $\iota$, we have $m_{(\iota,\mathbf{k})} \mid x^\mu m_{\mathbf{k}}$ and hence a unique exponent vector $\nu$ exists with $\mathrm{lt}_{\prec_q}(x^\nu \delta v_{(\iota,\mathbf{k})}) = x^\alpha v_{\mathbf{k}}$. It is trivial that $\iota$ is the smallest value with this property. □

As a trivial corollary of this result we find that the restriction of the contracting homotopy $\psi$ to the $\Bbbk$-vector space generated by the terms in $\mathrm{lt}_{\prec_q} \mathcal{J}_q$ is injective. Probably of more interest is the following observation.

Theorem 5.1: *The map $\psi \circ \delta : \mathcal{T}_q \to \mathcal{T}_q$ yields the normal form with respect to the Gröbner basis $\Delta_q$ and the map $\delta \circ \psi = 1 - \psi \circ \delta$ is a projector on $\mathcal{J}_q$.*

*Proof:* Let $v \in \mathcal{T}_q$ be an arbitrary element. Applying the division algorithm with respect to the Gröbner basis $\Delta_q$ of $\mathcal{J}_q$ yields according to (3) a representation $v = \sum_{|\mathbf{k}|=q+1} \delta(P_{\mathbf{k}} v_{\mathbf{k}}) + \hat{v}$ where $P_{\mathbf{k}} \in \mathcal{P}$ are some polynomials and where $\hat{v}$ is the normal form of $v$ with respect to $\Delta_q$. Recall that the normal form is unique (in contrast to the coefficients $P_{\mathbf{k}}$) and that it consists only of terms *not* contained in $\mathrm{lt}_{\prec_q} \mathcal{J}_q$.

Exploiting Lemma 5.1 and that $\psi$ is a contracting homotopy, we find

$$\psi(v) = \sum_{|\mathbf{k}|=q+1} \psi\delta(P_{\mathbf{k}} v_{\mathbf{k}}) = -\sum_{|\mathbf{k}|=q+1} \delta\psi(P_{\mathbf{k}} v_{\mathbf{k}}) + \sum_{|\mathbf{k}|=q+1} P_{\mathbf{k}} v_{\mathbf{k}} \,. \tag{13}$$

This implies immediately $\delta\psi(v) = \sum_{|\mathbf{k}|=q+1} \delta(P_{\mathbf{k}} v_{\mathbf{k}})$ and thus $\psi\delta(v) = \hat{v}$. □

In fact, we may extend this idea to a general principle for the construction of contracting homotopies in complexes over the polynomial ring $\mathcal{P}$ where Gröbner bases for the images of the differential are known.

Theorem 5.2: *Let $(\mathcal{C}, \delta)$ be a (not necessarily finite) exact complex of free polynomial modules with a $\mathcal{P}$-linear differential $\delta : \mathcal{C}_q \to \mathcal{C}_{q-1}$. Assume that for all degrees $q$ a Gröbner basis of the submodule $\delta(\mathcal{C}_q)$ is known. Thus every element $u \in \mathcal{C}_q$ may be written in the form $u = \delta v + \hat{u}$ where $\hat{u}$ is the unique normal form of $u$ with respect to the Gröbner basis of $\delta(\mathcal{C}_q)$. Then the map $\psi : \mathcal{C}_q \to \mathcal{C}_{q+1}$ defined by $\psi(u) = \hat{v}$ where $\hat{v}$ is the unique normal form of $v$ with respect to the Gröbner basis of $\delta(\mathcal{C}_{q+1})$ is a contracting homotopy of $(\mathcal{C}, \delta)$ satisfying $\psi^2 = 0$.*

*Proof:* We must first show that $\psi$ is well-defined. But this is trivial: if $v$ and $v'$ are two elements of $\mathcal{C}_{q+1}$ such that $\delta v = \delta v'$, then $v = v' + \delta w$ by the exactness of $(\mathcal{C}, \delta)$ and thus $\hat{v} = \hat{v}'$. Then we must prove that $\psi\delta + \delta\psi = \mathbb{1}$. Again this is very simple, as, by definition of $\psi$, we find $\delta\psi(u) = \delta\hat{v} = \delta v$ and $\psi\delta(u) = \hat{u}$. Hence $(\psi\delta + \delta\psi)(u) = \delta v + \hat{u} = u$ as required. The relation $\psi^2 = 0$ follows trivially from the definition. □

It follows from Theorem 5.1 that Fröberg's contracting homotopy is precisely the homotopy arising from this principle applied to the Gröbner bases $\Delta_q$ with respect to the term orders $\prec_q$, although it appears to be non-trivial (or at least rather tedious) to directly derive the explicit expression (2).

We may also introduce a "reverse" contracting homotopy $\psi_r$ as follows. For a given term $x^\mu v_{\mathbf{k}} \in \mathcal{T}_q$ let $\iota = \iota(x^\mu v_{\mathbf{k}})$ be the maximal value for $i$ such that $m_i \mid x^\mu m_{\mathbf{k}}$ and define

$$\psi_r(x^\mu v_{\mathbf{k}}) = [\iota > k_q]\frac{x^\mu m_{\mathbf{k}}}{m_{(\mathbf{k},\iota)}} v_{(\mathbf{k},\iota)} \; . \tag{14}$$

This corresponds to applying Theorem 5.2 to the Gröbner bases $\Delta_q$ with respect to the "reverse" term orders $\prec_q^r$. We leave the obvious details like the "reverse" form of Lemma 5.1 to the reader.

Finally, we consider the restriction of the map $\psi$ to the Lyubeznik complex $\mathcal{L}$. From its definition (2), it is not completely obvious that $\psi(\mathcal{L}) \subset \mathcal{L}$ and thus that $\psi$ is a contracting homotopy for $\mathcal{L}$, too; in fact, this was proven only very recently [Johansson et al., 2002]. Taking our approach, this becomes a simple corollary to Proposition 4.1 and Theorem 5.2.

Corollary 5.1: *The map $\psi$ defined by (2) is a contracting homotopy of the Lyubeznik complex $\mathcal{L}$.*

*Proof:* The subset $\Delta'_q = \{\delta(v_{\mathbf{k}}) \mid v_{\mathbf{k}} \in \mathcal{L}_{q+1}\} \subseteq \Delta_q$ is, by Proposition 4.1, still a Gröbner basis of the ideal generated by $\Delta_q$. Thus applying Theorem 5.2 yields exactly the same contracting homotopy as for the full Taylor complex $\mathcal{T}$. This immediately implies that $\psi(\mathcal{L}) \subset \mathcal{L}$. □

## 6. Strong Deformation Retracts

Barnes and Lambe [1991] introduced the notion of a *splitting homotopy* of a chain complex $(\mathcal{C}, \delta)$ over a ring $\mathcal{R}$. This is a graded $\mathcal{R}$-module homomorphism $\phi : \mathcal{C} \to \mathcal{C}$ such that (i) $\phi(\mathcal{C}_q) \subseteq \mathcal{C}_{q+1}$ for all $q$, (ii) $\phi^2 = 0$, and (iii) $\phi\delta\phi = \phi$. Such a map leads immediately to a *strong deformation retract*

$$\begin{tikzcd} \mathcal{S} \arrow[r, bend left, "\iota"] & \mathcal{C} \arrow[l, bend left, "\pi"] \arrow[loop right, "\phi"] \end{tikzcd} \tag{15}$$

where $\pi = 1 - \delta\phi - \phi\delta$, $\mathcal{S} = \pi(\mathcal{C}) \subseteq \mathcal{C}$ and $\iota$ is the inclusion map. Indeed, it is easy to see that our assumptions imply $\pi^2 = \pi$ and one easily checks that $\pi\iota = \mathbb{1}_{\mathcal{C}}$ and $\iota\pi = \pi$. One can show that any strong deformation retract arises this way.

We are interested in the special case that $\mathcal{J} \subseteq \mathcal{P}$ is a polynomial ideal and the chain complex $(\mathcal{C}, \delta)$ defines a free resolution of $\mathcal{J}$, i. e.

$$\cdots \longrightarrow \mathcal{C}_q \xrightarrow{\delta} \mathcal{C}_{q-1} \longrightarrow \cdots \longrightarrow \mathcal{C}_0 \xrightarrow{\epsilon} \mathcal{P}/\mathcal{J} \longrightarrow 0 \tag{16}$$

is an exact sequence of free $\mathcal{P}$-module. Here we assume for simplicity that $\mathcal{C}_0 \subseteq \mathcal{P}$ and $\epsilon$ is the canonical projection. Obviously, both the Taylor and the Lyubeznik resolution (or more generally any resolution constructed via Schreyer's theorem) is of this form. In such a situation, strong deformation retracts allow us to construct smaller resolutions [Lambe, 1991].

It is trivial to see that any contracting homotopy $\psi$ satisfying $\psi^2 = 0$ is also a splitting homotopy. Thus, if the assumptions of Theorem 5.2 are satisfied, then the map $\psi$ defined in it implies via the isomorphism $\mathcal{C}_0/\delta_1(\mathcal{C}_1) \cong \mathcal{P}/\mathcal{J}$ a strong deformation retract

$$\begin{tikzcd} \mathcal{P}/\mathcal{J} \arrow[r, bend left, "c"] & \mathcal{C} \arrow[l, bend left, "\epsilon"] \arrow[loop right, "\psi"] \end{tikzcd} \tag{17}$$

where the projection $\epsilon$ is extended to $\mathcal{C}$ by setting it zero outside of $\mathcal{C}_0$ and where the map $c$ yields the canonical representative of each equivalence class with respect to the Gröbner basis of $\delta_1(\mathcal{C}_1)$.

However, the retract (17) is not very interesting, as the left hand side is not really a resolution anymore. We show now how the contracting homotopy $\psi$ of Theorem 5.2 may be used to obtain another splitting homotopy $\phi$. For this construction we generalise a homotopy found by Johansson et al. [2002] (Proposition 3.9) which yields the Lyubeznik from the Taylor resolution.

Let the vectors $\mathbf{e}_\alpha^{(q)}$ with $1 \leq \alpha \leq \dim \mathcal{C}_q$ form a basis of the free module $\mathcal{C}_q$. Thus in the case of the Taylor resolution we may use the vectors $v_\mathbf{k}$ with $|\mathbf{k}| = q$. Then we define inductively a map $f : \mathcal{C} \to \mathcal{C}$ by setting $f(1) = 1$,

$$f(\mathbf{e}_\alpha^{(q)}) = \psi\bigl(f(\delta \mathbf{e}_\alpha^{(q)})\bigr) \tag{18}$$

and extending $\mathcal{P}$-linearly to the whole complex $\mathcal{C}$. The idea behind this definition may be conveniently expressed in the following lemma.

Lemma 6.1: *Let $\Delta_q$ be the Gröbner basis of $\delta(\mathcal{C}_{q+1})$ and assume that we have $\mathbf{e}_\alpha^{(q)} \notin \mathrm{lt}_\prec \Delta_q$ for all $1 \leq \alpha \leq \dim \mathcal{C}_q$ and all $q \geq 0$. Then $f$ is the identity map.*

*Proof:* Since $f(1) = 1$, we find that $f(\mathbf{e}_\alpha^{(1)}) = \psi\delta(\mathbf{e}_\alpha^{(1)})$. By definition of the contracting homotopy $\psi$, this means that $f(\mathbf{e}_\alpha^{(1)})$ is the normal form of $\mathbf{e}_\alpha^{(1)}$ with respect to $\Delta_1$. If $\mathbf{e}_\alpha^{(1)} \notin \mathrm{lt}_\prec \Delta_1$, then this normal form is again $\mathbf{e}_\alpha^{(1)}$. Now the assertion follows by induction. □

Note that $\mathbf{e}_\alpha^{(q)} \in \mathrm{lt}_\prec \Delta_q$ for some $\alpha$ and $q$ obviously implies that (16) is not the minimal resolution. The map $f$ projects to a subcomplex $\mathcal{S} \subseteq \mathcal{C}$ by eliminating all generators that appear in the leading ideals $\mathrm{lt}_\prec \Delta_q$. Thus if $f$ is not the identity map, we obtain a smaller resolution that is closer to the minimal one.

Above we showed how a suitable projection $\pi : \mathcal{C} \to \mathcal{S}$ is obtained from a splitting homotopy $\phi$. Now we have found a projection $f$ and would like to get a suitable splitting homotopy $\phi$. As we know a contracting homotopy for our complex, this is easily accomplished. We set $\phi(u) = 0$ for $u \in \mathcal{C}_0$. If we know $\phi$ restricted to $\mathcal{C}_{q-1}$, we can extend it to $\mathcal{C}_q$ by solving the "differential" equation $\delta\big(\phi(u)\big) = u - \phi\delta(u) - f(u)$. If we set $v = u - \phi\delta(u) - f(u)$, then obviously $\delta(v) = 0$ and we have $\delta\psi(v) = v$. Thus a possible solution is $\phi(u) = \psi(v)$. This leads to the following result.

THEOREM 6.1: *Let the map $\phi : \mathcal{C} \to \mathcal{C}$ be defined inductively by setting $\phi(u) = 0$ for $u \in \mathcal{C}_0$ and*

$$\phi(u) = \psi\big(u - \phi(\delta u) - f(u)\big) \tag{19}$$

*for $u \in \mathcal{C}_q$ with $q > 0$. Then $\phi$ is a splitting homotopy of the complex $(\mathcal{C}, \delta)$ and we have a strong deformation retract*

$$\begin{tikzcd} \mathcal{S} \arrow[r, bend left, "\iota"] & \mathcal{C} \arrow[l, bend left, "f"] \arrow[loop right, "\phi"] \end{tikzcd} \tag{20}$$

*where again $\iota$ is the inclusion map.*

*Proof:* We have $\psi^2 = 0$ by Theorem 5.2 and this implies at once $\phi^2 = 0$. The relation $\phi\delta\phi = \phi$ follows by a simple computation from the fact that $\psi$ is a contracting homotopy and again that $\psi^2 = 0$. □

It is now easy to see that applying this construction to the Taylor complex leads immediately to the Lyubeznik complex. Indeed, assume that $m_i \mid m_{\mathbf{k}}$ with $i < k_1$; then by Lemma 3.1 we find

$$\mathrm{lt}_{\prec_q} \delta v_{(i,\mathbf{k})} = \frac{m_{(i,\mathbf{k})}}{m_{\mathbf{k}}} v_{\mathbf{k}} = v_{\mathbf{k}} \ . \tag{21}$$

Thus the basis vector $v_{\mathbf{k}}$ is not in image of $f$. But $m_i \mid m_{\mathbf{k}}$ with $i < k_1$ is just Lyubeznik's condition for redundant generators in the Taylor resolution. Obviously, the converse holds, too: if $v_{\mathbf{k}} \in \mathrm{lt}_{\prec_q} \Delta_q$, then there exists an $i < k_1$ such that $m_{(i,\mathbf{k})} = m_{\mathbf{k}}$ which is equivalent to $m_i \mid m_{\mathbf{k}}$.

## 7. Conclusions

We have demonstrated that both the Taylor and the Lyubeznik resolution are directly derivable from the theory of Gröbner bases. The Taylor resolution is a syzygy resolution obtained by applying the Schreyer theorem on a Gröbner basis for the syzygy module. The Lyubeznik subresolution arises by invoking some instances of the Buchberger chain criterion.

Möller [1985], Mora and Möller [1983, 1986] studied in a series of articles the Taylor resolution and discussed strategies to obtain smaller resolutions from it. One of them lead to the Lyubeznik resolution. Most (if not all) of their reduction strategies may be homologically interpreted as splitting homotopies.†

†This observation is due to Larry Lambe.

The essential point in the proof of Proposition 3.1 that the sets $\Delta_q$ are in fact Gröbner bases of the images of the differential $\delta$ was to find the right ordering of the generators. Actually, we found two natural orderings on the bases $v_{\mathbf{k}}$ that both lead to the Taylor resolution.

The Schreyer theorem is often used to provide a simple proof of the Hilbert syzygy theorem that every ideal in $\mathcal{P} = \Bbbk[x_1, \dots, x_n]$ possesses a free resolution of length at most $n$ (see e. g. Adams and Loustaunau [1994]). Here yet another ordering of the generators is used. Obviously, the length of the Taylor resolution is in general much larger, as it is given by the number $s$ of monomials in $\mathcal{M}$. Thus in this respect our orderings based on the integer sequences labelling the generators are not "good". This is the price to be payed for the fact that the differential of the Taylor resolution has such a simple explicit representation which makes it very useful in theoretical considerations.

Theorems 5.2 and 6.1 demonstrate that the contracting homotopy for the Taylor complex found by Fröberg [1979] and the splitting homotopy of Johansson et al. [2002] leading to the Lyubeznik resolution, respectively, are not something particular but actually emerge from general principles applicable to any complex of free polynomial modules. Both theorems are not really surprising. One of the main tasks of a Gröbner basis of an ideal $\mathcal{J}$ is to distinguish a unique representative in each equivalence class in $\mathcal{P}/\mathcal{J}$ (this was the problem studied by Buchberger [1965]); a contracting homotopy does something fairly similar. Hence a relation between the two concepts has to be expected.

The splitting homotopy constructed in Theorem 6.1 removes only obstructions to the minimality of the resolution sitting in the leading terms of the Gröbner basis of $\delta(\mathcal{C}_q)$. Another explicit resolution of monomial ideals was recently derived with the help of Pommaret bases [Seiler, 2001, 2002]. While the construction yields automatically a Gröbner basis of each image $\delta(\mathcal{C}_q)$, obstructions to minimality never sit in their leading terms. Thus Theorem 6.1 yields only the identity map in this case.

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## References

W.W. Adams and P. Loustaunau. *An Introduction to Gröbner Bases.* Graduate Studies in Mathematics 3. AMS, Providence, 1994.

D.W. Barnes and L.A. Lambe. A fixed point approach to homological perturbation theory. *Proc. AMS*, 112:881–892, 1991.

Th. Becker and V. Weispfenning. *Gröbner Bases*. Graduate Texts in Mathematics 141. Springer-Verlag, New York, 1993.

B. Buchberger. *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal.* PhD thesis, Universität Innsbruck, 1965.

B. Buchberger. A criterion for detecting unnecessary reductions in the construction of Gröbner bases. In W. Ng, editor, *Proc. EUROSAM '79*, Lecture Notes in Computer Science 72, pages 3–21. Springer-Verlag, Berlin, 1979.

D. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms.* Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1992.

R. Fröberg. Some complex constructions with applications to Poincaré series. In *Séminaire d'Algèbre Paul Dubreil, 31ème année (Paris, 1977–1978)*, Lecture Notes in Mathematics 740, pages 272–284. Springer-Verlag, 1979.

R.L. Graham, D.E. Knuth, and O. Patashnik. *Concrete Mathematics.* Addison-Wesley, Reading, 1989.

L. Johansson, L.A. Lambe, and E. Sköldberg. On constructing resolutions over the polynomial algebra. *Homol. Homot. Appl.*, 4:315–336, 2002.

L.A. Lambe. Resolutions via homological perturbation. *J. Symb. Comp.*, 12:71–87, 1991.

L.A. Lambe and W.M. Seiler. Differential equations, Spencer cohomology, and computing resolutions. Preprint, 2002.

G. Lyubeznik. A new explicit finite resolution of ideals generated by monomials in an $R$-sequence. *J. Pure Appl. Alg.*, 51:193–195, 1988.

H.M. Möller. A reduction strategy for the Taylor resolution. In B.F. Caviness, editor, *Proc. EUROCAL '85, Vol. 2*, Lecture Notes in Computer Science 204, pages 526–534. Springer-Verlag, Berlin, 1985.

F. Mora and H.M. Möller. The computation of the Hilbert function. In J.A. van Hulzen, editor, *Proc. EUROCAL '83*, Lecture Notes in Computer Science 162, pages 157–167. Springer-Verlag, Berlin, 1983.

F. Mora and H.M. Möller. Computational aspects of reduction strategies to construct resolutions of monomial ideals. In A. Poli, editor, *Proc. AAECC-2*, Lecture Notes in Computer Science 228, pages 182–197. Springer-Verlag, Berlin, 1986.

F.O. Schreyer. Die Berechnung von Syzygien mit dem verallgemeinerten Weierstraßschen Divisionssatz. Master's thesis, Fakultät für Mathematik, Universität Hamburg, 1980.

W.M. Seiler. Involution — the formal theory of differential equations and its applications in computer algebra and numerical analysis. Habilitation thesis, Dept. of Mathematics, Universität Mannheim, 2001.

W.M. Seiler. A combinatorial approach to involution and $\delta$-regularity II: Structure analysis of polynomial modules with Pommaret bases. Preprint Universität Mannheim, 2002.

D. Taylor. *Ideals Generated by Monomials in an R-Sequence.* PhD thesis, University of Chicago, 1960.