Completion to Involution and Symmetry Analysis*

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Abstract

We discuss some applications of the theory of involutive systems in the Lie symmetry analysis of partial differential equations. Special emphasis is put on non-classical symmetries.

1 Introduction

Lie symmetry analysis represents one of the most important techniques for tackling differential equations, in particular non-linear ones. It often provides the only possibility to find solutions in closed form. But many other applications exist, too; perhaps best known is the construction of conservation laws via Noether's law. By now, a number of textbooks on symmetry analysis are available (see e. g. [3, 9, 13]); a survey of computer algebra tools for symmetry analysis was given by Hereman [5, 6].

The theory of involutive systems appears at many places in symmetry analysis. The best known application is, of course, the completion of the determining systems in order to facilitate its integration in closed form. In fact, this problem has been a major impetus for the renewal of interest in involution in the last, say, twenty years. In this article we will, however, concentrate on other, less known applications.

In the literature one can find several different notions of involutive (or passive or complete) systems. We will use the so-called formal theory [14, 18].

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It is a geometric approach and may be formulated completely intrinsically. This makes its application in symmetry theory very natural.

This article is organised as follows. The next section briefly reviews some basic notions of the formal theory. Then we study the role involution plays for the very definition of symmetries. Section 4 contains some formal results about measuring the size of the solution space of differential equations. Then we analyse symmetry reductions; this leads in a natural manner to non-classical symmetries. Finally, some conclusions are given.

2 Involution

For lack of space, we cannot give here a detailed introduction into the formal theory or the underlying jet bundle formalism. Our presentation follows [18]; a general reference is the book of Pommaret [14].

For simplicity, we will use local coordinates. Let $\boldsymbol{x} = (x^1, \dots, x^n)$ and $\boldsymbol{u} = (u^1, \dots, u^m)$ be an adapted coordinate system on the fibred manifold $\pi : \mathcal{E} \to \mathcal{X}$. Coordinates on the qth order jet bundle $J_q\mathcal{E}$ are obtained by adding all derivatives $p^\alpha_\mu = \partial^{|\mu|}u^\alpha/\partial(x^1)^{\mu_1}\cdots\partial(x^n)^{\mu_n}$ with $|\mu| \leq q$. Here $\mu = [\mu_1, \dots, \mu_n]$ is a multi index and $|\mu| = \mu_1 + \cdots + \mu_n$ its length. Thus the jet bundle $J_q\mathcal{E}$ may be interpreted as the space of all Taylor series truncated at order q.

We define a *differential equation*[1] of order q intrinsically as a fibred submanifold $\mathcal{R}_q \subseteq J_q\mathcal{E}$. Locally, $\mathcal{R}_q$ is described by a system of equations

$$\mathcal{R}_q \, : \, \left\{ \, \Phi^\tau(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = 0 \, , \quad \tau = 1, \dots, p \, . \right. \tag{1}$$

A *solution* is a (local) section $\sigma : \mathcal{X} \to \mathcal{E}$ such that the image of its prolongation $j_q\sigma : \mathcal{X} \to J_q\mathcal{E}$ completely lies in $\mathcal{R}_q$. Locally, a section is of the form $\sigma(\boldsymbol{x}) = (\boldsymbol{x}, \boldsymbol{s}(\boldsymbol{x}))$ with some functions $\boldsymbol{s} = (s^1, \dots, s^m)$. For its prolongation we differentiate these functions and σ is a solution, if the substitution $p^\alpha_\mu = \partial^{|\mu|}s^\alpha/\partial(x^1)^{\mu_1}\cdots\partial(x^n)^{\mu_n}$ in (1) yields an identity.

The *prolongation* $\mathcal{R}_{q+1} \subseteq J_{q+1}\mathcal{E}$ is obtained by formally differentiating every equation in (1) with respect to all independent variables $\boldsymbol{x}$, i.e. by adding the differential equations

$$D_i\Phi^\tau = \frac{\partial\Phi^\tau}{\partial x^i} + \frac{\partial\Phi^\tau}{\partial u^\alpha}p^\alpha_i + \frac{\partial\Phi^\tau}{\partial p^\alpha_\mu}p^\alpha_{\mu+1_i} = 0 \, , \qquad i = 1, \dots, n \, . \tag{2}$$

Iteration yields higher order prolongations $\mathcal{R}_{q+r} \subseteq J_{q+r}\mathcal{E}$. A (smooth) solution of $\mathcal{R}_q$ is automatically also a solution of $\mathcal{R}_{q+r}$ and vice versa.

[1] Note that we do not distinguish between scalar equations and systems!

The *symbol* $\mathcal{M}_q$ of the differential equation $\mathcal{R}_q$ locally represented by the system (1) is the solution space of the following linear system of (algebraic!) equations in the unknowns v^α_μ

$$\mathcal{M}_q\ :\ \left\{\sum_{\alpha=1}^{m}\sum_{|\mu|=q}\left(\frac{\partial\Phi^\tau}{\partial p^\alpha_\mu}\right)v^\alpha_\mu = 0\,. \right. \tag{3}$$

(By abuse of language, we will refer to both the linear system and its solution space as the symbol). The place-holders v^α_μ are coordinates of a finite-dimensional vector space; we introduce one for each derivative of order q. For a linear differential equation, the symbol is simply obtained by considering only the highest order or principal part and substituting v^α_μ for p^α_μ.

Let us make a power series ansatz for the solutions of the differential equation $\mathcal{R}_q$ expanding around some point $\boldsymbol{x}_0 \in \mathcal{X}$. Substituting into (1) and evaluating at $\boldsymbol{x} = \boldsymbol{x}_0$ yields a system of algebraic equations for the Taylor coefficients up to order q. Substituting the ansatz into $\mathcal{R}_{q+r}$ and evaluating again at $\boldsymbol{x} = \boldsymbol{x}_0$ yields an inhomogeneous linear system for the coefficients of order $q+r$. Its homogeneous part is given by the matrix of the prolonged symbol $\mathcal{M}_{q+r}$, i. e. the symbol of the prolonged equation $\mathcal{R}_{q+r}$.

This order by order construction of a formal power series solution fails, if integrability conditions occur. They pose additional conditions on coefficients of lower order and must all be known in order to pursue the above described procedure. A differential equation containing all its integrability conditions is called *formally integrable*, as it possesses a formal power series solution. Unfortunately, no finite criterion for formal integrability is known.

For an intrinsic formulation of integrability conditions we exploit the natural projection maps between jet bundles of different order (they simply forget the higher order derivatives). For $r > q$ we write $\pi^r_q : J_r\mathcal{E} \to J_q\mathcal{E}$ and define $\mathcal{R}^{(r)}_q = \pi^{q+r}_q(\mathcal{R}_{q+r}) \subseteq \mathcal{R}_q$, i. e. we first prolong r times and then project back r times. Integrability conditions appear during these prolongations, if and only if $\mathcal{R}^{(r)}_q \subsetneq \mathcal{R}_q$.

Definition 1 *The differential equation $\mathcal{R}_q$ is* formally integrable*, if for all integers $r > 0$ the equality $\mathcal{R}^{(1)}_{q+r} = \mathcal{R}_{q+r}$ holds.*

It follows from the construction above that the arbitrariness of the general formal power series solution is reflected by the dimensions of the prolonged symbols, because at each order $\dim\mathcal{M}_{q+r}$ coefficients are not determined by the differential equations but can be chosen freely. Formal integrability does, however, not suffice to determine these dimensions without explicitly constructing the prolonged symbols.

A jet variable p^α_μ is said to be of *class* k, if μ_k is the first non-vanishing entry of its multi index. We order the columns of the symbol $\mathcal{M}_q$ by class (highest class first) and compute a row echelon form. If v^α_μ is the leading term of an equation in this solved form of the symbol, then p^α_μ is a *principal* derivative. All other derivatives are *parametric*. We define the indices $\beta^{(k)}_q$ of the symbol $\mathcal{M}_q$ as the number of principal derivatives of class k.

Definition 2 *The symbol $\mathcal{M}_q$ is called* involutive*, if*

$$\operatorname{rank} \mathcal{M}_{q+1} = \sum_{k=1}^{n} k\beta^{(k)}_q \,. \tag{4}$$

The differential equation $\mathcal{R}_q$ is involutive*, if it is formally integrable and if its symbol $\mathcal{M}_q$ is involutive.*

In contrast to formal integrability, involution can be checked in a finite manner. To decide whether a symbol is involutive requires only some linear algebra. For differential equations we note the following very important criterion for involution that needs only one prolongation instead of infinitely many ones as the definition of formal integrability.

Theorem 3 *Let $\mathcal{R}_q$ be a differential equation with involutive symbol $\mathcal{M}_q$. Then $\mathcal{R}_q$ is involutive, if and only if $\mathcal{R}^{(1)}_q = \mathcal{R}_q$.*

Two important results on involutive equations are the *Cartan–Kuranishi* and the *Cartan–Kähler Theorem*. The former one states that any (regular) differential equation $\mathcal{R}_q$ can be completed in a finite number of steps to an equivalent differential equation $\mathcal{R}^{(s)}_{q+r}$ that is involutive; discussions of the algorithmic realisation of this completion can be found in [4, 17]. The latter is an existence and uniqueness theorem in the analytic category; if $\mathcal{R}_q$ is an analytic involutive differential equation, then a certain initial value problem for it has a unique analytic solution for analytic initial data. Thus it generalises the well-known Cauchy–Kovalevskaya Theorem.

3 Basic Symmetry Theory

Symmetry theory [3, 9, 13] provides us with some of the most important techniques for the analysis of differential equations, in particular for non-linear equations. A somewhat subtle and often overlooked point is the importance of the local solvability of the analysed differential equation. Essentially, this concerns the equivalence of two different definitions of what a symmetry of a differential equation actually is.

Definition 4 *A* symmetry *of a differential equation is a transformation that maps any of its solutions into another solution.*

This probably represents the most general definition of a symmetry of a differential equation. It is still somewhat vague, as it is not clearly specified what kinds of transformations or solutions are considered. We restrict in this article to point symmetries, i. e. to diffeomorphisms $\phi : \mathcal{E} \to \mathcal{E}$, and to formal power series solutions. Definition 4 is very hard to apply in practice, as it requires the knowledge of the full solution space. Thus usually the following, more geometric approach is taken.

Definition 5 *A diffeomorphism $\phi : \mathcal{E} \to \mathcal{E}$ is a* point symmetry *of the differential equation $\mathcal{R}_q \subseteq J_q\mathcal{E}$, if its prolongation $\phi^{(q)} : J_q\mathcal{E} \to J_q\mathcal{E}$ leaves the submanifold $\mathcal{R}_q$ invariant.*

The infinitesimal version of this definition forms the basis of all concrete computations with Lie symmetries. One considers a one-parameter group of diffeomorphisms ϕ_ϵ generated by a vector field X over $\mathcal{E}$. Given a local description of $\mathcal{R}_q$ in the form (1), a linearisation of the condition above leads to the well-known infinitesimal symmetry criterion

$$X^{(q)}\Phi^\tau\big|_{\mathcal{R}_q} = 0 \tag{5}$$

where the prolonged vector field $X^{(q)}$ on $J_q\mathcal{E}$ is the generator of $\phi_\epsilon^{(q)}$. $X^{(q)}$ and $\phi_\epsilon^{(q)}$ are easily determined by the chain rule.

With respect to our adapted coordinates, an arbitrary vector field X over $\mathcal{E}$ may be locally written in the form $X = \xi^i(\boldsymbol{x}, \boldsymbol{u})\partial_{x^i} + \eta^\alpha(\boldsymbol{x}, \boldsymbol{u})\partial_{u^\alpha}$. Evaluation of the criterion (5) yields an overdetermined linear system of partial differential equations for the coefficients ξ^i and η^α, the *determining system* $\mathcal{D}_q$ (it has the same order as the equation $\mathcal{R}_q$ to be analysed).

It is often assumed that Definitions 4 and 5 are equivalent, but this holds only under certain assumptions on the differential equation $\mathcal{R}_q$. We must assume that for every point $\rho \in \mathcal{R}_q$ at least one solution $\sigma : \mathcal{X} \to \mathcal{E}$ exists such that $\rho \in j_q(\sigma)$. This property is known as *local solvability.* It is trivial for scalar equations or systems in Cauchy–Kovalevskaya form but becomes an issue for overdetermined systems.

We illustrate the problem with a very simple example. Consider the following first order equation:

$$\mathcal{R}_1 : \begin{cases} u_z + yu_x = 0\,, \\ u_y = 0\,. \end{cases} \tag{6}$$

Since y appears explicitly, the manifold $\mathcal{R}_1$ does not remain invariant under y-translations and thus they are *not* symmetries in the sense of Definition 5. Cross-differentiations show that $\mathcal{R}_1^{(1)}$ is defined by the equations $u_x = u_y = u_z = 0$. Hence the general solution of our equation is $u(x, y, z) = const$ and y-translations *are* symmetries in the sense of Definition 4. The geometric approach "loses" this symmetry, as it requires that the whole manifold $\mathcal{R}_1$ should remain invariant and not only its submanifold $\mathcal{R}_1^{(1)}$. Since the image of any prolonged solution $j_1\sigma$ lies in $\mathcal{R}_1^{(1)}$, the second definition imposes a stronger condition than the first one.

A less trivial example are the incompressible Navier–Stokes equations. They are not formally integrable; the integrability condition is the well-known Poisson equation for the pressure. But in this special case, the hidden equation has no influence on the symmetry analysis. Thus the classical results on the symmetry group of the Navier–Stokes equations are correct, although their derivations neglect the integrability condition.

The problem disappears, as soon as we consider involutive differential equations. The Cartan–Kähler Theorem implies that any involutive equation is locally solvable. Note that as we are only considering formal power series solutions, the analyticity assumptions of this theorem play no role here.

4 Measuring the Size of Solution Spaces

It was already mentioned in Section 2 that the dimensions of the prolonged symbols $\dim \mathcal{M}_{q+r}$ measure the size of the formal solution space. For involutive equations these dimensions can be easily determined explicitly. Recall that the indices $\beta_q^{(k)}$ count the number of principal derivatives of order q and class k. Similarly, the *Cartan characters*

$$\alpha_q^{(k)} = m\binom{q+n-k-1}{q-1} - \beta_q^{(k)}\,, \quad k = 1, \dots, n \tag{7}$$

count the parametric derivatives of order q and class k.

Associating with each equation whose principal derivative is of class k its *multiplicative variables* $x^1, \dots, x^k$, one can show that if we prolong each equation with respect to its multiplicative variables only, we get algebraically independent equations. By (4), no further independent equations of order $q+1$ exist in the case of an involutive symbol. This implies a simple way to compute the indices of the prolonged symbol: $\beta_{q+1}^{(k)} = \sum_{i=k}^{n} \beta_q^{(k)}$ and correspondingly $\alpha_{q+1}^{(k)} = \sum_{i=k}^{n} \alpha_q^{(k)}$. Iteration yields

$$\alpha_{q+r}^{(k)} = \sum_{i=k}^{n} \binom{r+i-k-1}{r-1} \alpha_q^{(i)} . \tag{8}$$

Since $\dim \mathcal{M}_{q+r} = \sum_{k=1}^{n} \alpha_{q+r}^{(k)}$, we see that if (and only if) the symbol $\mathcal{M}_q$ is involutive, we can trivially determine the size of the formal solution space based on the Cartan characters $\alpha_q^{(k)}$. For more details see [19].

A trivial application of these results consists in measuring the size of the symmetry group. As the dimension of a Lie group coincides with the dimension of its Lie algebra, we perform an involution analysis of the determining system $\mathcal{D}_q$. The simplest case arises, if the symmetry group is finite-dimensional. It follows from the discussion above that then all Cartan characters of the involutive completion $\mathcal{D}_{q+r}^{(s)}$ must vanish and the dimension of the group is simply $\dim \mathcal{D}_{q+r}^{(s)}$. In the case of an infinite-dimensional group, the Cartan characters of $\mathcal{D}_{q+r}^{(s)}$ still encode much information about the size and, in particular, permit an easy comparison of different groups. These results are not restricted to Lie point symmetries but may obviously be applied in the same manner to generalised symmetries.

A less trivial application, which we cannot discuss here, is that one may deduce not only the size but also the structure of Lie symmetry algebra by a completely formal analysis. In the case of a finite-dimensional algebra this is rather simple; for infinite-dimensional algebras one needs deeper results from their structure theory. More details can be found in [7, 16].

One interesting feature of this approach is the possibility to formally "subtract" some effects. We will consider here only as a fairly trivial example superposition symmetries. A much more important application concerns gauge symmetries; formal methods allow for a simple analysis of gauge fixing conditions or for an intrinsic definition of the number of degrees of freedom. For lack of space we can only refer to the literature [21, 22].

All linear differential equations possess the superposition symmetry: if $u^\alpha = f^\alpha(\boldsymbol{x})$ is any solution of $\mathcal{R}_q$, then $X = f^\alpha(\boldsymbol{x})\partial_{u^\alpha}$ is an infinitesimal symmetry. This reflects the elementary fact that linear combinations of solutions are again solutions; the solution space has the structure of a vector space. Thus one always finds an infinite-dimensional symmetry algebra for linear equations. But usually the other symmetries are of much more interest. Thus one would like to know the size of the remaining algebra.

We illustrate the procedure for the heat equation, although it is trivial to compute its symmetries explicitly. The heat equation $u_t = u_{xx}$ defines a five-dimensional submanifold $\mathcal{R}_2$ in a six-dimensional jet bundle. Its Cartan characters are $\alpha_2^{(2)} = 0$ and $\alpha_2^{(1)} = 2$.

The determining system of the heat equation is also of second order:

$$\mathcal{D}_2 : \begin{cases} \tau_u = 0, & \tau_{ux} + \xi_u = 0, & \eta_{uu} - \xi_{ux} = 0, \\ \tau_x = 0, & \tau_{xx} + 2\xi_x - \tau_t = 0, & \eta_{ux} - \frac{1}{2}\xi_{xx} + \frac{1}{2}\xi_t = 0, \\ \tau_{uu} = 0, & \xi_{uu} = 0, & \eta_{xx} - \eta_t = 0. \end{cases} \tag{9}$$

Its involutive completion is $\mathcal{D}_3^{(4)}$ and generated by the following equations (plus all their prolongations up to third order):

$$\mathcal{D}_3^{(4)} : \begin{cases} \tau_{ttt} = 0, & \eta_{xx} - \eta_t = 0, & \xi_u = 0, \\ \eta_{uu} = 0, & \xi_{tt} = 0, & \xi_x - \frac{1}{2}\tau_t = 0, \\ \eta_{ux} + \frac{1}{2}\xi_t = 0, & \tau_u = 0, & \tau_x = 0, \\ \eta_{ut} + \frac{1}{4}\tau_{tt} = 0. \end{cases} \tag{10}$$

Thus $\mathcal{D}_3^{(4)}$ defines a 13-dimensional submanifold. Its Cartan characters are given by $\alpha_3^{(3)} = \alpha_3^{(2)} = 0$ and $\alpha_3^{(1)} = 2$.

As the last character does not vanish, we have an infinite-dimensional symmetry group. The question is whether an infinite-dimensional subgroup besides the superposition symmetry exists. But from a comparison of the Cartan characters of the determining system and the heat equation, we see that the non-vanishing ones are identical. Hence the infiniteness stems solely from the superposition symmetry.

In order to determine the number of remaining symmetries, we must compare the dimension of the heat equation and of its determining system as submanifolds, as the superposition symmetry manifests itself in the appearance of the equation $\eta_{xx} - \eta_t = 0$ in $\mathcal{D}_3^{(4)}$. As the involutive completion of the determining system is of third order, we must also prolong the heat equation to third order.[2] The number of remaining symmetries is hence given by $\dim \mathcal{D}_3^{(4)} - \dim \mathcal{R}_3$ and one readily computes that this yields in our example $13 - 7 = 6$. Thus we recover the well-known result that besides superposition the heat equation has a six-dimensional symmetry group.

Another simple application of such formal considerations consists in measuring how "special" solutions are that have been obtained with symmetry methods. It is well-known that in the case of partial differential equations, it is not possible to construct the general solution using symmetry theory. Thus one would like to know how large the part of the solution space is that one has constructed with the help of symmetry reductions.

[2]In principle, such an adjustment of the order would also have been necessary for the above comparison of the Cartan characters. In our example we could ignore this, as only the last Cartan character did not vanish and thus the Cartan characters are not affected by a prolongation.

For lack of space, we cannot go here into details. The basic idea is to perform an involution analysis of the combined system consisting of the original equation and the relevant invariant surface condition (see below). An example for the application of this technique to the case of generalised symmetries of a first order differential equation in Cauchy–Kovalevskaya form can be found in [20] (see also [8]).

5 Symmetry Reductions

The fundamental idea behind symmetry reductions is to construct solutions of a given differential equation $\mathcal{R}_q$ that are invariant under the action of a group. Given a symmetry of $\mathcal{R}_q$, one usually rewrites the differential equation in terms of group invariants. Geometrically, this corresponds to "factoring" the submanifold $\mathcal{R}_q$ by the group action. Computationally, this leads to equations in a lower number of independent variables.

For our purposes another approach is more useful. A solution remains invariant under the action of the one-parameter group generated by the vector field $X = \xi^i(\boldsymbol{x}, \boldsymbol{u})\partial_{x^i} + \eta^\alpha(\boldsymbol{x}, \boldsymbol{u})\partial_{u^\alpha}$, if and only if it is a solution of the corresponding *invariant surface condition*

$$\xi^i(\boldsymbol{x}, \boldsymbol{u})\frac{\partial u^\alpha}{\partial x^i} = \eta^\alpha(\boldsymbol{x}, \boldsymbol{u}), \qquad \alpha = 1, \dots, m. \tag{11}$$

Obviously, this represents a first order system of differential equations. In the sequel we will assume $\xi^n(\boldsymbol{x}, \boldsymbol{u}) \not\equiv 0$, so that we can normalise it to one. As the numbering of the independent variables x^i is arbitrary, this assumption amounts to a transversality condition: not all coefficients ξ^i may vanish identically or, equivalently, the action of the group generated by X must be transversal to the fibration of $\mathcal{E}$.

If the differential equation $\mathcal{R}_q$ has the local representation (1), then any solution of it invariant under the group generated by X must solve the augmented differential equation $\bar{\mathcal{R}}_q$ generated by the equations in (1) and in (11) (and the prolongations of the latter ones up to order q)

$$\bar{\mathcal{R}}_q \, : \left\{ \begin{array}{ll} \Phi^\tau(\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{p}) = 0\,, & \tau = 1, \dots, p\,, \\ \xi^i(\boldsymbol{x}, \boldsymbol{u})p_i^\alpha = \eta^\alpha(\boldsymbol{x}, \boldsymbol{u})\,, & \alpha = 1, \dots, m\,. \end{array} \right. \tag{12}$$

Even if $\mathcal{R}_q$ is a scalar equation or a system in Cauchy–Kovalevskaya form, the augmented equation $\bar{\mathcal{R}}_q$ will be an overdetermined equation. Hence the question whether it possesses solutions or equivalently whether $\mathcal{R}_q$ has group-invariant solution is non-trivial.

Following the discussion in Section 3, we assume throughout that $\mathcal{R}_q$ is an involutive equation. It is easy to see that (under the made transversality assumption) the system (11) is involutive, too. Hence integrability conditions in (12) may only arise from cross-derivatives between equations of these two subsystems. In a straightforward calculation one shows [15, 21] that these conditions can be concisely expressed in the form $X^{(q)}\Phi^\tau = 0$.

Assume now that the vector field X generates a one-parameter group of Lie point symmetries of $\mathcal{R}_q$. Then (5) entails that all integrability conditions vanish modulo the equations contained in (1). Hence we always find group-invariant solutions for transversal point symmetries.

Obviously, more possibilities for group-invariant solutions exist. Assume for example that the integrability conditions $X^{(q)}\Phi^\tau = 0$ do not vanish modulo (1) alone, thus X does *not* generate symmetries of $\mathcal{R}_q$ in the sense of Definition 5, but modulo (1) *and* (11). Then the augmented equation $\bar{\mathcal{R}}_q$ is formally integrable and has at least formal power series solutions. Hence $\mathcal{R}_q$ does possess solutions invariant under the group generated by X, although $\mathcal{R}_q$ itself does not remain invariant under the group. In fact, this is nothing but the *non-classical method* of Bluman and Cole [2].

But the non-classical method does not end here — in contrast to what is claimed in most articles on it. Olver and Rosenau [12] were probably the first to point this out introducing the concept of a *weak symmetry group* as a group of transformations such that invariant solutions exist. Later, this observation was again emphasised by Pucci and Saccomandi [15] and one of the present authors [21]. In the language of the formal theory we may formulate it as follows: it is *not* necessary that $\bar{\mathcal{R}}_q$ is formally integrable (as required by Bluman and Cole), but it must be possible to complete $\bar{\mathcal{R}}_q$ to a formally integrable equation without encountering inconsistencies.

We distinguish non-classical symmetries of different *levels*. The symmetries of level 0 are the classical ones and those of Bluman–Cole; here $\bar{\mathcal{R}}_q$ is already formally integrable. The vector field X generates a one-parameter group of symmetries of level s, if s is the minimal number of projections required to render $\bar{\mathcal{R}}_q$ formally integrable; i. e. the completion leads to an equation of the form $\bar{\mathcal{R}}^{(s)}_{q+r_s}$.

Let us consider a simple example of a non-classical symmetry of level 1 for the heat equation (a number of further examples can be found in [12, 15]). It is generated by the vector field $X = t\partial_t - x\partial_x - 3x^3\partial_u$. The augmented system (12) has now the following form (we omit to write down the prolongations of the second equation to second order)

$$\bar{\mathcal{R}}_2 \,:\, \begin{cases} u_t - u_{xx} = 0\,, \\ tu_t - xu_x + 3x^3 = 0\,. \end{cases} \tag{13}$$

$\bar{\mathcal{R}}_2$ is not formally integrable because of the appearance of the integrability condition $u_t = 6x$ (after simplification); hence X generates neither a classical point symmetry nor a non-classical symmetry in the sense of Bluman–Cole. Note that $X^{(2)}(u_t - u_{xx}) = 18x - u_t - 2u_{xx}$ and modulo the heat equation this is indeed equivalent to our integrability condition. As $\bar{\mathcal{R}}_2^{(1)}$ is formally integrable (in fact, even involutive), X generates a non-classical symmetry of level 1. The direct integration of the completed system $\bar{\mathcal{R}}_2^{(1)}$ is trivial and yields the corresponding group-invariant solutions of the heat equation: $u(x,t) = x^3 + 6xt + c$ with an arbitrary constant c.

6 Conclusions

The goal of this article was to exhibit some applications of the formal theory of differential equations within the realm of symmetry methods. We demonstrated the importance of completing a given differential equation before it is analysed, as otherwise one might overlook important symmetries. The formal analysis of the determining systems as studied in Section 4 is probably of interest only in some special situations; symmetry methods lose a lot of their charm, if one is not able to explicitly determine the symmetries.

Any solution of a differential equation $\mathcal{R}_q$ is invariant under some Lie group of point transformations. Thus, in principle, *any* solution can be constructed via non-classical symmetry reductions [12, Theorem 5]. Unfortunately, in practice this seems hardly possible. The determining systems of the non-classical method are non-linear and for higher levels the systems comprise less equations which are, in addition, more complicated. Thus the brute force approach of setting up the determining system and solving it explicitly will almost always fail. Furthermore, it is a priori not clear up to which level one needs non-classical symmetries.

Nevertheless, non-classical symmetries of higher level have already been applied successfully to compute explicit solutions. The method is probably most useful when one has already some ideas what groups might be of particular interest (for example for physical reasons). Then one can impose strong restrictions on the form of the generator X and has a chance to obtain manageable determining systems.

Apparently, the `DETools` library [1] of *MuPAD* represents currently the only available computer algebra software for setting up the determining systems for non-classical symmetries of higher level. It allows for prescribing special ansätze for the symmetry generators. The library also contains procedures for the completion to involution. Detailed descriptions of these packages will appear elsewhere.

The non-classical method may be extended to generalised symmetries. The role of the invariant surface condition is then played by the characteristic of the symmetry generator [12, Lemma 2]. Basically, this means that we augment $\mathcal{R}_q$ by some arbitrary differential equations. Thus the non-classical method for generalised symmetries coincides with what is often called the *method of differential constraints* (comprising also many other reduction methods [10]). Again the practical applicability of the approach is greatly reduced by the unsolved problem of finding reasonable ansätze for the differential constraints.

Our analysis underlines again the correctness of a remark made already in 1986 by Olver and Rosenau [11]:

> The most important conclusion to be drawn from this approach is that the unifying theme behind finding special solutions to partial differential equations is *not*, as is commonly supposed, group theory, but rather the more analytic subject of overdetermined systems of partial differential equations.

The importance of group theory lies in the fact that it permits the systematic construction of "good" differential constraints.

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