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An Efficient Algebraic Algorithm for the Geometric Completion to Involution

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Abstract We describe an adaption of a differential algebraic completion algorithm for linear systems of partial differential equations that allows us to deduce intrinsic differential geometric information like the number of prolongations and projections needed for the completion. This new hybrid algorithm represents a much more efficient realisation of the classical Cartan–Kuranishi completion than previous purely geometric ones.

A classical problem in geometric completion theory is the existence of δ-singular coordinate systems in which the algorithms do not terminate. We develop a new and a very simple criterion for δ-singularity based on a comparison of the Janet and the Pommaret division. This criterion can also be used for the direct construction of δ-regular coordinates.

Key words Partial differential equation, involution, completion, Cartan–Kuranishi theorem, involutive basis, δ-regularity

1 Introduction

Completion is a fundamental technique for dealing with general systems of differential equations. For such systems it is usually not possible to make any reasonable statements about their properties before they have been completed. One may even say that whatever one wants to do with such a system, it is probably better to start with a completion! The importance of completion is also evident from the fact that it may be considered as a generalisation to differential equations of the concept of Gröbner bases.

Completion is a rather general but somewhat vague concept. Intuitively, it is clear that one means something like constructing all integrability conditions of a given system of differential equations. However, the term "integrability condition" is not rigorously defined. So completion acquires a precise meaning only if it is embedded in a concrete approach to differential equations like for example differential algebra. Typically, this requires the introduction of further ingredients like rankings or involutive divisions.

An idea underlying most completion theories is the construction of formal power series solutions. Integrability conditions are then obstructions to this process. As there exist many different ways for building power series solutions, this leads to different notions of integrability conditions and thus of formal integrability, although this is rarely explicitly stated in the literature. In all approaches, the main difficulty is less to find some integrability condition, this is usually straightforward, but to detect that one has found all of them, i. e. a termination criterion for the completion process.

In most approaches to completion, one can distinguish between integrability conditions and further equations that are added during the completion process in order for the termination criterion to work. We will see examples for this later on. These additional equations typically lead to further special properties of the completed system besides the mere formal integrability. Thus different completion approaches lead in general to different results possessing different properties.

The probably oldest algorithmic approach to the completion of (partial) differential equations is the *Janet–Riquier theory* established around 1900 [23,24,33,40,56]. It is based on two fundamental ideas. Firstly, one divides the coefficients of the general power series solution into *parametric* and *principal* ones. The former ones represent the arbitrariness of the general solution; the latter ones are determined by the differential equations. This distinction is usually based on a *ranking* for selecting the *leading derivative* of each equation. Secondly, one assigns to each equation a subset of all independent variables as *multiplicative variables*. This requires the choice of what is nowadays called an *involutive division*.

A system of differential equations is complete or *passive*, if all principal derivatives are obtained by differentiating the differential equations with respect to the multiplicative variables only. Thus integrability conditions are non-multiplicative prolongations that cannot be reduced to zero using multiplicative prolongations. Concrete realisations of these ideas in computer algebra systems are described for example in [37,39,55].

The ideas behind the Janet–Riquier theory have recently been taken up by Blinkov, Gerdt and Zharkov in a series of papers [9,11,12,59] and put into a rigorous algebraic form. Using the close relation between linear systems of differential equations with constant coefficients and sets of polynomials, they showed that this theory can be used to compute a special kind of Gröbner bases: *involutive bases*. In fact, the Janet–Riquier theory provides us with an alternative to Buchberger's algorithm for the construction of Gröbner bases which seems to be highly competitive [14]. A brief review of some basic definitions of this approach will be given in Subsection 5.2.

A geometric alternative to the purely algebraic Janet–Riquier theory is the *Cartan–Kähler theory* based on exterior systems [3,6,25,35]. An important feature of this approach is that it can be formulated completely coordinate-free; one may even work with anholonomic frames which is a great advantage in some geometric problems. An implementation in the computer algebra system REDUCE was presented in [17].

If a problem possesses naturally a formulation in terms of differential equations, it is usually not of advantage to rewrite it first as an exterior system and then to analyse it with the Cartan–Kähler theory. The *formal theory* [7,15,36,43,49,54] offers an intrinsic geometric approach working directly on the differential equation considered as a submanifold of a jet bundle. Its equivalence to the Cartan–Kähler approach can be shown but is highly non-trivial [16].

In principle, the geometric approaches do not require any additional ingredients. Using a terminology apparently introduced by Lie the goal of the completion is an *involutive* system, a completely intrinsic concept, whereas rankings or involutive divisions are obviously coordinate dependent. The *Cartan–Kuranishi theorem* asserts that any system of differential equations can be completed by a finite number of projections and prolongations to an equivalent involutive system. Its proof is constructive and yields a concrete method for the completion. We described an implementation of it in the computer algebra system AXIOM in [42,44].

A well-known problem in geometric completion algorithms is the question of *δ-regular coordinates.* Any concrete computation takes place in some coordinate system. Certain coordinates make problems during the completion; in particular the termination criterion may fail to hold in them. This is often considered as a purely technical nuisance, which is, however, not quite correct. These coordinates correspond to intrinsically defined directions having a special meaning for the given differential equation; in the simplest case one is dealing with *characteristic* coordinates. We will assume most of the time that we are working in δ-regular coordinates. Section 5 contains a more detailed discussion of this issue and in particular presents a new and very cheap test for δ-singularity.

The main purpose of this article is to present for linear systems of differential equations an efficient realisation (developed in the diploma thesis of the first author [18]) of the Cartan–Kuranishi completion using ideas of the Janet–Riquier theory. Our new algorithm produces the full intrinsic information provided by the Cartan–Kuranishi theorem but does not require more steps than the Janet–Riquier theory. This is in marked contrast to previous implementations requiring many redundant computations, as all prolongations of lower order equations had to be determined explicitly. Already for small examples like the classical one of Janet (see Section 6) this leads to tens if not hundreds of equations that are not really needed.

We could mention here only some major lines of research. A number of further approaches to the completion of differential equations exists. In particular, there have been some attempts to extend Gröbner bases to differential equations with polynomial nonlinearities. One central problem of this generalisation lies in the fact that the ring of differential polynomials is no longer Noetherian which makes termination proofs very hard. Another big problem is that many alternative characterisations of algebraic Gröbner bases exist. Many of them can be generalised to differential polynomials, but they cease to be equivalent. Thus it is still an open question how one should define a differential Gröbner basis. As we concentrate in this article on linear systems, we do not discuss these difficult questions but only list some basic references [2, 5, 8, 26, 27, 30]. A related interesting approach is the theory of Wu [58] with applications in automated theorem proving.

Mansfield [31] studied the relation between her concept of differential Gröbner bases and the intrinsic definition of involution. She showed that such a basis is a local representation of a formally integrable equation and that one can determine via the syzygy module of the symbol the number of prolongations needed to complete a differential Gröbner basis to an involutive equation. Her approach uses a differential version of the Buchberger algorithm (sometimes called the Kolchin–Ritt algorithm) based on differential S-polynomials whereas we follow the Janet–Riquier strategy of non-multiplicative prolongations. Her syzygy criterion is a simple consequence of the combinatorial interpretation of involutive symbols.

There is no such thing as the "best" completion algorithm. As we already indicated above, different approaches produce different results. It depends on the concrete application which theory is suited best. For example, much of the recent interest in completion stems from the integration of the determining systems in Lie symmetry theory (see [22] for a survey of this subject). Here one hopes that the arising integrability conditions simplify the system so far that some heuristics find its closed form solution. While our algorithm surely can be applied to this problem, the additional intrinsic information it provides are usually of no interest. So a purely algebraic approach is probably better for this particular application.

In other fields the situation is different. In recent times, we have studied some applications in mathematical physics, mainly in geometric mechanics [45, 48, 53], where problems are usually formulated in an intrinsic manner and so it appears only natural to apply an intrinsic completion procedure. Somewhat surprisingly, the geometric approach has also proved very fruitful in the numerical analysis of overdetermined systems [21, 29, 47, 57]. In particular, we could show in [52] that obstructions to involution become integrability conditions after a semi-discretisation. This indicates that the starting point for the numerical integration of a system of differential equations should preferable be its involutive completion; mere formal integrability is not sufficient here. The use of the Janet–Riquier theory in this context was investigated in [38], however, less results were achieved.

Some definitions in this article, in particular of lexicographic rankings and involutive divisions, differ from the usual conventions in the literature. The standard definitions are obtained by simply inverting the order of the independent variables: $x^1, \dots, x^n \longleftrightarrow x^n, \dots, x^1$. Our definitions fit better with the conventions in the formal theory of differential equations.

This article is organised as follows. The next section introduces in an intrinsic language the notion of an involutive differential equation and describes the Cartan–Kuranishi completion algorithm in abstract form. Section 3 discusses some aspects of its computational realisation. Our new algorithm is developed in the following section and we prove that for linear systems it represents a realisation of the Cartan–Kuranishi completion. Section 5 discusses the problem of δ-regularity and presents a new simple criterion for detecting δ-singular coordinates. The next section studies a few concrete examples in detail. Finally, some conclusions are given.

2 Geometric Theory of Differential Equations

The formal theory represents a powerful geometric framework for analysing differential equations based on the jet bundle formalism [41]. In this article we can only briefly review some basic notions; for more details we refer to the literature, see e. g. [7,36,49]. The description in this section is completely intrinsic and we do not discuss any computational aspects.

2.1 Jet Bundles and Differential Equations

Let $\pi : \mathcal{E} \to \mathcal{B}$ be a fibred manifold. Geometrically, the first-order jet bundle $\pi^1_0 : J_1\mathcal{E} \to \mathcal{E}$ is most easily described as an *affine* bundle where the fibre over a point $\xi = (x, u) \in \mathcal{E}$ is the affine space

$$(J_1\mathcal{E})_\xi = \left\{\gamma \in T^*_x\mathcal{B} \otimes T_\xi\mathcal{E} \mid T\pi \circ \gamma = \mathrm{id}_{T_x\mathcal{B}}\right\} \tag{1}$$

modelled on the vector space $T^*_x\mathcal{B} \otimes V_\xi\mathcal{E}$. As $J_1\mathcal{E}$ may again be considered as a fibred manifold over $\mathcal{B}$, we can iterate this construction. Higher-order jet bundles are then obtained by identifying in an obvious manner $J_{q+r}\mathcal{E}$ with a subbundle of $J_r(J_q\mathcal{E})$. For any $q, r \geq 0$ they possess bundle structures $\pi^{q+r}_q : J_{q+r}\mathcal{E} \to J_q\mathcal{E}$ where we set $J_0\mathcal{E} = \mathcal{E}$ and the structure of a fibred manifold $\pi^q : J_q\mathcal{E} \to \mathcal{B}$.

A section $\sigma : \mathcal{B} \to \mathcal{E}$ is *prolonged* to a section $j_1(\sigma) : \mathcal{B} \to J_1\mathcal{E}$ by setting $j_1(\sigma)(x) = (\sigma(x), T_x\sigma)$. As above, this construction can naturally be extended to higher-order jet bundles. We define a *differential equation*[1] (of order q) as a fibred submanifold $\mathcal{R}_q$ of the bundle $\pi^q : J_q\mathcal{E} \to \mathcal{B}$. A (local) section $\sigma : \mathcal{U} \subseteq \mathcal{B} \to \mathcal{E}$ is a *(local) solution* of the equation $\mathcal{R}_q$, if $\mathrm{Im}\, j_q(\sigma) \subseteq \mathcal{R}_q$.

There exist two natural geometric operations which can be applied to a differential equation $\mathcal{R}_q$: *projection* and *prolongation*. The first one is induced by the canonical projections between jet bundles of different order. If $\mathcal{R}_{q+r}$ is an equation of order $q + r$, we define the projected equation of order q by $\mathcal{R}^{(r)}_q = \pi^{q+r}_q(\mathcal{R}_{q+r})$. Conversely, a qth-order differential equation $\mathcal{R}_q$ can be prolonged to one of order $q + r$: $\mathcal{R}_{q+r} = J_r(\mathcal{R}_q) \cap J_{q+r}\mathcal{E}$ (the intersection is understood to take place in $J_r(J_q\mathcal{E})$ with obvious identifications). In general, we cannot expect that either projection or prolongation

[1] We do not distinguish between a scalar equation and a system.

leads again to a fibred manifold. However, for simplicity, we will make this assumption in the sequel, i. e. we restrict to so-called *regular* equations. In particular, linear equations are always regular.

For our purposes, equations of the form $\mathcal{R}^{(s)}_{q+r} = \pi^{q+r+s}_{q+r}(\mathcal{R}_{q+r+s})$, i. e. equations that are first prolonged $r+s$ times and then projected back s times, are especially important. Note that in general prolongation and projection are not inverse: we only get that $\mathcal{R}^{(s)}_{q+r} \subseteq \mathcal{R}_{q+r}$. A proper subset indicates the presence of *integrability conditions.*

Definition 1 *The differential equation $\mathcal{R}_q$ is* formally integrable*, if the equality $\mathcal{R}^{(1)}_{q+r} = \mathcal{R}_{q+r}$ holds for all $r \geq 0$.*

In other words, $\mathcal{R}_q$ is formally integrable, if at no order of prolongation integrability conditions occur. The name stems from the fact that for such equations it is straightforward to construct formal power series solutions. Unfortunately, no finite criterion for formal integrability is known so far.

2.2 Symbol and Spencer Cohomology

A very important property of jet bundles is that for any order $q > 0$ the projection by only one order $\pi^q_{q-1} : J_q\mathcal{E} \to J_{q-1}\mathcal{E}$ defines an *affine* bundle modelled on the vector bundle $S_qT^*\mathcal{B} \otimes V\mathcal{E}$ (considered as bundle over $J_{q-1}\mathcal{E}$ via appropriate pull-backs) with S_q denoting the q-fold symmetric product. This leads naturally to the concept of the *symbol*[2] $\mathcal{M}_q$ of a differential equation $\mathcal{R}_q$. It is a family of vector spaces over $\mathcal{R}_q$; at a point $\xi \in \mathcal{R}_q$ we define $(\mathcal{M}_q)_\xi = V^{(q-1)}_\xi \mathcal{R}_q \subseteq V^{(q-1)}_\xi J_q\mathcal{E}$ where $V^{(q-1)}_\xi J_q\mathcal{E} = \ker T_\xi \pi^q_{q-1}$ is the vertical space with respect to the projection π^q_{q-1}. We may thus identify the symbol with a subspace of $S_qT^*\mathcal{B} \otimes V\mathcal{E}$. For simplicity, we will assume in the sequel that the symbol is not just a family of vector spaces but actually a vector bundle over $\mathcal{R}_q$.

Consider the map $\delta : S_{r+1}T^*\mathcal{B} \to T^*\mathcal{B} \otimes S_rT^*\mathcal{B}$ defined by the composition of the natural inclusion $S_{r+1}T^*\mathcal{B} \hookrightarrow T^*\mathcal{B} \otimes \bigotimes^r T^*\mathcal{B}$ with the canonical projection $T^*\mathcal{B} \otimes \bigotimes^r T^*\mathcal{B} \to T^*\mathcal{B} \otimes S_rT^*\mathcal{B}$. By wedging with $\Lambda_sT^*\mathcal{B}$ (the s-fold exterior product of $T^*\mathcal{B}$) and tensoring with $V\mathcal{E}$ we can extend δ to a map $\Lambda_sT^*\mathcal{B} \otimes S_{r+1}T^*\mathcal{B} \otimes V\mathcal{E} \to \Lambda_{s+1}T^*\mathcal{B} \otimes S_rT^*\mathcal{B} \otimes V\mathcal{E}$. This leads to the δ-sequences

$$\begin{aligned} 0 \longrightarrow S_rT^*\mathcal{B} \otimes V\mathcal{E} \xrightarrow{\delta} T^*\mathcal{B} \otimes S_{r-1}T^*\mathcal{B} \otimes V\mathcal{E} \xrightarrow{\delta} \cdots \\ \cdots \xrightarrow{\delta} \Lambda_sT^*\mathcal{B} \otimes S_{r-s}T^*\mathcal{B} \otimes V\mathcal{E} \xrightarrow{\delta} \cdots \\ \cdots \xrightarrow{\delta} \Lambda_nT^*\mathcal{B} \otimes S_{r-n}T^*\mathcal{B} \otimes V\mathcal{E} \longrightarrow 0 \end{aligned} \tag{2}$$

[2] Note that while our definition of a symbol is closely related to the standard one in text books on differential equations, it is not the same! Written out in local coordinates, our symbol corresponds to a much larger matrix; the classical (principal) symbol is obtained by a contraction with a one-form $\chi \in T^*\mathcal{B}$.

where we set $S_iT^*\mathcal{B} = 0$ for $i < 0$ and where $n = \dim \mathcal{B}$. The *formal Poincaré lemma* states that these sequences are exact for all $r \geq 0$.

The prolongation of a symbol $\mathcal{M}_q \subseteq S_qT^*\mathcal{B} \otimes V\mathcal{E}$ is directly computed as the intersection $\mathcal{M}_{q+r} = (S_rT^*\mathcal{B} \otimes \mathcal{M}_q) \cap (S_{q+r}T^*\mathcal{B} \otimes V\mathcal{E})$ (which is understood to take place in $\bigotimes^{q+r} T^*\mathcal{B} \otimes V\mathcal{E}$). Setting $\mathcal{M}_i = 0$ for $i < 0$ and $\mathcal{M}_i = S_iT^*\mathcal{B} \otimes V\mathcal{E}$ for $0 \leq i < q$ the δ-sequence (2) may be restricted to a sequence

$$0 \longrightarrow \mathcal{M}_{q+r} \xrightarrow{\delta} T^*\mathcal{B} \otimes \mathcal{M}_{q+r-1} \xrightarrow{\delta} \cdots \xrightarrow{\delta} \Lambda_n T^*\mathcal{B} \otimes \mathcal{M}_{q+r-n} \longrightarrow 0 \quad (3)$$

which is still a complex but in general no longer exact. Its (bigraded) cohomology is called the *Spencer cohomology* of the symbol $\mathcal{M}_q$. We denote by $H^{s,r}(\mathcal{M}_q)$ the cohomology group at $\Lambda_s T^*\mathcal{B} \otimes \mathcal{M}_r$.

Definition 2 *The symbol $\mathcal{M}_q$ is* involutive*, if $H^{s,q+r}(\mathcal{M}_q) = 0$ holds for all $0 \leq s \leq n$ and all $r \geq 0$. The differential equation $\mathcal{R}_q$ is* involutive*, if it is formally integrable and if its symbol $\mathcal{M}_q$ is involutive.*

The above introduction of the map δ and thus of the Spencer cohomology appears rather ad hoc. Another approach embedding the Spencer cohomology into standard constructions in homological algebra and clarifying its meaning will appear in [28].

Let $\{e_1, \ldots, e_n\}$ be an ordered basis of the tangent space $T\mathcal{B}$. We define for $1 \leq k \leq n$ the subspaces

$$\begin{aligned} \mathcal{M}_{q,k} = \big\{\rho \in \mathcal{M}_q \mid \rho(e_i, v_1, \ldots, v_{q-1}) = 0\,, \\ \forall\, 1 \leq i \leq k\,,\ \forall\, v_1, \ldots, v_{q-1} \in T\mathcal{B}\big\} \end{aligned} \quad (4)$$

and $\mathcal{M}_{q,0} = \mathcal{M}_q$. We call the basis *$\delta$-regular* for the symbol $\mathcal{M}_q$, if

$$\dim \mathcal{M}_{q+1} = \sum_{k=0}^{n-1} \dim \mathcal{M}_{q,k}\ . \quad (5)$$

One can show that a symbol is involutive, if and only if δ-regular bases exist for it. Given such a basis for the symbol $\mathcal{M}_q$, we can compute its *Cartan characters* $\alpha_q^{(k)}$ as the differences

$$\alpha_q^{(k)} = \dim \mathcal{M}_{q,k-1} - \dim \mathcal{M}_{q,k}\ . \quad (6)$$

We introduce the *Hilbert function* $h(s)$ of the differential equation $\mathcal{R}_q$ as the number of arbitrary coefficients of order s of the general formal power series solution of $\mathcal{R}_q$. It probably represents the most natural and useful measure for the size of the formal solution space. If $\mathcal{R}_q$ is involutive, the Hilbert function is a polynomial $H(s)$ for $s \geq q$ given by $H(s) = \dim \mathcal{M}_s$. One can derive a closed form expression for the Hilbert polynomial in terms of the Cartan characters of $\mathcal{M}_q$:

$$H(q+r) = \sum_{k=1}^{n} \alpha_q^{(k)} \binom{q+r}{k}\,, \quad \forall r \geq 0\ . \quad (7)$$

It is important to note that all these constructions based on the symbol are to be understood pointwise on $\mathcal{R}_q$. In general, one must expect that the dimensions of the cohomology classes vary from point to point and so it may happen that $\mathcal{R}_q$ is involutive at some points and not at other ones. Again we will assume for simplicity that this does not happen.

Involutive equations possess a number of special properties. An important one is the existence and uniqueness of analytical solutions for the (non-characteristic) Cauchy problem, the *Cartan–Kähler theorem*. It generalises the Cauchy–Kovalevskaya theorem covering only normal equations.

2.3 Cartan–Kuranishi Completion

Many differential equations appearing in applications are not involutive. However, one can show that under some mild regularity conditions every equation can be completed to an equivalent involutive one. Equivalent means here that the formal solution space remains unchanged (actually, all smooth solutions remain unaffected).

Theorem 1 (Cartan–Kuranishi) *Two integers $r, s \geq 0$ exist for every regular differential equation $\mathcal{R}_q$ such that $\mathcal{R}^{(s)}_{q+r}$ is involutive.*

The proof is based on three fundamental properties of involutive symbols which we summarise in the following proposition without proof (a coordinate version of the first part is a simple corollary to Proposition 7 below).

Proposition 1

(i) *Every symbol $\mathcal{M}_q$ becomes involutive after a finite number of prolongations; i. e. there exists an integer $r \geq 0$ such that $\mathcal{M}_{q+r}$ is involutive.*
(ii) *Let the symbol $\mathcal{M}_q$ of the differential equation $\mathcal{R}_q$ be involutive. If $\mathcal{R}^{(1)}_q = \mathcal{R}_q$, then $\mathcal{R}_q$ is involutive.*
(iii) *If the symbol $\mathcal{M}_q$ of the differential equation $\mathcal{R}_q$ is involutive, then the equality $\bigl(\mathcal{R}^{(1)}_q\bigr)_{+1} = \mathcal{R}^{(1)}_{q+1}$ holds.*

Using these results, we can even provide a constructive proof of the Cartan–Kuranishi theorem, namely a concrete completion method. Algorithm 1 gives its pseudo code formulation.

The algorithm consists of two nested loops. The inner one (lines 5–7) prolongs the equation until the symbol becomes involutive; the variable r is a counter for the total number of prolongations needed in the completion. By Part (i) of the proposition above, this loop always terminates.

Lines 8–11 realise the criterion for an involutive equation of Part (ii). We noted above that the definition of formal integrability requires infinitely many checks whether $\mathcal{R}^{(1)}_{q+r} = \mathcal{R}_{q+r}$ for all $r \geq 0$. But if the symbol $\mathcal{M}_q$ is involutive, it suffices to check this condition only for $r = 0$: if it holds there, it will hold for all $r > 0$, too!

Algorithm: Cartan–Kuranishi completion
Input: differential equation $\mathcal{R}_q$
Output: involutive differential equation $\mathcal{R}_{q+r}^{(s)}$

```
begin
    r := 0; s := 0;
    repeat
        equationInvolutive := TRUE;
        while M_{q+r}^{(s)} is not involutive do
            r := r + 1;
        end_while;
        if R_{q+r}^{(s+1)} ⊊ R_{q+r}^{(s)} then
            s := s + 1;
            equationInvolutive := FALSE;
        end_if;
    until equationInvolutive end_repeat;
    return R_{q+r}^{(s)};
end;
```

Algorithm 1 *Cartan–Kuranishi completion*

The termination of the outer loop follows by a simple Noetherian argument. Part (iii) of the proposition above guarantees that the output of the algorithm is indeed of the form $\mathcal{R}_{q+r}^{(s)}$, as projections are always performed when the symbol is involutive (the total number of projections is counted by the variable s). Together with Part (ii) this ensures the correctness of the algorithm.

3 Computational Realisation

The last section gave a brief introduction to involutive differential equations and outlined in an abstract language a simple completion algorithm based on the Cartan–Kuranishi theorem. In this section we discuss how this theory can be made effective and applied to concrete equations.

3.1 Formal Integrability

As a first step, we need local coordinates on all used manifolds. A natural choice for a coordinate system on the jet bundle $J_q\mathcal{E}$ is $(x^i; u^\alpha; p^\alpha_\mu)$ where x^i with $1 \leq i \leq n$ denote the independent variables, u^α with $1 \leq \alpha \leq m$ the dependent variables and p^α_μ with multi indices $\mu = [\mu_1, \dots, \mu_n] \in \mathbb{N}_0^n$ (where $|\mu| = \sum_{i=1}^n \mu_i \leq q$) represent derivatives of the dependent variables with respect to the independent ones. This choice also reflects the interpretation of a point in $J_q\mathcal{E}$ as a truncated power series: x^i marks the expansion point, u^α the values of the expanded functions at this point and p^α_μ the values of their derivatives up to order q.

Locally, any fibred submanifold $\mathcal{R}_q \subseteq J_q\mathcal{E}$ may be described as the zero[3] set of a fibred map $\Phi : J_q\mathcal{E} \to \mathcal{E}'$ for some vector bundle $\pi' : \mathcal{E}' \to \mathcal{B}$. Thus for any local computation we assume that the differential equation is represented in the form

$$\mathcal{R}_q \;:\; \left\{ \; \Phi^\tau(x^i, u^\alpha, p^\alpha_\mu) = 0 \; , \qquad \tau = 1, \dots, p \right. \tag{8}$$

and we recover the traditional form of a system of differential equations.

In order to construct such a local representation for the prolonged equation $\mathcal{R}_{q+1}$, we need the *formal derivative*

$$D_i\Phi^\tau = \frac{\partial \Phi^\tau}{\partial x^i} + \sum_{\alpha=1}^{m} \frac{\partial \Phi^\tau}{\partial u^\alpha}\, p^\alpha_{1_i} + \sum_{\alpha=1}^{m} \sum_{|\mu|\leq q} \frac{\partial \Phi^\tau}{\partial p^\alpha_\mu}\, p^\alpha_{\mu+1_i} \tag{9}$$

where 1_i represents the multi index where all entries are zero besides the ith which is one. Note that $D_i\Phi^\tau$ is a function defined on $J_{q+1}\mathcal{E}$ and that it is quasi-linear, i. e. linear in the derivatives p^α_μ with $|\mu| = q + 1$. Now we get for the prolonged equation the local representation

$$\mathcal{R}_{q+1} \;:\; \left\{ \begin{array}{ll} \Phi^\tau(x, u, p) = 0 \; , & \tau = 1, \dots, p \; , \\ D_i\Phi^\tau(x, u, p) = 0 \; , & i = 1, \dots, n \; . \end{array} \right. \tag{10}$$

Similarly, a local representation of $\mathcal{R}_{q+r}$ is obtained by adding all formally differentiated equations $D_\mu\Phi^\tau = 0$ with $|\mu| \leq r$. Here we use again standard multi index notation: $D_\mu = (D_1)^{\mu_1} \cdots (D_n)^{\mu_n}$.

For arbitrary equations it may be very difficult to determine a local representation of a projected equation, as this requires the elimination of the derivatives of highest order. Whether or not this can be done effectively depends on the structure of the functions Φ^τ: if they are linear, Gaussian elimination is sufficient; for polynomials Gröbner techniques may be applied; for general functions no algorithm is known.

Fortunately, our completion algorithm requires only to project prolonged equations. As the formal derivative (9) always produces functions which are linear in the highest-order derivatives, the construction of integrability conditions is a linear operation. But to decide whether such a condition represents truly a new equation is difficult generally, as we must decide whether it is algebraically independent of the remaining equations.[4] Again algorithmic solutions exist only in the linear and the polynomial case.

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[3] We assume here that zero is a regular value of the map Φ.

[4] At first sight, this might not appear as a big problem: one "only" has to compute the rank of a Jacobian. The difficulty arises from the fact that this rank has to be determined *on* the manifold described by the equations, i. e. the decision whether or not a pivot is zero has to be made modulo these equations.

3.2 Involutive Symbols

The symbol $\mathcal{M}_q$ of the differential equation $\mathcal{R}_q$ locally represented by (8) is the solution space of the following linear system of equations on the vector space $S_qT^*\mathcal{B} \otimes V\mathcal{E}$

$$\mathcal{M}_q \,:\, \left\{ \sum_{\alpha=1}^{m} \sum_{|\mu|=q} \frac{\partial \Phi^\tau}{\partial p_\mu^\alpha}\, v_\mu^\alpha = 0\,, \qquad \tau = 1, \dots, p\,. \right. \tag{11}$$

Note that the summation is only over the derivatives of order q; this reflects the fact that the symbol is defined as the vertical space with respect to the fibration π^q_{q-1}. The unknowns v_μ^α represent fibre coordinates on the vector bundle $S_qT^*\mathcal{B} \otimes V\mathcal{E}$ (with respect to the basis $dx^\mu \otimes \frac{\partial}{\partial u^\alpha}$).

In the last section we introduced involutive symbols in terms of the Spencer cohomology, as this provides an intrinsic definition. However, for concrete calculations it is not so well suited. Thus one typically uses a combinatorial approach instead.

Let M_q be the matrix of the linear system (11). It has p rows, one for each equation in (8), and $\dim S_qT^*\mathcal{B} \otimes V\mathcal{E} = m\binom{n-1+q}{n-1}$ columns, one for each coordinate v_μ^α. We now sort the columns in a special manner. The *class* $\mathrm{cls}(\mu)$ of the multi index $\mu = [\mu_1, \dots, \mu_n]$ is defined as the smallest i such that μ_i is not zero.[5] If v_μ^α and v_ν^β are such that $\mathrm{cls}(\mu) > \mathrm{cls}(\nu)$, then we sort the column corresponding to v_μ^α to the left of that for v_ν^β. If $\mathrm{cls}(\mu) = \mathrm{cls}(\nu)$, the order of the columns does not matter.

A simple way to achieve this sorting is to use a *class respecting ranking*, i. e. a ranking where for derivatives of the same order a derivative of higher class is always greater than one of lower class. An obvious example is the *(degree) reverse lexicographic ranking*[6] which we define as follows: derivatives of higher order are greater than those of lower order and for derivatives of the same order $p_\mu^\alpha \prec p_\nu^\beta$, if the first non-vanishing entry of $\mu - \nu$ is positive or if $\mu = \nu$ and $\alpha < \beta$.

Next we determine a row echelon form of M_q (using only row operations) and analyse the positions of the pivot elements. We define $\beta_q^{(k)}$ as the number of rows where the pivot sits in a column corresponding to a variable of

[5] For consistency we define the class of the zero index $[0, \dots, 0]$ as n.

[6] In the case of one dependent variable, the reverse lexicographic ranking is actually the *only* class respecting ranking. Indeed, the reverse lexicographic ranking is uniquely characterised by the following property: if the leading term $lt(f)$ of any homogeneous polynomial $f \in k[x^1, \dots, x^n]$ is contained in the ideal $\langle x^1, \dots, x^k\rangle$ generated by the first k variables, then $f \in \langle x^1, \dots, x^k\rangle$ [1, Exercise 1.4.9]. But one easily shows that this property is equivalent to the requirement that the ranking respects classes. In the case of several dependent variables the only class respecting rankings are *TOP* lifts of the reverse lexicographic rankings comparing first the multi indices μ, ν by the reverse lexicographic ranking and then breaking ties by comparing α, β (see [1, Definition 3.5.2]).

class k; these numbers are sometimes called the *indices* of the symbol. The symbol $\mathcal{M}_q$ is involutive, if and only if

$$\operatorname{rank} M_{q+1} = \sum_{k=1}^{n} k\beta_q^{(k)} \ . \tag{12}$$

Here M_{q+1} is the matrix associated with the prolonged symbol $\mathcal{M}_{q+1}$. Condition (12) is equivalent to (5). Indeed, the indices $\beta_q^{(k)}$ and the Cartan characters $\alpha_q^{(k)}$ are related by

$$\alpha_q^{(k)} = \binom{q+n-k-1}{q-1} - \beta_q^{(k)} \ . \tag{13}$$

$\beta_q^{(k)}$ gives the number of principal derivatives of order q and class k; $\alpha_q^{(k)}$ does the same for the parametric derivatives.

This approach has one obvious caveat: the values of the $\beta_q^{(k)}$ may depend on the chosen coordinate system. Thus (12) cannot be intrinsic. We will discuss this problem in more detail in Section 5. Here we only state that if our coordinates x^i are δ-regular, i.e. if the tangent vectors ∂_{x^i} form a δ-regular basis of $T^*\mathcal{B}$, the combinatorial and the homological approach are equivalent (for a proof see [36]). Fortunately, generic coordinates are δ-regular. In the sequel we will always assume that we are working in such a coordinate system.

As the determination of a row echelon form requires only linear operations, we may assume without loss of generality for linear differential equations that the equations $\Phi^\tau = 0$ in the local representation (8) are chosen such that (11) yields the symbol matrix directly in this *solved form*. In our algorithm this will always be the case. The pivot corresponds then to the *leading derivative* of the equation.

3.3 Combinatorial Interpretation

The definition of the indices and criterion (12) for an involutive symbol look rather bizarre at first sight. But a simple combinatorial interpretation exists having its origin in the Janet–Riquier theory and leading directly to the theory of *involutive bases*. We present here only some elementary aspects of this theory needed for our algorithm and for understanding (12); a few more details can be found in Subsection 5.2.

We associate to each equation in the symbol matrix M_q a subset of the independent variables x^i, its *multiplicative variables*. It is selected by a simple rule. Let the symbol matrix be in solved form as described above. If the leading term of an equation is v^α_μ, its multiplicative variables are $x^1, \dots, x^k$ where $k = \operatorname{cls}(\mu)$ is the class of the multi index μ. In the language of involutive bases this defines the *Pommaret division*.

Now we study what happens if we prolong each equation in the symbol only with respect to its multiplicative variables. One easily sees by just looking at the leading terms that all equations produced this way are independent and again in solved form. The question is whether or not this produces already the full matrix M_{q+1} of the prolonged symbol. If yes, $\mathcal{M}_q$ is involutive. Now the criterion (12) becomes obvious: on the left hand side we have the number of independent rows in M_{q+1}; the right hand side gives the number of rows produced by prolonging only with respect to the multiplicative variables. For general symbols the left hand side is larger; for involutive symbols both sides are equal.

Mansfield's syzygy criterion [31] is a simple consequence of this interpretation. If a symbol is involutive, a generating set of the syzygy module of the symbol equations is obtained by expressing all *non-multiplicative* prolongations as linear combinations of the multiplicative ones. Obviously, all these syzygies are of first order.

Two simple examples may clarify these combinatorial ideas. Consider first the following second-order symbol consisting of two equations

$$\mathcal{M}_2 \;:\; \Big\{\; v_{x^2x^2} = 0\;, \quad v_{x^1x^2} = 0\;. \tag{14}$$

Obviously, it is in solved form. The first equation is of class 2; the second one of class 1. Thus following our rule we prolong the second equation only with respect to x^1 and not with respect to x^2. But this has no consequences: the equation $v_{x^1x^2x^2} = 0$ in the prolonged symbol $\mathcal{M}_3$ can also be obtained by prolonging the first equation with respect to x^1. So in fact our rule has resolved this ambiguity and we have an involutive symbol: $\operatorname{rank} M_3 = 3$ and the indices of $\mathcal{M}_2$ are $\beta_2^{(1)} = \beta_2^{(2)} = 1$; thus (12) is satisfied.

Now we consider another second-order symbol:

$$\mathcal{M}_2 \;:\; \Big\{\; v_{x^2x^2} = 0\;, \quad v_{x^1x^1} = 0\;. \tag{15}$$

It only differs slightly in the second equation from (14). But this has a significant effect. The classes of the equations are the same as above. However, prolonging the second equation with respect to its non-multiplicative variable x^2 yields now the equation $v_{x^1x^1x^2} = 0$ which cannot be obtained from the first equation. Thus this symbol is not involutive: while the indices have the same values as above, $\operatorname{rank} M_3 = 4$ this time.

For easier reference, we summarise this approach to involution in form of a proposition. We formulate it at the level of the differential equation and not the symbol, as in this form we will need it later.

Proposition 2 *Let the local representation (8) of the differential equation $\mathcal{R}_q$ be in solved form and the coordinate system δ-regular. The symbol $\mathcal{M}_q$ is involutive, if and only if all independent equations of order $q+1$ in a local representation of the prolongation $\mathcal{R}_{q+1}$ can be obtained by formally differentiating each equation in (8) of order q with respect to its multiplicative variables only.*

It is important to note that this proposition concerns only the equations of order $q+1$ in $\mathcal{R}_{q+1}$, as only these equations contribute to the prolonged symbol $\mathcal{M}_{q+1}$. No statement is made about lower-order equations, so it may well be that integrability conditions occur during the prolongation.

4 A Hybrid Algorithm

We want to develop for linear differential equations an effective algebraic realisation of the Cartan–Kuranishi completion algorithm based on the combinatorial ideas of the last section. A naive realisation is immediate: one explicitly determines local representations for all differential equations $\mathcal{R}^{(s)}_{q+r}$ and symbols $\mathcal{M}^{(s)}_{q+r}$ appearing in the course of the algorithm. For linear or polynomial equations it is comparatively straightforward to perform all required operations algorithmically using Gröbner bases. Our earlier works [42,44] followed this strategy.

Of course, such a brute force approach is not very efficient; in general, many unnecessary prolongations (which are usually rather expensive to compute) are performed and one has to deal with rather large matrices (see the Janet example in Section 6). Pure algebra in form of involutive bases leads to fairly fast algorithms for linear equations [9], but all geometric information is lost. For example, it is not possible to determine the number of performed prolongations and projections. We combine the geometric and the algebraic approach, so that we maintain the efficiency of pure algebra and still obtain the full geometric information.

4.1 Linear Differential Equations

In the sequel we will assume that the coefficients of our equations stem from a function field $\Bbbk$ in that all required operations (arithmetics and differentiations) can be performed effectively and that contains the coordinates x^i. Thus a typical example of $\Bbbk$ is a finite algebraic extension of $\mathbb{Q}(x^1,\dots,x^n)$.

Definition 3 *A* system of linear partial differential equations F *of order* q *with* n *independent variables* x^i *and* m *dependent variables* u^α *is given by* p *equations of the form*[7]

$$\Phi^\tau(x,p) = \sum_{\alpha=1}^{m} \sum_{0\leq|\mu|\leq q} a^\tau_{\alpha,\mu}(x)p^\alpha_\mu - c^\tau(x) = 0\ , \qquad \tau = 1,\dots,p \qquad (16)$$

where the coefficients $a^\tau_{\alpha,\mu}$ *and* c^τ *are elements of* $\Bbbk$. *Such a system yields a local representation of a (linear) differential equation* $\mathcal{R}_q$ *in the sense of Sections 2 and 3.*

[7] Here and in the sequel we use the convention that $p^\alpha_{[0,\dots,0]} = u^\alpha$.

We strictly distinguish between a *differential equation*, which is as a submanifold $\mathcal{R}_q$ a *geometric* object, and its algebraic representation as a *system* of equations. In the sequel, we drop the 0 on the right and call the remaining left hand side of an equation a *row* of the system. After fixing an orderly class respecting ranking $\prec$ on the jet coordinates $\{p^\alpha_\mu\}$, we define $\mathrm{ld}(f)$, the *leading derivative* of a row f, as the highest p^α_μ effectively occurring in f with respect to this ranking. Notions like *order*, *class*, *multiplicative* and *non-multiplicative variables* of a row f are defined in terms of the corresponding attributes of $\mathrm{ld}(f)$.

One may view F as an inhomogeneous system of linear equations with the unknowns p^α_μ, the $p \times \binom{n+q}{q}$ coefficient matrix $A = (a^\tau_{\alpha,\mu})$, and the right hand sides $\mathbf{c} = (c^\tau)$. Then $\mathcal{R}_q$ coincides with the solution space of this system. Hence we can perform elementary row operations on $(A, \mathbf{c})$ while leaving $\mathcal{R}_q$ unchanged. It is convenient to formulate this process in the language of reductions and normal forms.

Definition 4 *A row* $f = a(x)\mathrm{ld}(f) + \sum_{p^\alpha_\mu \prec \mathrm{ld}(f)} a_{\alpha,\mu}(x) p^\alpha_\mu - c(x)$ *is* (head) reducible *modulo a row* $g = b(x)\mathrm{ld}(g) + \sum_{p^\alpha_\mu \prec \mathrm{ld}(g)} b_{\alpha,\mu}(x) p^\alpha_\mu - d(x)$*, if and only if* $\mathrm{ld}(f) = \mathrm{ld}(g)$*. We have the* reduction $f \xrightarrow{g} h = \mathrm{rem}(f,g)$ *where*

$$\mathrm{rem}(f,g) := \sum_{p^\alpha_\mu \prec \mathrm{ld}(f)} \left(a_{\alpha,\mu}(x) - \frac{a(x)}{b(x)} b_{\alpha,\mu}(x) \right) p^\alpha_\mu - d(x) + \frac{a(x)}{b(x)} c(x)\ . \quad (17)$$

f reduces to h modulo a set F of rows, if there exists a row $g \in F$ such that $h = \mathrm{rem}(f,g)$. f is in normal form *modulo the set F, if f is not reducible modulo F. The set F is called* (head) autoreduced*, if each $f \in F$ is in normal form modulo $F \setminus \{f\}$.*

It is obvious how one can obtain a normal form of f modulo a set $F = \{g_1, \dots, g_\ell\}$ by repeatedly reducing f by elements g_k of F, until this is no longer possible. This process terminates, since the leading derivative becomes smaller with respect to $\prec$ in each step, and yields a representation

$$f = \sum_{k=1}^{\ell} a_k(x) g_k + h \quad (18)$$

where h is in normal form modulo F. We write $h = \mathrm{rem}(f, F)$. We should stress that we perform only *head reductions*, since this is sufficient for our aims, and that thus a normal form is not unique.

As we admit inhomogeneous equations, it may happen that a reduction leads to a row of the form $f = c(x)$ with $c \in \Bbbk$. In this case we stop all computations, as this implies that our system is inconsistent: an equation of the form $c(x) = 0$ restricts the values of the independent variables which is not allowed for a differential equation.

In the course of our algorithm, we are interested in computing a *triangular form* of the system F currently under consideration. This is achieved

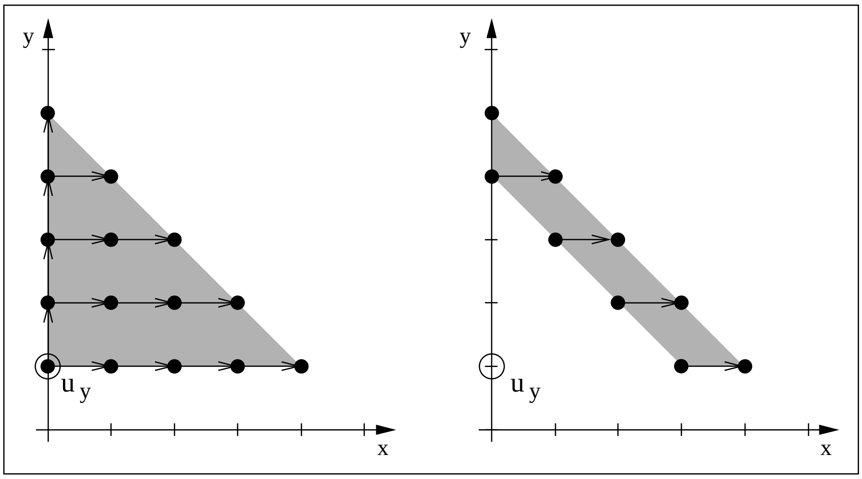

Figure 1 *Left:* truncated cone of a real row $\hat{C}_4\big((u_y)_{(0)}\big)$. *Right:* truncated cone of a phantom row $\hat{C}_4\big((u_y)_{(0,2)}\big)$.

by choosing from F a row f with maximal leading derivative and replacing it by its normal form modulo $F \setminus \{f\}$ or eliminating it entirely if it reduces to 0. Again, it is trivial to see that this process terminates and produces a system in which different rows have different leading derivatives and which describes the same differential equation $\mathcal{R}_q$ as the original system.

4.2 Skeletons of Systems

The basic idea of our completion algorithm is to follow the steps of the Cartan–Kuranishi algorithm stated in Subsection 2.3 and at the same time to use the strategy from the completion algorithms proposed in [11], i. e. to compute only the non-multiplicative prolongations of a row and reducing them with multiplicative prolongations from the system. This approach is based on Proposition 2: if the symbol is involutive, the multiplicative prolongations generate everything in the next order; thus it suffices to analyse what is happening in non-multiplicative prolongations.

Definition 5 *Let f denote a row from a system F. We can make f into an* indexed row*, either a* single indexed row *(or* real row*) $f_{(k)}$ or a* double indexed row *(or* phantom row*) $f_{(k,l)}$ by assigning to it one respectively two nonnegative integers k and l where $0 \leq k < l$ in the second case. We call k the* initial level *and l the* phantom level *of the indexed row. For a global level $\lambda \geq k$, we define the* truncated involutive cone $\hat{C}_\lambda$ *of an indexed row to be the following set of rows:*

$$\hat{C}_\lambda(f_{(k)}) := \big\{D_\mu f \mid 0 \leq |\mu| \leq (\lambda - k);\ \forall i > \mathrm{cls}(f) : \mu_i = 0\big\}\,, \tag{19}$$

$$\hat{C}_\lambda(f_{(k,l)}) := \big\{D_\mu f \mid (\lambda - l) < |\mu| \leq (\lambda - k);\ \forall i > \mathrm{cls}(f) : \mu_i = 0\big\}\,. \tag{20}$$

The truncated involutive cone of an indexed row determined by the numbers λ, k and possibly l is the set of all multiplicative prolongations whose orders fulfil the conditions in the above definition (see Figure 1 for an example). With the help of this notion, we define the skeleton of a system.

Definition 6 *A* skeleton $\mathcal{S}_\lambda$ *with the global level* λ *is the union of a set* $\mathcal{O}_\lambda = \{f^\sigma_{(k_\sigma)}\}$ *of single indexed rows and a set* $\mathcal{P}_\lambda = \{f^\sigma_{(k_\sigma,l_\sigma)}\}$ *of double indexed rows such that for all* σ *the inequalities* $k_\sigma \leq \lambda$ *and* $k_\sigma < l_\sigma \leq \lambda$*, respectively, hold. It defines the linear system*

$$\bar{\mathcal{S}}_\lambda := \{\hat{C}_\lambda(f_{(k)}) \mid f_{(k)} \in \mathcal{O}_\lambda\} \cup \{\hat{C}_\lambda(f_{(k,l)}) \mid f_{(k,l)} \in \mathcal{P}_\lambda\} . \tag{21}$$

This means that we obtain the system $\bar{\mathcal{S}}_\lambda$ corresponding to a skeleton $\mathcal{S}_\lambda$ by simply adding all permitted multiplicative prolongations and dropping the indices. The meaning and usefulness of these concepts will become clear in the next subsection.

4.3 Prolongation and Triangulation of Skeletons

In each iteration of our algorithm, the system is prolonged by one order. Naively, this can be done by adding all prolongations of the rows currently contained in the system. Since we are interested in autoreduced systems, we afterwards perform a *triangulation* resulting in a system in which different rows have different leading derivatives. We show next how these two operations are carried out more efficiently on the level of skeletons.

We may assume that the system F for the differential equation $\mathcal{R}_q$ to be completed is already given in triangular, i. e. (head) autoreduced, form. We turn F into a skeleton $\mathcal{S}_0$ by simply setting the initial level k_σ of every row f^σ to 0. The global level λ is always equal to the number of the current iteration step; if a new row enters the skeleton in this step, its initial level becomes λ. Double indexed rows arise, if during the triangulation process a row is eliminated which has produced multiplicative prolongations before. In this case, the phantom level is set to the current global level. Thus we produce the following sequence of skeletons:

$$\mathcal{S}_0 = \mathcal{S}_0^\triangle \longrightarrow \mathcal{S}_1 \longrightarrow \mathcal{S}_1^\triangle \longrightarrow \mathcal{S}_2 \longrightarrow \mathcal{S}_2^\triangle \longrightarrow \mathcal{S}_3 \longrightarrow \cdots \tag{22}$$

where the $^\triangle$ indicates a skeleton in triangular form.

Algorithm 2 merely computes the non-multiplicative prolongations of those single indexed rows that have been newly added in the previous iteration. Due to the increase of the global level, multiplicative prolongations of *all* rows from the skeleton are present in the system it defines. No rows are explicitly removed; the new global level causes the elimination of the least-order multiplicative prolongations of all phantom rows in the system (see Figure 2). We will show below that these operations yield the desired skeleton for the prolonged system.

Algorithm: Prolong Skeleton
Input: skeleton $\mathcal{S}_\lambda = \mathcal{O}_\lambda \cup \mathcal{P}_\lambda$
Output: prolonged skeleton $\mathcal{S}_{\lambda+1} = \mathcal{O}_{\lambda+1} \cup \mathcal{P}_{\lambda+1}$

n: number of independent variables
begin
    $\mathcal{O}_{\lambda+1} := \mathcal{O}_\lambda$; $\mathcal{P}_{\lambda+1} := \mathcal{P}_\lambda$
    for $f^\tau_{(\lambda)}$ **in** $\mathcal{O}_\lambda$ **do**
        for i **from** $\operatorname{cls}(f^\tau)$ **to** n **do**
            $\mathcal{O}_{\lambda+1} := \mathcal{O}_{\lambda+1} \cup \{(D_i f^\tau)_{(\lambda+1)}\}$;
        end_for;
    end_for;
    return $\mathcal{S}_{\lambda+1} = \mathcal{O}_{\lambda+1} \cup \mathcal{P}_{\lambda+1}$;
end;

Algorithm 2 *Prolong Skeleton*

Algorithm: Triangulise Skeleton
Input: skeleton $\mathcal{S}_\lambda = \mathcal{O}_\lambda \cup \mathcal{P}_\lambda$
Output: skeleton $\mathcal{S}^\triangle_\lambda = \mathcal{O}^\triangle_\lambda \cup \mathcal{P}^\triangle_\lambda$ with $\bar{\mathcal{S}}_\lambda$ in triangular form

begin
    $\mathcal{O}^\triangle_\lambda := \emptyset$; $\mathcal{P}^\triangle_\lambda := \mathcal{P}_\lambda$;
    while $\mathcal{O}_\lambda \neq \emptyset$ **do**
        choose $f^\sigma_{(k_\sigma)} \in \mathcal{O}_\lambda$ with $\operatorname{ld}(f^\sigma)$ maximal w. r. t. $\prec$;
        $\mathcal{O}_\lambda := \mathcal{O}_\lambda \setminus \{f^\sigma_{(k_\sigma)}\}$; $f' := f^\sigma$;
        while $\exists f^\tau_{(k_\tau)} \in \mathcal{O}_\lambda \cup \mathcal{O}^\triangle_\lambda$ with $\operatorname{ld}(f') \in \hat{C}_\lambda(\operatorname{ld}(f^\tau_{(k_\tau)}))$
            or $\exists f^\tau_{(k_\tau,l_\tau)} \in \mathcal{P}^\triangle_\lambda$ with $\operatorname{ld}(f') \in \hat{C}_\lambda(\operatorname{ld}(f^\tau_{(k_\tau,l_\tau)}))$ **do**
            $p^\alpha_{\mu_1} := \operatorname{ld}(f')$; $p^\alpha_{\mu_2} := \operatorname{ld}(f^\tau)$; $\mu := \mu_1 - \mu_2$;
            $g := D_\mu f^\tau$;
            $f' := \operatorname{rem}(f', g)$;
        end_while;
        if $f^\sigma = f'$ **then**
            $\mathcal{O}^\triangle_\lambda := \mathcal{O}^\triangle_\lambda \cup \{f^\sigma_{(k_\sigma)}\}$;
        else
            if $f' \neq 0$ **then**
                $\mathcal{O}^\triangle_\lambda := \mathcal{O}^\triangle_\lambda \cup \{f'_{(\lambda)}\}$;
            end_if;
            if $k_\sigma < \lambda$ **then**
                $\mathcal{P}^\triangle_\lambda := \mathcal{P}^\triangle_\lambda \cup \{f^\sigma_{(k_\sigma,\lambda)}\}$;
            end_if;
        end_if;
    end_while;
    return $\mathcal{S}^\triangle_\lambda = \mathcal{O}^\triangle_\lambda \cup \mathcal{P}^\triangle_\lambda$;
end;

Algorithm 3 *Triangulise Skeleton*

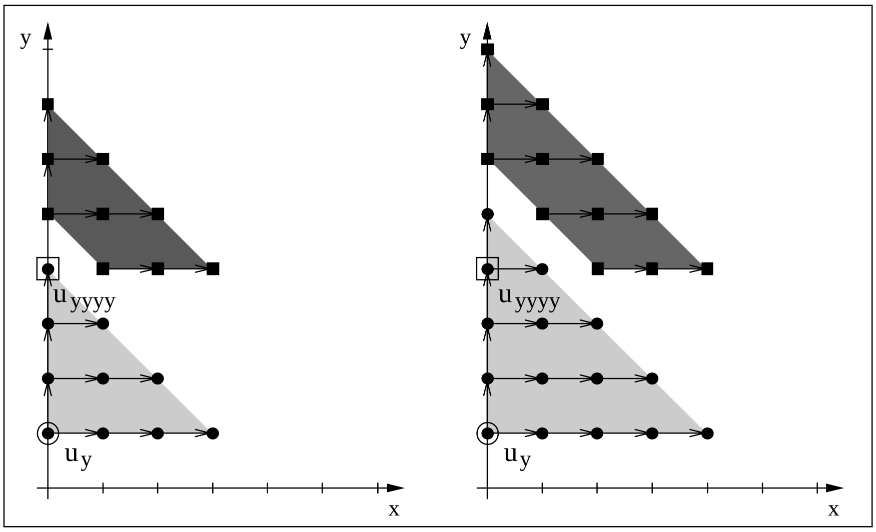

Figure 2 Prolongation of a real and a phantom row shown for the skeleton $\mathcal{S}_3 = \{(u_y)_{(0)}, (u_{yyyy})_{(0,3)}\}$ (left) and its prolongation $\mathcal{S}_4$ (right).

The triangulation operation described in Algorithm 3 is slightly more complicated. Nevertheless, it is very similar to the autoreduction of a system as mentioned in Subsection 4.1. It suffices to reduce only the single indexed rows, as the multiplicative prolongations of the phantom rows obviously have different leading derivatives. In the outer while loop (line 3–21), we take a row from $\mathcal{O}_\lambda$ with maximal leading derivative and try to reduce it modulo the rows contained in the system $\bar{\mathcal{S}}_\lambda$. Hence we have to check whether a reduction with a row lying in one of the truncated involutive cones is possible (line 6). In the inner while loop (line 6–10), the actual reduction is performed. If no change occurs, the row is written back into the skeleton (line 12); otherwise the reduced row is added with initial level λ (line 15). The next part is a bit tricky: if already multiplicative prolongations of the eliminated row have been computed, its removal would cause them to disappear. Therefore we have to keep track of their existence: we turn the row into a phantom row (line 18), assuring that in future iterations its multiplicative prolongations are still taken into account.

Proving correctness of the above procedure is equivalent to showing that the system $\bar{\mathcal{S}}_\lambda^\triangle$ defined by the skeleton $\mathcal{S}_\lambda^\triangle$ yields a local representation for the differential equation $\mathcal{R}_{q+\lambda}$, if the initial differential equation was $\mathcal{R}_q$. The first step towards achieving this result is the following proposition.

Proposition 3 *Consider the sequence (22) derived from a skeleton $\mathcal{S}_0$ by repeatedly applying Algorithms 2 and 3. For $\lambda \geq 0$, let g be a linear combination of rows of the system $\bar{\mathcal{S}}_\lambda^\triangle$. Then each prolongation of g by one order can be written as a linear combination of rows of the system $\bar{\mathcal{S}}_{\lambda+1}$.*

Proof We first slightly modify Algorithm 3 by replacing line 18 with

$$\mathcal{O}_\lambda^\triangle := \mathcal{O}_\lambda^\triangle \cup \{f_{(k_\sigma)}^\sigma\} . \tag{23}$$

This implies in particular that phantom rows never occur; we can ignore, for the moment, all references to the set $\mathcal{P}$. Any row of which multiplicative prolongations have been computed is contained in all subsequent skeletons. Of course, one thereby loses the triangular form of the systems $\bar{\mathcal{S}}_\lambda^\triangle$.

So let the skeleton $\mathcal{S}_\lambda^\triangle$ consist of the single indexed rows $\{f_{(k_\sigma)}^\sigma\}$ and

$$g := \sum_{p=1}^{\ell} a_p(x) D_{\mu_p} f^{\sigma_p} \tag{24}$$

be a linear consequence of the rows in the system $\bar{\mathcal{S}}_\lambda^\triangle$. The rows in $\bar{\mathcal{S}}_\lambda$ are of course multiplicative prolongations of rows in the corresponding skeleton. We prolong (24) in the direction of an arbitrary independent variable x^j obtaining

$$D_j g = \sum_{p=1}^{\ell} \left[a_p(x) D_{\mu_p + 1_j} f^{\sigma_p} + \frac{\partial a_p(x)}{\partial x^j} D_{\mu_p} f^{\sigma_p}) \right] . \tag{25}$$

We have to show that all the rows of $\bar{\mathcal{S}}_\lambda^\triangle$ *and* their prolongations by one order are members of $\bar{\mathcal{S}}_{\lambda+1}$. By the above remark, this is obvious for the unprolonged rows. In the other case, we proceed by induction on the global level λ. For $\lambda = 0$, Algorithm 2 computes all non-multiplicative prolongations (since for each row $f_{(k_\sigma)}^\sigma$, $k_\sigma = \lambda = 0$ holds); the multiplicative prolongations are being taken care of by the incrementation of the global level λ. Now assume $\lambda > 0$. Let $h = D_{\mu+1_j} f$ be a prolongation with $D_\mu f \in \bar{\mathcal{S}}_\lambda^\triangle$ and $f_{(k)} \in \mathcal{S}_\lambda^\triangle$. We prove $h \in \bar{\mathcal{S}}_{\lambda+1}$ by a Noetherian argument. The terminal cases are:

- If x^j is *multiplicative* for $f_{(k)}$, we are done: $\mu + 1_j$ is multiplicative for f and, as the global level is raised during a prolongation, $h \in \bar{\mathcal{S}}_{\lambda+1}$.
- If $|\mu| = 0$ and x^j is *non-multiplicative* for $f_{(k)}$, the prolongation is explicitly computed and therefore contained in the next system.

In all other cases, we look for the leftmost non-vanishing entry in $\mu + 1_j$ and denote its position by t. One can rewrite h as $D_t(D_{\mu+1_j-1_t} f)$, and because of $|\mu| \leq (\lambda - k)$, the total derivative in parentheses is an element of $\bar{\mathcal{S}}_{\lambda'}$ for some $\lambda' \leq \lambda$. Thus it is also contained in $\bar{\mathcal{S}}_\lambda$ and we can apply the induction hypothesis writing

$$h = D_t \Big[\sum_{p=1}^{\ell'} b_p(x) D_{\nu_p} f^{\sigma_p} \Big] . \tag{26}$$

We observe that for the summand h' possessing the highest leading derivative with respect to the ranking $\prec$ (it has the multi index $\mu + 1_j - 1_t$) the variable x^t is by construction multiplicative. Hence $D_t h' \in \bar{\mathcal{S}}_{\lambda+1}$ as desired. With each of the remaining summands, we repeat the above process. Note that the multi indices of their leading derivatives are all less than $\mu + 1_j - 1_t$ with respect to $\prec$, so applying the described substitution sufficiently many times, the process must terminate. This proves the statement above.

Finally, we must include the phantom rows, i.e. we restore line 18 of Algorithm 2. Observe that from (25) it follows that if a row is dependent of the other rows in $\bar{\mathcal{S}}_\lambda$ and therefore eliminated, all its multiplicative prolongations by one order reduce to 0 modulo $\bar{\mathcal{S}}_{\lambda+1}$, i.e. they are dependent of the rows in $\bar{\mathcal{S}}_{\lambda+1}$. Therefore, it does not matter for the system (however, it does matter for its triangular form) whether we carry out these reductions or not. Non-multiplicative prolongations of phantom rows never need to be computed, as for $f_{(k,l)}$ always $k < l \leq \lambda$ holds. $\square$

Theorem 2 *Let the system F be a local representation for the linear differential equation $\mathcal{R}_q$. If we turn F into a skeleton $\mathcal{S}_0$ as described above and form the sequence (22), then for all $\lambda \geq 0$ the system $\bar{\mathcal{S}}_\lambda^\triangle$ is triangulised and represents the equation $\mathcal{R}_{q+\lambda}$.*

Proof We denote by $F_0 = F$, F_1, F_2,... the prolonged systems of F, each of which is a system for the corresponding differential equation $\mathcal{R}_q$, $\mathcal{R}_{q+1}$, $\mathcal{R}_{q+2}$,... Now because of $F_0 = \bar{\mathcal{S}}_0^\triangle$, we have also equality of the linear hulls: $[F_0] = [\bar{\mathcal{S}}_0^\triangle]$. Repeatedly applying Proposition 3, we get for $\lambda \geq 0$: $[F_\lambda] = [\bar{\mathcal{S}}_\lambda]$. As we perform only elementary row operations during Algorithm 3, it follows that $\bar{\mathcal{S}}_\lambda^\triangle$ is a representation of $\mathcal{R}_{q+\lambda}$.

All that remains to show is that the systems $\bar{\mathcal{S}}_\lambda^\triangle$ are really in triangular form. First, the rows in $\mathcal{O}_\lambda^\triangle$ obviously possess different leading derivatives. So suppose that we have $\mathrm{ld}(D_{\mu_1} f^1) = \mathrm{ld}(D_{\mu_2} f^2)$ for two rows contained in the system $\bar{\mathcal{S}}_\lambda^\triangle$ with $f^1_{(k_1,l_1)}$ and $f^2_{(k_2,l_2)}$ being the rows contained in the skeleton $\mathcal{S}_\lambda^\triangle$ from which they have been derived by multiplicative prolongations. If either f^1 or f^2 is a single indexed row, we simply ignore all references made to the phantom level.

It follows from the definition of the class of a multi index that if truncated two involutive cones intersect, the tip of one lies in the other one [11,49]. So without loss of generality assume $\mathrm{ld}(f^2) = \mathrm{ld}(D_\nu(f^1))$ for some multiplicative prolongation D_ν. This means that in step $k_1 + |\nu|$ the row f^2 was reduced and became a phantom row. In the next $|\mu_2|$ steps, all multiplicative prolongations of f^2 up to this order are reduced; from $|\nu| + |\mu_2| = |\mu_1| \leq (\lambda - k_1)$ it follows that $D_{\mu_2} f^2 \notin \bar{\mathcal{S}}_\lambda^\triangle$, and we have arrived at a contradiction. $\square$

4.4 Projection, Involution and Dimensions

It is convenient at this point to pause for a moment and to review what we call from now on the "algebraic part" of the algorithm. By working with skeletons instead of full systems, we have to compute the multiplicative

$$
\begin{tikzcd}
\mathcal{S}_0^{\triangle} \arrow[r, Rightarrow] \arrow[d, dash] & \mathcal{S}_1^{\triangle} \arrow[r, Rightarrow] \arrow[d, dash] & \mathcal{S}_2^{\triangle} \arrow[r, Rightarrow] \arrow[d, dash] & \cdots \\
\mathcal{R}_q \arrow[r] \arrow[d, dash] \arrow[dr, equal, "?"] & \mathcal{R}_{q+1} \arrow[r] \arrow[d, dash] \arrow[dr, equal, "?"] & \mathcal{R}_{q+2} \arrow[r] \arrow[d, dash] & \cdots \\
\mathcal{R}^{(1)}_{q-1} \arrow[r, dashed] \arrow[d, dash] \arrow[dr, equal, "?"] & \mathcal{R}^{(1)}_{q} \arrow[r, dashed] \arrow[d, dash] \arrow[dr, equal, "?"] & \mathcal{R}^{(1)}_{q+1} \arrow[r, dashed] \arrow[d, dash] & \cdots \\
\mathcal{R}^{(2)}_{q-2} \arrow[r, dashed] \arrow[d, dash] & \mathcal{R}^{(2)}_{q-1} \arrow[r, dashed] \arrow[d, dash] & \mathcal{R}^{(2)}_{q} \arrow[r, dashed] \arrow[d, dash] & \cdots \\
\vdots & \vdots & \vdots &
\end{tikzcd}
$$

Figure 3 Skeletons and their respective systems

prolongations only when they are needed for a reduction. In each step of the algorithm, we prolong the skeleton by exactly one order and compute a triangular form. However, as we want to realise the Cartan–Kuranishi algorithm, we have to relate these ideas to the geometric theory.

We have already seen that the skeleton $\mathcal{S}_\lambda^{\triangle}$ yields a representation for the differential equation $\mathcal{R}_{q+\lambda}$. In fact, it contains even more information. Projecting into jet bundles of lower order, one can extract from $\mathcal{S}_\lambda^{\triangle}$ representations for all differential equations $\mathcal{R}^{(s)}_{q+r}$ with $\lambda = r + s$:

$$
\mathcal{S}_\lambda^{\triangle} \; : \; \Big\{ \mathcal{R}_{q+\lambda} \overset{\pi^{q+\lambda}_{q+\lambda-1}}{\longrightarrow} \mathcal{R}^{(1)}_{q+\lambda-1} \overset{\pi^{q+\lambda-1}_{q+\lambda-2}}{\longrightarrow} \mathcal{R}^{(2)}_{q+\lambda-2} \overset{\pi^{q+\lambda-2}_{q+\lambda-3}}{\longrightarrow} \cdots \; . \tag{27}
$$

Because of the triangular form of $\bar{\mathcal{S}}_\lambda^{\triangle}$, this amounts to dropping the rows of appropriate orders, although we will never do this explicitly. Instead, we identify the differential equation $\mathcal{R}^{(s)}_{q+r}$ currently considered by the values of the parameters r and s.

If we juxtapose all the sequences of the form (27), we arrive at the grid pictured in Figure 3. Starting with $\mathcal{R}_q$, the Cartan–Kuranishi algorithm moves along the horizontal lines searching for an equation with involutive symbol. If one is found, it is checked for integrability conditions by comparing it with the equation one row below and one column to the right. If the two equations differ, the process is continued in the row below, i.e. after adding the integrability conditions. Note that in general $\big(\mathcal{R}^{(s)}_{q+r}\big)_{+1} \neq \mathcal{R}^{(s)}_{q+r+1}$; to ensure equality we need the involution of $\mathcal{M}^{(s-1)}_{q+r}$ (cf. Part (iii) of Proposition 1). So two tasks remain to be solved:

1. provide a method for deciding which of the differential equations represented by a skeleton have involutive symbols;
2. determine whether $\mathcal{R}^{(s)}_{q+r}$ and $\mathcal{R}^{(s+1)}_{q+r}$ are equal. Since the second is a submanifold of the first, it suffices to compare their dimensions.

Proposition 4 *Let $\mathcal{S}^{\triangle}_{\lambda}$ be a skeleton for the differential equation $\mathcal{R}_{q+\lambda}$ in δ-regular coordinates and let t be the maximal order of a single indexed row in $\mathcal{S}^{\triangle}_{\lambda+1}$. Then the symbol $\mathcal{M}^{(s)}_{q+r}$ is involutive for all values r, s such that $r+s=\lambda$ and $q+r \geq t$.*

Proof We choose r, s with $r+s=\lambda$. A representation for $\mathcal{R}^{(s)}_{q+r}$ can be obtained from $\bar{\mathcal{S}}^{\triangle}_{\lambda}$ by dropping all rows of order greater than $q+r$. We prolong this system by one order and compute a triangular form. If no non-multiplicative prolongations of order $q+r+1$ remain, Proposition 2 may be applied, proving the involution of the symbol $\mathcal{M}^{(s)}_{q+r}$. As by assumption, even after performing prolongation and triangulation with the full skeleton no single indexed rows of this or a greater order exist in $\mathcal{S}^{\triangle}_{\lambda+1}$, signalling the existence of an independent non-multiplicative prolongation, the involution of the examined symbol can be read off from this skeleton. □

As a useful side effect, this proposition allows us to carry out additional projections inside a given skeleton: we trace a vertical line in Figure 3 belonging to some skeleton $\mathcal{S}^{\triangle}_{\lambda}$ downwards as long as the symbols of the corresponding differential equations are involutive and continue the completion with the last such equation. This is shown in Figure 4 in comparison to the strategy followed in the Cartan–Kuranishi algorithm (boxed equations possess involutive symbols).

Definition 7 *For the skeleton $\mathcal{S}_{\lambda}$, we denote by $\#\mathcal{S}_{\lambda,t}$ the number of rows of order t in the system $\bar{\mathcal{S}}_{\lambda}$. The* rank vector *of $\mathcal{S}_{\lambda}$ (if the initial equation was $\mathcal{R}_q$) is the list*

$$\mathbf{r}_{\mathcal{S}_\lambda} := [\#\mathcal{S}_{\lambda,0}, \ldots, \#\mathcal{S}_{\lambda,q+\lambda}] \,. \tag{28}$$

As the triangulation guarantees algebraic independence of the rows, the numbers $\#\mathcal{S}^{\triangle}_{\lambda,t}$ allow us to compute the dimensions of the differential equations represented by $\mathcal{S}^{\triangle}_{\lambda}$. With a simple combinatorial consideration we can derive from the (fibre) dimensions of the jet bundle $J_q\mathcal{E}$ and the vector bundle $S_qT^*\mathcal{B} \otimes V\mathcal{E}$ the following formulae for $\lambda = r+s$

$$\dim \mathcal{R}^{(s)}_{q+r} = m\binom{q+r+n}{q+r} - \sum_{i=0}^{q+r} \#\mathcal{S}^{\triangle}_{\lambda,i} \,, \tag{29}$$

$$\dim \mathcal{M}^{(s)}_{q+r} = m\binom{q+r+n-1}{q+r} - \#\mathcal{S}^{\triangle}_{\lambda,q+r} \,. \tag{30}$$

It is straightforward how the Algorithms 2 and 3 have to be modified in order to update the rank vector. Computing it for the initial skeleton is trivial. During the prolongation from $\mathcal{S}_{\lambda}$ to $\mathcal{S}_{\lambda+1}$ the following corrections are necessary for each row $f \in \mathcal{S}_{\lambda}$:

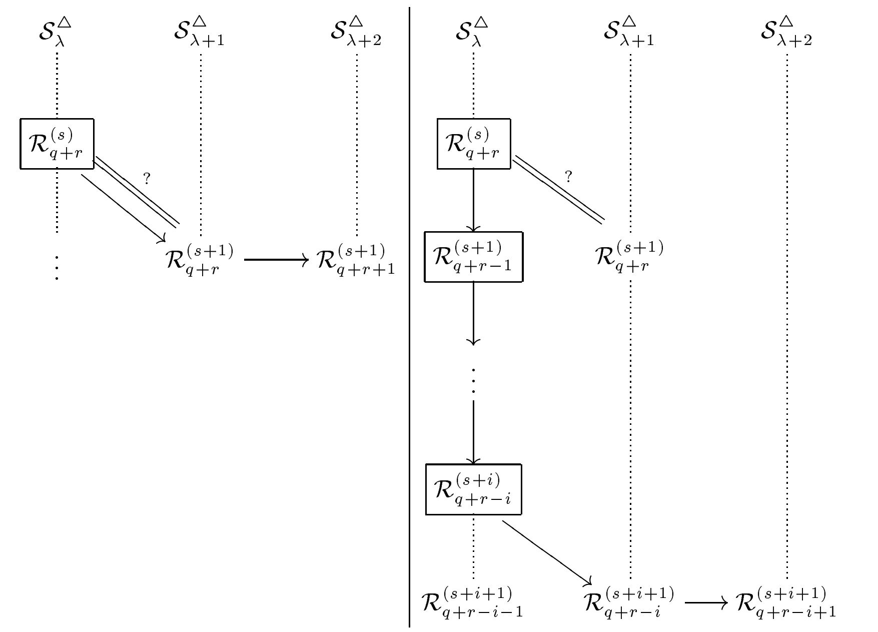

Figure 4 Analysis of symbols and systems in the Cartan–Kuranishi algorithm (left) and the new algorithm (right).

- If $f_{(k)}$ is a single indexed row of order t with level $k = \lambda$, prolongations in the direction of all independent variables are computed, so the entry for order $t + 1$ is increased by n, the number of independent variables.
- If the initial level of $f_{(k)}$ with order t and class j is less than λ, $\binom{(\lambda-k)+j}{j-1}$ new rows of order $t + (\lambda - k) + 1$ enter the system.
- If $f_{(k,l)}$ is a double indexed row of order t and class j, there are $\binom{(\lambda-k)+j}{j-1}$ new rows of order $t+(\lambda-k)+1$, and $\binom{(\lambda-l)+j}{j-1}$ rows of order $t+(\lambda-l)+1$ are removed.

For a triangulation, the corrections are even simpler: if a row is reduced to a new row of lower order, the entry in the rank vector for the old order has to be decreased and the one for the new order has to be increased by one. We have now all necessary tools for our hybrid algorithm at hand and give below its pseudocode (Algorithm 4).

Note that, strictly speaking, Algorithm 4 is *not* a realisation of the Cartan–Kuranishi completion because of the additional projections (in a strict realisation we would execute line 17 always only once). But these additional projections offer a great advantage. The Cartan–Kuranishi completion provides *some* equation $\mathcal{R}^{(s)}_{q+r}$ that is involutive. Obviously, all equations $\mathcal{R}^{(s')}_{q+r'}$ with $r' + s' > r + s$ are involutive, too. Furthermore, some equations with $r' + s' = r + s$ are involutive, in particular all with $r' > r$.

Algorithm: Geometric Algebraic Completion
Input: skeleton $\mathcal{S}_0^{\triangle}$ of the differential equation $\mathcal{R}_q$
Output: skeleton $\mathcal{S}_\lambda^{\triangle}$ and $r, s \in \mathbb{N}_0$ such that $\mathcal{R}_{q+r}^{(s)}$ is involutive

begin
$r := 0$; $s := 0$; $\lambda := 0$;
repeat
systemInvolutive := FALSE;
$\mathbf{r}' := \mathbf{r}_{\mathcal{S}_\lambda^{\triangle}}$; $\lambda := \lambda + 1$;
$\mathcal{S}_\lambda :=$ ProlongSkeleton($\mathcal{S}_{\lambda-1}^{\triangle}$);
$\mathcal{S}_\lambda^{\triangle} :=$ TrianguliseSkeleton($\mathcal{S}_\lambda$);
analyse first the symbol
if $\exists f_{(k)} \in \mathcal{O}_\lambda^{\triangle}$ of order $q + r + 1$ **then**
symbol not involutive
$r := r + 1$;
else
$\mathcal{M}_{q+r}^{(s)}$ involutive; check integrability of $\mathcal{R}_{q+r}^{(s)}$
if $\forall i \in \{0, \dots, q + r\} : \mathbf{r}_{\mathcal{S}_\lambda^{\triangle}}[i] = \mathbf{r}'[i]$ **then**
systemInvolutive := TRUE;
else
$r := r + 1$;
end_if;
project as far as possible
while not $\exists f_{(k)} \in \mathcal{O}_\lambda^{\triangle}$ of order $q + r$ **do**
$r := r - 1$; $s := s + 1$;
end_while;
end_if;
until systemInvolutive **end_repeat**;
return$(r, s, \mathcal{S}_\lambda^{\triangle})$;
end;

Algorithm 4 *Geometric Algebraic Completion*

Our algorithm determines the minimal r' such that $\mathcal{R}_{q+r'}^{(s')}$ is involutive. In applications, this fact is of considerable interest, as one prefers to work with equations of lower order.

Given the triangulised skeleton $\mathcal{S}_\lambda^{\triangle}$ of a differential equation $\mathcal{R}_q$, we refine the above considerations in order to determine the indices $\beta_q^{(i)}$ of the symbol $\mathcal{M}_q$. A row $f_{(k,l)} \in \mathcal{S}_\lambda^{\triangle}$ of order t and class c obviously contributes to them, if and only if (i) $\lambda - k + t \geq q$ and (ii) $l = 0$ or $\lambda - l + t < q$. Its contribution $\beta_q^{(i)}(f_{(k,l)})$ is then given by

$$\beta_q^{(i)}(f_{(k,l)}) = \begin{cases} B(c - i + 1, q - t, 1) & \text{for } 1 \leq i \leq c\,, \\ 0 & \text{for } c < i \leq n \end{cases} \tag{31}$$

where $B(n,q,c) = \binom{n-c+q-1}{q-1}$ denotes the number of multi indices of length n, order q and class c. The Cartan characters $\alpha_q^{(i)}$ of the equation are then computed by (13).

5 A Constructive Approach to δ-Regularity

5.1 The Problem of δ-Regularity

δ-regularity concerns our criterion for deciding involution of the symbol. In (12) the indices $\beta_q^{(k)}$ of the symbol $\mathcal{M}_q$ appear. Their definition is obviously coordinate dependent, as the concept of the class of a multi index is not invariant under changes of the independent variables. In fact, it suffices to consider linear transformations, as by the chain rule only the Jacobian enters the transformation law for the derivatives.

Coordinates are called δ-regular for a symbol $\mathcal{M}_q$ with indices $\beta_q^{(k)}$, if the sum $\sum_{k=1}^{n} k\beta_q^{(k)}$ is maximal, i. e. if one cannot achieve a larger value by a change of coordinates. A linear transformation of the independent variables is characterised by an invertible $n \times n$ matrix. The set of all such matrices corresponding to a transformation to δ-singular coordinates is described by a system of polynomial equations for the matrix entries. Thus it is of measure zero and a generic coordinate system is δ-regular.

Nevertheless, one encounters in applications δ-singular coordinates. This is often not by chance but on purpose, as such coordinates have a special meaning for the given differential equation. We cannot go here into details; we only mention as a simple example that the hypersurface $x^n = 0$ is characteristic for a given differential equation, if and only if $\beta_q^{(n)}$ (and thus also the sum $\sum_{k=1}^{n} k\beta_q^{(k)}$) has *not* its maximal value.

Proposition 5 *Let the used coordinates be δ-singular for a symbol $\mathcal{M}_{q+r}^{(s)}$ appearing during the completion of the differential equation $\mathcal{R}_q$. Then Algorithm 4 does not terminate. Conversely, if the algorithm terminates, then the used coordinates have been δ-regular for all appearing symbols.*

Proof If coordinates are δ-singular for a symbol $\mathcal{M}_q$, they are also δ-singular for all its prolongations $\mathcal{M}_{q+r}$. Indeed, any prolongation of an equation of class k yields an equation whose class is at most k. Criterion (12) for an involutive symbol fails in δ-singular coordinates, as the value on its left hand side is independent of the used coordinate system and the values on the right hand side is smaller than in a δ-regular system.

In the inner loop of the Cartan–Kuranishi algorithm the differential equation is prolonged, until its symbol becomes involutive. As we fail to detect this termination criterion in δ-singular coordinates, Algorithm 4 prolongs infinitely often in this case. Conversely, if the algorithm terminates, it has correctly recognised all arising involutive symbols which is only possible, if the coordinates have been δ-regular for all symbols. □

Different possibilities exist to deal with this problem. A simple solution consists of choosing a random coordinate system. With probability one it is generic and thus δ-regular for all appearing symbols. There are two caveats: it is still possible that the coordinates are δ-singular for a symbol and one destroys the sparsity typically present in differential equations. This makes all subsequent computations more expensive.

The brute force approach (proposed in [36]) consists of performing an indeterminate coordinate transformation which leads of course always to a generic coordinate system. However, this approach introduces n^2 parameters into the system and, even worse, completely eliminates any sparsity: all derivatives up to order q are present in the transformed differential equation. Thus from a computational point of view this approach is very expensive.

In [43] we presented some major improvements to this approach. We showed how one can determine step by step the correct values of the $\beta_q^{(k)}$ independent of the used coordinate system. Indirectly, the method is also based on coordinate transformations which are, however, not explicitly performed so that the sparsity is not affected. Furthermore, if $\beta_q^{(k)} = 0$ for $k < \ell$, then only $(n - \ell)n$ parameters have to be introduced. As in many applications ℓ is close to n, much less parameters are needed than in the above brute force approach. But the method is still computationally quite expensive, as parametric matrices must be analysed.

We develop in the remainder of this section a new and very cheap method to check on the fly whether the used coordinates are δ-regular for the current symbol. It is inspired by the work of Gerdt [10] on the relation between Pommaret and Janet bases.

5.2 Involutive Systems

For lack of space we cannot go into details of the theory of involutive bases; we only include some basic definitions. Besides the articles mentioned in the Introduction we refer to the introductory survey [4] and the forthcoming articles [50, 51] for more information.

Definition 8 *An* involutive division L *is given on the set of all derivatives* $\mathbb{D} = \{p^\alpha_\mu \mid 1 \leq \alpha \leq m,\ \mu \in \mathbb{N}_0^n\}$, *if a subset* $X_L(\delta, \mathcal{D}) \subseteq \{x^1, \dots, x^n\}$ *of* multiplicative variables *is associated to every derivative* δ *in a finite set* $\mathcal{D} \subset \mathbb{D}$ *such that the following two conditions on the* involutive cones $C_L(\delta, \mathcal{D}) = \{\bar\delta \in \mathbb{D} \mid \bar\delta = D_\nu\delta;\ \nu_i = 0 \text{ if } x^i \notin X_L(\delta, \mathcal{D})\}$ *are satisfied.*

1. *If two elements* $\delta_1, \delta_2 \in \mathcal{D}$ *exist with* $C_L(\delta_1, \mathcal{D}) \cap C_L(\delta_2, \mathcal{D}) \neq \emptyset$, *then either* $C_L(\delta_1, \mathcal{D}) \subseteq C_L(\delta_2, \mathcal{D})$ *or* $C_L(\delta_2, \mathcal{D}) \subseteq C_L(\delta_1, \mathcal{D})$ *holds.*
2. *If* $\mathcal{D}' \subset \mathcal{D}$, *then* $X_L(\delta, \mathcal{D}) \subseteq X_L(\delta, \mathcal{D}')$ *for all* $\delta \in \mathcal{D}'$.

An involutive division is called global, *if the sets* $X_L(\delta, \mathcal{D})$ *do not depend on* $\mathcal{D}$. *In this case we write* $X_L(\delta)$.

Thus the involutive cone of a derivative $\delta \in \mathcal{D}$ contains all prolongations of it with respect to its multiplicative variables only. The two conditions above state that if two involutive cones intersect, then one is completely contained in the other one, and if we remove elements of the set $\mathcal{D}$, then all multiplicative variables of any remaining derivative remain multiplicative but some non-multiplicative variables may become multiplicative.

The *Pommaret division* we have used so far is an example of a global division: the assignment of multiplicative variables to one derivative δ does not depend on the other ones in $\mathcal{D}$; if $\delta = p^\alpha_\mu$ with $\mathrm{cls}(\mu) = k$, then its multiplicative variables are $X_P(\delta) = \{x^1, \dots, x^k\}$.

As a second example we introduce the *Janet division*. It is not global but takes all derivatives in $\mathcal{D}$ into account. This makes its definition somewhat more complicated. Collect in the set $\mathcal{D}_\alpha \subseteq \mathcal{D}$ all derivatives of the dependent variable u^α, i.e. which are of the form p^α_μ for some multi index μ. We introduce the following subsets of $\mathcal{D}_\alpha$

$$(d_k, \dots, d_n)_\alpha = \bigl\{p^\alpha_\mu \in \mathcal{D}_\alpha \mid \mu_i = d_i \text{ for } k \leq i \leq n\bigr\} . \tag{32}$$

Let $\delta = p^\alpha_\mu$. Then x^n is multiplicative for it, i.e. $x^n \in X_J(\delta, \mathcal{D})$, if

$$\mu_n = \max_{p^\alpha_\nu \in \mathcal{D}_\alpha} \{\nu_n\} , \tag{33}$$

i.e. if no other derivative in $\mathcal{D}_\alpha$ has a higher entry at the last position of the multi index. For $1 \leq k < n$ we have $x^k \in X_J(\delta, \mathcal{D})$, if

$$\mu_k = \max_{p^\alpha_\nu \in (\mu_{k+1}, \dots, \mu_n)_\alpha} \{\nu_k\} . \tag{34}$$

Thus this time we look for the maximal value in the kth entry, however, only in the subset $(\mu_{k+1}, \dots, \mu_n)_\alpha \subseteq \mathcal{D}_\alpha$.

Given a linear system F, we assign the multiplicative variables according to the leading derivatives, i.e. we set $\mathcal{D} = \mathrm{ld}(F)$. We stress again that we determine the leading derivatives always with the help of a class respecting ranking. For brevity, we write $X_L(f, F)$ instead of $X_L\bigl(\mathrm{ld}(f), \mathrm{ld}(F)\bigr)$ where $L = P$ for the Pommaret and $L = J$ for the Janet division.

Definition 9 *A row $f \in F$ is* involutively head reducible *by another row $g \in F$ with respect to the involutive division L, if $\mathrm{ld}(f) \in C_L\bigl(\mathrm{ld}(g), \mathrm{ld}(F)\bigr)$. The system F is* involutively head autoreduced, *if no row in it is involutively head reducible by another row in F.*

The involutive span *of an involutively head autoreduced system F is*

$$\langle F \rangle_L = \bigcup_{f \in F} C_L(f, F) . \tag{35}$$

Its involutive size $|\langle F \rangle_L|$ *is $\sum_{f \in F} \bigl|X_L\bigl(\mathrm{ld}(f), \mathrm{ld}(F)\bigr)\bigr|$, i.e. the total number of multiplicative variables. The system F is* involutive *with respect to the involutive division L, if any prolongation $D_\mu f$ of a row $f \in F$ is a linear combination of rows contained in $\langle F \rangle_L$.*

If Algorithm 4 terminates with the skeleton $\mathcal{S}_\lambda^\triangle$, then the corresponding system $\bar{\mathcal{S}}_\lambda^\triangle$ is involutive according to this definition. The proof of this statement is straightforward but requires more material on involutive bases than we can present here. The intermediate skeletons are in general not involutively head autoreduced, as our reductions use only the truncated involutive cones $\hat{C}_\lambda(f)$ and not the full ones.

5.3 A Criterion for δ-Singularity

Above we defined δ-regularity only for symbols $\mathcal{M}_q$. Now we give a purely algebraic definition that applies to systems. If we take an involutively head autoreduced system F and perform a change of variables $x^i \to \tilde{x}^i$, we obtain a new system $\tilde{F}$ which after a triangulation yields an involutively head autoreduced system $\tilde{F}^\triangle$. Using in any coordinate system the reverse lexicographic order to select the leading derivatives, we compare the involutive sizes of the respective Pommaret spans.

Definition 10 *The coordinates x^i are* δ-regular *for a Pommaret head autoreduced system F, if after any change of coordinates $x^i \to \tilde{x}^i$ the inequality $|\langle F\rangle_P| \geq |\langle\tilde{F}^\triangle\rangle_P|$ holds.*

Our criterion for δ-singular coordinates founds on a comparison of the sets of multiplicative variables obtained with respect to the Pommaret and the Janet division, respectively. The following result on their relation for Pommaret head autoreduced systems can already be found in [11].

Proposition 6 *Let the system F be Pommaret head autoreduced. Then the inclusion $X_P(f,F) \subseteq X_J(f,F)$ holds for all $f \in F$.*

Proof Let p_μ^α be the leading derivative of a row $f \in F$ with $\mathrm{cls}(\mu) = k$. Thus $\mu = [0,\dots,0,\mu_k,\dots,\mu_n]$ with $\mu_k > 0$. We must show that the variables $x^1,\dots,x^k$ are multiplicative for p_μ^α with respect to the Janet division. In order to decide whether $x^k \in X_J\bigl(p_\mu^\alpha, \mathrm{ld}(F)\bigr)$ we study the set $(\mu_{k+1},\dots,\mu_n)_\alpha$. If it contained a leading derivative p_ν^α with $\nu_k > \mu_k$, then the system F would not be Pommaret autoreduced. Thus x^k is multiplicative for the Janet division, too. The same argument can be applied for $x^{k-1},\dots,x^1$ where the sets $(0,\dots,0,\mu_k,\dots,\mu_n)_\alpha$ with an increasing number of zeros are considered. Hence $X_P(f,F) \subseteq X_J(f,F)$. $\square$

The following theorem, which is the main result of this section, asserts that a coordinate system is δ-singular for a given system F, if the Janet division yields more multiplicative variables for at least one row of F.

Theorem 3 *Let the system F be Pommaret head autoreduced. The coordinates x^i are δ-singular for F, if the involutive size of its Janet span $\langle F\rangle_J$ is larger than that of its Pommaret span $\langle F\rangle_P$.*

Proof By the proposition above, we have $X_P(f,F) \subseteq X_J(f,F)$ for all rows $f \in F$. Assume there exists a row $h \in F$ such that $X_P(h,F) \subset X_J(h,F)$. This implies the existence of a variable x^ℓ which is Janet multiplicative for h but where ℓ is greater than the class of $\mathrm{ld}(h)$. We change to new independent coordinates $\tilde{x}^i$ defined by $\tilde{x}^i = x^i$ for $i \neq \ell$ and $\tilde{x}^\ell = x^\ell + ax^k$ with an arbitrary parameter a. This induces the following transformation of the derivatives

$$p^\alpha_\mu = \sum_{j=0}^{\mu_k} \binom{\mu_k}{j} a^j \tilde{p}^\alpha_{\mu - j_k + j_\ell} \tag{36}$$

where $\mu - j_k + j_\ell$ denotes the multi index obtained by subtracting j from μ_k and adding j to μ_ℓ. Obviously, this transformation does not affect any derivative with a multi index of class greater than k.

Let $\mathrm{ld}(h) = p^\alpha_\mu$. Thus $\mu = [0,\dots,0,\mu_k,\dots,\mu_n]$ with $\mu_k > 0$. We set $\nu = \mu - (\mu_k)_k + (\mu_k)_\ell$; obviously, $\mathrm{cls}(\nu) > k$ by construction. Applying our transformation to the row h introduces the derivative $\tilde{p}^\alpha_\nu$ in it. Note that p^α_ν cannot be an element of $\mathrm{ld}(F)$. Indeed, if it was, it would be an element of the same set $(\mu_{\ell+1},\dots,\mu_n)_\alpha$ as p^α_μ. This contradicts the assumption that x^ℓ is Janet multiplicative for p^α_μ, as $\nu_\ell > \mu_\ell$.

We apply the transformation (36) to all rows. This yields a system $\tilde{F}$ on which we perform an involutive head autoreduction in order to obtain the system $\tilde{F}^\triangle$. The leading derivatives are again selected according to the reverse lexicographic ranking, though now of course with respect to the new coordinates. We must determine the size of the Pommaret span $\langle \tilde{F}^\triangle \rangle_P$.

We can choose the parameter a such that after the transformation each row $\tilde{f} \in \tilde{F}$ has at least the same class as the corresponding row $f \in F$. This is a simple consequence of (36): only for a finite number of values of a cancellations of derivatives could occur in some rows. It follows from the definition of the Pommaret division that if $\mathrm{ld}(f_2) \in C_P\bigl(\mathrm{ld}(f_1)\bigr)$, then $\mathrm{cls}(\mathrm{ld}(f_1)) \geq \mathrm{cls}(\mathrm{ld}(f_2))$. Thus even after the involutive head autoreduction the size of the involutive span $\langle \tilde{F}^\triangle \rangle_P$ cannot be smaller than that of $\langle F \rangle_P$.

Now consider again the row h. The leading derivative of the transformed equation $\tilde{h} \in \tilde{F}$ must be greater than or equal to $\tilde{p}^\alpha_\nu$. Thus its class is greater than k. This remains true even after an involutive head autoreduction with all those rows $\tilde{f} \in \tilde{F}$ where the class of f was greater than k, as p^α_ν was not an element of $\mathrm{ld}(F)$. Hence the only possibility to obtain a leading derivative of class less than or equal to k consists of an involutive reduction with respect to a row $\tilde{f}$ where $\mathrm{cls}(\mathrm{ld}(f)) \leq k$. But this implies that $\mathrm{cls}(\mathrm{ld}(\tilde{f})) > k$. In any case we may conclude that after the transformation we have at least one row more whose class is greater than k. So the coordinates x^i cannot be δ-regular. □

5.4 Termination of Algorithm 4

We want to exploit these results to give an effective sufficient criterion for the termination of Algorithm 4. As a first step we study a simplified situation:

the completion to involution of a symbol $\mathcal{M}_q$. Thus we consider a local representation of it in form of a linear system F_q with the following special properties: the rows are homogeneous and all appearing derivatives are of the same order q. Furthermore, we assume that the system has constant coefficients. This assumption has the following reason. In the prolongation of a system with variable coefficients, lower order terms appear due to the Leibniz rule. These lower order terms do not affect the prolonged symbol. Thus as long as we are only interested in the symbol, we may treat any system like a system with constant coefficients.

Proposition 7 *Assume that during the application of Algorithm 4 to the local representation F_q of the symbol $\mathcal{M}_q$ the Janet and the Pommaret span of the skeleton coincide at any iteration. Then the algorithm terminates and the symbol $\mathcal{M}_{q+r}$ is involutive where r is the number of iterations required by the algorithm.*

Proof Under the made assumptions on the system F_q, our algorithm performs precisely the same steps as the standard involutive completion algorithm. Indeed, as we are dealing with homogeneous equations with constant coefficients where all derivatives are of the same order, integrability conditions cannot occur but only obstructions to involution. The Janet division is Noetherian, i. e. that any system possesses a finite Janet basis [11]. Thus the algorithm terminates after a finite number of steps. As we assume that the Janet and the Pommaret span always coincide, the result is also a Pommaret basis. Prolonging all lower order rows in the final skeleton to order $q+r$ yields a local representation of $\mathcal{M}_{q+r}$. It is trivial to see that this system is also a Pommaret basis which means that the equality (12) is satisfied. Thus the symbol $\mathcal{M}_{q+r}$ is involutive. □

Note that this result implies Part (i) of Proposition 1: given δ-regular coordinates, any symbol becomes involutive after a finite number of prolongations. Furthermore, it is obvious that $\mathcal{M}_{q+r}$ is in fact the lowest order prolongation of $\mathcal{M}_q$ that is involutive. The extension of the proposition to the general case requires some preparations. In particular, we must identify those rows in a skeleton that contribute to the symbol.

Definition 11 *Let $\mathcal{S}_\lambda^\triangle = \mathcal{O}_\lambda^\triangle \cup \mathcal{P}_\lambda^\triangle$ be a skeleton for the differential equation $\mathcal{R}_{q+r}^{(s)}$ with $\lambda = r+s$. We write $\Sigma_{q+r}^{(s)}$ for the set of all rows contributing to a local representation of the symbol $\mathcal{M}_{q+r}^{(s)}$*

$$\begin{aligned}\Sigma_{q+r}^{(s)} := \Big\{ f \mid f_{(k)} \in \mathcal{O}_\lambda^\triangle;\ (q+r) \le |f| + (\lambda - k) \Big\} \cup \\ \Big\{ f \mid f_{(k,l)} \in \mathcal{P}_\lambda^\triangle;\ |f| + (\lambda - l) < (q+r) \le |f| + (\lambda - k) \Big\}\ .\end{aligned} \tag{37}$$

Here, $|f|$ denotes the order of a row.

The set $\hat{\Sigma}^{(s)}_{q+r}$ contains all multiplicative prolongations of rows in $\Sigma^{(s)}_{q+r}$ to order $q+r$, i. e. the principal parts of its elements define a local representation of $\mathcal{M}^{(s)}_{q+r}$:

$$\hat{\Sigma}^{(s)}_{q+r} := \left\{ D_\mu f \mid f \in \Sigma^{(s)}_{q+r};\ |\mu| + |f| = q + r;\ \forall i > \mathrm{cls}(f) : \mu_i = 0 \right\} . \quad (38)$$

Theorem 3 is formulated for involutively head autoreduced systems. In order to apply it to $\Sigma^{(s)}_{q+r}$ or $\hat{\Sigma}^{(s)}_{q+r}$ we must first show that these sets are involutively head autoreduced. In the case of $\hat{\Sigma}^{(s)}_{q+r}$ this is trivial, as all rows contained in it are of the same order and their leading derivatives are different by construction.

Proposition 8 *The set $\Sigma^{(s)}_{q+r}$, derived from the skeleton $\mathcal{S}^{\triangle}_{r+s}$, is involutively head autoreduced with respect to the Pommaret division.*

Proof We first note that, as the skeleton $\mathcal{S}^{\triangle}_{r+s}$ is in triangular form, $\mathcal{O}_{r+s}$ must also be in triangular form and hence involutively head autoreduced (even with respect to the rows in $\mathcal{P}_{r+s}$). Thus if f and g are two rows in $\Sigma^{(s)}_{q+r}$ with initial levels k_f and k_g, respectively, such that f involutively head reduces g, then g must be a phantom row (with phantom level l_g). This implies that $|g|$ (and hence also $|f|$) is less than $q+r$, as otherwise g could not contribute to $\Sigma^{(s)}_{q+r}$.

If the reduction with f has turned g into a phantom row, this must have taken place in the l_gth iteration, so $l_g = |g| - |f| + k_f$. But inserting this into $|f| + (\lambda - k_f) \geq q + r$ yields $|g| + (\lambda - l_g) \geq q + r$, a contradiction. If instead another row h has involutively reduced g, this must have happened before f had been able to do so. Hence in this case even $l_g \leq |g| - |f| + k_f$ holds, leading again to a contradiction. □

In concrete computations, we do not want to use the set $\hat{\Sigma}^{(s)}_{q+r}$, as the main point of our whole approach is to avoid as much as possible multiplicative prolongations. So we show next that if $\langle \Sigma^{(s)}_{q+r} \rangle_J = \langle \Sigma^{(s)}_{q+r} \rangle_P$, then the same holds for $\hat{\Sigma}^{(s)}_{q+r}$.

Proposition 9 *Assume that for the system $\Sigma^{(s)}_{q+r}$ the Janet and the Pommaret division yield identical multiplicative variables. Then the same is true for the system $\hat{\Sigma}^{(s)}_{q+r}$.*

Proof Suppose that the Janet span of $\hat{\Sigma}^{(s)}_{q+r}$ is larger than its Pommaret span. Thus in the set $\Sigma^{(s)}_{q+r}$ a row exists with leading derivative p^α_μ such that for the leading derivative p^α_ν of one of its multiplicative prolongation to order $q+r$ a variable x^ℓ with $\ell > \mathrm{cls}(\nu)$ is Janet multiplicative with respect to the set $\mathrm{ld}(\hat{\Sigma}^{(s)}_{q+r})$. If $\ell > \mathrm{cls}(\mu)$, then, because of $\mu_i = \nu_i$ for all $i > \mathrm{cls}(\mu)$, the variable x^ℓ is also multiplicative for p^α_μ by the definition of the Janet division. As this contradicts our assumption, we must have

$k = \mathrm{cls}(\nu) \le \ell < \mathrm{cls}(\mu)$. But in this case $\hat{\Sigma}^{(s)}_{q+r}$ also contains a row with the leading derivative $p_{\nu-1_k+1_\ell}$. Its multi index has a higher entry at the ℓth position as ν, therefore x^ℓ cannot be Janet multiplicative for p^α_ν. □

Theorem 4 *Algorithm 4 terminates, if at each iteration of it the condition* $\langle \Sigma^{(s)}_{q+r} \rangle_J = \langle \Sigma^{(s)}_{q+r} \rangle_P$ *is satisfied.*

Proof Our algorithm is designed as an algebraic realisation of the Cartan–Kuranishi Algorithm. Recall that this algorithm consists of two nested loops. The termination of the outer one follows from a simple Noetherian argument and is independent of the used coordinates. Problems can only arise in the inner loop where we prolong until an involutive symbol is reached. Thus we must only show that under the made assumption our algorithm correctly detects all involutive symbols. But this is a simple consequence of the propositions above.

By Proposition 9, our assumption implies that $\langle \hat{\Sigma}^{(s)}_{q+r} \rangle_J = \langle \hat{\Sigma}^{(s)}_{q+r} \rangle_P$. The principal parts of the elements of this set define the symbol $\mathcal{M}^{(s)}_{q+r}$. Obviously, taking the principal part does not affect the leading derivatives and thus the size of the involutive spans. Now we may apply Proposition 7 which tells us that after a finite number of iterations an involutive symbol is detected. This implies the termination of Algorithm 4. □

5.5 Algorithmic Realisation

In order to incorporate the above ideas into Algorithm 4 we need first of all an efficient method to determine the multiplicative variables with respect to the Janet division. Gerdt et al. [13] recently studied this problem (and related problems like the efficient search for an involutive divisor) in detail and proposed a special data structure in form of a binary tree, the *Janet tree*. For our purposes, a simpler approach is sufficient.

Algorithm 5 proceeds by first sorting *inverse lexicographically*[8] the multi indices of the derivatives to be analysed and then comparing the positions at which two consecutive multi indices differ. The variables p_1 and p_2 keep track of these. From the fact that I is sorted, we know in each iteration of the **for**-loop that all x^i with $i \le p_2$ are multiplicative for $\mu^{(r)}$. Which x^i are multiplicative for $i > p_2$ depends on whether p_1 is greater than p_2 or not and is straightforwardly deduced from the definition of the Janet division.

In the language of Janet trees [13], this corresponds to the following method for determining the multiplicative variables: we start at the root of the tree with all variables being multiplicative and perform a depth-first traversal; at each branching node, the variable to which the right hand pointer points becomes non-multiplicative for all the leaves in the right hand subtree. In this way, also the inverse lexicographical ordering from above can be retrieved from the Janet tree.

[8] We have $\mu \prec_{\mathrm{ilex}} \nu$, if the rightmost non-vanishing entry of $\mu - \nu$ is negative.

Algorithm: Multiplicative Variables for the Janet Division
Input: list $I = [\mu^{(1)}, \dots, \mu^{(k)}]$ of pairwise different multi indices of length n
Output: list $M = [X_J(\mu^{(1)}, I), \dots, X_J(\mu^{(k)}, I)]$ of Janet multiplicative variables

begin
  $I := \mathrm{sort}(I, \prec_{\mathrm{ilex}})$;
  $\nu := I[1]$; $p_1 := n$; $X := \{1, \dots, n\}$;
  $M[1] := X$;
  for r **from** 2 **to** k **do**
    $p_2 := \max\{i \mid (\nu - I[r])_i \neq 0\}$;
    if $p_2 \leq p_1$ **then**
      $X := X \setminus \{p_2\}$;
    else
      $X := (X \setminus \{p_2\}) \cup \{p_1, \dots, p_2 - 1\}$;
    end_if;
    $M[r] := X$; $\nu := I[r]$; $p_1 := p_2$;
  end_for;
  return(I, M);
end;

Algorithm 5 *Multiplicative Variables for the Janet Division*

The above algorithm has the nice advantage that in case a multi index is added to or removed from a list for which the multiplicative variables have already been determined, no complete recomputation is necessary. A new multi index, for instance, is inserted into the ordered list I at, say, the ℓth position; this causes a change in the multiplicative variables of multi indices only from this position on. So it suffices to invoke algorithm 5 for the sublist $[\mu_{\ell-1}, \mu_\ell, \dots, \mu_k]$ where $M[\ell - 1]$ is known and p_1 can be computed from $\mu_{\ell-2}$ and $\mu_{\ell-1}$. For the removal of a multi index, one proceeds accordingly.

The computational costs of applying our criterion for δ-singular coordinates are negligible. The determination of the set $\mathrm{ld}(\Sigma^{(s)}_{q+r})$ and its multiplicative variables with respect to the Janet division require only some simple list manipulations. They are extremely cheap compared with the costs of a prolongation or a triangulation with their many differentiations and arithmetical operations.

Besides from recognising a δ-singular coordinates, the proof of Theorem 3 implies a method how to transform them into δ-regular coordinates. Assume that for some set $\Sigma^{(s)}_{q+r}$ appearing in the course of the algorithm, our criterion fails. Then we collect in a list $\mathcal{L}$ all pairs (i, j) with $i < j$ such that a row $f \in \Sigma^{(s)}_{q+r}$ exists with $\mathrm{cls}(f) = i$ and $j \in X_J(f, \Sigma^{(s)}_{q+r})$. Based on this list we construct the following matrix A: we set $A_{ii} = 1$ for $1 \leq i \leq n$ and $A_{ij} = 1$ for all $(i, j) \in \mathcal{L}$; all other entries are zero.

Now we apply the transformation $\tilde{x} = Ax$ to the current skeleton $\mathcal{S}^{\triangle}_\lambda$ obtaining the transformed skeleton $\tilde{\mathcal{S}}_\lambda$. Most probably some of the truncated

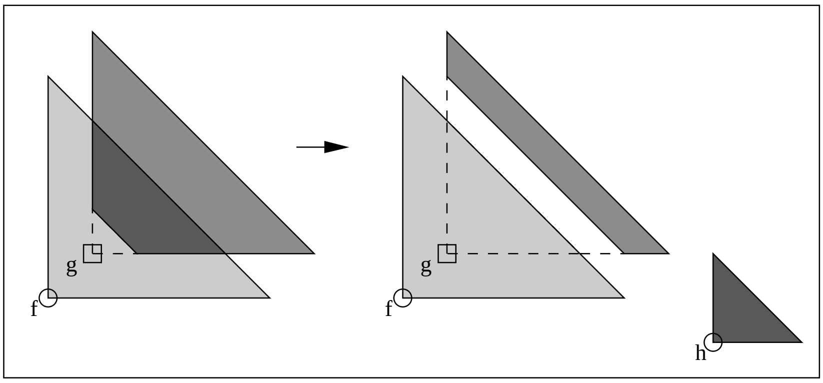

Figure 5 Involutive autoreduction after a coordinate transformation.

multiplicative cones of the rows intersect, i.e. $\tilde{\mathcal{S}}_\lambda$ is no longer triangular. Furthermore, the partition into real and phantom rows has to be checked, as some prolongations we have considered as non-multiplicative may now suddenly be multiplicative after the transformation. Therefore, a triangulation similar to the one outlined in Algorithm 3 becomes necessary.

Regardless of whether the original row $g_{(k_g)}$ was a real or a phantom row, we must check for each transformed row $\tilde{g}_{(k_g)}$, whether it lies in the truncated cone $\hat{C}_\lambda(\tilde{f}_{(k_f)})$ of some other transformed row[9] $\tilde{f}_{(k_f)}$. If this is the case, the involutive normal form $\tilde{h}$ of $\tilde{g}$ is computed and $\tilde{g}$ becomes a phantom row. The new levels of $\tilde{g}$ and $\tilde{h}$ are determined as follows: $\tilde{g}$ is assigned the new phantom level $l'_g = |g| - |f| + k_f$; if this is equal to or greater than the initial level k_g of g, the row $\tilde{g}$ is completely removed from the skeleton. If $g_{(k_g,l_g)}$ has been a phantom row, $\tilde{h}$ gets the initial level $l'_g - l_g$ and otherwise simply l'_g. Figure 5 visualises these considerations.

After the new triangular skeleton $\tilde{\mathcal{S}}_\lambda^\triangle$ has been obtained, we must repeat our test for δ-regularity: due to cancellations during the triangulation it might happen that we have still not reached a δ-regular coordinate system (recall that the proof of Theorem 3 contained a parameter a which we have simply set to one). But such cancellations may only happen for a finite number of parameter values. Thus after a finite number of transformations constructed as above, we end up with a skeleton in δ-regular coordinates with which we continue the algorithm.

6 Examples

The hybrid algorithm derived in Section 4 has been implemented in the computer algebra system *MuPAD*; details will be published elsewhere [20].

[9] $\tilde{f}$ and $\tilde{g}$ may also be double indexed (with phantom levels l_f and l_g).

In this system an object-oriented programming environment for differential equations [19] exists which is very convenient for the implementation of such algorithms. All examples in this section have been computed with our *MuPAD* programs.

A very classical example is the following linear second-order equation due to Janet showing many facets of the completion algorithm:

$$\mathcal{R}_2 \,:\, \begin{cases} u_{zz} + y u_{xx} = 0\,, \\ u_{yy} = 0\,. \end{cases} \tag{39}$$

Especially, one encounters here several times non-involutive symbols.

We choose the ordering $z > y > x$ on the independent variables. (39) is turned into a skeleton by assigning the initial level 0 to both equations. The initial rank vector is $\mathbf{r}_{\mathcal{S}_0^\triangle} = [0,0,2]$. From now on, we omit the right hand sides of equations.

1. *Iteration*: There is only one non-multiplicative prolongation: u_{yyz}. It cannot be reduced and is added to the skeleton with initial level 1. Since $\mathcal{S}_1^\triangle$ contains a third-order row, the symbol $\mathcal{M}_3$ is not involutive.
2. *Iteration*: u_{yyzz} is the only new non-multiplicative prolongation and it reduces to u_{xxy}. As no row of order 4 remains in the skeleton, $\mathcal{M}_4$ is involutive. We must now compare the rank vectors: $\bar{\mathcal{S}}_1^\triangle$ contains 6 equations of order 3, whereas $\bar{\mathcal{S}}_2^\triangle$ contains 7. Thus there exists one integrability condition (which is of course u_{xxy}) and the algorithm continues with the equation $\mathcal{R}_3^{(1)}$.
3. *Iteration*: The computed non-multiplicative prolongations are u_{xxyy} and u_{xxyz}. Only the first row can be reduced to zero, while the second one inhibits the involution of $\mathcal{M}_3^{(1)}$.
4. *Iteration*: Again there are two non-multiplicative prolongations: u_{xxyyz} can be reduced to zero; u_{xxyyz} is replaced by u_{xxxx}. Due to the lack of an equation of order 5 in $\mathcal{S}_4^\triangle$, $\mathcal{M}_4^{(1)}$ is involutive. The rank vectors that must be compared are: $\mathbf{r}_{\mathcal{S}_3^\triangle} = [0,0,2,7,13,17]$ and $\mathbf{r}_{\mathcal{S}_4^\triangle} = [0,0,2,7,14,19,24]$. Only the first five components are taken into account, but the difference between the entries 13 and 14 signals another integrability condition (u_{xxxx}). The next equation to be examined is $\mathcal{R}_4^{(2)}$.
5. *Iteration*: Of the two non-multiplicative prolongations, u_{xxxxy} is reduced to zero, whereas u_{xxxxz} prevents $\mathcal{M}_4^{(2)}$ from being involutive. The current skeleton is now

$$\mathcal{S}_5^\triangle \,:\, \begin{cases} (u_{xxxxz})_{(5)} & (u_{yyz})_{(1)} \\ (u_{xxxx})_{(4)} & (u_{zz} + y u_{xx})_{(0)} \\ (u_{xxyz})_{(3)} & (u_{yy})_{(0)} \\ (u_{xxy})_{(2)} & \end{cases} \tag{40}$$

with rank vector $\mathbf{r}_{\mathcal{S}_5^\triangle} = [0,0,2,7,14,21,26,32]$.

6. *Iteration*: Both non-multiplicative prolongations u_{xxxxyz} and u_{xxxxzz} can be reduced to zero; $\mathcal{M}_5^{(2)}$ is involutive, and the comparison of the rank vectors shows that there are no new integrability conditions. Thus the algorithm stops and outputs the skeleton $\mathcal{S}_6^{\triangle}$, which is identical to (40), for the differential equation $\mathcal{R}_5^{(2)}$. A local representation is obtained by prolonging all equations in the skeleton multiplicatively to fifth order.

Thus the path of the Cartan–Kuranishi completion can be summarised in the following sequence of differential equations:

$$\mathcal{R}_2 \to \mathcal{R}_3 \to \mathcal{R}_3^{(1)} \to \mathcal{R}_4^{(1)} \to \mathcal{R}_4^{(2)} \to \mathcal{R}_5^{(2)} \ . \tag{41}$$

In order to demonstrate the superior efficiency in comparison with a direct implementation of the Cartan–Kuranishi completion we do some statistics. The new algorithm computes 10 non-multiplicative and 18 multiplicative prolongations (5 of the latter ones are used several times). In contrast, a full determination of all systems requires the prolongation of 132 equations – almost five times more. In addition, the old approach requires to determine row echelon forms for a number of matrices (Jacobians and symbols), the largest being an 86×84 matrix.

One may wonder why we need rank vectors, as all integrability conditions in the Janet example stem from non-multiplicative prolongations that do not reduce to zero. This is, however, only one type of integrability condition; there exists a second type for systems comprising equations of different orders. As a trivial example we consider the equation $\mathcal{R}_2$ locally represented by $u_{yy} = u_{xy} = u_x = 0$. Obviously, the symbol $\mathcal{M}_2$ is involutive. But $\mathcal{R}_2$ is not formally integrable: prolongation of the lower order equation $u_x = 0$ with respect to the *multiplicative* variable x yields the additional equation $u_{xx} = 0$ required to represent $\mathcal{R}_2^{(1)}$.

In our algorithm the skeletons $\mathcal{S}_0^{\triangle}$ and $\mathcal{S}_1^{\triangle}$ contain exactly the same rows. But their rank vectors differ: $\mathbf{r}_{\mathcal{S}_0^{\triangle}} = [0,1,2]$ and $\mathbf{r}_{\mathcal{S}_1^{\triangle}} = [0,1,3,3]$. The increase in the third entry is due to the integrability condition $u_{xx} = 0$. Our algorithm never explicitly determines this type of integrability conditions; they are solely visible through changes in the rank vector.

Phantom equations do not occur in the Janet example. The next example demonstrates why they are necessary. It represents the determining system for the generators of Lie point symmetries of the heat equation $u_t = u_{xx}$ and consists of nine equations

$$\mathcal{R}_2 : \begin{cases} \tau_u = 0, & \tau_{ux} + \xi_u = 0, & \eta_{uu} - \xi_{ux} = 0, \\ \tau_x = 0, & \tau_{xx} + 2\xi_x - \tau_t = 0, & \eta_{ux} - \frac{1}{2}\xi_{xx} + \frac{1}{2}\xi_t = 0, \\ \tau_{uu} = 0, & \xi_{uu} = 0, & \eta_{xx} - \eta_t = 0 \end{cases} \tag{42}$$

with the independent variables $u > x > t$ and the dependent variables $\eta > \xi > \tau$. As above, we get $\mathcal{S}_0^{\triangle}$ by dropping the right hand side of each equation and assigning an initial level of 0. During the triangulation of $\mathcal{S}_1$, the row $\tau_{xx} + 2\xi_x - \tau_t$ can be reduced by the (multiplicative) prolongation with respect to x of the row τ_x. This yields the row $\xi_x - \frac{1}{2}\tau_t$, which is added to $\mathcal{S}_1^{\triangle}$. However, the system it defines also contains multiplicative prolongations of $\tau_{xx} + 2\xi_x - \tau_t$, so dropping this row completely would lead to a loss of information and to wrong values for the dimensions. Instead, its phantom level is set to 1. Thus in the λth iteration, all rows obtained by prolonging λ times the row $\tau_{xx} + 2\xi_x - \tau_t$ are present in $\bar{\mathcal{S}}_\lambda$ and available for reducing other rows.

The algorithm continues to produce the following sequence of equations (note the double projection in the third iteration step):

$$\mathcal{R}_2 \to \mathcal{R}_3 \to \mathcal{R}_3^{(1)} \to \mathcal{R}_2^{(3)} \to \mathcal{R}_2^{(4)} \to \mathcal{R}_3^{(4)} \;. \tag{43}$$

The final skeleton $\mathcal{S}_5^{\triangle}$ contains the following equations:

$$\mathcal{S}_5^{\triangle} \;:\; \begin{cases} \tau_{ttt} = 0, & \eta_{xx} - \eta_t = 0, \quad \xi_u = 0, \\ \eta_{uu} - \xi_{ux} = 0, & \xi_{tt} = 0, \quad \xi_x - \frac{1}{2}\tau_t = 0, \\ \eta_{ux} - \frac{1}{2}\xi_{xx} + \frac{1}{2}\xi_t = 0, & \tau_u = 0, \quad \tau_x = 0, \\ \eta_{ut} + \frac{1}{2}\xi_{xt} = 0. & \end{cases} \tag{44}$$

Since only head reductions have been performed during the algorithm, there exist further simplifications. Comparing the Hilbert polynomials and dimensions of the completed determining system and the heat equation, one easily finds that besides the obvious superposition symmetry the heat equation admits a six-dimensional Lie symmetry group [46].

Finally, we consider an example where δ-regularity is an issue. Here it is really just a technical nuisance: the used coordinates are δ-singular only for an intermediate symbol; they are δ-regular for both the original and the final involutive system.

We consider the following second-order equation for two independent variables $y > x$ and two dependent variables u, v.

$$\mathcal{R}_2 \;:\; \begin{cases} u_{yy} = 0 \;, \\ u_{xy} + v = 0 \;, \\ v_x = 0 \;. \end{cases} \tag{45}$$

Because of the simplicity of the equations there is no need to distinguish between the skeletons $\mathcal{S}_\lambda$ and their triangulations $\mathcal{S}_\lambda^{\triangle}$. Thus we always give at once the triangulised skeletons.

The skeleton $\mathcal{S}_0^{\triangle}$ is obtained as usual by dropping the right hand sides of the equations in (45) and adding the initial level 0. Relevant for the analysis of the symbol $\mathcal{M}_2$ are the sets $\Sigma_2 = \hat{\Sigma}_2 = \{u_{yy}, u_{xy} + v\}$. One easily checks that their Janet and Pommaret spans coincide and that our coordinates are δ-regular for $\mathcal{M}_2$.

The skeleton $\mathcal{S}_1^{\triangle}$ contains the two additional rows v_{xy} (non-multiplicative prolongation of v_x) and v_y (integrability condition). This leads to the sets $\Sigma_2^{(1)} = \Sigma_2 \cup \{v_x, v_{xy}\}$ and $\hat{\Sigma}_2^{(1)} = \hat{\Sigma}_2 \cup \{v_{xx}, v_{xy}\}$. For both the Janet span is larger than the Pommaret span, as the Janet division assigns to v_x (and v_{xy}, respectively) the multiplicative variables $\{x, y\}$ whereas y is not multiplicative for the Pommaret division. Thus our algorithm does not terminate for this system, as it cannot detect that the symbol $\mathcal{M}_2^{(1)}$ is involutive because of δ-singular coordinates.

We may nevertheless continue with the skeleton $\mathcal{S}_2^{\triangle}$; it differs from $\mathcal{S}_0^{\triangle}$ by the row v_y (v_{xy} has now been turned into a phantom row by a multiplicative prolongation of v_y). Our algorithm would analyse the symbol $\mathcal{M}_3^{(1)}$, as it did not recognise that $\mathcal{M}_2^{(1)}$ was involutive. But if we study instead the symbol $\mathcal{M}_2^{(2)}$ with the help of $\Sigma_2^{(2)} = \Sigma_2 \cup \{v_x, v_y\}$, the Janet and the Pommaret span coincide and one easily finds that $\mathcal{M}_2^{(2)}$ (and $\mathcal{R}_2^{(2)}$) is involutive.

Our results of Section 5 suggest to apply the transformation $\tilde{x} = x$ and $\tilde{y} = x + y$. One easily sees that these coordinates are δ-regular for all arising symbols and thus our algorithm terminates for the transformed equation. In this case, the change of variables hardly affects the sparsity; the transformed equations contain one term more than the original ones.

One may think that it is not necessary to construct the subsets $\Sigma_{q+r}^{(s)}$, as one could directly compare the Janet and the Pommaret span of the full skeleton. In the example above this would signal problems with the δ-regularity already in the first step, as obviously the Janet span of $\mathcal{S}_0^{\triangle}$ is larger than its Pommaret span. However, this approach may lead to false alarms. If we modify the second row in (45) to $u_{xy} + v_y = 0$, the Janet span of the skeleton is still larger than the Pommaret span. Nevertheless, no problems with δ-regularity appear during the whole completion; our algorithm terminates with the involutive equation $\mathcal{R}_2^{(1)}$. This is due to the fact that at the time when the row v_x affects the symbol the integrability condition v_{yy} is already present making the coordinates δ-regular.

7 Conclusions

Geometric and algebraic approaches to differential equations have a complementary nature. Geometry typically yields more insight; its results are intrinsic and do not depend on artifical ingredients like rankings or divisions. On the other hand, it is not very algorithmic and there is usually a price to pay for the intrinsicness in form of inefficiency. In contrast, algebra allows generally for a concise and rigorous formulation of algorithms.

In this article we combined algebraic and geometric theory in order to obtain a completion algorithm that is efficient and nevertheless returns intrinsic results. If the used coordinates are δ-regular for all appearing symbols, our algorithm performs essentially the same steps than the standard computation of a Pommaret basis in the differential algebraic approach, however in a special order. The geometric information requires only a careful bookkeeping, especially during the reductions.

Nevertheless, one must clearly say that this bookkeeping incurs a certain overhead. Furthermore, not all optimisations of the involutive completion algorithm fit easily into our hybrid approach. A simple example is

Buchberger's second criterion: we cannot apply such techniques for avoiding normal form computations, as in general the reduction to zero happens only at a later stage of our algorithm. In such a situation we would get wrong dimensions of intermediate equations by neglecting the corresponding non-multiplicative prolongations. However, the importance of Buchberger's criteria is much smaller for the involutive completion algorithm than for Buchberger's algorithm, as to a large extent they are already encompassed by considering only non-multiplicative prolongations.

In concrete computations, at least three different ways exist to deal with the problem of δ-regularity. In many cases already a simple inspection of the original system indicates δ-singular coordinates. In particular, if the differential equation to complete is the central object of the analysis (and not just a derived object like the determining systems in symmetry analysis), characteristic directions are of importance anyway. For most equations in mathematical physics this approach suffices in our experience.

A straightforward solution consists of using random coordinates. Our *MuPAD* completion package offers this as an option; a similar approach was used in the implementation [17] of the Cartan–Kähler theory where the same problem appears. As mentioned above, for large equations this is not an optimal solution, as the sparsity is destroyed.

Our results in Section 5 offer a third alternative. First of all, it allows us to detect very cheaply δ-singularity, so that one can distinguish whether a completion really needs so many prolongations or whether only bad coordinates are used. Secondly, we may even use the results to find appropriate changes of coordinates (even algorithmically). Because of the simplicity of the transformations they do not destroy much sparsity.

Not much is known about the complexity of completion algorithms for partial differential systems. As the completion of a linear system with constant coefficients is equivalent to the determination of a Gröbner basis of a polynomial module, the complexity of Gröbner bases computation represents a lower bound for any such algorithm. By a classical result of Mayr and Meyer [32] the worst case complexity is doubly exponential. Recall that this result is independent of any particular algorithm, as it describes the *size* of the basis. Thus it applies to both geometric and algebraic approaches.

Involutive bases are non-reduced Gröbner bases, thus they are in general larger. Gerdt et al. [14] compared runtimes and sizes of Janet and reduced Gröbner bases for most classical benchmark problems. Often the Janet bases are not much larger but there are exceptions. Not surprisingly, in such cases the classical algorithms are usually faster. Up to now, it is not known what factors are decisive for the size of involutive bases.

Finally, we want to discuss the restriction to linear equations. The linearity is important only for the reduction, as it ensures that we can decide whether a row reduces to zero. A straightforward extension of our algorithm to differential equations with polynomial nonlinearities in the derivatives would consist of using *pseudo-reductions* [34]. However, this implies that one multiplies rows by differential polynomials which may introduce spurious zeros, i.e. the skeletons correspond to larger submanifolds than $\mathcal{R}^{(s)}_{q+r}$. This is a classical problem and encountered by essentially every approach to non-linear completion.

Using the language of differential algebra [26,27], the involutive completion algorithm does not necessarily produce a basis of the differential ideal generated by the original system but only a coherent autoreduced set. Thus it may be considered as an alternative to the Kolchin–Ritt algorithm. In particular, it could be incorporated into Mansfield's algorithm for the determination of differential Gröbner bases [30]. Based on the experiences with polynomial ideals, in many cases this might even be faster than the Kolchin–Ritt algorithm! Of course, in this context one would not need our geometric variant.

The use of our algorithm is trivial, if the differential equation is quasi-linear (and remains so in the course of the algorithm), as it is the case for many important equations in mathematical physics. In this situation we obtain without problems the correct involutive completion. As a simple prototypical example, we consider the *incompressible Navier-Stokes equations* which are even semi-linear. In vector notation they read:

$$\mathcal{R}_2 : \begin{cases} \mathbf{u}_t + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} - \Delta \mathbf{u} + \boldsymbol{\nabla} p = 0 \,, \\ \qquad\qquad \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \,. \end{cases} \tag{46}$$

Here the dependent variables are the fluid velocity $\mathbf{u} = (u, v, w)$ and the pressure p; the independent variables are (t, x, y, z). In order to avoid problems with the δ-regularity it is important here to take the time t as the first independent variable, as the hyperplanes $t = Const$ are obviously characteristic surfaces. The gradient $\boldsymbol{\nabla}$ is to be understood with respect to the spatial variables (x, y, z) only.

The completion produces the sequence $\mathcal{R}_2 \to \mathcal{R}_2^{(1)} \to \mathcal{R}_2^{(2)}$ (so the symbol remains involutive throughout). The first projection does not affect the skeleton, as only the multiplicative prolongations of the incompressibility constraint are added. In the second projection the well-known Poisson equation for the pressure arises

$$\Delta p + \boldsymbol{\nabla} \cdot (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = 0 \tag{47}$$

which is of considerable importance in the numerical integration of (46).

Our *MuPAD* completion package [20] can handle polynomial nonlinearities, too. It stores all factors used in pseudo-reductions in a global list so that after the completion the user may check where problems with shrinking the given differential ideal might occur.

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