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Review: The formal theory provides a very general framework for studying differential equations combining geometric and algebraic elements. The author of the book has already published a number of articles on the theory and especially on its application to the calculus of variations in one and higher dimensions. The book is a self-contained exposition of his view of the field comparable with similar books by Pommaret (*Systems of partial differential equations and Lie pseudogroups*, Gordon & Breach, London 1978) or Krasilschik et al. (*Geometry of jet spaces and nonlinear partial differential equations*, Gordon & Breach, London 1986).
The main difference to previous works in this field is that the author dispenses with the use of jet bundles; he uses instead $\mathbb{R}^\infty$. The basic concept is that of a *diffiety*. Originally introduced by Vinagradov as the infinite prolongation of a fibred submanifold of some jet bundle, it replaces the classical notion of a differential equations. The author defines it as a flat submodule Ω of the module of all differential one-forms on $\mathbb{R}^\infty$. The contact structure of the jet bundle is replaced by the requirement that a so-called “good” filtration of this submodule by finite-dimensional submodules exists. This definition also leads to close relations between the presented material and Cartan's theory of exterior systems (for a modern exposition see Bryant et al. *Exterior Differential Systems*, Springer-Verlag, 1991, which covers many of the topics treated in this book).
The book contains nine chapters and an appendix. Chapter I introduces the geometry of and calculus on $\mathbb{R}^\infty$ which forms the basis of the subsequent chapters. Especially the introduction of vector fields requires some care. Chapter II discusses the important Frobenius theorem stating that finite-dimensional flat submodules are completely integrable. It also introduces transformation groups and their infinitesimal generators. The study of differential equations starts in Chapter III with the case of ordinary differential equations. It contains a first definition of diffities, introduces morphisms between them and factor and subdiffities. As a special case contact forms are treated showing the relation to the classical jet bundle approach. Finally, symmetries of diffieties are introduced. Throughout the main emphasis is on underdetermined systems.
In Chapter IV as a first application the Monge problem is treated extensively. In this rather forgotten problem one tries to express the general solution of one system of differential equations by that of another one. Typically, the second system is taken to be empty; in this case the problem can be rephrased as looking for some arbitrary functions parametrizing (together with their derivatives up to a finite order) the solution space. In the language of the author this means that one wishes to determine whether a given diffiety is isomorphic to a factor diffiety of a contact diffiety. As a second application Chapter V studies the calculus of variations in one dimension. The main point is the introduction of a generalized Poincaré-Cartan form omitting the introduction of Lagrange multipliers. No general theory is developed; rather the use of formal methods is demonstrated on some particular problems, especially on the inverse problem.
In order to extend the theory from ordinary to partial differential equations one needs a number of basic concepts from commutative algebra which are recalled in Chapter VI. This includes the Koszul complex, Hilbert resolutions and function, homology, and (quasi-)regular sequences. A number of analytic concepts like prolongation or characteristics are defined completely algebraically. Finally, the Koszul, the Dedecker, and the Spencer differential are introduced. Chapter VII on partial differential equations gives then the general definition of a diffiety. It is shown that it can indeed completely replace the classical notion of a system of differential equations. Heavy use is made of the algebraic machinery of the previous chapter in order to define acyclicity and involution which is the basis for the existence theory of general diffieties. The important compatibility algorithm is only briefly discussed. Other topics are an extensive discussion of

characteristics or point and generalized symmetries (especially for evolution equations).
The topic of Chapter VIII are Lie-Poisson pseudogroups. Firstly it is shown how they naturally arise in the context of the equivalence problem and the notion of a diffiety is slightly generalized (allowing for infinite-dimensional submodules) in order to introduce groupieties. This generalizes the usual point of view of pseudogroups that their transformations are defined as solutions of differential equations. The classical machinery for the study of Lie groups like Maurer-Cartan forms, invariants, sub- and factor groups, or homogeneous spaces is then generalized to cover groupieties. Chapter IX extends the results of Chapter V to multi-dimensional variational integrals. As an important tool the variational bicomplex is introduced and some links to the commutative algebra of Chapter VI are pointed out. The final chapter contains six appendices devoted to miscelleaneous topics: a review of the basic results of Lie on Lie groups, a comparison of Lie groups and pseudogroups, an introduction to spectral sequences, a discussion of diffieties with symmetries, the lowering of the order of a system of differential equations, and finally some more examples.
The book contains many highly interesting results that cannot easily be found elsewhere in the literature. However, the author follows the rather unfortunate "tradition" in this field to develop a completely individual and highly non-standard language. This makes the book very hard to read. Especially, it is almost impossible to read just a single section, if one is only interested in a specific topic. It does not help that the book has a very dense layout with most formulae inline and not displayed; it also would have profited from some language editing. The writing is very concise with only few examples. Thus the reader should have some previous acquaintance with the ideas of the formal theory of differential equations.
Remarks: