```
> restart:
  read "qFPS.mpl":
> with(qFPS):
```

Sitzung 01 (q-Ableitungen und q-Shifts)

q-Ableitung bzw. q-Shift von Potenzen

```
> qdiff(x^k,x,q);
```

$$\frac{(q^k-1)\,x^{k-1}}{q-1} \tag{1}$$

```
> qshift(x^k,x,q);
```

$$(q\,x)^k \tag{2}$$

Zweite q-Ableitung bzw. doppelter q-Shift des q-Pochhammersymbols

```
> qdiff(qpochhammer(x,q,k),[x$2],q);
```

$$-\frac{(q^k-1)\,(-q^k+q)\,qpochhammer(x,q,k)}{(q-1)^2\,(-1+x)\,(-1+q\,x)} \tag{3}$$

```
> qshift(qpochhammer(x,q,k),[x$2],q);
```

$$\frac{(-1+q\,x\,q^k)\,(-1+x\,q^k)\,qpochhammer(x,q,k)}{(-1+q\,x)\,(-1+x)} \tag{4}$$

q-Ableitung bzw. q-Shift der kleinen q-Exponentialfunktion

```
> qdiff(qexp(x,q),x,q);
```

$$-\frac{qexp(x,q)}{q-1} \tag{5}$$

```
> qshift(qexp(x,q),x,q);
```

$$(1-x)\,qexp(x,q) \tag{6}$$

```
> qdiff(qexp((1-q)*x,q),x,q);
```

$$qexp((1-q)\,x,q) \tag{7}$$

Inverser q-Shift

```
> qshift(qpochhammer(x,q,k),x,1/q);
```

$$\frac{(-q+x)\,qpochhammer(x,q,k)}{-q+x\,q^k} \tag{8}$$

Kompliziertere, höhere q-Ableitungen

```
> qdiff(sinq(x,q)*f(x,y)+cosq(y*x,q)*g(y),[x,y,y],q);
```

$$\begin{aligned}&-y^2\,q^2\,x\,(q-1)\,(q+1)\,cosq(y\,x,q)\,Dq_{y,y}(g(y))+y\,q^2\,(y^2\,q^3\,x^2-2\,q^2\,y^2\,x^2+y^2\,q\,x^2\\&\quad-1)\,Dq_{y,y}(g(y))\,sinq(y\,x,q)+(q-1)\,x\,cosq(x,q)\,Dq_{x,y,y}(f(x,y))+sinq(x,\\&\quad q)\,Dq_{x,y,y}(f(x,y))+cosq(x,q)\,Dq_{y,y}(f(x,y))-x\,(q+1)\,cosq(y\,x,q)\,g(y)\\&\quad+y\,x^2\,q^2\,sinq(y\,x,q)\,g(y)+(q+1)\,(y^2\,q^3\,x^2-q^2\,y^2\,x^2-1)\,Dq_y(g(y))\,sinq(y\,x,q)\\&\quad-y\,x\,(q+1)\,(q^2+q-1)\,cosq(y\,x,q)\,Dq_y(g(y))\end{aligned} \tag{9}$$

Sitzung 02 (q-Holonome Differential- und Rekursionsgleichungen)

Beispiele q-holonomer Differential- und Rekursionsgleichungen q-holonomer Funktionen

Potenzen

```
> qHolonomicDE(x^k,F(x));
```

$$(q-1)\,x\,Dq_x(F(x)) + \left(-q^k+1\right) F(x) = 0 \quad \textbf{(10)}$$

```
> qHolonomicRE(x^k,F(x));
```

$$F(x)\,q^k - Sq_x(F(x)) = 0 \quad \textbf{(11)}$$

Kleine q-Exponentialfunktion

```
> qHolonomicDE(qexp(x,q),F(x));
```

$$(q-1)\,Dq_x(F(x)) + F(x) = 0 \quad \textbf{(12)}$$

```
> qHolonomicRE(qexp(x,q),F(x));
```

$$Sq_x(F(x)) + (-1+x)\,F(x) = 0 \quad \textbf{(13)}$$

Große q-Cosinusfunktion

```
> qHolonomicDE(qCos(x,q),F(x));
```

$$q\,x\,(q-1)\,(q+1)\,Dq_x(F(x)) + (q-1)^2\left(1+q^2x^2\right) Dq_{x,x}(F(x)) + q\,F(x) = 0 \quad \textbf{(14)}$$

```
> qHolonomicRE(qCos(x,q),F(x));
```

$$(-1-q)\,Sq_x(F(x)) + q\,F(x) + \left(1+q^2x^2\right) Sq_{x,x}(F(x)) = 0 \quad \textbf{(15)}$$

q-Orthogonales Polynom: q-Laguerre

```
> qHolonomicDE(qLaguerreL(n,a,x,q),F(x));
```

$$(q-1)\left(q^a q^2 x - q\,q^n q^a x + q^a q + q\,q^a x - 1\right) Dq_x(F(x)) + x\,q^a q\,(q-1)^2\,(q\,x + 1)\,Dq_{x,x}(F(x)) - q^a q\left(q^n - 1\right) F(x) = 0 \quad \textbf{(16)}$$

```
> qHolonomicRE(qLaguerreL(n,a,x,q),F(x));
```

$$\left(-q\,q^n q^a x - q^a - 1\right) Sq_x(F(x)) + F(x) + q^a\,(q\,x+1)\,Sq_{x,x}(F(x)) = 0 \quad \textbf{(17)}$$

Sitzung 03 (q-Holonome Rekursionsgleichung für die verallgemeinerte q-hypergeometrische Funktion)

```
> qHolonomicRE(qphihypergeom([a,b,c],[d,e],x,q),F(x));
```

$$q\,(q\,x\,c + q\,x\,b + q\,x\,a - q - e - d)\,Sq_x(F(x)) - q^2\,(-1+x)\,F(x) + \left(x\,q^2\,a\,b\,c - d\,e\right) Sq_{x,x,x}(F(x)) + \left(-x\,q^2\,b\,c + d\,e - x\,q^2\,a\,c - x\,q^2\,a\,b + q\,e + d\,q\right) Sq_{x,x}(F(x)) = 0 \quad \textbf{(18)}$$

Sitzung 04 (Summen-, Produkt und Kompositionsalgorithmus)

```
> RE1:=qHolonomicRE(qsin(x,q),F(x));
```

$$RE1 := (-1-q)\,Sq_x(F(x)) + q\left(x^2+1\right) F(x) + Sq_{x,x}(F(x)) = 0 \quad \textbf{(19)}$$

```
> RE2:=qHolonomicRE(qpochhammer(x,q,k),F(x));
```

$$RE2 := (-1+x)\,Sq_x(F(x)) + \left(1 - x\,q^k\right) F(x) = 0 \quad \textbf{(20)}$$

```
> qSumRE(RE1,RE2,F(x));
```

$$q\Big(-1 + x - 2q + q^2x^3 + q\,x - q^2x\left(q^k\right)^2 + x\,q^2q^k + q\,x\,q^k + q^k - q\,x^2 - 2q^2x^2 - q^3x^4 - q^4x^4 + q^4x^3 + x^3q^5 - q^4x^2 - q^5x^4 + q^5x^5 + 3q^3x - \left(q^k\right)^2 + 2x\,q^2 - 2q^2 - q^3x\left(q^k\right)^2 + x\,q\left(q^k\right)^3 + q^4x^3\left(q^k\right)^2 - 2q^3x^2 + x\,q^4 - q^5x^2 - q^3 + 2q^kq - q^2\left(q^k\right)^2 + 2q^kq^2 + q^kq^3 - q\left(q^k\right)^2 + 2q^3x^3 - q\,x\left(q^k\right)^2 - x^2q^kq^3 - x^2q^kq^4 + x\,q^2\left(q^k\right)^3 - x^2q^2\left(q^k\right)^2\Big)\,Sq_x(F(x)) - q^3\left(x^2+1\right)\left(-1+x\,q^k\right)\left(q^3x^2 - x\,q^2 - q\,x + q - q^kq - q^k + \left(q^k\right)^2 + 1\right) F(x) + \left(-1+x\,q^2\right)\left(q\,x^2 - x - q\,x + 1 + q + \left(q^k\right)^2\right.$$

$$- q^k q - q^k\big)\, Sq_{x,x,x}(F(x)) + \big(1 - x + 2q - 2qx - q^k + qx^2 + 2q^2x^2 - q^4x^3 + q^4x^2 - 3q^3x + (q^k)^2 - 2xq^2 + 2q^2 + 2q^3x^2 + xq^kq^4 - 2xq^4 + q^5x^2 + q^3 - 2q^kq + q^2(q^k)^2 - 2q^kq^2 - q^kq^3 + q(q^k)^2 - q^3x^3 + x^2q^kq^3 + x^2q^kq^4 - x^3q^kq^5 + xq^kq^3 - xq^2(q^k)^3\big)\, Sq_{x,x}(F(x)) = 0$$

```
> qProductRE(RE1,RE2,F(x));
```

$$-(q+1)(-1+x)(-1+qxq^k)\, Sq_x(F(x)) + q(x^2+1)(-1+xq^k)(-1+qxq^k)\, F(x) + (-1+qx)(-1+x)\, Sq_{x,x}(F(x)) = 0 \quad (22)$$

```
> qCompositionRE(RE1,F(x),a*x^2);
```

$$(1+q^2)(q^3a^2x^4-1)\, Sq_x(F(x)) + q^2(1+a^2x^4)(q^2a^2x^4+1)\, F(x) + Sq_{x,x}(F(x)) = 0 \quad (23)$$

```
> qHolonomicRE(qsin(a*x^2,q),F(x));
```

$$(1+q^2)(q^3a^2x^4-1)\, Sq_x(F(x)) + q^2(1+a^2x^4)(q^2a^2x^4+1)\, F(x) + Sq_{x,x}(F(x)) = 0 \quad (24)$$

Sitzung 05 (q-Petkovšek-Algorithmus)

```
> RE1:=(q^(k+2)-1)*A(k+2)+(q^(2*k+2)*(1+q)-q^(k+1))*A(k+1)+q^(3*
  k+2)*A(k)=0;
```

$$RE1 := (q^{k+2}-1)\, A(k+2) + (q^{2+2k}(q+1) - q^{k+1})\, A(k+1) + q^{2+3k} A(k) = 0 \quad (25)$$

```
> qPetkovsek(RE1,A(k));
```

$$\left[(-1)^k q^{\mathrm{binomial}(k,\,2)},\ \frac{q^{k^2}}{qpochhammer(q, q, k)} \right] \quad (26)$$

```
> RE2:=A(k+2)-(1+q)*A(k+1)+q*(1-q^(2*k+1))*A(k)=0;
```

$$RE2 := A(k+2) - (q+1)\, A(k+1) + q\,(1 - q^{1+2k})\, A(k) = 0 \quad (27)$$

```
> qPetkovsek(RE2,A(k));
```

$$[\ ] \quad (28)$$

```
> qPetkovsek(RE2,A(k),quadratic);
```

$$\left[qpochhammer(-\sqrt{q}, q, k),\ qpochhammer(\sqrt{q}, q, k) \right] \quad (29)$$

Sitzung 06 (q-FPS-Algorithmus)

```
> convert(qexp(x,q),qFPS);
```

$$\sum_{k=0}^{\infty} \frac{x^k}{qpochhammer(q, q, k)} \quad (30)$$

```
> infolevel[qFPS]:=4:
> convert(qsin(x,q),qFPS);
convert/qFPS: q-holonomic recurrence equation of order 2
```

$$(-q-1)\, Sq_x(F(x)) + q(x^2+1)\, F(x) + Sq_{x,x}(F(x)) = 0$$

```
convert/qFPS: q-holonomic recurrence equation for series
coefficients of order 2
```

$$A(k) + (q^kq^2-1)(q^kq-1)\, A(k+2) = 0$$

```
convert/qFPS: solution of recurrence equation
```

$$\left[0,\ \frac{(-1)^k x^{1+2k}}{qpochhammer(q, q, 1+2k)} \right]$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{1+2k}}{qpochhammer(q, q, 1+2k)} \tag{31}$$

```
> infolevel[qFPS]:=0:
> convert(qCos(x^2,q),qFPS);
```

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{k(-1+2k)} \left(x^2\right)^{2k}}{qpochhammer(q, q, 2k)} \tag{32}$$

```
> convert(qphihypergeom([a,b],[c,d],x,q),qFPS,x);
```

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{1}{2}(k-1)k}\, qpochhammer(a, q, k)\, qpochhammer(b, q, k)\, x^k}{qpochhammer(c, q, k)\, qpochhammer(d, q, k)\, qpochhammer(q, q, k)} \tag{33}$$

Anwendungsbeispiel: Beweis von q-Identitäten

```
> convert(qsin(x,1/q),qFPS);
```

$$\sum_{k=0}^{\infty} \left(-\frac{(-1)^k q^{(k+1)(1+2k)} x^{1+2k}}{qpochhammer(q, q, 1+2k)} \right) \tag{34}$$

```
> convert(-qSin(q*x,q),qFPS);
```

$$\sum_{k=0}^{\infty} \left(-\frac{(-1)^k q^{(k+1)(1+2k)} x^{1+2k}}{qpochhammer(q, q, 1+2k)} \right) \tag{35}$$

oder

```
> convert(x/(1-q)*qphihypergeom([0,0],[q^3],-x^2,q^2),qFPS);
```

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{1+2k}}{qpochhammer(q, q, 1+2k)} \tag{36}$$

```
> convert(qsin(x,q),qFPS);
```

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{1+2k}}{qpochhammer(q, q, 1+2k)} \tag{37}$$

```
>
```