Orthogonal Polynomials and Special Functions

SIAM Activity Group on Orthogonal Polynomials and Special Functions

★ ★ ★ ★ **Newsletter** ★ ★ ★ ★

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Contents

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The News

During the final two days of the 1994 SIAM Annual Meeting, held last July in San Diego, we sponsored a Minisymposium on *Asymptotics and Special Functions.* Altogether twelve talks were given and we are printing summaries of eight of those talks here. The remaining four should appear in the next issue of the Newsletter.

This issue is late due to a substantial number of technical contributions plus other factors beyond our control. We will therefore streamline production. Methods for doing this will be found below, in the section called A Faster Newsletter, which contributors are encouraged to read. Incidentally, the deadline for submission of material for the Winter Issue is November 10. Material received after November 10 will be held until the Spring Issue.

According to a rumor picked up at the Airport in San Diego, our Activity Group charter has been renewed for another three years, but we have no official confirmation of this. Relevant data on the nine Activity Groups within SIAM is given later in this issue. As of August 31 our membership has grown to 171.

A few ideas have been floating around recently on devoting the 1995 Minisymposium in Charlotte to numerical and software issues. You might send your thoughts on this, or anything else, to Martin Muldoon, our Program Director.

For those who still need reassurance that our subject is alive and well, we note that there will be a substantial amount of orthogonal polynomials and special functions activity in Holland, France

(continues on p. 3)

THE PURPOSE *of the Activity Group is*

—to promote basic research in orthogonal polynomials and special functions; to further the application of this subject in other parts of mathematics, and in science and industry; and to encourage and support the exchange of information, ideas, and techniques between workers in this field, and other mathematicians and scientists.

Poolside in San Diego

Editorial Comments from the Chair

This may be a good time to review some recent history: Several years ago this group had small membership and an annual newsletter. These two facts may well be related. However the group did attract some energetic persons who got involved and wondered why there wasn't more activity. One of these, of course, was Eugene Tomer, the hard-working and dedicated Editor of this quarterly Newsletter. I cannot overemphasize how crucial Eugene's contributions are to the present good health of our group.

Although our group has grown, we are still experiencing some minor irritants such as imperfect subscription fulfillment, errors in membership listings, and slights in the allocation of meeting rooms and time-slots at annual SIAM meetings. We are making progress in the first two areas—Marta Lafferty at SIAM has been designated as contact person for problems in regard to membership listings. We will continue to lobby SIAM for more consideration at meetings, and we think we have made a start in approaching Bart Ng, a Vice President. It may be a good idea for individuals to write him regarding concerns with the relationship our group has with SIAM headquarters.

One more item needs to be emphasized—our Editor has no assistants, no staff, and no technical typist. Contributors to the Newsletter are asked to help as much as possible, especially in the submission of technical mathematical material. See the later part of this edition for specific recommendations. As Chair, I ask you to follow these closely. We are all grateful for the work that Eugene puts into this professional production but there are limits on what we can ask him to do.

To conclude my first column, I want to express my appreciation to every member of the group, the officers, editors, speakers, and Newsletter contributors, and I look forward to years of rewarding collaboration.

Charles Dunkl

Addresses, Phone Numbers, e-mails

The *SIAM Activity Group on Orthogonal Polynomials and Special Functions* consists of a broad set of mathematicians, both pure and applied. The Group also includes engineers and scientists, students as well as experts. We now have around 170 members scattered about in more than 20 countries. Whatever your specialty might be, we welcome your participation in this classical, and yet modern, topic.

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Current members should send any address corrections to lafferty@siam.org.

The News *(continued)*

and Canada this year and the next. For details you can refer to the Meetings and Conferences section in this issue.

Alberto Grünbaum would like to make a slight correction to Problem #3. It should read like this: A proof is known for the interval $1/2 \leq \alpha < 1$. Can one extend this to $0 < \alpha < 1$?

Alan Law has moved from University of Regina to Newfoundland. His new address is

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St. John's, NFLD A1B 3X7 Canada
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alaw@kean.ucs.mun.ca

S.B. Yakubovich, formerly of Minsk, would like us to know about his two recent books:

1. *Hypergeometric Approach to Integral Transforms and Convolutions,* S.B. Yakubovich and Yu.F. Luchko. Kluwer 1994, 336 pp.
2. *The Double Mellin-Barnes Type Integrals and Their Applications to Convolution Theory,* Nguyen Thanh Hai and S.B. Yakubovich. World Scientific Co., Pte., Ltd. 1992, 295 pp.

Yakubovich gives this address where he is spending one year with Prof. Saigo:

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sm037952@cc.fukuoka-u.ac.jp

We have had reports that some members have not received their three Ramanujan booklets, which were advertised in the Summer 1993 Newsletter, page 6, having waited many months. If you have also encountered this problem, please write to

Prof G. Rangan
Ramanujan Institute for
Advanced Study in Mathematics
University Buildings, Chepauk
Triplicane P.O. Madras 600 005, India

(continues on p. 4)

The News *(continued)*

Rangan already knows about the problem and he will look into the matter. You can also try an e-mail to him at the address

unimad!chari@iitm.ernet.in

plus variations on this. The Editor would also like to know your story so please contact him.

Some people said at the San Diego meeting that they have not been getting their Newsletters. Of course if you are reading this, then you received the current edition. If you have had problems, please inform SIAM as well as the Editor.

The Activity Group Membership Directory still contains some errors. We now have a designated individual at SIAM who will make the corrections. Contact

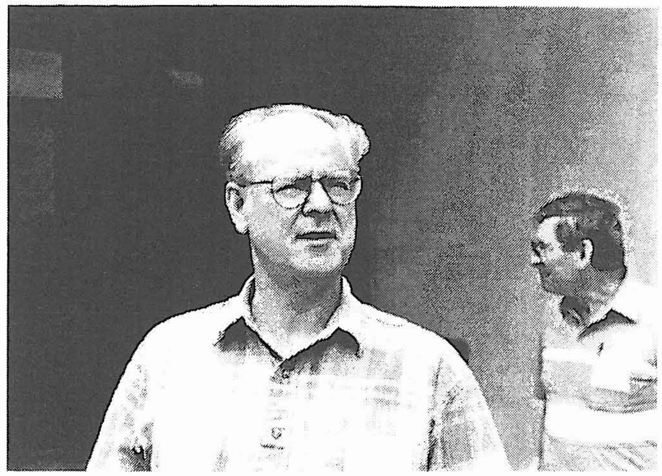
Marta Lafferty (lafferty@siam.org).

At the same time you should check your listing in the Combined Membership List (CML). If you see an error, then you may want to inform AMS or MAA as well. The three organizations evidently have different databases. It is not clear how the CML arrives at a listing when the databases do not agree. So you have to inform everybody of address changes.

Vickie Kearn at SIAM would like to remind us that the membership renewals were mailed out in August and that SIAM administration encourages renewal as early as possible.

You are also reminded to e-mail any news items for the OP-SF Net to the address: poly@siam.org. The Net reaches some people who are not members of the Activity Group and who therefore do not get the printed Newsletter.

Finally, it would be helpful if you would inform us of any planned visits you or your colleagues may be undertaking (say for more than a week) so we can announce this in the Newsletter and the OP-SF Net. For example, we know that Eric Opdam will be in Ann Arbor, Michigan during what they call the winter semester.



George Gasper (left) emerges from an afternoon session while Martin Muldoon (right) waits for the others.

Data on the SIAM Activity Groups

Below is some data we received from SIAM on the nine activity groups.

Activity Group	*Est.*	*Members*	*Q.N.*	*Net*
Linear Algebra	1982	632	sa	Yes
Supercomputing	1984	999	No	Yes
Discrete Math.	1984	635	Yes	No
Optimization	1985	646	o	No
Control & Sys.	1986	453	Yes	No
Dynamical Sys.	1988	592	No	Yes
Geo. Design	1990	217	No	No
OP & SF	1990	171	Yes	Yes
Geosciences	1991	284	o	No

Legend ——

Est. = Year Established *sa = semiannual*
Q.N. = Quarterly Newsletter *o = occasional*
Net = Electronic News Net

There seems to be some correlation between the date of approval and the total membership. Only three SIAGs have a quarterly newsletter. Perhaps more significant is the total number of memberships. If one assumes the groups are pretty much disjoint, then at least 1/2 of SIAM belongs to one group or another.

Data as of August 31, 1994, from Bernadetta Di Lisi at SIAM (e-mail: bdilisi@siam.org).

A Faster Newsletter

Since the introduction of the current Newsletter format two years ago, various news items and articles have been submitted by various modes: by e-mail, by fax, telephone, US Mail, even notes scribbled at a meeting somewhere. This seems altogether appropriate and there have been no complaints so far. With this Issue, however, the Editor was a bit overwhelmed by the *variety* of files and manuscripts received, not to mention typing input files from hard copy.

We must therefore impose requirements on submission of articles and news items. Otherwise the Newsletter will be delayed or scaled back, or even eliminated.

Accordingly, we have adopted the following:

- All submissions of material must go *directly* from the author to the Editor (without an intermediary), and the same goes for corrections, additions and deletions.

- Items without mathematical symbols can be submitted in any form, for example as a textfile or as a typewritten manuscript.

- Newsletter items *containing substantial mathematical symbols* must be submitted as a *standard* LaTeX file by e-mail or by disc. There is only one exception:

- *Plain* TeX files can be accepted by making special arrangements with the Editor.

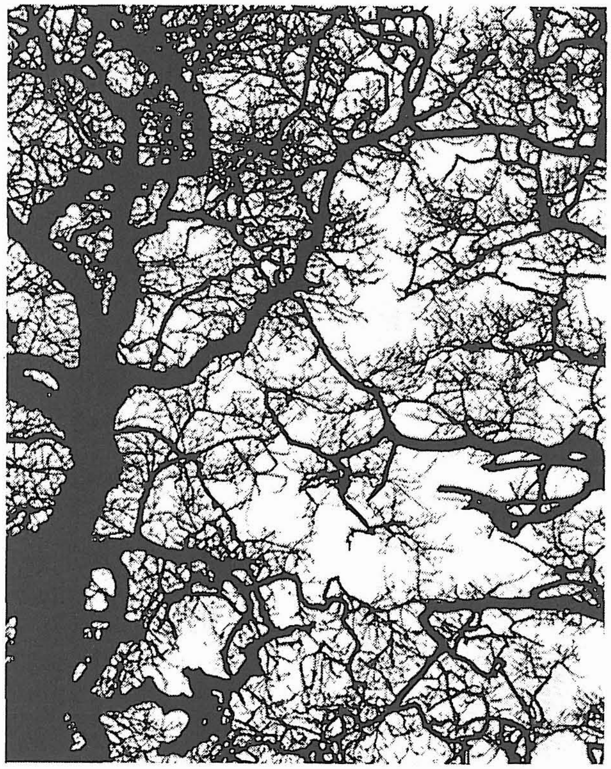
- AMS-TeX and AMS-LaTeX files cannot be accepted. You must translate them into standard LaTeX.

- Whatever the content of your LaTeX article, you must use the *preamble* given in the last section of every Newsletter. You can obtain this preamble by e-mail from the Editor. Other preambles, macros, and style files obtained from archives are not allowed.

- To accommodate special situations, a limited use of `\newcommand`, `\renewcommand`, and `\newenvironment` is allowed provided sufficient explanatory comments are written in the file.

Material submitted for the Newsletter can also include photos and drawings. Photos must have good definition and contrast, black and white preferred, minimum size 10 by 15 cm. Later on we may experiment with electronic transmission of photos and drawings but at this stage, all photos and drawings must be *hard copy.*

These instructions will be summarized in the last section of every issue, including this one.



The Newsletter is assembled from submitted material by Eugene Tomer in San Francisco. Work is done on a Macintosh Quadra 800 using the latest Textures software from Blue Sky Research. With two monitors, one for the LaTeX input and one for the output, this gives essentially instant typesetting of small to medium files, equivalent to a *wysiwyg* setup. A second computer provides backups. Drawings are usually done with MacDraw Pro which was also used to design the AG logo. A host of other software is also available. Photos are taken with Nikon professional equipment, carefully processed, then scanned with a basic Apple OneScanner and inserted into the Textures file. Then a first draft is printed on a LaserWriter Pro.

At this stage the contributors are contacted to verify the accuracy and completeness of the results, after which iterations are made. What appears to be draft $\# n-1$ is sent to Charles Dunkl and George Gasper who are the other two members of the editorial board. When agreement has been reached, draft $\# n$ (the final manuscript) is sent to SIAM in Philadelphia where the ads are supposed to be inserted and final checks are made. Then SIAM reproduces the Newsletters and mails them out. This happens four times a year.

We therefore encourage authors to submit their contributions in a form that minimizes the editorial work so the Newsletter will be produced on time.

Workshop in Leganes, Spain

Marcel de Bruin of the Technical University of Delft sent this report on a recent gathering in Madrid.

A four day "Workshop on Orthogonal Polynomials on the Unit Circle: Theory and Applications" was hosted by the Departamento de Ingeniería, Universidad Carlos III, in its new building in Leganés, near Madrid, Spain. This took place during June 27–30, 1994.

Organization was in the able hands of Paco Marcellán who invited a group of experts. Each expert gave two one-hour lectures to form the core of the workshop:

- Adhemar Bultheel (Katholieke Universiteit, Leuven, Belgium): *1) Orthogonality on the unit circle: generalizations and new developments 2) Orthogonal rational functions and applications.*
- Leonid Golinskii (Ukrainian Academy of Sciences, Kharkov, Ukraine): *1) Measures on the unit circle, orthogonal polynomials and reflection coefficients 2) Schur functions, Schur parameters and orthogonal polynomials on the unit circle.*
- Xin Li (University of Central Florida, Orlando, USA): *Asymptotics of polynomials orthogonal with varying measures I & II.*
- Franz Peherstorfer (Johannes Kepler Universität, Linz, Austria): *Perturbed orthogonal polynomials: explicit representations and asymptotics I & II.*
- Lothar Reichel (Kent State University, Kent, USA): *1) The eigenvalue problem for unitary matrices and applications to signal theory 2) Least squares approximation by trigonometric polynomials.*
- Walter Van Assche (Katholieke Universiteit, Leuven, Belgium): *1) Spectral theory and Rakhmanov's theorem for orthogonal polynomials on the unit circle 2) Orthogonal matrix polynomials on the unit circle and applications.*

Then the program was rounded out with eight one-hour lectures given by:

A. Cachafeiro (Vigo, Spain)
D. Calvetti (Hoboken, USA)
P. González-Vera (La Laguna, Spain)
F. Marcellán (Madrid, Spain)
D. Peña (Madrid, Spain)
P.E. Ricci (Roma, Italia)
R. Steinbauer (Linz, Austria)
F.H. Szafraniec (Kraków, Poland)

In addition, Xin Li chaired a session on open problems. This workshop was attended by approximately fifty people from eleven countries.

Report From Leiden-Amsterdam

A "Concentration Period on Representation Theory and q-special Functions" was the topic of a special program in Leiden and in Amsterdam last spring. The organizers, Gerrit van Dijk and Eric Opdam at Leiden University, and Tom Koornwinder at the University of Amsterdam, hosted 15 guests who stayed for various intervals. Seminars were held each Tuesday. The program was sponsored, in part, by a Stimulus Grant from NWO, the Dutch national research organization. This grant was given to the inter-university Stieltjes Research School for Mathematics. Tom Koornwinder sent this report on the activities:

First of all, three guests who taught short courses deserve a special mention. Charles F. Dunkl (Charlottesville, Virginia) spoke about his Dunkl operators, and their connections with special functions and representation theory. Then Ian G. Macdonald (London, England) treated his celebrated Macdonald polynomials; these are are orthogonal polynomials in several variables associated with root systems, and which generalize the q-ultraspherical polynomials. The second half of Macdonald's course explained Ivan Cherednik's important recent work on an approach to Macdonald polynomials by representations of double affine Hecke algebras, where certain q-analogues of the Dunkl operators appear. Finally, Masatoshi Noumi (Tokyo, Japan) lectured on quantum groups, in particular on an interpretation of a certain class of Macdonald polynomials as spherical functions on a quantum homogeneous space.

These three main courses were quite representative of the topics discussed during this special period. Root systems are basic. In fact, a host of objects is associated with them: special functions; differential, difference, and reflection operators, or a mixture of these; Lie groups and homogeneous spaces or their quantum analogues; discrete groups generated by reflections, their group algebras, and deformations of these algebras.

The Dutch school, including the organizers mentioned earlier, and certainly also Gert Heckman from Nijmegen, has been active in this area for a long time. We were happy to have so many visitors sharing our excitement for this subject.

The number of guests in Leiden reached its peak at the beginning of May, while the first week of June was also very busy in Amsterdam. And, needless to say, there were many pleasant social activities during this special period.

The July Minisymposium in San Diego

The 1994 Annual Meeting of SIAM took place during July 25–29 at the Sheraton Harbor Island East Hotel in San Diego. The meeting was dedicated to I. Edward Block on the occasion of his retirement as the managing director of SIAM.

The varied and interesting program included several invited addresses, minisymposia, contributed papers, and presentation of awards and prizes. Martin D. Kruskal gave the John von Neumann Lecture, with the title "Surreal Numbers". A George Polya Prize was given to Gregory V. Chudnovsky, followed by a talk on "Fast Computational Methods in Pure and Applied Mathematics" given jointly by Gregory and his brother, David.

Another feature of the meeting was the larger number of sessions on applied mathematics education. And in addition to the usual exhibit of books and software, there was a Poster-Video session combined with a reception on the first evening.

Among the contributed paper sessions was one with the title "Applied Analysis; Asymptotics and Special Functions", with seven speakers.

Our Activity Group organized a successful Minisymposium "Special Functions and Asymptotics" with talks by Roderick Wong, Adri Olde Daalhuis, Mark Dunster, Frank Stenger, Richard Askey, Bruce Berndt, Ronald Evans, George Gasper, Audrey Terras, Jeff Geronimo, Renato Spigler and André Ronveaux. Below we present summaries of eight of these twelve talks. The remaining four should appear in the next issue.

The announced business meeting of our Activity Group did not take place because too few people showed up. However, organisational matters were discussed informally, such as our sponsorship of a Minisymposium at ICIAM '95 in Hamburg. This matter will be decided soon.

The next SIAM Annual Meeting will be held October 23–26, 1995 in Charlotte, North Carolina, (and not in Louisville, as announced earlier). The 1996 Annual Meeting will be held in Kansas City, Missouri, in mid-July.

Uniform Asymptotic Expansion of an Oscillatory Integral

by Roderick Wong

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In elastic waves, integrals of the form

$$I(x,t) = \int_{-\infty}^{\infty} a(k)\, e^{i(\omega t - kx)}\, dk$$

arise frequently, where k is the wave number and ω is the wave frequency. The dispersion relation, $\omega = \omega(k)$, is usually given implicitly by a transcendental equation. To obtain the long time behavior (large t) of $I(x,t)$, one holds x/t fixed, and applies Kelvin's principle of stationary phase. This principle asserts that the major contribution to the integral $I(x,t)$ comes from points where the phase function $\omega t - kx$ is stationary, that is, the points where $d\omega/dk = x/t$. Let k_0 be such a point. Clearly k_0 depends on the ratio x/t. If k_0 is finite and $\omega''(k_0) \neq 0$, then it is well known that as $t \to +\infty$,

$$I(x,t) \sim a(k_0)\sqrt{\frac{2\pi}{t|\omega''(k_0)|}}\; e^{-ik_0x + i\omega(k_0)t + i(\pi/4)\operatorname{sgn}\omega''(k_0)}.$$

If k_0 is infinite or if $\omega''(k_0)$ vanishes as k_0 varies, then this formula fails to hold.

In studying the shear-wave front of transient waves in a layer, we encounter the integral $I(x,t)$, where the dispersion relation is given implicitly by a transcendental equation involving ω^2 and k^2. Furthermore, for large values of k, we have

$$\omega^2(k) = \gamma_{-1}k^2 + \gamma_0 + \frac{\gamma_1}{k^2} + \cdots,$$

where $\gamma_{-1} > 0$. As a consequence $\omega(k)$ has different expansions for large positive values of k and for large negative values of k, and we have a stationary point k_0 tending to $+\infty$ as $x/t \to \gamma_{-1}^{1/2}$. Since $\omega(k)$ is a function of k^2, $d\omega/dk$ is an odd function of k, and therefore $-k_0$ cannot be a stationary point.

Here we present an asymptotic expansion for the integral $I(x,t)$ as $t \to +\infty$, which holds uniformly for x/t near $\gamma_{-1}^{1/2}$ as long as their difference is $O(1/t)$.

Reference

H.-H. Day and R. Wong, "A uniform asymptotic expansion for the shear-wave front in a layer". Wave Motion **19**, 1994, pp. 293–308.

Left to right: Frank Stenger, Adri Olde Daalhuis, T. Mark Dunster, George Gasper, and Roderick Wong.

Hyperasymptotics

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1. Introduction

As was observed by Dingle in [2], in many cases the coefficients a_s of a divergent ordinary (Poincaré) asymptotic expansion $f(z) \sim \sum_{s=0}^{\infty} a_s z^{-s}$ grow like $a_s = \Gamma(s)\,\mathcal{O}(1)$, as $s \to \infty$. If we truncate such an expansion after N terms, then the remainder $R_N(z) = f(z) - \sum_{s=0}^{N-1} a_s z^{-s}$ is of order z^{-N} as $|z| \to \infty$ and N fixed. But when we minimize $|R_N(z)|$ as a function of N, the optimal number of terms is given by $N = N_0 = |z| + \mathcal{O}(1)$, and the minimal remainder is of order $z^{-1/2}\exp(-|z|)$. The minimal remainder can be expanded in a new (divergent) asymptotic expansion. Again, this new expansion can be truncated at an optimal number of terms. Repetition of this procedure at successive levels leads to a *hyperasymptotic expansion*, where at the nth level the minimal remainder is of order $\exp(-\lambda_n|z|)$, with $1 < \lambda_1 < \lambda_2 < \cdots$.

Instead of giving an extended summary of my presentation at the Minisymposium in San Diego, I show in §2 how to obtain a hyperasymptotic expansion for the K_0-Bessel function. I show in §3 how the method of §2 can be used to obtain hyperasymptotic expansions for solutions of ODE's.

2. Hyperasymptotics for the K_0-Bessel function

Let $u(z) = \pi^{-1/2} z^{1/2} e^{z/2} K_0(z/2)$. This function has the asymptotic expansion

$$u(z) = \sum_{s=0}^{N-1} \frac{a_s}{z^s} + R_N^{(0)}(z), \qquad (1)$$

where $a_s = (-)^s (2s!)^2 16^{-s} (s!)^{-3} \sim \pi^{-1}(-)^s \Gamma(s)$, and $R_N^{(0)}(z) = z^{-N}\mathcal{O}(1)$ when N is fixed and $z \to \infty$ in $|\mathrm{ph}\, z| < 3\pi/2$. We use the integral representation

$$R_N^{(0)}(z) = \frac{(-)^N}{z^{N-1}\pi} \int_0^\infty \frac{e^{-t}\, t^{N-1}}{t+z}\, u(t)\, dt, \qquad (2)$$

(see [1]) to find the optimal N. It is not difficult to show that the right-hand side of (2) is minimal for $N = N_0 = |z| + \mathcal{O}(1)$, and we have

$$R_{N_0}^{(0)}(z) = e^{-|z|} z^{-1/2}\,\mathcal{O}(1), \quad z \to \infty \quad \text{in } |\mathrm{ph}\, z| \le \pi. \quad (3)$$

To obtain a re-expansion for $R^{(0)}_{N_0}(z)$, we substitute (1) into (2):

$$R^{(0)}_{N_0}(z) = \frac{(-)^{N_0}}{z^{N_0-1}\pi}\sum_{s=0}^{N_1-1} a_s G^{(1)}(z; N_0 - s) + R^{(1)}_{N_1}(z), \quad (4)$$

where $G^{(1)}(z; M_0) = \int_0^\infty e^{-t}\, t^{M_0-1}(t+z)^{-1}\, dt$, and

$$R^{(1)}_{N_1}(z) = \frac{(-)^{N_0+N_1}}{z^{N_0-1}\pi^2}\int_0^\infty\int_0^\infty \frac{e^{-t-t_1}t^{N_0-N_1}t_1^{N_1-1}}{(t+z)(t_1+t)}u(t_1)dt_1dt. \quad (5)$$

Thus our first re-expansion is in terms of generalized exponential integrals. Again, we want to find the minimal remainder. If we take $N_0 = |z| + \mathcal{O}(1)$, then the optimal N_1 is $N_1 = |z|/2 + \mathcal{O}(1)$, and we have $R^{(1)}_{N_1}(z) = \exp(-1.69\cdots|z|)\,\mathcal{O}(1)$. But if we minimize $R^{(1)}_{N_1}(z)$ as a function of both N_0 and N_1, then we find that the optimal numbers of terms are $N_0 = 2|z| + \mathcal{O}(1)$, $N_1 = |z| + \mathcal{O}(1)$, and the minimal remainder at "level 1" is given by

$$R^{(1)}_{N_1}(z) = e^{-2|z|}\,\mathcal{O}(1), \quad z\to\infty \quad \text{in } |\mathrm{ph}\, z| \le \pi. \quad (6)$$

To obtain the next re-expansion we substitute (1) into (5):

$$R^{(1)}_{N_1}(z) = \frac{(-)^{N_0+N_1}}{z^{N_0-1}\pi^2}\sum_{s=0}^{N_2-1} a_s G^{(2)}(z; N_0 - N_1, N_1 - s) + {} + R^{(2)}_{N_2}(z), \quad (7)$$

where $G^{(2)}(z; M_0, M_1) = \int_0^\infty\int_0^\infty e^{-t-t_1}t^{M_0}t_1^{M_1-1}(t+z)^{-1}\cdot$ $\cdot(t_1+t)^{-1}\, dt_1\, dt$. Now the optimal numbers of terms are $N_0 = 3|z| + O(1)$, $N_1 = 2|z| + O(1)$, $N_2 = |z| + \mathcal{O}(1)$, and the minimal remainder is given by

$$R^{(2)}_{N_2}(z) = e^{-3|z|}z^{-1}\,\mathcal{O}(1), \quad z\to\infty \quad \text{in } |\mathrm{ph}\, z| \le \pi. \quad (8)$$

We can repeat this procedure, and we obtain a hyperasymptotic expansion in terms of multiple integrals, with a minimal remainder at level m of the order $R^{(m)}_{N_m}(z) = \exp(-m|z|)\,\mathcal{O}(1)$.

3. Hyperasymptotics for solutions of ODE's

The homogeneous linear differential equation of the second order is given by

$$\frac{d^2w}{dz^2} + f(z)\frac{dw}{dz} + g(z)\, w = 0. \quad (9)$$

We suppose that infinity is an irregular singularity of rank one. After normalization we may assume, without loss of generality, that (9) has unique solutions $w_1(z)$ and $w_2(z)$ such that as $|z|\to\infty$

$$\begin{aligned} w_1(z) &\sim e^{-z}z^{-\omega}\sum_{s=0}^\infty \frac{a_{s,1}}{z^s}, \\ w_2(ze^{\pi i}) &\sim \sum_{s=0}^\infty \frac{a_{s,2}}{z^s}. \qquad |\mathrm{ph}\, z| < 3\pi/2, \end{aligned} \quad (10)$$

With these expansions, and with the connection formulas

$$\begin{aligned} w_1(z) &= e^{-2\pi i\omega}w_1(ze^{-2\pi i}) + C_1w_2(z), \\ w_2(z) &= w_2(ze^{+2\pi i}) + C_2w_1(z), \end{aligned} \quad (11)$$

we obtain in [3] new Stieltjes-type integral representations for $w_1(z)$ and $w_2(z)$. Beginning with these integral representations we use a method that is very similar to the method of §2, and we obtain in [4] and [5] hyperasymptotic expansions for $w_1(z)$ and $w_2(z)$.

As a side result we obtain

$$\begin{aligned} C_1 &= \frac{(-)^s 2\pi i e^{-\omega\pi i}a_{s,1}}{\sum_{j=0}^{m-1} a_{j,2}\,\Gamma(s+\omega-j)} + \mathcal{O}(s^{-m}), \\ C_2 &= \frac{(-)^{s+1}2\pi i a_{s,2}}{\sum_{j=0}^{m-1} a_{j,1}\,\Gamma(s-\omega-j)} + \mathcal{O}(s^{-m}), \end{aligned} \quad (12)$$

as $s\to\infty$, where m is a fixed integer. These two expansions furnish powerful numerical tools to compute the connection coefficients C_1 and C_2 of (11).

References

[1] W.G.C. Boyd, "Stieltjes transforms and the Stokes phenomenon". Proc. Roy. Soc. London, Ser. A, **429**, 1990, pp. 227–246.

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New Uniform Asymptotic Approximations For Jacobi Polynomials

by T.M. DUNSTER

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1. Introduction

Jacobi functions are solutions of the differential equation

$$(1-x^2)y''+[(\beta-\alpha)-(\alpha+\beta+2)x]y'+n(n+\alpha+\beta+1)y=0.$$

Consider the solution $P_n^{(\alpha,\beta)}(x)$. This solution is recessive [2, pp. 48 and 155] at $x=1$. It is also a polynomial when $n=0,1,2,\ldots$ although we do not require that. In the present investigation a uniform asymptotic approximation is given for nonnegative values of the parameters, with α fixed, and for the case

$$u \equiv n+\frac{\alpha+\beta+1}{2} \to \infty.$$

We observe that

$$w(u,\beta,x) \equiv (1-x)^{(\alpha+1)/2}(1+x)^{(\beta+1)/2}P_n^{(\alpha,\beta)}(x)$$

satisfies

$$\frac{d^2w}{dx^2}=\left[u^2\frac{x-(1-a)}{(x-1)(x+1)^2}+\frac{\alpha^2-1}{4(1-x^2)}+\right.$$
$$\left.+\frac{\alpha^2-1}{4(1-x)^2}-\frac{1}{4(x+1)^2}\right]w, \qquad (1)$$

where $a=2-\frac{1}{2}\beta^2u^{-2}$. Equation (1) has a turning point at $x_t=1-a$, and regular singularities at $x=\pm1$. For the dominant term involving u^2 we observe that $x=1$ is a simple pole and $x=-1$ is a double pole.

Note that $a\to2$ as $\beta\to0$, or as $u\to\infty$ with β fixed, and so the turning point coalesces with the double pole because $x_t=1-a\to-1$ in these circumstances. If $\beta\to\infty$, then $a\to0$ (since $\frac{1}{2}\beta^2u^{-2}\to2$) and the turning point coalesces with the simple pole, i.e., $x_t=1-a\to1$. We seek asymptotic approximations for $0\le a\le2-\delta$, where we denote by δ an arbitrary positive constant, so that the turning point can coalesce with the simple pole at $x=1$ but not with the double pole at $x=-1$.

2. Results

The Liouville transformation $W(\xi)=(d\xi/dx)^{1/2}w(x)$ with

$$(\xi')^{-2}\frac{x-(1-a)}{(x-1)(x+1)^2}=\frac{\xi-\sigma}{\xi}, \qquad (2)$$

transforms (1) into the form

$$\frac{d^2W}{d\xi^2}=\left[u^2\frac{\xi-\sigma}{\xi}+\frac{\alpha^2-1}{4\xi^2}+\frac{\Psi(\sigma,\xi)}{\xi}\right]W, \qquad (3)$$

where the function $\Psi(\sigma,\xi)$, which can be given explicitly, will be analytic at $\xi=0$ and σ. Integration of (2) yields

$$\int_0^\xi\left[\frac{\sigma-\tau}{\tau}\right]^{1/2}d\tau=\int_x^1\left[\frac{t-1+a}{(1-t)(t+1)^2}\right]^{1/2}dt, \qquad (4)$$

where the arbitrary integration limits are chosen so that $\xi=0$ corresponds to $x=1$. For the turning points to correspond, i.e., for $x=1-a$ to map to $\xi=\sigma$, one finds that $\sigma=2-(\beta/u)$. The branches of the square roots in (4) are chosen so that ξ is real and positive for $1-a<x<1$, and continuous in $\mathrm{Re}(x)\ge0$ (at least). Actually $\xi(x)$ is analytic in $-\pi<\arg(x+1)<\pi$.

If we neglect the $\Psi(\sigma,\xi)/\xi$ term in (3), we have an equation with the exact solution $M_{u\sigma/2,\alpha/2}(2u\xi)$, Whittaker's confluent hypergeometric function. We have chosen this particular solution since it (and it alone) is recessive at the singularity $\xi=0$ corresponding to $x=1$.

Dunster [1] has established asymptotic solutions of the transformed equation (3), uniformly valid in certain unbounded complex domains containing the pole and the turning point, with $\sigma\in[0,\Lambda]$. One such solution is $M_{u\sigma/2,\alpha/2}(2u\xi)+\epsilon(u,\sigma,\xi)$, where $\epsilon(u,\sigma,\xi)$ is explicitly bounded, and is $O(M_{u\sigma/2,\alpha/2}(2u\xi))\cdot O((u\sigma+1)^{1/3}u^{-1+\delta})$ uniformly in a domain containing the critical points, except near the zeros of $M_{u\sigma/2,\alpha/2}(2u\xi)$. On matching this asymptotic solution with the Jacobi function and employing the series expansion for $M_{u\sigma/2,\alpha/2}(2u\xi)$ as in [2, p.255], we arrive at the following

Theorem. *Under all the above conditions we have the uniform asymptotic approximation*

$$P_n^{(\alpha,\beta)}(x)=\frac{C_n^{(\alpha,\beta)}}{(1+x)^{\beta/2}}\times$$
$$\times\left(\frac{2n+\alpha+1-u\xi}{x-1+a}\right)^{1/4}\left(\frac{\xi}{1-x}\right)^{(2\alpha+1)/4}\times$$
$$\times\left\{e^{-u\xi}\sum_{s=0}^{\infty}\frac{(-n)_s}{(\alpha+1)_s}\frac{(2u\xi)^s}{s!}+(2u\xi)^{-(\alpha+1)/2}\epsilon(u,\sigma,\xi)\right\},$$

where

$$C_n^{(\alpha,\beta)}=\frac{\binom{n+\alpha}{n}2^{(4\alpha+2\beta+3)/4}}{(2n+\alpha+2\beta+1)^{(\alpha/2)}(2n+\alpha+\beta+1)^{(2\alpha-1)/4}},$$

uniformly valid for complex x lying in (at least) the right half-plane.

3. Remark

The parameter α is fixed, and β and n satisfy

$$0\le1-\beta^2(2n+\alpha+\beta+1)^{-2}\le1-\delta.$$

Therefore the approximation is valid either for large n with β satisfying

$$\delta(1-\delta)^{-1}(2n+\alpha+1) \leq \beta < \infty,$$

or for large β with n satisfying $0 \leq n \leq \delta^{-1}\beta$.

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Applications of Sums and Integrals of Squares of Special Functions

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It is well known that the solutions to many problems depend on being able to prove that a specific function, for example a kernel, is positive or at least nonnegative. In this talk we consider how sums and integrals of squares of special functions can be used to prove the positivity or nonnegativity of some important functions that arise in mathematical analysis.

In particular, we consider their use in proving the positivity of certain sums, for example the Cesàro kernels, and integrals of products of orthogonal polynomials and of other special functions. We also discuss applications to absolutely monotonic, and completely monotonic, functions.

Then we show how this method can be used to prove that certain entire functions

$$F(x+iy), \qquad -\infty < x, y < \infty,$$

such as the classical orthogonal polynomials, the cosine transform of $(1-t^2)_+^{\alpha}$, for $\alpha > -1$, and the cosine transform of $\exp(-a\cosh(t))$, for $a > 0$, have only real zeros. It is shown that the derived formulas also lead to families of inequalities for the first and second order partial derivatives of $|F(x+iy)|^2$ with respect to y, which give new proofs of the reality of the zeros of $F(x+iy)$.

For example, consider the Meixner-Pollaczek polynomials

$$P_n^{\lambda}(x;\alpha) = c_n^{\lambda}\, e^{-in\alpha}{}_2F_1(-n, \lambda - ix;\, 2\lambda; 1-e^{2i\alpha}),$$

which are orthogonal with respect to the weight function $e^{-(\pi-2\alpha)x}\,|\Gamma(\lambda+ix)|^2$ on $(-\infty,\infty)$ when $0<\alpha<\pi$ and $\lambda>0$. For these polynomials we derived a sum of squares expansion of the form

$$|P_n^{\lambda}(x+iy;\alpha)|^2 = \left(P_n^{\lambda}(x;\alpha)\right)^2 + \\ + y^2\sum_{k=1}^{n} a_{n,k}^{\lambda}(x,y,\alpha)\left(P_{n-k}^{\lambda+k}(x;\alpha)\right)^2,$$

with each $a_{n,k}^{\lambda}(x,y,\alpha) \geq 0$, from which it immediately follows that $P_n^{\lambda}(x+iy;\alpha)$ has only real zeros.

For the Macdonald functions

$$K_{iz}(a) = \int_0^{\infty} e^{-a\cosh(t)}\cos(zt)\,dt, \qquad a>0,$$

we derived the formula

$$|K_{i(x+iy)}(a)|^2 = (K_{i(x+iy)}(a))^2 + \\ + y^2\int_0^1 t^{y-1}{}_2F_1(y+1,y+1;2;1-t)(K_{ix}(at^{-1/2}))^2\,dt,$$

which shows that the left side is positive when $y \neq 0$, and hence that the functions $K_{iz}(a)$ have only real zeros in z when $a>0$.

To explain the origin of Problem 2 in the Problems section of this Newsletter, we made the following observation. Take the second partial with respect to y of both sides of the above identity to obtain that

$$\frac{\partial^2}{\partial y^2}|K_{i(x+iy)}(a)|^2 = \\ = \int_0^1 \frac{\partial^2}{\partial y^2}[y^2t^y{}_2F_1(y+1,y+1;2;1-t)](K_{ix}(at^{-1/2}))^2\frac{dt}{t},$$

and then observe that in order to use this formula to prove that $\frac{\partial^2}{\partial y^2}|K_{i(x+iy)}(a)|^2 \geq 0$, which would give another proof of the reality of the zeros of $K_{iz}(a)$, it suffices to prove that $y^2t^y{}_2F_1(y+1,y+1;2;1-t)$ is a convex function of y whenever $-\infty<y<\infty$ and $0<t<1$.

Comparison of Special Functions for Finite and Continuous Symmetric Spaces

by AUDREY TERRAS

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Much of this work has been done jointly with Jeff Angel, Nancy Celniker, Steve Poulos, Cindy Trimble and Elinor Velasquez, and it has appeared in various volumes of Contemporary Mathematics, e.g., Vols. 138 and 143. Some of the background will be found in A. Terras *Fourier Analysis on Finite Groups and Applications* (book in preparation).

Definition.

A *symmetric space* is a quotient G/K, where K is a subgroup of the group G such that the algebra $L^1(K\backslash G/K)$ of K-bi-invariant functions on G is a commutative algebra under convolution on G.

We consider two basic examples.

Example 1.
The Poincaré Upper Half Plane over $\mathbf{R}$.

Here $G = SL(2, \mathbf{R})$, $K = SO(2)$, $G/K \cong H$, the Poincaré upper half plane with arc length given by

$$ds^2 = y^{-2}(dx^2 + dy^2),$$

and $g \in G$ acting by fractional linear transformation

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad gz = \frac{az+b}{cz+d}.$$

The G-invariant Poincaré arc length corresponds to the Laplacian

$$\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

The special functions under consideration are the eigenfunctions of Δ with additional invariance properties. They are very well known and essential to the study of harmonic analysis on H. See my book *Harmonic Analysis on Symmetric Spaces and Applications, Vol. 1, Chapter 3.* We have in mind, in particular, the spherical functions $h_s(z)$, which are the K-invariant eigenfunctions of Δ on H such that $h_s(i) = 1$. These functions can be identified with the Legendre function $P_{s-1}(\cosh r)$, where r is the geodesic radial coordinate of z; i.e., $z = ke^{-r}i$, for $k \in K = SO(2)$, $r > 0$ [Terras, *loc. cit.*, p.141]. Note that the level curves for these spherical functions are circles.

It is also possible to consider K-Bessel functions on H, but there is no room to do so here. The finite analogues are related to Kloosterman sums.

Example 2.
Upper Half Planes over Finite Fields, and Corresponding Ramanujan Graphs.

Replace $\mathbf{R}$ in the preceding example by the finite field with q elements $\mathbf{F}_q$, where $q = p^r$ and p is an odd prime. Take $G = GL(2, \mathbf{F}_q)$ and K the subgroup of matrices in G of the form

$$\begin{pmatrix} a & b\delta \\ b & a \end{pmatrix},$$

where δ is a nonsquare element of $\mathbf{F}_q$. Then we can think of G/K as the set H_q of $z = x + y\sqrt{\delta}$, for $x, y \in \mathbf{F}_q$, with $y \neq 0$. Again G acts on H_q by fractional linear transformation. And we have an analogue of the Poincaré distance given by

$$d(z,w) = \frac{\mathrm{N}(z-w)}{\mathrm{Im}(z)\,\mathrm{Im}(w)},$$

where we define for $z = x + y\sqrt{\delta}$,

$$y = \mathrm{Im}(z), \qquad \mathrm{N}(z) = x^2 - y^2\delta.$$

We can associate a graph $X_q(\delta, a)$ to H_q by taking the vertices to be the elements of H_q and connecting 2 vertices z, w iff $d(z, w) = a$. Here a is any fixed element of $\mathbf{F}_q$. If $a \neq 0$ or 4δ, we find that the graph $X_q(\delta, a)$ is regular of degree $q + 1$. It is also the Cayley graph of the affine group of all matrices in $GL(2, \mathbf{F}_q)$ with lower row (0,1) and generating set $S_q(\delta, a)$ the set of matrices

$$\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix},$$

such that $x^2 = ay + \delta(y-1)^2$. For example, the graph $X_3(2,1)$ is the octahedron. And the graph $X_5(2,1)$ may be viewed as lying on a dodecahedron with edges given by connecting each vertex of the dodecahedron to the two opposite vertices on each face. This produces a star on each face.

A G-invariant operator corresponding to Δ is

$$\mathrm{A}_a - (q+1)\mathrm{I}$$

where A_a is the adjacency operator of the graph $X_q(\delta, a)$. Here we take δ to be a fixed generator of the multiplicative group of $\mathbf{F}_q$ and let a vary over $\mathbf{F}_q$. So the special functions of interest are the simultaneous eigenfunctions of all the A_a, $a \in \mathbf{F}_q$. Since G/K is a symmetric space, the algebra of these operators A_a is, in fact, a commutative algebra of self-adjoint operators and thus we can find a simultaneous diagonalization.

It is easy to find an eigenfunction of A_a analogous to the function y^s on the Poincaré upper half plane. Take χ to be a character of the multiplicative group of $\mathbf{F}_q$. Then

Left to right: Charles Dunkl, Jeffrey Geronimo, Eugene Tomer, Audrey Terras, Renato Spigler, and André Ronveaux.

form $p_\chi(z) = \chi(\mathrm{I}m(z))$. By averaging this function over K, one obtains half the spherical functions on H_q. The other half are obtained by use of the discrete series representations of G. Here there is a big difference with Example 1 where we did not need representation theory; in particular, the discrete series representations did not come into play.

Using the formula which says spherical functions are averages over K of characters of G, J. Soto-Andrade [Proc. Symp. Pure Math. **47**, Amer. Math. Soc., 1987, pp.305–316] wrote down the explicit exponential sum giving the discrete series spherical functions of H_q.

N. Katz [J. für die Reine und Angew. Math. **438**, pp. 143–161 (1993)], estimated these sums to show that the graphs $X_q(\delta, a)$, $a \neq 0, 4\delta$, are Ramanujan graphs. This means that if λ is an eigenvalue of A_a, $|\lambda| \neq q+1$, then $|\lambda| \leq 2\sqrt{q}$. This definition was made by Lubotsky, et al [Combinatorica **8**, 1988, pp.261–277]. Ramanujan graphs are of interest in applied mathematics because they are good expander graphs and thus might be used to build communications networks. The proof of N. Katz used étale cohomology and Weil's proof of the Riemann hypothesis for zeta functions attached to curves. Recently Winnie Li has eliminated the étale cohomology from the proof.

Level curves of the finite spherical functions are quite chaotic finite analogues of circles. The chaos increases rapidly as $q \to \infty$.

Next Questions.

1) What is the distribution of the eigenvalues of A_a? Does it look like the semi-circle or the Sato-Tate distribution as $q \to \infty$? Recently, Bernadette Shook has produced histograms of eigenvalues that start to look like semi-circles.

2) What happens if you replace $\mathbf{F}_q$ by $\mathbf{Z}/q\mathbf{Z}$, $q = p^r$? Then for $p \geq 5$ the analogous graphs are not Ramanujan. See the preprint "Graph spectra for finite upper half planes over rings", by J. Angel, B. Shook, A. Terras, and C. Trimble.

3) Are there finite analogues of other symmetric spaces? In joint work with Perla Myers, we are investigating some other examples.

Multidimensional q-beta Integrals

by Ronald Evans

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Let $0 < q < 1$. For $a, x \in \boldsymbol{C}$, define

$$(x)_a = \prod_{m=0}^{\infty} \frac{(1-xq^m)}{(1-xq^{m+a})},$$

and

$$\Gamma_q(a) = \frac{(q)_{a-1}}{(1-q)^{a-1}}.$$

For a continuous function $f : \boldsymbol{C} \to \boldsymbol{C}$, and $\alpha, \beta, \in \boldsymbol{C}$, define the q-integral

$$\int_\alpha^\beta f(t)\, d_qt = (1-q)\sum_{m=0}^{\infty}\{f(\beta q^m)\beta q^m - f(\alpha q^m)\alpha q^m\}.$$

Fix a positive integer n, and define the n-dimensional q-measure

$$d_qT = \prod_{1\le i<j\le n} (t_i - t_j)\, d_qt_1 \cdots d_qt_n.$$

Let $a, b \in \boldsymbol{C}$, $c \in \boldsymbol{Z}$, with $\mathrm{Re}(a) > 0$, $\mathrm{Re}(b) > 0$, $c > 0$. We prove the following

Theorem.

$$\frac{1}{n!}\int_\alpha^\beta \cdots \int_\alpha^\beta \prod_{i=1}^{n} \left(\frac{qt_i}{\alpha}\right)_{a-1} \left(\frac{qt_i}{\beta}\right)_{b-1} \times$$

$$\times \prod_{1\le i<j\le n} \prod_{1-c\le k\le c-1} (t_i - q^k t_j)\, d_qT =$$

$$= K \prod_{j=0}^{n-1} \frac{\Gamma_q(a+jc)\,\Gamma_q(b+jc)\,\Gamma_q(c+jc)}{\Gamma_q(c)\,\Gamma_q(a+b+c(n+j-1))} \times$$

$$\times \frac{(\alpha/\beta)_{b+jc}\,(\beta/\alpha)_{a+jc}\,(\alpha\beta)^{1+jc}}{\alpha-\beta},$$

where

$$K = (-1)^{c\binom{n}{2}}\, q^{c^2\binom{n}{3} - \binom{c}{2}\binom{n}{2}}.$$

This result proves R. Askey's conjectured q-extension (1980) of A. Selberg's n-dimensional beta integral formula. The case $n = 1$ was proved in 1981 by G. Andrews and R. Askey. The case $\alpha = 0$, $\beta = 1$ was proved in 1944 by A. Selberg for $q = 1$, and it was proved for general q by K. Kadell and L. Habsieger, independently, in 1988.

We prove the theorem by setting up a $(2n+1)$-dimensional q-integral I, with n variables interlaced between $(n+1)$ other variables. We show how to change the order of q-integration, and thus obtain two separate evaluations of I. The evaluations are facilitated by another n-dimensional extension of the Andrews-Askey formula, given in 1992 by R. Evans. Equating the two evaluations of I, we obtain a proof of the Theorem by induction on n.

For references and details of the proof, see R. Evans, "Multidimensional beta and gamma integrals", Contemporary Math. **166**, pp. 341–357, 1994.

Extensions of Hermite Polynomials and Other Orthogonal Polynomials

by Richard Askey

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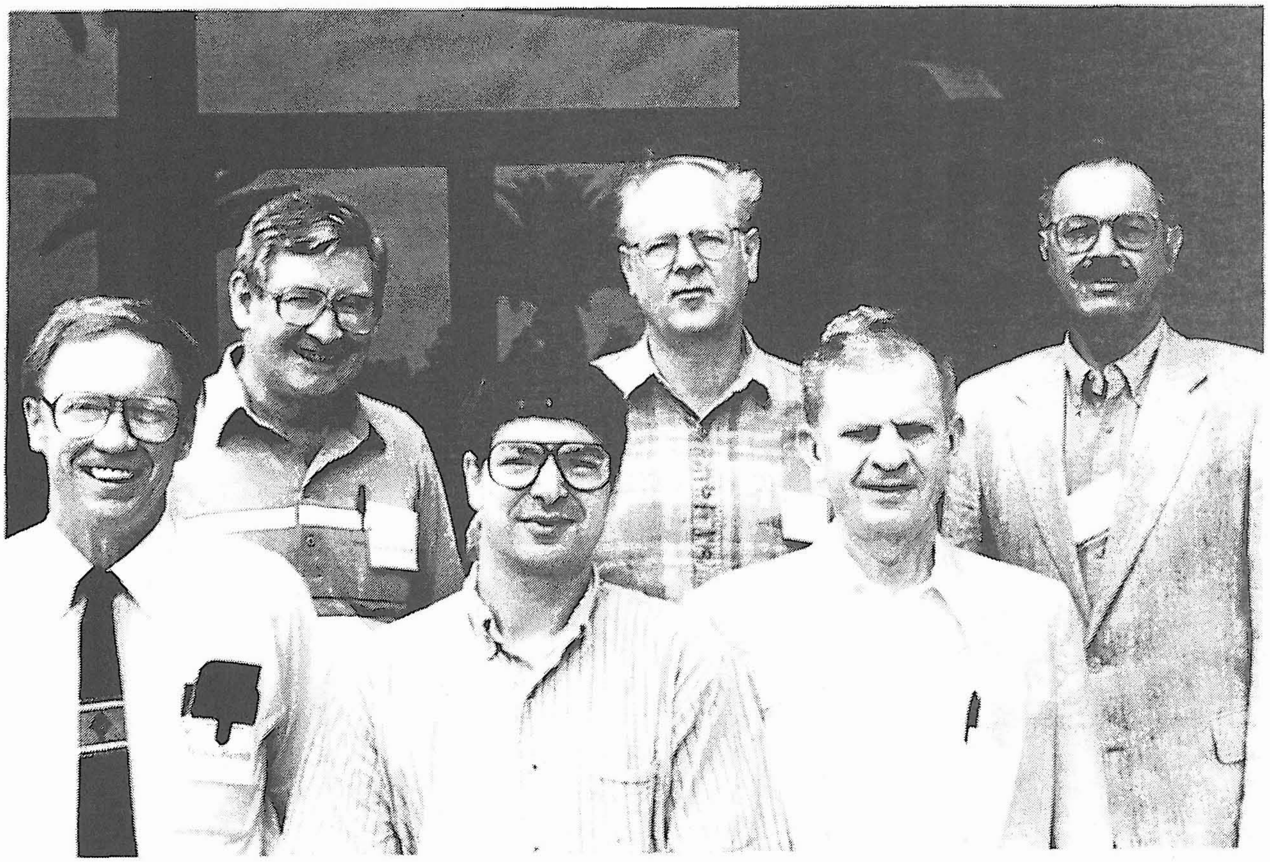
Recent work on nonclassical harmonic oscillators (some q, some not) convinced me that the essential property of Hermite polynomials is that there is an operator which acts on a set of orthogonal polynomials taking $p_n(x)$ to a multiple of $p_{n-1}(x)$. For Hermite polynomials this operator is the derivative, for Charlier polynomials it is a finite difference operator.

There are various q-extensions of both of these. I explained the classical work on the harmonic oscillator, but considering the audience I did not do any of the extensions, but gave a derivation of the finite q-binomial theorem in the noncommutative setting, where $yx = qxy$, and then used a device discovered twenty years ago by George Andrews to obtain the commutative form from the noncommutative form. This can be done by replacing y by xy and factoring out the x on the left. The noncommutative q-binomial theorem can be derived easily by considering $(x+y)^{n+1}$ as both $(x+y)(x+y)^n$ and $(x+y)^n(x+y)$.

In the commutative case, all one gets is Pascal's triangle, while in the noncommutative case one gets two versions of Pascal's triangle, and so can obtain a first-order recurrence relation for the coefficients on a given row.



Left to right: Bruce Berndt, Martin Muldoon, Ronald Evans, George Gasper, Richard Askey, and Charles Dunkl.

Heun's Equations and the Schrödinger Equation

by ANDRÉ RONVEAUX

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The origin of the Equation of Heun, a second order linear differential equation with four regular singularities, goes back to the problem of separation of variables for the Laplace equation in ellipsoidal coordinates.

After a quick review of successive changes of variable involving elliptic integrals, we present the Lamé equations. These equations are the building blocks. They allow us to derive, by confluence, and in the spirit of E.L. Ince [1], all possible differential equations starting with the Ince symbol $(8,0,0)$.

The basic Heun Equation (HE) can be written as

$$P_3(z)\,y''(z) + P_2(z)\,y'(z) + P_1(z)\,y(z) = 0,$$

where $P_i(z)$ is a polynomial of degree i. The HE with four regular singularities can generate irregular singularities by confluence in many ways. In the Confluent Heun Equation (CHE), one regular singular point is lost but the point at infinity becomes irregular. From the latter equation, two processes can now be considered: 1) the two finite singularities coalesce, giving the Double Confluent Heun Equation $(DCHE)$, or 2) one of the two finite singular points can join the already singular point at infinity giving the Biconfluent Heun Equation (BHE). The last possible process sends the three finite singular points of HE to infinity, giving the Triconfluent Heun Equation (THE).

Following this survey, we describe the forthcoming book

Heun's Differential Equations, Oxford Univ. Press,

which has been written in five parts, in collaboration with F.M. Arscott, S. Yu. Slavyanov, D.Schmidt and G. Wolf, P. Maroni, A. Duval. Then we indicate some applications, in particular to the Schrödinger equation.

This book, the first devoted entirely to Heun's equations, is quickly described. The book begins with a biography of Heun written by M. von Renteln. The contents follow.

Contents

An Appendix of 30 pages, written by S.Yu. Slavyanov, W. Lay and A. Seeger, treats in detail the processes of confluence and the underlying classification aspect. The book ends with a bibliography of more than 300 entries covering many titles containing problems from physics, chemistry and engineering.

Reference

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Meetings and Conferences

Some of these items have been extracted from the OP-SF Net Vol. 1, #7.

1. In commemoration of T.J. Stieltjes there will be a conference with the title "Orthogonality, Moment Problems, and Continued Fractions" to be held at the Delft University of Technology, October 31–November 4, 1994.

Each of four days will feature a different aspect of the work of Stieltjes, from continued fractions, rational approximation, moment problems, orthogonal polynomials, and asymptotics, to the properties of zeros and Gaussian quadrature. The format will consist of an invited lecture in the morning followed by short communications.

For more information, send a letter to

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The Proceedings of the conference will be published as a special issue of the Journal of Computational and Applied Mathematics.

2. At the AMS meeting in Richmond, Virginia, November 11–13, 1994 Doron Zeilberger will give an invited address with the title '='. Rodica E. Simion and Doron Zeilberger will also organize a Special Session there on "Identities and Enumeration". Here is a selected list of other speakers:

George Andrews
o "Pfaff's method and the WZ method"

Stephen Milne and Gaurav Bhatnagar
o "$U(n+1)$ bibasic summation formulas"

Gaurav Bhatnagar and Stephen Milne
o "$U(n+1)$ bibasic hypergeometric series"

Gert Almkvist
o "The coefficients of $\prod_{n=1}^{\infty}(1-x^n)^{\mu(n)}$"

Richard Askey
o "Identities and education"

Shaun Cooper
o "An inverse of the Wilson difference operator"

Robert Gustafson and Christian Krattenthaler
o "Heine transformations for a new kind of basic hypergeometric series in $U(n)$"

Charles Dunkl and Philip Hanlon
○ "Integrals of polynomials associated with tableaux"

Miller Maley
○ "A new formula for Hall polynomials"

Lynne Butler
○ "The monotonicity conjecture for Macdonald's two variable Kostka functions"

3. An International Symposium on "Methods and Applications of Analysis" will be held in Hong Kong, during December 16–19, 1994. The Program Organizers are

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Symposium topics will include asymptotics, integral equations, perturbation methods, special functions, and wave propagation. The symposium will not only provide a forum for an exchange of ideas among experts, but it will also disseminate information on recent advances made. There will be expository addresses, specialized talks, and poster sessions.

The Plenary Speakers are:

D.J. Benney, MIT
M.V. Berry, University of Bristol
C.K.R.J. Jones, Brown University
D.S. Jones, University of Dundee
M.D. Kruskal, Rutgers University
L. Lorch, York University
J.B. McLeod, University of Pittsburgh
M. Mimura, Tokyo University
F.W.J. Olver, University of Maryland
R.E. O'Malley, University of Washington

The deadline for abstracts has already passed. For additional information please contact Raymond Chan, Department of Mathematics, The Chinese University of Hong Kong. e-mail: rchan@euler.math.cuhk.hk.

4. A Special Session on "Approximation Theory and Special Functions" is being arranged for the AMS southeastern meeting, to be held at The University of Central Florida, in Orlando, March 17–18, 1995. The organizers are Ram N. Mohapatra and Xin Li. Anyone interested in giving a 20 minute talk should contact them at

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Here is a partial list of the people who have agreed to give a talk at this Special Session: A. Govil, Matthew He, Mourad E. H. Ismail, T. Kilgore, F. Marcellan, D. Masson, K. Pan, Q. I. Rahman, E. B. Saff, D. Stanton, and A. K. Varma.

5. The Stieltjes Colloquium, which has been planned for Toulouse during March 20-22, 1995, is one of the events commemorating Thomas J. Stieltjes, who died in Toulouse one hundred years ago, on the 31st of December, 1894.

The Colloquium will highlight the later development of topics on which Stieltjes worked: specifically, continued fractions and orthogonal polynomials; measures on the real line, moment problems; Laplace, Fourier and Stieltjes transforms; approximation and quadratures. Six plenary talks will be delivered:

C. Berg
○ "Moment problems and polynomial density"

J. Dhombres and J.-B. Pecot
○ "Les relations d'orthogonalité depuis 1750"

J.-P. Kahane
○ "La multiplication des series de Dirichlet, depuis Stieltjes"

T.W. Korner
○ "Via measures to universal Fourier series"

J. Korevaar
○ "Electrostatic fields due to distribution of electrons"

H. Stahl
○ "Padé approximants to algebraic functions"

There will also be contributed lectures. Abstracts, of at most two pages, must be sent before November 1, 1994 to the local organizing committee. The final selection will be made by the International Scientific Committee consisting of C. Berg, C. Brezinski, M.G. de Bruin, J. Dhombres, J.P. Kahane, G. van Dijk and W.T. van Est.

The address to use for the submission of abstracts, and also for information about participation, is:

Prof. J.-B. Hiriart-Urruty
Groupe d'Histoire des Mathématiques
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118, Route de Narbonne
31062 Toulouse, France
Fax: (33) 61 55 61 83 jbhu@cict.fr

6. Martin Muldoon recently forwarded this announcement of a conference in Canada.

A conference on "Special Functions and Related Topics in Analysis" will be held at York University, North York (Metropolitan Toronto), Ontario, Canada on Friday and Saturday, June 9–10, 1995. It will be dedicated to Lee Lorch, in honour of his forthcoming 80th birthday which will occur on September 20, 1995.

This meeting will follow the 50th Anniversary Meeting of the Canadian Mathematical Society to be held at the University of Toronto during June 5–9, and it will precede the Workshop on q-Series and Special Functions to be held at the Fields Institute, Toronto during June 12–23.

The Lorch Conference will be devoted to those topics in analysis, such as Fourier Analysis, Summability Theory, Special Functions, Ordinary Differential Equations and so on, to which Lee Lorch has made particular contributions.

Here is a tentative and incomplete list of speakers:

Richard Askey, University of Wisconsin
Chandler Davis, University of Toronto
James A. Donaldson, Howard University
Mary Gray, American University
Mourad Ismail, University of South Florida
Jean-Pierre Kahane, Paris-Orsay
A. McD. Mercer, University of Guelph
Angelo Mingarelli, Carleton University
Donald J. Newman, Temple University
Cora Sadowsky, Howard University
Walter Van Assche, Katholieke Universiteit Leuven
Roderick Wong, City Polytechnic of Hong Kong

The organizing committee consists of

Mourad Ismail, University of South Florida
David Masson, University of Toronto
Martin Muldoon, York University
Roderick Wong, City Polytechnic of Hong Kong
Asia Ivic Weiss, York University

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7. The following preliminary announcement has been copied from the OP-SF Net Vol.1, #7:

In the summer of 1995 here will be a two week program on "Special Functions, q-Series and Related Topics" at the Fields Institute, University of Toronto, Canada. During the first week, June 12–16, there will be five Minicourses. In the second week, June 19–23, there will be a Workshop on various research topics related to special functions and q-series, with a different emphasis for each day.

The members of the Conference Advisory Committee are G.E. Andrews, R. Askey, C.F. Dunkl, M.E.H. Ismail, T.H. Koornwinder, D.R. Masson, M. Rahman, S.K. Suslov, and D. Zeilberger.

Minicourses, June 12–16, 1995

- *Orthogonal Polynomials*
  Walter Van Assche, Katholieke Universiteit Leuven
- *Asymptotics of Orthogonal Polynomials*
  Paul Nevai, Ohio State University
- *q-series*
  George Gasper, Northwestern University
- *Quantum Groups and q-special Functions*
  Tom Koornwinder, University of Amsterdam
- *Computer Algebra and Special Functions*
  Doron Zeilberger, Temple University

The Workshop, June 19–23, 1995

Orthogonal Polynomials and Special Functions
Multivariable Theory
Number Theory and Combinatorics
Computer Algebra
Quantum Groups and Representation Theory

For each Workshop topic there are three invited speakers as well as a selection of contributed papers. The invited speakers are:

Krishnaswami Alladi, University of Florida
Bruce C. Berndt, University of Illinois
Peter Borwein, Simon Fraser University
David and Gregory Chudnovsky, Columbia University
Pavel I. Etingof, Harvard University
I. M. Gelfand, Rutgers University
R. Wm. Gosper, Symbolics, Inc.
Robert A. Gustafson, Texas A & M University
D. M. Jackson, University of Waterloo
A. Klimyk, Academy of Sciences of the Ukraine
Hendrik T. Koelink, Katholieke Universiteit, Belgium
Christian Krattenthaler, University of Vienna
M. Noumi, University of Tokyo
E. Opdam, University of Leiden, The Netherlands
Peter Paule, Johannes Kepler University, Austria

For information and registration material, contact:

Sheri Albers, The Fields Institute
185 Columbia Street West
Waterloo, Ontario N2L 5Z5 Canada
Tel: (519) 725-0096 Fax: (519) 725-0704
spfunct@fields.uwaterloo.ca

This will be a busy period in Canada. To sum up:

June 5–9, The Canadian Mathematical Society's 50th Anniversary Meeting, University of Toronto.

June 9–10, Conference on "Special Functions and Related Topics in Analysis" at York University, dedicated to Lee Lorch.

June 12–23, The Fields Institute Program.

Problems

Thus far eight problems have been submitted while three have been solved (#**1**, **4**, **6**). A printout of all the problems and the solutions is available from the Editor.

2. Is it true that

$$x^2 t^x \, {}_2F_1(x+1,\, x+1;\, 2;\, 1-t)$$

is a convex function of x whenever $-\infty < x < \infty$ and $0 < t < 1$?

Submitted by George Gasper, August 19, 1992.
(g-gasper@nwu.edu)

3. The following Toeplitz matrix arises in several applications. Define for $i \neq j$

$$A_{ij}(\alpha) = \frac{\sin \alpha\pi(i-j)}{\pi(i-j)},$$

and set $A_{ii} = \alpha$. Conjecture: the matrix

$$M = (I - A)^{-1}$$

has positive entries. A proof is known for $1/2 \leq \alpha < 1$. Can one extend this to $0 < \alpha < 1$?

Submitted by Alberto Grünbaum, November 3, 1992.
(grunbaum@math.berkeley.edu)

5. The result of Problem #4 can be generalized to

$$\begin{aligned} S_m &= \sum_{n=0}^{\infty} \frac{(-1)^n (mn+1/2)!}{\sqrt{\pi}\,(mn+1)!} \\ &= \frac{1}{m} \sum_{k=0}^{m-1} \frac{\sin\left(5(2k+1)\pi/(4m) + \pi/4\right)}{\left[2\sin\left((2k+1)\pi/(2m)\right)\right]^{1/2}} \end{aligned}$$

valid for integral $m \geq 2$.

Submitted by J. Boersma and P.J. de Doelder, July 12, 1993.
(wstanal@win.tue.nl)

7. The incomplete Airy integral given by [1]

$$I_0(\sigma, \gamma; k) = \int_{\gamma}^{\infty} e^{jk(\sigma z + z^3/3)}\, dz \qquad (1)$$

serves as a canonical integral for some sparsely explored diffraction phenomena involving the evaluation of high frequency EM fields [2] near terminated caustics and composite shadow boundaries. In equation (1), k is the wavenumber of the propagation medium and is assumed to be the large parameter. Both the parameters σ and γ are real.

The desired task is to derive a complete asymptotic expansion for I_0 in inverse powers of $k \to \infty$ for the case when the saddle points of the integrand satisfying

$$\begin{aligned} z^2 + \sigma &= 0 &(2)\\ z_{1,2} &= \pm(-\sigma)^{1/2} &(3) \end{aligned}$$

are real and widely separated ($\sigma \ll -1$). The asymptotic expansion should be of the form

$$I_0(\sigma, \gamma; k) \sim \sum_{n=0}^{\infty} k^{-n} f(\sigma, \gamma, n) \qquad (4)$$

in which $f(\sigma, \gamma, n)$ is expressed in terms of known and easily computed functions. The asymptotic expansion in (4) should also hold uniformly as the endpoint γ approaches, or coincides with, one of the saddle points.

Submitted by E.D. Constantinides and R.J. Marhefka, August 11, 1993.
(rjm@tiger.eng.ohio-state.edu)

8. Can the real and imaginary parts of a hypergeometric series of type ${}_pF_q$ with one complex parameter (either in the numerator or the denominator) be expressed by means of multiple hypergeometric series?

Submitted by Ernst D. Krupnikov, July 25, 1993.
(ernst@net.neic.nsk.su)

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[1] Electrical engineers use j for $\sqrt{-1}$, reserving $i = v/r$ for current.

[2] See their brief article on electromagnetic (EM) diffraction in the Fall, 1993 issue of the Newsletter.

How to Contribute to the Newsletter

Send your Newsletter contributions directly to the *Editor:*

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etomer@netcom.com

The publication schedule is approximately:

Edition	*Draft #1*	*Mailing* ≈
Fall	August 10	September 1
Winter	November 10	December 1
Spring	February 10	March 1
Summer	May 10	June 1

Items without mathematical symbols can be submitted in any convenient form, for example as a textfile or as a typewritten manuscript. Items *containing substantial mathematical symbols* must be submitted as a *standard* LaTeX file by e-mail or by disc. There is only one exception: *plain* TeX files can be accepted by making special arrangements with the Editor.

AMS-TeX and AMS-LaTeX files cannot be accepted. You must translate them into standard LaTeX.

Whatever the content of your LaTeX article, you must use the *preamble* below. You can obtain this preamble by e-mail from the Editor. Other preambles, macros, and style files obtained from archives are not allowed.

```
\documentstyle[twocolumn]{article}
\topmargin -0.5in
\oddsidemargin -0.375in
\evensidemargin -0.375in
\textheight 9.2in
\textwidth 7.25in
\columnsep 0.25in
\parskip 0.02in
\begin{document}
\title{Draft}
\author{Name of Author(s)}
\date{\today}
\maketitle
%  Text Input.   Do not use macros.   * *
\vfill
\end{document}
```

To accommodate special situations, a *very limited use* of `\newcommand`, `\renewcommand`, and `\newenvironment` is allowed provided sufficient explanatory comments appear in the file.

Newsletter material can also include photos and drawings but they must be submitted as *hard copy.* Photos must have good definition and contrast, black and white preferred, minimum size 10 by 15 cm.

The Activity Group also sponsors an electronic news net, called the **OP-SF Net**, which is transmitted periodically by SIAM. The Net provides a rather fast turnaround compared to the Newsletter. To *receive* transmissions, just send your name and e-mail address to poly-request@siam.org (as with other nets, nonmembers can also receive the transmissions). Your OP-SF Net *contributions* should be sent to poly@siam.org.

The Net is organized by Tom Koornwinder (thk@fwi.uva.nl).