Solving Differential Equations in Terms of Bessel Functions

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ABSTRACT

For differential operators of order 2, this paper presents a new method that combines generalized exponents to find those solutions that can be represented in terms of Bessel functions.

Categories and Subject Descriptors

I.1.2 [**Symbolic and Algebraic Manipulation**]: Algorithms—*Algebraic algorithms*; G.4 [**Mathematical Software**]: Algorithm design and analysis

General Terms

Algorithms

1. INTRODUCTION

Consider a differential operator $L = \sum_{i=0}^{n} a_i \partial^i$ with coefficients in some differential field K and $\partial = \frac{d}{dx}$. We search for solutions of $L(y) = 0$. In a first step, if $K = \mathbb{C}(x)$, then one can try to factor [2, 9, 10] the differential operator. From then on we will only consider irreducible $L \in K[\partial]$.

If B is a special function that satisfies a differential equation $L_B(B) = 0$, then the question if we can solve $L(y) = 0$ in terms of B is equivalent to the question whether there exist certain transformations that send L_B to L.

There are three types of transformations in $K[\partial]$ that preserve order two, namely: (i) change of variables[1] $x \to f(x)$, (ii) an exp-product $y \to \exp(\int r) \cdot y$, and (iii) a gauge[2] transformation $y \to r_0 y + r_1 y'$, where $'$ is $\frac{d}{dx}$. We take the parameters f, r, r_0, r_1 in K so that the result is again in $K[\partial]$.

Our algorithm can decide if an operator L can be obtained from the Bessel operator under these three transformations. Thus, given L, it can find all solutions of the form

$$\exp\left(\int r\, dx\right) \left(r_0 B_\nu(f(x)) + r_1 B_\nu(f(x))'\right), \qquad (1)$$

where $r, r_0, r_1, f \in \mathbb{C}(x)$ and $B_\nu(x)$ is a Bessel function.

One could argue that this is only a minor addition to prior work. Prior algorithms treat more than just Bessel functions, and they also already treat two of these transformations; [3, 4, 5, 11, 15] treat (i)+(ii), and [1, 14] treat (ii)+(iii). However, we argue that unless all three transformations are covered simultaneously, the work can not be considered to be complete: take equations that the algorithm can solve, and then apply the non-treated transformation, and one encounters solvable equations that the algorithm does not solve.

In contrast, the set of equations solvable by our algorithm is closed under all three transformations. This closure property means that if L is an operator that the algorithm can solve, and if one applies the above "order preserving transformations", then the result is again an operator that the algorithm can solve. This closure property is the key novelty in our algorithm. It requires (main task in this paper) solving a combinatorial problem introduced in Section 3.

Prior papers do not have this combinatorial problem; if one does not treat (iii) then there exists a *global* invariant [3, 4, 5, 15] that fully determines f and there is no need to combine *local* invariants such as (generalized) exponents. The papers [1, 14] treat (ii)+(iii) but not (i). Adding a restricted version of (i) to this (with f restricted to Möbius transformations $\frac{ax+b}{cx+d}$) requires only a little bit of extra code because such f is determined by 3 points that are easily obtained from L without combining (generalized) exponents. However, adding (i), for general f, to (ii)+(iii) is much more work; it is the above mentioned combinatorial problem and takes up most of our implementation.

In summary: Treating (i)+(ii)+(iii) simultaneously, for arbitrary r, r_0, r_1, $f \in \mathbb{C}(x)$, is the new result in this paper. Handling two of these transformations has already been done before (and: for more than just Bessel functions).

Our method can also be adapted to work for other special functions; our implementation in *Maple* can solve differential operator in terms of Bessel, Whittaker and Kummer functions. Due to page limitations we only treat the Bessel case here; the first author's master thesis [7] explains the algorithm in more detail including the Whittaker case[3]. Two

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[1] This always refers to a change of the *independent* variable.

[2] Terminology from [12, Definition 8].

[3] The master thesis and the *Maple* implementation can be obtained from `http://www.mathematik.uni-kassel.de/~debeerst/master/`.

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important cases that are not yet treated are: the ${}_2F_1$ special function (for general f), and the Bessel case where f is not a rational function but the square root of a rational function. The combinatorial problem is more difficult for these cases.

2. PRELIMINARIES

2.1 Differential Operators

We denote by $K[\partial]$ the ring of differential operators with coefficients in K. Mostly we have $K = k(x)$ but sometimes we will also need power series coefficients $K = k((x))$. Here k will be $\mathbb{C}$ until Section 4.2 which will treat non-algebraically closed k.

A point $p \in \mathbb{C} \cup \{\infty\}$ is called a singular point of $L \in K[\partial]$ if p is a zero of the leading coefficient or a pole of one of the other coefficients. Otherwise, p is called regular.

By the solutions of L we mean the solutions of the differential equation $L(y) = 0$. If p is regular, we can express all solutions locally around p as power series $\sum_{i=0}^{\infty} b_i t_p^i$ where t_p denotes the local parameter which is $t_p = \frac{1}{x}$ if $p = \infty$ and $t_p = x - p$ otherwise.

2.2 Formal Solutions and Generalized Exponents

Definition 1. A *universal extension* U of $K = \mathbb{C}((x))$ is a minimal differential ring in which every operator $L \in K[\partial]$ has precisely $\deg(L)$ $\mathbb{C}$-linear independent solutions.

We denote the solution space of L by $V(L) := \{y \in U \mid L(y) = 0\}$. There exists a universal extension U of $\mathbb{C}((x))$, so $V(L)$ has dimension $\deg(L)$ for every nonzero operator $L \in \mathbb{C}((x))[\partial]$. For the construction of U we refer to [13], Chapter 3.2, where U is denoted by $\mathrm{UnivR}_{\widehat{K}}$.

Most importantly we know that the fundamental system of local solutions at $x = 0$ can be represented as

$$\exp\left(\sum_{i=1}^{n} c_i x^{-1/m}\right) x^{\lambda} S$$

for some $c_i, \lambda \in \mathbb{C}, n, m \in \mathbb{N}$ and where $S \in \mathbb{C}[[x^{1/m}]][\log(x)]$ has a nonzero constant term. This can also be written as $\exp(\int e/x\, dx)S$ where $e \in E = \cup_{m\in\mathbb{N}} E_m$ and $E_m = \mathbb{C}[x^{-1/m}]$. If $e \in \mathbb{C}$, we get a solution $x^e S$ and e is called an *exponent*. For $e \in \mathbb{N}$ we get power series solutions. In general $e \in E$ is called *generalized exponent* and can depend on x. Solutions that involve a logarithm will be called *logarithmic solutions*.

This construction can be done at any point p, for which we just have to replace x by the local parameter t_p. Then a local solution at p has the representation

$$\exp\left(\int \frac{e}{t_p}\, dt_p\right) S,\ e \in \mathbb{C}[t_p^{-\frac{1}{m}}],\ S \in \mathbb{C}[[t_p^{\frac{1}{m}}]][\log(t_p)] \quad (2)$$

If $m = 1$ (when no fractional powers of t_p occur in e and S) then e is called *unramified*.

At any point p there are n generalized exponents $e_1, \ldots, e_n$ corresponding to a basis $\exp(\int e_i/t_p)S_i(p)$, $i = 1, \ldots, n$ of $V(L)$. These generalized exponents can be computed [9].

2.3 Bessel Functions

The solutions of the operators $L_1 = x^2\partial^2 + x\partial + (x^2 - \nu^2)$ and $L_2 = x^2\partial^2 + x\partial - (x^2 + \nu^2)$ are called Bessel functions. The two linearly independent solutions $J_\nu(x)$ and $Y_\nu(x)$ of L_1 are called Bessel functions of first and second kind, respectively. Similarly the solutions $I_\nu(x)$ and $K_\nu(x)$ of L_2 are called the modified Bessel functions of first and second kind.

The Bessel functions with parameter $\nu \in \frac{1}{2} + \mathbb{Z}$ are hyperexponential functions and in that case L_1 and L_2 are reducible. Since we only consider irreducible operators, we can assume $\nu \notin \frac{1}{2} + \mathbb{Z}$.

The transformation $x \to \sqrt{-1} \cdot x$ sends L_1 to L_2 and vice versa. Since we will allow such transformations later, we only have to deal with one of the two cases. We choose the modified Bessel case and we denote $L_B := L_2$.

For Bessel functions, the generalized exponents are unramified (i.e. $m = 1$ in the previous section) so no fractional exponents (Puiseux series) are needed. The operator L_B has generalized exponents $\pm\nu$ at 0 and $\pm t_\infty^{-1} + \frac{1}{2}$ at ∞.

3. TRANSFORMATIONS

From now on we will restrict ourselves to irreducible operators of degree two (so the formula $r_0 y + r_1 y'$ in item (iii) below describes any K-linear combination of $y, y', y'', \ldots$).

Definition 2. Let $K = k(x)$. A *transformation* between two differential operators $L_1, L_2 \in K[\partial]$ is a map from the solution space $V(L_1)$ onto the solution space $V(L_2)$. We will address the following transformations:

(i) **change of variables** $y(x) \to y(f(x))$, $\quad f(x) \in K \setminus k$.
(ii) **exp-product** $y \to \exp(\int r\, dx) \cdot y$, $\quad r \in K$.
(iii) **gauge transformation** $y \to r_0 y + r_1 y'$, $\quad r_0, r_1 \in K$.

For the resulting operator $L_2 \in K[\partial]$ we write $L_1 \xrightarrow{f}_C L_2$, $L_1 \xrightarrow{r}_E L_2$, and $L_1 \xrightarrow{r_0,r_1}_G L_2$, respectively. Furthermore, we write $L_1 \longrightarrow L_2$ if there exists a sequence of transformations that sends L_1 to L_2.

The rational functions f, r, r_0 and r_1 will be called parameters of the transformation, and in case (ii) the function $\exp(\int r)$ is a hyperexponential function.

Lemma 1 *Let $L_1 \in K[\partial]$ be irreducible of degree two. If the parameters of the transformations above are given, we can always find $L_2 \in K[\partial]$ with $\deg(L_2) = 2$ such that $V(L_1)$ is mapped onto $V(L_2)$ by the given transformations.*

PROOF. If $y \in V(L_1)$ is mapped to $z \in V(L_2)$, then z, z' and z'' can be rewritten in terms of y and y' using L_1. An ansatz for L_2 yields a system of two equations with three variables (see [7, Theorem 2.4] for details). $\square$

It is clear that if $L_1 \longrightarrow L_2$ and $L_1 = L_B$, the solutions of L_2 can be expressed by Bessel functions. Conversely, from examples studied so far, it appears that any operator $L \in K[\partial]$ that has Bessel solutions satisfies $L_B \longrightarrow L$ provided that we generalize (i) to allow f for which $f^2 \in K$. This converse statement remains to be proven/disproven (first the converse statement would need to be made precise by defining the phrase "has Bessel solutions").

The algorithm presented in this paper finds solutions in all cases where $L_B \longrightarrow L$ is satisfied (with $f \in K$; the case $f \notin K$, $f^2 \in K$ is not handled in this paper).

Since we are now interested in finding those transformations for a given operator L we study them more precisely.

The relations $\longrightarrow_E$ and $\longrightarrow_G$ are equivalence relations (see [1] or [7]) but $\longrightarrow_C$ is not (if f is not a Möbius transformation, then (i) is not invertible, unless we generalize to allow algebraic functions for f).

An important question when searching for transformations between two operators L_1 and L_2 is whether we can restrict our search to a specific order of the transformations $\longrightarrow_C$, $\longrightarrow_E$ and $\longrightarrow_G$.

Lemma 2 *Let $L_1, L_2, L_3 \in K[\partial]$ be three differential operators such that $L_1 \longrightarrow_G L_2 \longrightarrow_E L_3$. Then there exists a differential operator $M \in K[\partial]$ such that $L_1 \longrightarrow_E M \longrightarrow_G L_3$. Similarly, if $L_1 \longrightarrow_E L_2 \longrightarrow_G L_3$ we find M such that $L_1 \longrightarrow_G M \longrightarrow_E L_3$.*

Proof. Let $L_1 \xrightarrow{r_0,r_1}_G L_2 \xrightarrow{r}_E L_3$ and we denote $R = \exp\left(\int r\right)$. Then the solution space of L_3 is

$$\begin{aligned} V(L_3) &= \left\{R(r_0 y + r_1 y') \mid y \in V(L_1)\right\} \\ &= \left\{(r_0 - r_1 r)Ry + r_1 (Ry)' \mid y \in V(L_1)\right\}. \end{aligned}$$

Hence, $L_1 \xrightarrow{r}_E M \xrightarrow{\bar{r}_0,r_1}_G L_3$ for some $M \in K[\partial]$ with $\bar{r}_0 = r_0 - r_1 r$. The converse follows by the same computation. □

Since the order of $\longrightarrow_E$ and $\longrightarrow_G$ can be switched we write $\longrightarrow_{EG}$ for any sequence of those. The order concerning a change of variables can be changed as follows.

Lemma 3 *Let $L_1, L_2 \in K[\partial]$ such that $L_1 \longrightarrow L_2$. Then there exists $M \in K[\partial]$ such that $L_1 \longrightarrow_C M \longrightarrow_{EG} L_2$.*

Proof. As in the last proof one simply rewrites the solution space of $V(L_3)$ to show that

(i) $L_1 \xrightarrow{r}_E L_2 \xrightarrow{f}_C L_3 \Rightarrow \exists M \in K[\partial]$ such that $L_1 \xrightarrow{f}_C M \xrightarrow{\bar{r}}_E L_3$, with $\bar{r} = R(f)'$ and $R(x) = \int r\,dx$.

(ii) $L_1 \xrightarrow{r_0,r_1}_G L_2 \xrightarrow{f}_C L_3 \Rightarrow \exists M \in K[\partial]$ such that $L_1 \xrightarrow{f}_C M \xrightarrow{\bar{r}_0,\bar{r}_1}_G L_3$ with $\bar{r}_0 = r_0(f)$ and $\bar{r}_1 = r_1(f)/f'$.

The rest follows immediately. □

We conclude: If $L_1 \longrightarrow L_2$ for any sequence of transformations then $L_1 \xrightarrow{f}_C M \longrightarrow_{EG} L_2$ for some $M \in K[\partial]$. Assume $L_2 = \partial^2 + a_1\partial + a_0$ is given and we search for transformations where $L_1 = L_B$. If no gauge transformation occurred, then it is easy to recover f from L_2 by using the fact that $I := a_0 - a_1^2/4 - a_1'/2$ remains invariant under $\longrightarrow_E$, and the fact that I completely determines f (see [3, 4, 5, 15] for details). But there is no simple formula for an invariant under $\longrightarrow_{EG}$ that can determine f completely.

There are, however, local invariants for $\longrightarrow_{EG}$, based on exponent differences introduced in the next section. Each local invariant only yields partial information about f, and our main task will be to:

Combinatorial Problem: recover f (and the Bessel parameter ν) by combining these partial pieces of data.

After f and ν are found, we can find the $M \longrightarrow_{EG} L_2$ transformation using algorithms presented in [1].

3.1 The Exponent Difference

From here on, the word "exponent" refers to "unramified generalized exponent" (i.e. $m = 1$ in Section 2.2).

Lemma 4 *Let $L, M \in K[\partial]$ be two differential operators such that $M \xrightarrow{r}_E L$ and let e be an exponent of M at the point p. Furthermore, let r have the series representation*

$$r = \sum_{i=m}^{\infty} r_i t_p^i, \quad m \in \mathbb{Z}, m \leq -1.$$

Then $e + \sum_{i=m}^{-1} r_i t_p^{i+1}$ is an exponent of L at p.

If $M \longrightarrow_G L$ and e is an exponent of M at the point p, then L has an exponent $\bar{e}$ with $\bar{e} \equiv e$ mod $\mathbb{Z}$.

Proof. Let t be the local parameter t_p. Since e is an exponent, M has a solution of the form

$$y = \exp\left(\int \frac{e}{t}\,dt\right) S,$$

for some $S \in \mathbb{C}[[t]][\ln(t)]$ with non-zero constant term. The exp-product converts this solution into

$$z = \exp\left(\int r\,dt\right) \exp\left(\int \frac{e}{t}\,dt\right) S.$$

In order to determine the exponent at p we have to rewrite this expression into the form (2). We have to handle the positive and negative powers of t in r separately. For the power series part $\bar{r} = \sum_{i=0}^{\infty} r_i t^i$ we get

$$\exp\left(\int \bar{r}\,dt\right) = \exp\left(\sum_{i=0}^{\infty} \frac{r_i}{i+1} t^{i+1}\right).$$

With $\exp(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!}$ we can rewrite this as a power series in t such that $\exp\left(\int \bar{r}\,dt\right) = \sum_{i=0}^{\infty} a_i t^i$ with $a_0 = 1$.

The negative powers of t in the series expansion of r become a part of the exponent:

$$\exp\left(\int \sum_{i=m}^{-1} r_i t^i\,dt\right) = \exp\left(\int \frac{1}{t} \sum_{i=m}^{-1} r_i t^{i+1}\,dt\right).$$

Combining the two results we get

$$z = \exp\left(\int \frac{1}{t}\left(e + \sum_{i=m}^{-1} r_i t^{i+1}\right)\,dt\right) \bar{S},$$

where $\bar{S} \in \mathbb{C}[[t]][\ln(t)]$ has a non-zero constant term.

For a gauge transformation with parameters r_0 and r_1, the result follows from the following facts: the exponents of r_0 and r_1 are integers, taking derivatives of unramified power series changes exponents only by integers, and finally, adding unramified power series can only change exponents by integers. Hence, gauge transformations only change exponents by integers (note: for ramified generalized exponents one would get $\bar{e} \equiv e$ mod $\frac{1}{m}\mathbb{Z}$ with m as in Section 2.2). □

If $\deg(L) = 2$, we have two exponents e_1, e_2 at a point p and we call $e_1 - e_2$ an exponent difference. The exponent difference is defined up to a $\pm$ sign and we denote $\Delta(L, p) = \pm(e_1 - e_2)$. It follows from the previous lemma that Δ mod $\mathbb{Z}$ is invariant under exp-products and gauge transformations, i.e. $\Delta(L_1, p)$ mod $\mathbb{Z} = \Delta(L_2, p)$ mod $\mathbb{Z}$ if $L_1 \longrightarrow_{EG} L_2$.

Since we will be interested in singularities of L which are logarithmic or whose exponent difference is not an integer, the following theorem will be important.

Theorem 1 *Let $L \in K[\partial]$ be a differential operator and let p be a point. If there exists an operator $M \in K[\partial]$ where p*

is regular such that $M \longrightarrow_{EG} L$, then the solutions of L are not logarithmic and $\Delta(L,p) \in \mathbb{Z}$.

PROOF. Let M and p be as required. Then there exist rational functions $r_0, r_1, r_2 \in K$ and $\tilde{M} \in K[\partial]$ such that $M \xrightarrow{r_0}_E \tilde{M} \xrightarrow{r_1,r_2}_G L$. Furthermore, let p be a regular point of M. The generalized exponents at p are 0 and 1. Hence, $\Delta(M,p) \in \mathbb{Z}$ and from the previous lemma it follows that $\Delta(L,p) \in \mathbb{Z}$.

Since p is regular, the local solutions of M at p do not have logarithms. The local solutions of L at p can be derived from these by an exp-product and a gauge transformation. Neither of these transformations brings in logarithms. Hence, solutions of L at p are not logarithmic. $\square$

In the following we will use Δ to find the parameter f of the change of variables as well as the Bessel parameter ν that is involved. Assume that $K = \mathbb{C}(x)$; so we can factor every polynomial into linear factors. We will later treat $K = k(x)$ for finitely generated extensions k of $\mathbb{Q}$.

Theorem 2 *Let $M \in K[\partial]$ such that $L_B \xrightarrow{f}_C M$, $f \in K$.*

(i) If p is a zero of f with multiplicity $m \in \mathbb{N}$, then p is a regular singularity of M and $\Delta(M,p) = \pm 2m\nu$.

(ii) If p is a pole of f with multiplicity $m \in \mathbb{N}$ such that $f = \sum_{i=-m}^{\infty} f_i t_p^i$, then p is an irregular singularity of M and

$$\Delta(M,p) = \pm 2 \sum_{i=-m}^{-1} i f_i t_p^i. \tag{3}$$

PROOF. Let t be the local parameter t_p. To compute the generalized exponents of M at the point p, we start with a solution y of L_B, replace x by f to get a solution z of M and rewrite z into the form (2).

(a) Let p be a zero of f with multiplicity $m > 0$, then f has the representation $f = t^m \sum_{i=0}^{\infty} f_i t^i$ with $f_i \in k$ and $f_0 \neq 0$. Furthermore, let $y \in V(L_B)$ be a local solution at $x = 0$ of the form

$$y = x^{\nu} \sum_{i=0}^{\infty} a_i x^i, \quad a_i \in k, a_0 \neq 0.$$

If we now replace x by f, we get a local solution $z = f^{\nu} \sum_{i=0}^{\infty} a_i f^i$ of M. To compute the generalized exponent at p we rewrite z such that $z = \exp(\int e/t\, dt) \sum_{i=0}^{\infty} b_i t^i$ for some $e \in E, b_i \in k, b_0 \neq 0$. The fact that $f^i = t^{mi} \bar{f}$, where the constant coefficient of $\bar{f} \in k[[t]]$ is non-zero, simply yields $e_1 = m\nu$.

Similarly, for the second independent local solution of L_B at $x = 0$, which has exponent $-\nu$, we obtain the generalized exponent $e_2 = -m\nu$. Hence, the singularity p is regular and $\Delta(M,p) = \pm(e_1 - e_2) = \pm 2m\nu$.

If $\nu \in \mathbb{Z}$ the second independent solution contains a logarithm $\ln(x)$. However, we can still do the same computations. The solution z would then involve $\ln(t)$ and the result for the exponent is still true.

(b) A similar approach works in second case. Let p be a pole of f with multiplicity $m \in \mathbb{N}$. Then f can also be written as $f = t^{-m} \sum_{i=0}^{\infty} f_{i-m} t^i$ with $f_i \in k, f_{-m} \neq 0$.

We start with the local solution y of L_B at $x = \infty$ corresponding to the exponent $e := \frac{1}{t_\infty} + \frac{1}{2}$. There exists a series $S \in k[[t_\infty]]$ such that

$$y = \exp\left(\int \frac{e}{t_\infty} dt_\infty\right) S = \exp\left(-\frac{1}{t_\infty}\right) t_\infty^{1/2} S \tag{4}$$

is a solution of L_B. In order to get a solution z of M we have to replace x by f, i.e. $t_\infty = \frac{1}{x}$ by $\frac{1}{f}$. Hence, we do the following substitutions:

$$\begin{aligned} t_\infty &\longrightarrow \frac{1}{f} = t^m \sum_{i=0}^{\infty} \tilde{f}_i t^i, \quad \tilde{f}_i \in k, \tilde{f}_0 \neq 0, \\ \frac{1}{t_\infty} &\longrightarrow f, \\ \text{and} \quad t_\infty^{1/2} &\longrightarrow \frac{1}{f^{1/2}} = t^{m/2} \sum_{i=0}^{\infty} \bar{f}_i t^i, \quad \bar{f}_i \in k, \bar{f}_0 \neq 0. \end{aligned} \tag{5}$$

We apply these substitutions to (4) and get a local solution z of M at $x = p$: $z = \exp(-f)\, t^{m/2} \tilde{S}, \tilde{S} \in k[[t]]$, where $\tilde{S}$ combines all the new series that we obtain from (5). As in the proof of Lemma 4 we can rewrite $\exp(\sum_{i=0}^{\infty} f_i t^i)$ as power series in t. The negative powers of t remain in the exponential part, which then becomes

$$\exp\left(-\sum_{i=-m}^{-1} f_i t^i\right) t^{m/2} = \exp\left(\int \frac{1}{t}\left(\sum_{i=-m}^{-1} -i f_i t^i + \frac{m}{2}\right) dt\right).$$

Thus, z has the generalized exponent $-(\sum_{i=-m}^{-1} i f_i t^i) + \frac{m}{2}$.

If we start with the second independent solution with generalized exponent $-\frac{1}{t_\infty} + \frac{1}{2}$ we similarly get $(\sum_{i=-m}^{-1} i f_i t^i) + \frac{m}{2}$. Hence, p is an irregular singularity of M and $\Delta(L,p) = \pm 2 \sum_{i=-m}^{-1} i f_i t^i$. $\square$

The last two theorems illustrate the following definitions.

Definition 3. A point p of $L \in K[\partial]$ for which $\Delta(L,p) \in \mathbb{Z}$ and L is not logarithmic at p is called an *exp-apparent* point. If p is not exp-apparent, p is called

(i) *exp-regular* $\Leftrightarrow \Delta(L,p) \in \mathbb{C} \backslash \mathbb{Z}$ or L is logarithmic at p,

(ii) *exp-irregular* $\Leftrightarrow \Delta(L,p) \in \mathbb{C}[1/t_p] \backslash \mathbb{C}$.

We denote the set of singularities that are exp-regular by S_{reg} and those that are exp-irregular by S_{irr}.

Note that regular points are also exp-apparent and that every point which is not exp-apparent must be a singularity.

If we have an operator L such that $L_B \xrightarrow{f}_C M \longrightarrow_{EG} L$, exp-apparent points of L are singularities of L which might have been introduced by exp-products and gauge transformations. So they are unimportant when searching for f. Exp-irregular singularities of L are also irregular singularities of M and correspond exactly to the poles of f. Finally, every exp-regular singularity is a zero of f (this is not a one–to–one correspondence; a zero of f need not be a singularity of L when $\nu \in \mathbb{Q}$). We combine these very important results in the following corollary.

Corollary 1 *If $L_B \xrightarrow{f}_C M \longrightarrow_{EG} L$, the following holds:*

(i) $p \in S_{irr} \Leftrightarrow p$ is a pole of f.

(ii) $p \in S_{reg} \Rightarrow p$ is a zero of f.

PROOF. (i) If p is a pole of f, it follows from Theorem 2(ii) that $p \in S_{irr}$. If p is not a pole, $\Delta(M,p) \in \mathbb{C}$ and $p \notin S_{irr}$ by Theorem 1 and Theorem 2(i).

(ii) Using $S_{reg} \cap S_{irr} = \emptyset$ and (i), the only thing that remains to be proven is that $p \notin S_{reg}$ for any ordinary point p of f. So let p be neither a zero nor a pole of f. Then p is a regular point of M, i.e. $\Delta(M,p) \in \mathbb{Z}$ and the solutions of M at p are not logarithmic. From Theorem 1 it follows that this also holds for L. $\square$

The sets S_{reg} and S_{irr} can be computed easily. So we already know all poles of f. But since we have no equivalence in (ii), we might not see *all* zeros of f. However, from Theorem 2 we know more information about the poles of f: we can compute every polar part of f (up to a $\pm$ sign).

By the polar part of f at a point p we mean the negative power part of the series representation of f at p. So if $f = \sum_{i=m}^{\infty} f_i t_p^i$, $m \in \mathbb{Z}$, the polar part at p is $f = \sum_{i=m}^{-1} f_i t_p^i$. Hence, the polar part of f at $p \in \mathbb{C}$ is non-zero if and only if p is a pole of f. Considering the partial fraction decomposition, the polar parts at all $p \in \mathbb{C}$ determine f uniquely up to a polynomial in $k[x]$, e.g. $a_0 + g(x), g(x) \in xk[x]$. Furthermore, the polynomial $g(x)$ is the polar part of f at $p = \infty$, which can be seen when representing f at $p = \infty$ (i.e. in terms of $t_\infty = \frac{1}{x}$). Thus, the sum of the polar parts determines f up to a constant a_0.

Theorem 2(ii) shows how to compute those polar parts from the exponent differences $\Delta(L,p)$ at the exp-irregular points. Since the $\Delta(L,p)$ are defined up to $\pm$ signs, we obtain each polar part up to a sign as well. If S_{irr} has n elements, our algorithm checks all 2^n combinations[4] of $\pm$ signs, yielding a set of candidates denoted by $\mathcal{F}$. One of these candidates will be f up to a constant.

If we know at least one zero of f we can use it to compute this constant. In this case we can find f by trying all candidates in $\mathcal{F}$. But if $S_{reg} = \emptyset$, then we will need an additional method (Section 4.1) to find f and the Bessel parameter ν.

3.2 The Parameter ν

An important property for Bessel functions is that the space $\mathbb{C}(x)B_\nu(x) + \mathbb{C}(x)B_\nu(x)'$, i.e. the space generated by all gauge transformations of a Bessel function $B_\nu(x)$, is invariant under $\nu \to \nu + 1$. In other words, it is sufficient to find ν modulo an integer. If we take a 'wrong' ν that is off by an integer, then this is caught by the gauge transformation, which is computed afterwards.

For the Bessel parameter ν we will have to consider several cases. We may assume $\nu \notin \frac{1}{2}+\mathbb{Z}$ (otherwise L_B is reducible). The following lemma handles the case $\nu \in \mathbb{Z}$.

Lemma 5 *Let $L \in K[\partial]$ be a differential operator. Assume $L_B \longrightarrow L$. Then the following are equivalent:*

(i) The Bessel parameter is an integer, i.e. $\nu \in \mathbb{Z}$.

(ii) There is an exp-regular singularity p of L such that L is logarithmic at p.

PROOF. (i) $\Rightarrow$ (ii) If ν is an integer, the functions in the solution space $V(L_B)$ are gauge transformations of the functions in $V(L_B, \nu = 0)$. So it is sufficient to prove the case $\nu = 0$. Then L_B has a logarithmic solution at $x = 0$. This solution is transformed into a solution z of L. This works as in the proofs of Theorem 2 and Lemma 4. If p is a zero of the parameter f in the change of variables, the logarithm changes as follows:

$$\begin{aligned}\ln(x) \to \ln(f) &= \ln\left(t_p^m c \sum_{i=0}^{\infty} f_i t_p^i\right)\\ &= \ln(c) + m\ln(t_p) + \ln\left(1 + \sum_{i=1}^{\infty} f_i t_p^i\right),\end{aligned}$$

where $f_0 = 1$. So L has a logarithmic solution at the point p and since it was a zero of f, it is also exp-regular.

(ii) $\Rightarrow$ (i) If $\nu \notin \mathbb{Z}$ then the local solutions of L_B do not involve logarithms[5] and hence the same is true for L. $\square$

We will use the fact from Theorem 2(i) that $\Delta(L,p)$ is $2m\nu \bmod \mathbb{Z}$ for exp-regular singularities $p \in S_{reg}$.

Lemma 6 *These statements are true for all $s \in S_{reg}$:*

(i) logarithmic case:	L *logarithmic at* s	$\Leftrightarrow$	$\nu \in \mathbb{Z}$
(ii) integer case:	$S_{reg} = \emptyset$	$\Rightarrow$	$\nu \in \mathbb{Q}\backslash\mathbb{Z}$
(iii) rational case:	$\Delta(L,s) \in \mathbb{Q} \setminus \mathbb{Z}$	$\Rightarrow$	$\nu \in \mathbb{Q}\backslash\mathbb{Z}$
(iv) base field case:	$\Delta(L,s) \in k\backslash\mathbb{Q}$	$\Leftrightarrow$	$\nu \in k\backslash\mathbb{Q}$
(v) irrational case:	$\Delta(L,s) \notin k$	$\Leftrightarrow$	$\nu \notin k$

Exactly one case applies (we assume $L_B \longrightarrow L$). Case (v) only occurs if k is not algebraically closed (in Section 4.2).

PROOF. Case (i) has been proven in Lemma 5. By Theorem 2(i) $\Delta(L,s) = 2m_s\nu + z_s$ for all zeros s of the parameter f. Hereby, m_s is the multiplicity and $z_s \in \mathbb{Z}$. Since f has at least one zero (possibly at ∞) there is at least one such equation from which we can deduce cases (ii) to (v). $\square$

Recall that if $S_{reg} \neq \emptyset$, we can pick a zero $s \in S_{reg}$ of f. In that case we can compute candidates $\mathcal{F}$ for the parameter in the change of variables. For each candidate f we compute a set of candidates $\mathcal{N}$ for ν.

Definition 4. Let $s \in S_{reg} \neq \emptyset$ be a zero of the parameter $f \in K$. Let m_s be the multiplicity of s. We define

$$\mathcal{N}_s := \left\{ \frac{\Delta(L,s)+i}{2m_s} \;\middle|\; 0 \le i \le 2m_s - 1 \right\}$$

and $$\mathcal{N} := \Big\{ \pm\nu \bmod \mathbb{Z} \;\Big|\; \forall s \in S_{reg}\ \exists z_s \in \mathbb{Z} : \nu + z_s \in \mathcal{N}_s \text{ or } -\nu + z_s \in \mathcal{N}_s \Big\}.$$

Both sets are finite and it is easy to see:

Corollary 2 *If $L_B \longrightarrow L$ and ν is the Bessel parameter in L_B then* $\pm\nu \bmod \mathbb{Z} \in \mathcal{N}$.

4. THE ALGORITHM

The input of our algorithm is a differential operator L_{in} and we want to know whether the solutions can be expressed in terms of Bessel functions. We assume that $L_B \longrightarrow L_{in}$ for some transformations. If we find a solution to that problem, then we also find the solution space of L_{in}.

We will first assume $k = \mathbb{C}$ and will deal with more general fields k in the next section. Let L_{in} be a differential operator of degree two with coefficients in $K = \mathbb{C}(x)$.

Let's summarize the steps of the algorithm that we have deduced in the previous sections:

1. **(Singularities)** We can compute the singularities S of L_{in} by factoring the leading coefficient of L_{in} and the denominators of the other coefficients into linear factors.

2. **(Generalized exponents)** For each $s \in S$ we compute $d_s = \Delta(L_{in}, s)$, isolate exp-apparent points with $d_s \in \mathbb{Z}$, and differ between exp-regular singularities S_{reg} with $d_s \in \mathbb{C}$ and exp-irregular singularities S_{irr} with $d_s \in \mathbb{C}[t_s^{-1}]\backslash\mathbb{C}$.

[4]Actually: 2^{n-1} combinations, see Example 1.

[5]In general, if $\Delta(L,p) \notin \mathbb{Z}$ then no logarithm appears at p.

3. **(Polar parts)** We can use the exponent differences d_s for $s \in S_{irr}$ to compute candidates $\mathcal{F}$ for the parameter f up to a constant $c \in k$.

4. **(Constant term of f)** In all cases but the integer case we know at least one zero of f by picking some $s_0 \in S_{reg}$. So we can also compute the missing constant c for each $\tilde{f} \in \mathcal{F}$.

5. **(The set $\mathcal{N}$)** The set $\mathcal{N}$ is a set of candidates for ν. When not in the integer case, we compute this finite set as in Section 3.2; $\mathcal{N}$ might depend on f.

6. **(Compute M)** For each $f \in \mathcal{F}$ and each $\nu \in \mathcal{N}$ compute $M = M(\nu, f) \in k(x)[\partial]$ such that $L_B \xrightarrow{f}_C M$.

7. **(Exp-product and gauge transformation)** For each M decide (e.g. using [1]) whether $M \longrightarrow_{EG} L_{in}$, and if so, compute the transformation.

The only case in which this algorithm does not yet work is when $S_{reg} = \emptyset$, which we will handle in the next section. Note that one can also use the case separation of Lemma 6 to reduce the number of candidates that we obtain from steps 4 and 5. More details can be found in [7].

Example 1: Let

$$\begin{aligned}L := {} & 9(3x^2 - 14x + 7)(x-2)^2(x-5)^2(x-1)^4\partial^2 + \\ & 9(3x^4 - 28x^3 + 82x^2 - 52x - 21)(x-2)(x-5)(x-1)^3\partial - \\ & (3x^2 - 14x + 7)^3(9x^8 - 198x^7 + 1845x^6 - 9540x^5 + \\ & 30060x^4 - 59328x^3 + 71860x^2 - 48968x + 14404).\end{aligned}$$

We apply the algorithm above to L step by step:

Step 1: The zeros of the leading coefficient are $1, 2, 5$ and $(7 \pm 2\sqrt{7})/3$. Furthermore, ∞ is also a singularity of L.
Step 2: The generalized exponents[6] at the points $p = (7 \pm 2\sqrt{7})/3$ are 0 and 2; those at $p = 2$ are -2 and 2. No logarithm appears at these three points either, so they are exp-apparent and are not considered anymore. At the other points we compute the following exponent differences:

$$\begin{aligned}&\Delta(M, 1) = 8/t_1 = 8/(x-1)\\ &\Delta(M, 5) = 4/3\\ &\Delta(M, \infty) = 6/t_\infty^3 - 40/t_\infty^2 + 64/t = 6x^3 - 40x^2 + 64x\end{aligned}$$

Hence, $S_{irr} = \{1, \infty\}$ and $S_{reg} = \{5\}$.
Step 3: Using equation (3) in Theorem 2 we can compute the polar parts corresponding to the exp-irregular points:

$$f_1 = \frac{4}{x-1}, \quad f_\infty = x^3 - 10x^2 + 32x.$$

The set of candidates is $\mathcal{F} = \{f_1 + f_\infty,\ f_1 - f_\infty,\ -f_1 + f_\infty,\ -f_1 - f_\infty\}$. Since $L_B \xrightarrow{-x}_C L_B$ we can ignore the latter two candidates.
Step 4: The point $5 \in S_{reg}$ must be a zero of f. Evaluating the candidates at this point yields the candidates

$$g_1 = f_1 + f_\infty - 36 \quad \text{and} \quad g_2 = f_1 - f_\infty + 34$$

for the parameter in the change of variables.
Step 5: To get candidates for ν we determine $\mathcal{N}$ which is equal to $\mathcal{N}_5$ because we just have one exp-regular point. Therefore we compute the multiplicity of the zero 5 of g_1 and g_2; it is 1 in both cases. Therefore:

$$\mathcal{N} = \mathcal{N}_5 = \left\{\frac{2}{3}, \frac{7}{6}\right\}.$$

Step 6: For each pair f, ν compute M such that $L_B \xrightarrow{f}_C M$.
Step 7: Finally, for each M check if $M \longrightarrow_{EG} L$.

In practice we combine steps 6 and 7. At the end we will get $L_B\,|_{\nu=2/3} \xrightarrow{f}_C M$ for $f = (x-5)(x-2)^3/(x-1)$ and the solution space

$$C_1 I_{2/3}\left(\frac{(x-5)(x-2)^3}{x-1}\right) + C_2 K_{2/3}\left(\frac{(x-5)(x-2)^3}{x-1}\right).$$

One can solve Example 1 with prior algorithms [3, 4, 5, 15] because no gauge transformation is involved. An example not solved by prior algorithms will be given in Example 2.

4.1 Integer Case

Let $S_{reg} = \emptyset$, i.e. $\Delta(L_{in}, s) \in \mathbb{Z}$ for all $s \notin S_{irr}$. Then $\nu \in \mathbb{Q}$ and we define

$$\mathcal{N}(m) := \left\{\frac{i}{2m},\ i = 1, \dots, 2m-1\right\} \setminus \left\{\frac{1}{2}\right\}.$$

Then the following holds:

Lemma 7 *(a) Let n be the degree of the numerator of f. Then there exists $p, \ell \in \mathbb{Z}$ such that $p \mid n$ and $\nu + \ell \in \mathcal{N}(p)$.*
(b) For the degree of the numerator we know:

(i) If $\infty \in S_{irr}$, then
$\deg(\mathrm{numer}(f)) = \deg(\mathrm{numer}(f + c))$ *for all* $c \in \mathbb{C}$.
(ii) If $\infty \notin S_{irr}$, then
$p \mid \deg(\mathrm{numer}(f)) \Leftrightarrow p \mid \deg(\mathrm{denom}(f))$.

(c) Let $p \in \mathbb{N}$ and $p > 1$, $f \in k[x]$. With linear algebra we can decide whether there exists g such that $f = g^p$ and compute g explicitly.

We will use the lemma as follows: If we know the degree n of the numerator of f then by (a) we can compute candidates $\mathcal{N}(p)$ for ν for every $p \mid n$. By (b) it is enough to know candidates $\mathcal{F}$ for f modulo the constant term to get all $p \mid n$ we have to consider. In our case we still have an undetermined constant c. But if the numerator of f is a p-th power ($p > 1$), we can determine g by (c) such that $\mathrm{numer}(f)$ *can be* equal to g^p. From $\mathrm{numer}(f) - g^p = 0$ we can then compute candidates for c.

Proof. (a) Let s be a zero of the parameter f of the change of variables, then $\Delta(L_{in}, s) = 2m\nu$ modulo $\mathbb{Z}$.[7] Since $\nu \in \mathbb{Q}$ can be written as $\nu = z + \frac{\nu_1}{2p}$ with $z \in \mathbb{Z}, \nu_1, p \in \mathbb{N}$ and $0 < \nu_1 < 2p$, $\gcd(\nu_1, p) = 1$. Then $\nu - z \in \mathcal{N}(p)$ and $2m\nu = 2mz + \frac{m\nu_1}{p} \in \mathbb{Z}$. Since $z, m \in \mathbb{Z}$ and $\gcd(\nu_1, p) = 1$ this is equivalent to $p \mid m$. So p divides all multiplicities of the zeros of f. The degree of the numerator of f is equal to the sum of these multiplicities and we get $p \mid \deg(\mathrm{numer}(f))$. Hence, p and $\ell = -z$ satisfy the statement.

(b) After using the exp-irregular points S_{irr} to find the polar parts, f has the form $f = \frac{f_1}{f_2} + c + f_3$, where $f_1, f_2, f_3 \in k[x]$ and $\deg(f_1) < \deg(f_2)$ or $f_1 = 0$. The polar parts for

[6]E.g. using the command `gen_exp` in *Maple*.

[7]Since $\Delta(L_{in}, s) \in \mathbb{Z}$ in this section, we need no $\pm$ signs when working modulo $\mathbb{Z}$.

$s \in S_{irr} \backslash \{\infty\}$ are combined in $\frac{f_1}{f_2}$. The polynomial f_3 is the polar part of $\infty \in S_{irr}$.

(i) In this case $\infty \in S_{irr}$ and hence $f_3 \neq 0$. So c does not affect the degree of the numerator of f.

(ii) Since $\infty \notin S_{irr}$, $f_3 = 0$ and $f = \frac{f_1}{f_2} + c$ with $\deg(f_1) < \deg(f_2)$.

Case 1: If $c \neq 0$, then $\deg(\mathrm{numer}(f)) = \deg(\mathrm{denom}(f))$ and nothing remains to be proven.

Case 2: If $c = 0$, then ∞ is a zero of f. The multiplicity m must be a multiple of p. Otherwise $\Delta(L_{in}, \infty) \notin \mathbb{Z}$ since $\Delta(L_{in}, \infty) = 2m\nu + z$ for some $z \in \mathbb{Z}$ and $2\nu = \frac{\nu_1}{p}$. Hence, $m = kp$ for some $k \in \mathbb{N}$.

This multiplicity of the point ∞ is $m = \deg(\mathrm{denom}(f)) - \deg(\mathrm{numer}(f))$. This can be seen if $f(\frac{1}{x})$ is written as power series at the point 0. In total we get $\deg(\mathrm{numer}(f)) = \deg(\mathrm{denom}(f)) - kp$ for some $k \in \mathbb{N}$ and this proves (ii).

(c) Comparing the highest coefficients of g^p and f in an ansatz for g yields the result. $\square$

Concluding, the integer case works as follows. For each $f \in \mathcal{F}$ that is determined up to a constant term c we perform the following steps:

$n := \deg(\mathrm{numer}(f))$ if $\infty \in S_{irr}$
and $n := \deg(\mathrm{denom}(f))$ otherwise
for each $p \mid n$, $p \neq 1$
  compute candidates $\mathcal{C}$ for the constant term c
  for each $c \in \mathcal{C}$ and each $\nu \in \mathcal{N}(p)$
    compute M such that $L_B \xrightarrow{f+c}_C M$

4.2 Solving Over a General Field k

Until now, we were working over the constant field $k = \mathbb{C}$ and we haven't thought of the speed of the algorithm yet. We started by computing all the singularities of L and did some computations with them. So what we actually did is factor the leading coefficient $l(x)$ of L into linear factors. This can be very expensive and can lead to a huge extension of $\mathbb{Q}$, in which all the other computations take place. In this section we will discuss how we can work over a much smaller extension of $\mathbb{Q}$.

We will use the following setting. Let k be a finitely generated extension of $\mathbb{Q}$ such that the input operator L has coefficients in $k[x]$ and let $K = k(x)$. For each irreducible factor $q(x)$ of $l(x)$ in $k[x]$ we pick one zero s. Furthermore, let $\sigma \in \mathrm{Hom}_k(k(s), \bar{k})$ be an embedding of $k(s)$ in an algebraic closure $\bar{k}$ that keeps k fixed and we denote the *trace* of a element $a \in k(s)$:

$$\mathrm{Tr}(a) = \sum_{\sigma \in \mathrm{Hom}_k(k(s),\bar{k})} \sigma(a).$$

We will now focus on each of the seven steps of the algorithm and explain the changes that have to be made.

1. (**Singularities**) When we factor the coefficients of L in $k[x]$ we get irreducible factors whose degree can be greater than one. For each irreducible factor, we fix one zero. The finite singularities then are

$$S = \left\{ \sigma(s) \;\middle|\; s \text{ zero of irred. factor}, \sigma \in \mathrm{Hom}_k(k(s), \bar{k}) \right\}.$$

Now fix an irreducible factor $q(x)$ of $l(x)$ and let s be a zero of $q(x)$ and $\sigma \in \mathrm{Hom}_k(k(s), \bar{k})$.

The strategy is to use the computations at the singularity s for other singularities $\sigma(s)$, which are zeros of the same irreducible factor.

2. (**Generalized exponents**) In the computation of the generalized exponent at the point $x = s$ the field $k(s)$ is taken as the field of constants. If e_1, e_2 are the exponents at s then the exponents at $\sigma(s)$ are $\sigma(e_1), \sigma(e_2)$. Note: e_1, e_2 may be defined over an extension field of $k(s)$, in which case the σ's need to be extended as well.

Similarly, if y is a local solution at the point $x = s$, then $\sigma(y)$ is a local solution at $x = \sigma(s)$ because the operator cannot distinguish between the points s and $\sigma(s)$. Hence, $\Delta(L, \sigma(s)) = \sigma(\Delta(L, s))$.

Since all our results were based on generalized exponents and exponent differences we can now transfer results for s to $\sigma(s)$. So for each irreducible factor $q(x)$, only one of its zeros is needed in S_{reg} or in S_{irr}. In the terminology of [6], the singularities are computed *up to conjugation over* k (see Section 5.1 in [6] for more details).

3. (**Polar parts**) Let $s \in S_{irr}$. We can compute the polar part f_s corresponding to s. Assume for now that $f \in k(x)$ (we will explain below what to do if $f \in \overline{k}(x) \setminus k(x)$). Then the polar part corresponding to $\sigma(s)$ is $f_{\sigma(s)} = \sigma(f_s)$. So the trace of f_s is:

$$\mathrm{Tr}(f_s) = \sum_{\sigma \in \mathrm{Hom}_k(k(s),\bar{k})} \sigma(f_s) = \sum_{\sigma \in \mathrm{Hom}_k(k(s),\bar{k})} f_{\sigma(s)}.$$

The result is the polar part of f corresponding to the irreducible polynomial $q(x)$. Hence, we computed the whole polar part corresponding to all zeros of $q(x)$ by just using one zero of $q(x)$.

4. (**Constant term of** f) Let $f = \tilde{f} + c$ for some $\tilde{f} = \tilde{f}(x) \in \mathcal{F}$ be a candidate for the parameter in the change of variables. If $S_{reg} \neq \emptyset$, then we know at least one zero of f. Assume $s \in S_{reg}$, then we compute c such that $f(s) = 0$. If $s \notin k$, we would get $c \notin k$ in general. However, the $\sigma(s)$ must be zeros of f as well. So $q(x)$ divides the numerator of f, which translates into a system of linear equations (in one unknown: c) defined over k.

The integer case needs no change.

5. (**The set** $\mathcal{N}$) For these computations we only used exp-regular points $s \in S_{reg}$ with $\Delta(L_{in}, s) = 2m_s\nu$. If $\nu \in k$ then $\Delta(L_{in}, \sigma(s)) = \sigma(\Delta(L_{in}, s)) = \Delta(L_{in}, s)$ for all $s \in S_{reg}$. If $\nu \notin k$ then we compute ν^2 instead, which will be in k. In either case, we need to use only one root s for each irreducible $q(x)$.

6./7. (**Compute M, exp-product and gauge transformation**) From here, everything works as before.

One problem remains when generalizing the algorithm as we did above. For computing the polar parts in step 3 we used that f is defined over k. However, we take k as the smallest field for which $L \in k(x)[\partial]$, in which case f need not be in $k(x)$.

For example $L_B \xrightarrow{f}_C L$ with $f = \sqrt{2}(x^2 - 2)$ and $L = (x^2-2)^2 x\partial^2 + (x^4 - 4)\partial - 4(2x^4 - 8x^2 + 8 + \nu^2)x^3$ is more complicated. Here $\sqrt{2}$ does not appear in L. More generally, if $L_B \xrightarrow{cx}_C L$ for a constant c, then just c^2 appears in L.

If $L_{in} \in k(x)[\partial]$ is given, we will thus restrict constant factors c of f to elements in quadratic extensions of k for which $c^2 \in k$. This is enough to combine all Bessel functions in one algorithm. However, we still have to prove that this is really sufficient in all cases.

Let c be the constant factor we search for and let $s \in S_{irr}$ be a singularity. Since a constant factor of f is also a factor of its polar parts, c must be a factor of $\Delta(L_{in}, s)$. For each point s we have the constant fields $k \subseteq k(c) \subseteq k(c,s) =: k_s$. The exponent difference $\Delta(L_{in}, s)$ is defined over k_s and we can read off k_s from $\Delta(L_{in}, s)$. So we have to find algebraic extensions $\tilde{k}$ of k of degree two such that $\tilde{k} \subseteq k_s$ for all s. Then for each $\tilde{k}$, take a constant $c \in \tilde{k}$ satisfying $c^2 \in k$. This gives a finite list of candidates for the constant factor c. If $k(c,s) = k(s)$ for each s then add $c = 1$ to this list as well.

For each c in the list, we can divide all exponent differences (for $s \in S_{irr}$) by c and apply the algorithm as before. This effectively divides f by c, and if we picked the correct c, the quotient will be in $k(x)$ so then the algorithm will work correctly. If we try all candidates c, at some time we will get the right one and we will find a solution.

Example 2: Consider the differential equation

$$\left(x\, y''(x)\right)'' - \left(\left(\frac{9}{x} + \frac{8}{M}x\right) y'(x)\right)' = \lambda^2 \left(\lambda^2 + \frac{8}{M}\right) x\, y(x)$$

for all $x \in (0, \infty)$, whereas M and λ are constant parameters. Factoring the corresponding differential operator (for example in *Maple*) gives two operators L_1 and L_2 of degree two. This example occurred in research of W. N. Everitt. Although it has been solved before (see [8]) by the second author[8], our implementation is the first that can solve L_1 and L_2 completely automatically. One of them is

$$L_1 = \partial^2 + \frac{\left(\lambda^4 M^2 x^2 + 8\,\lambda^2 M x^2 + 16\,x^2 - 48\,M\right)}{x\left(\lambda^4 M^2 x^2 + 8\,\lambda^2 M x^2 + 16\,x^2 - 16\,M\right)}\partial + \frac{\lambda^2\left(-4\,\lambda^2 M^2 - 32\,M + 16\,x^2 + 8\,\lambda^2 M x^2 + \lambda^4 M^2 x^2\right)}{\lambda^4 M^2 x^2 + 8\,\lambda^2 M x^2 + 16\,x^2 - 16\,M}$$

and our implementation yields the solutions

$$\frac{C_1}{x}\left(-2\lambda M\, J_1(x\lambda) + \left(\lambda^2 M + 4\right) x\, J_0(x\lambda)\right) + \frac{C_2}{x}\left(-2\lambda M\, Y_1(x\lambda) + \left(\lambda^2 M + 4\right) x\, Y_0(x\lambda)\right).$$

Likewise our implementation solves L_2 as well.

Note: A referee alerted us that the soon to be released Maple 12 can solve L_1 and L_2 in terms of Heun functions. For completeness, we give here another equation that to our knowledge is not solved by other algorithms:

$$\begin{aligned} L = {} & (4x^6 - 6x^4 - 5x^2 + 3)(x^4+1)^4 x\partial^2 \\ & - 3(4x^{10} - 2x^8 + 11x^6 - 13x^4 - 5x^2 + 1)(x^4+1)^3\partial \\ & + (12x^{18} + 18x^{16} - 83x^{14} + 83x^{12} + 534x^{10} - 714x^8 \\ & - 299x^6 + 315x^4 + 224x^2 - 58)x^3. \end{aligned}$$

5. CONCLUSION

We gave an algorithm to detect the Bessel type solutions of a second order differential operator with rational function coefficients and implemented it in *Maple*.

The method can in principle be applied to the case of solutions in terms of the Gauß hypergeometric function. However, this case is significantly more difficult if we aim to solve it in general (if we restrict f to Möbius transformations then it is an easy addition to [1]). In particular, the important case where all exponent differences are in $\mathbb{Q}$ poses interesting but difficult problems that are as of yet not solved.

Another task for future work is to generalize our algorithm to allow f with $f^2 \in K$ (instead of $f \in K$ as in this paper) and then to prove completeness for that algorithm, i.e. for a definition of "Bessel type" that is both natural and general, prove that every Bessel type solution would always be found.

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