Some New Classes of Orthogonal Polynomials and Special Functions: A Symmetric Generalization of Sturm-Liouville Problems and its Consequences

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Chapter 1

Outline of dissertation

This thesis includes 8 chapters. The chapter 2 specifies how to generalize the usual Sturm-Liouville problems with symmetric solutions to a larger class. In other words, if the symmetric sequence $y = \Phi_n(x)$ satisfies a usual Sturm-Liouville problem as

$$\frac{d}{dx}\left(k(x)\frac{dy}{dx}\right) + (\lambda_n\,\rho(x) - q(x))y = 0 \quad ; \quad k(x) > 0, \quad \rho(x) > 0, \tag{1.1}$$

with the boundary conditions

$$\begin{cases} \alpha_1\, y\,(a) + \beta_1\, y\,'(a) = 0 \\ \alpha_2\, y\,(b) + \beta_2\, y\,'(b) = 0 \end{cases} \quad ; \quad a < x < b\,, \tag{1.1.1}$$

then, under some specific conditions, it can be generalized to a more extensive problem as

$$\frac{d}{dx}\left(k(x)\frac{dy}{dx}\right) + (\lambda_n\,\rho(x) - q(x) + \frac{1-(-1)^n}{2}E(x))y = 0\,, \tag{1.2}$$

with the individual condition

$$\alpha_3\, y\,(\theta) + \beta_3\, y\,'(\theta) = 0 \;\; ; \;\; -\theta < x < \theta\,. \tag{1.2.1}$$

It should be noted that the main advantage of this extension is that the generalized solutions preserve the orthogonality property. In this sense, by using the above theorem, the well-known symmetric associated Legendre functions (having extensive applications in physics and engineering) are generalized and it is shown that the orthogonality interval is the same as [-1,1].

Another important consequence of the extension of usual Sturm-Liouville problems with symmetric solutions is given in the chapter 3. In this chapter, by using the mentioned theme, a main class of symmetric orthogonal polynomials (MCSOP) with four free parameters is introduced and all its standard properties, such as a generic second order differential equation of the form (1.3)

$$x^2(px^2+q)\,\Phi_n''(x) + x(rx^2+s)\,\Phi_n'(x) - \left(n(r+(n-1)p)x^2 + (1-(-1)^n)\,s/2\right)\Phi_n(x) = 0\,,$$

together with its explicit polynomial solution in the form

$$S_n\begin{pmatrix} r & s \\ p & q \end{pmatrix}\Bigg| x\Bigg) = \sum_{k=0}^{[n/2]} \binom{[n/2]}{k} \left(\prod_{i=0}^{[n/2]-(k+1)} \frac{\left(2i+(-1)^{n+1}+2[n/2]\right)p+r}{\left(2i+(-1)^{n+1}+2\right)q+s} \right) x^{n-2k} \,, \qquad (1.3.1)$$

a generic orthogonality relation, a generic three term recurrence relation and so on are obtained. Moreover, it is shown that four main sub-classes of symmetric orthogonal polynomials, i.e. the generalized ultraspherical polynomials, generalized Hermite polynomials and two new sequences of finite symmetric orthogonal polynomials are all special sub-cases of MCSOP. In this way, two half-trigonometric sequences of MCSOP are introduced.

But, usually the finite orthogonal polynomials are less known (and discussed) in the literature. In chapter 4, a comprehensive treatment about finite classical orthogonal polynomials that are specific solutions of the well-known differential equation of Sturm-Liouville type

$$\sigma(x)y_n''(x)+\tau(x)y_n'(x)-\lambda_n y_n(x)=0 \,, \qquad (1.4)$$

in which $\sigma(x)=ax^2+bx+c$, $\tau(x)=dx+e$ and $\lambda_n=n(n-1)a+nd$ is given.
In this chapter, three sequences of hypergeometric polynomials, which are finitely orthogonal with respect to the F, inverse gamma and generalized-T distribution functions are reintroduced in detail and their application in functions approximation and numerical integration are investigated.

It is interesting to know that the weight function of the third class of finite orthogonal polynomials corresponds to a generalization of T distribution, which as far as we know has not appeared in the mathematical statistics branches. For this reason, chapter 5 is allocated to a large generalization of Student's *t*-distribution with four free parameters and a comprehensive discussion of mathematical properties of this new distribution is investigated from special functions point of view. It is also shown that, similar to usual normal distribution, the generalized t-distribution converges to the normal distribution again when the number of samples tends to infinity. In this sense, since the Fisher *F*-distribution has a close relationship with the *t*-distribution, a generalization of the *F*-distribution is also introduced and shown that it similarly converges to the chi-square distribution as the number of samples tends to infinity. At the end of the chapter, some special cases of the generalized distributions are studied.

But as we observe, the introduced equation (1.3) has a generic polynomial solution as (1.3.1). Now, is it possible to find a generic polynomial solution for the differential equation (1.4) similarly? Chapter 6 replies this question in detail. In chapter 6, it is shown that the equation (1.4) has a generic monic polynomial solution as

$$y_n(x)=\bar{P}_n\begin{pmatrix} & d & e \\ a & b & c \end{pmatrix}\Big| x\Big) = \sum_{k=0}^{n}\binom{n}{k} G_k^{(n)}(a,b,c,d,e)\,x^k \,, \qquad (1.4.1)$$

where

$$G_k^{(n)} = \left(\frac{2a}{b+\sqrt{b^2-4ac}}\right)^{k-n} {}_2F_1\left(\left.\begin{matrix} k-n \quad \dfrac{2ae-bd}{2a\sqrt{b^2-4ac}}+1-\dfrac{d}{2a}-n \\ 2-d/a-2n \end{matrix}\right| \frac{2\sqrt{b^2-4ac}}{b+\sqrt{b^2-4ac}}\right).$$

Then, it is shown that all six sequences of infinite and finite classical orthogonal polynomials described in chapter 4 can be indicated by the general derived formula.
Some general properties of the formula (1.4.1), such as the affection of a linear change of variables, a generic three-term recurrence equation, a generic formula for the norm square value of the polynomials and so on are also given. At the end of the chapter, it is explained how to derive a generic formula for the values at the boundary points of monic classical orthogonal polynomials.

Here is interesting to know that classical orthogonal polynomials have a direct use in the explicit computation of inverse Laplace transforms. In other words, if the Mobius transform $x = pz^{-1}+q$, $p \neq 0$, $q \in R$ is employed in each six sequences of classical orthogonal polynomials, then the generated rational orthogonal polynomials can be used to compute the inverse Laplace transform directly, with no additional calculation for finding their roots. In chapter 7, by achieving this purpose, three basic examples are given to explicitly obtain the inverse Laplace transforms.

Finally, chapter 8 introduces some new sequences of special functions that have application in the solutions of classical potential, heat and wave equations in spherical coordinate.
In other words, it is known that in the solution of Sturm-Liouville problems, usually zero eigenvalue is ignored. Now, if the foresaid eigenvalue is considered to be zero, one of the solutions will be pre-assigned. This approach causes to appear new sequences of special functions in the solutions of the classical equations of potential, heat and wave in spherical coordinate. By noting these comments, a class of special functions is introduced in the first part of chapter 8 and is applied in the mentioned classical equations. In this way, some applied examples are given.
And eventually, in the second part of last chapter, two new classes of orthogonal hypergeometric functions are introduced and it is shown that using Fourier transforms and Paraseval identity they are finitely orthogonal with respect to two specific functions of Gamma type. Moreover, as the third new hypergeometric class, first a complicated integral is explicitly evaluated and then it is conjectured that the integrand of this integral may be finitely orthogonal with respect to the Ramanujan weight function on the real line.

Chapter 2

A symmetric generalization of Sturm - Liouville Problems

2.1. Introduction

In this chapter, we present some conditions under which the usual Sturm-Liouville problems with symmetric solutions can be extended to a larger class. The main advantage of this extension is that the corresponding solutions preserve the orthogonality property. As a sample, we investigate a basic example of generalized Sturm-Liouville problems and obtain their orthogonal solutions. The foresaid example generalizes the well-known associated Legendre functions (having extensive applications in physics and engineering) and preserves the orthogonality interval [-1,1] for the generalized functions.

2.2. Boundary value problems

When partial differential equations are solved by the method of separation of variables (see also chapter 8 section 8.2), the problem reduces to the solution of ordinary differential equations. The solutions of these equations can, in many interesting problems of mathematical physics, be expressed in terms of special functions. In order to obtain such solutions of the partial differential equations in specific cases, we have to impose additional conditions on the solutions such that the problems will have unique solutions. These conditions in turn lead to conditions on the solutions of the ordinary differential equations and thus to boundary value problems. We here intend to first have a survey on the solution of boundary value problems by the method of separation of variables.

2.2.1. Usual Sturm-Liouville problems

It has been proved that the method of separation of variables can extensively be applied for solving partial differential equations of the form

$$\rho(x,y,z)\left(A^*(t)\frac{\partial^2 u}{\partial t^2}+B^*(t)\frac{\partial u}{\partial t}\right)=Lu\,, \tag{2.1}$$

where

$$Lu=\operatorname{div}(k(x,y,z)\operatorname{grad}u)-q(x,y,z)\,u\ , \tag{2.1.1}$$

$$\operatorname{div}\vec{\mathrm{F}}=\text{Divergence of vector F}\ , \tag{2.1.2}$$

$$\operatorname{grad} \mathrm{F} = \frac{\partial F}{\partial x}\vec{i} + \frac{\partial F}{\partial y}\vec{j} + \frac{\partial F}{\partial z}\vec{k}\ . \tag{2.1.3}$$

The equation (2.1) describes the propagation of a vibration such as electromagnetic or acoustic waves if $A^*(t) = 1$ and $B^*(t) = 0$ and describes transfer processes such as heat transfer or the diffusion of particles in a medium when $A^*(t) = 0$ and $B^*(t) = 1$. Finally it describes the corresponding time-independent processes if $A^*(t) = 0$ and $B^*(t) = 0$ [56, p. 299].

To have a unique solution of equation (2.1), which corresponds to an actual physical problem, some supplementary conditions must be imposed. The most typical conditions are initial or boundary conditions. The initial conditions for equation (2.1) are usually the values of $u(x, y, z, t)$ and $\partial u(x, y, z, t)/\partial t$, while the simplest boundary condition is in the form

$$\left[\alpha(x, y, z)u + \beta(x, y, z)\frac{\partial u}{\partial \eta}\right]\Bigg|_S = 0\,, \tag{2.2}$$

where $\alpha(x, y, z)$ and $\beta(x, y, z)$ are given functions; S is the surface bounding the domain where (2.1) is to be solved; $\partial u / \partial \eta$ is the derivative in the direction of the outward normal to *S*.

But the particular solutions of (2.1) under the boundary conditions (2.2) will be found if one looks for a solution of the form

$$u(x, y, z, t) = T(t)\, v(x, y, z)\,. \tag{2.3}$$

By substituting (2.3) into the main equation (2.1) one respectively gets

$$A^*(t)T'' + B^*(t)T' + \lambda T = 0\,, \tag{2.4}$$

$$L\, v + \lambda\, \rho\, v = 0\,, \tag{2.5}$$

where λ is a constant. Clearly (2.4) can be solved for typical problems in mathematical physics. However, to solve (2.5) we should use a boundary condition that follows from (2.2), namely $(\alpha(x, y, z)\, v + \beta(x, y, z)\frac{\partial v}{\partial \eta})\Big|_S = 0$. The described problem is a known as multidimensional boundary value problem. Nevertheless, it can be simplified to a one-dimensional problem if (2.5) is reduced to an equation of the form

$$L\, y + \lambda\ \rho(x)\, y = 0\,, \tag{2.6}$$

where

$$L\, y = \frac{d}{dx}\left(k(x)\frac{dy}{dx}\right) - q(x)y\,, \quad k(x) > 0\,, \quad \rho(x) > 0\,. \tag{2.6.1}$$

The equation (2.6) should be considered on an open interval, say (a,b), with boundary conditions in the form

$$\begin{aligned} \alpha_1 y(a) + \beta_1 y'(a) = 0 \ , \\ \alpha_2 y(b) + \beta_2 y'(b) = 0 \ , \end{aligned} \tag{2.7}$$

where α_1, α_2 and β_1, β_2 are given constants and $k(x), k'(x), q(x)$ and $\rho(x)$ in (2.6) and (2.6.1) are to be assumed continuous for $x \in [a,b]$.

The simplified boundary value problem (2.6)-(2.7) is called a regular Sturm-Lioville problem. Moreover, if one of the boundary points a and b is singular (i.e. $k(a) = 0$ or $k(b) = 0$), the problem will be transformed to a singular Sturm-Liouville problem. In this case, one can ignore boundary conditions (2.7) and obtain the orthogonality property directly.

Now, let $y_n(x)$ and $y_m(x)$ be two solutions (eigenfunctions) of equation (2.6). According to the Sturm-Liouville theory [18, 56], these functions are orthogonal with respect to the weight function $\rho(x)$ on (a,b) under the given conditions (2.7), i.e.

$$\int_a^b \rho(x) y_n(x) y_m(x)\, dx = \left(\int_a^b \rho(x) y_n^2(x)\, dx \right) \delta_{n,m} \quad \text{if} \quad \delta_{n,m} = \begin{cases} 0 & if \quad n \neq m, \\ 1 & if \quad n = m\ . \end{cases} \tag{2.8}$$

Many important special functions in theoretical and mathematical physics are the solutions of regular or singular Sturm-Liouville problems that satisfy the orthogonality condition (2.8). For instance, the associated Legendre functions [18], Bessel functions [18, 56], trigonometric sequences related to Fourier analysis [13], Ultraspherical functions [27, 70], Hermite functions [27, 70] and so on are particular solutions of some Sturm-Liouville problems. Fortunately, most of these mentioned functions have the symmetry property, namely $\Phi_n(-x) = (-1)^n \Phi_n(x)$, and because of this they have found various applications in physics and engineering, see e.g. [18, 56] for more details. Now, if one can extend the mentioned examples symmetrically and preserve their orthogonality property, it seems that one will be able to find some new applications in physics and engineering that logically extend the previous applications. In the next sections, by achieving this matter, we generalize some classical symmetric orthogonal functions and obtain their orthogonality property.

2.3. A symmetric generalization of Sturm-Liouville problems [12]

Without loss of generality, let $y = \Phi_n(x)$ be a sequence of symmetric functions that satisfies the following differential equation

$$A(x)\Phi_n''(x) + B(x)\Phi_n'(x) + \left(\lambda_n C(x) + D(x) + \frac{1-(-1)^n}{2} E(x) \right) \Phi_n(x) = 0\,, \tag{2.9}$$

where $A(x)$, $B(x)$, $C(x)$, $D(x)$ and $E(x)$ are independent functions and $\{\lambda_n\}$ is a sequence of constants. Clearly choosing $E(x) = 0$ in (2.9) is equivalent to the one-dimensional Sturm-Liouville equation (2.6). Since $\Phi_n(x)$ is a symmetric sequence, by substituting the symmetry property $\Phi_n(-x)=(-1)^n\Phi_n(x)$ into (2.9), one can immediately conclude that $A(x)$, $C(x)$, $D(x)$ and $E(x)$ are *even* functions, while $B(x)$ must be an *odd* function. This note will frequently be used in this section.

To prove the orthogonality property of the sequence $\Phi_n(x)$, first both sides of equation (2.9) should be multiplied by the positive function

$$R(x)=\exp\left(\int\frac{B(x)-A'(x)}{A(x)}dx\right)=\frac{1}{A(x)}\exp\left(\int\frac{B(x)}{A(x)}dx\right), \tag{2.10}$$

in order to convert in the form of a self-adjoint differential equation. Note that $R(x)$ in (2.10) is an even function, because $B(x)$ is odd and $A(x)$ is even. Therefore, the self-adjoint form of equation (2.9) becomes

$$\frac{d}{dx}\left(A(x)\,R(x)\frac{d\,\Phi_n(x)}{dx}\right)=-(\lambda_n\,C(x)+D(x))\,R(x)\Phi_n(x)-\frac{1-(-1)^n}{2}E(x)R(x)\Phi_n(x). \tag{2.11}$$

Since $A(x)\,R(x)$ is an even function, the orthogonality interval corresponding to equation (2.11) should be considered symmetric, say $[-\theta,\theta]$. Hence, by assuming that $x=\theta$ is a root of the function $A(x)\,R(x)$ and applying the Sturm-Liouville theorem on (2.11) we have

$$\begin{aligned}&\left[A(x)\,R(x)\left(\Phi'_n(x)\Phi_m(x)-\Phi'_m(x)\Phi_n(x)\right)\right]_{-\theta}^{\theta}=\\&(\lambda_m-\lambda_n)\int_{-\theta}^{\theta}C(x)\,R(x)\,\Phi_n(x)\,\Phi_m(x)\,dx-\left(\frac{(-1)^m-(-1)^n}{2}\right)\int_{-\theta}^{\theta}E(x)\,R(x)\,\Phi_n(x)\,\Phi_m(x)\,dx.\end{aligned} \tag{2.12}$$

Obviously the left-hand side of (2.12) is zero. So, to prove the orthogonality property, it is enough to show that the value

$$F(n,m)=\frac{(-1)^m-(-1)^n}{2}\int_{-\theta}^{\theta}E(x)\,R(x)\,\Phi_n(x)\,\Phi_m(x)\,dx, \tag{2.13}$$

is always equal to zero for every $m,n\in Z^+$. For this purpose, four cases should generally be considered for *m* and *n*:

a) If both *m* and *n* are even (or odd), then $F(n,m)=0$, because we have $F(2i,2j)=F(2i+1,2j+1)=0$.

b) If one of the two mentioned values is odd and the other one is even (or conversely) then

$$F(2i,2j+1)=-\int_{-\theta}^{\theta}E(x)\,R(x)\,\Phi_{2i}(x)\,\Phi_{2j+1}(x)\,dx. \tag{2.14}$$

But in above relation $E(x)$, $R(x)$ and $\Phi_{2i}(x)$ are even functions and $\Phi_{2j+1}(x)$ is odd. Thus the integrand of (2.14) is an odd function and consequently $F(2i, 2j+1)=0$. This issue also holds for the case $n=2i+1$ and $m=2j$, i.e. $F(2i+1, 2j)=0$.
By noting the above comments, the main theorem of section (2.3) can here be expressed.

Theorem 1. [12]
The symmetric sequence $\Phi_n(x)=(-1)^n\,\Phi_n(-x)$, as a specific solution of differential equation (2.9), satisfies the orthogonality relation

$$\int_{-\theta}^{\theta} W^*(x)\,\Phi_n(x)\,\Phi_m(x)\,dx=\left(\int_{-\theta}^{\theta} W^*(x)\,\Phi_n^2(x)\,dx\right)\delta_{n,m}, \tag{2.15}$$

where

$$W^*(x)=C(x)R(x)=C(x)\exp\left(\int\frac{B(x)-A'(x)}{A(x)}dx\right), \tag{2.16}$$

denotes the corresponding weight function and is a positive and even function on $[-\theta,\theta]$.
It is now a good position to propound a practical example that generalizes the well-known symmetric associated Legendre functions and preserves their orthogonality property.

2.3.1. A symmetric generalization of the associated Legendre functions [12]

When Laplace's equation $\nabla^2 u=0$ (or Helmholtz's equation $\nabla^2 u=\lambda u$) is solved in spherical coordinates r,θ,φ, the following results appear [56]

$$\nabla^2 u=\Delta_r u+\frac{1}{r^2}\Delta_{\theta,\varphi}u\,, \tag{2.17}$$

where

$$\Delta_r u=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right), \tag{2.17.1}$$

$$\Delta_{\theta,\varphi}u=\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}. \tag{2.17.2}$$

As was explained in the section (2.2), by separating $u=R(r)\Psi(\theta)\Omega(\varphi)$ and substituting it into Laplace's equation (Potential equation), three following ordinary differential equations are derived

$$\begin{aligned}&(r^2R')'=\mu\,R\;,\\ &\Omega''+\upsilon\,\Omega=0\;,\\ &\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\Psi}{d\theta}\right)+(\mu\sin^2\theta-\upsilon)\Psi=0\;,\end{aligned} \tag{2.18}$$

where μ and υ are two constant values. For $x=\cos\theta$, the last equation of (2.18) changes to

$$(1-x^2)\frac{d^2\Psi}{dx^2}-2x\frac{d\Psi}{dx}+(\mu-\frac{\upsilon}{1-x^2})\Psi=0\ , \tag{2.19}$$

which is called the associated Legendre differential equation [18] and its solutions are naturally known as the associated Legendre functions. These functions play a key role in the theory of potential. In general, there are three types of solution for equation (2.19) depending on different values of μ and υ:

i) If $\mu=(n+\alpha)(n+\alpha+1)$ and $\upsilon=\alpha^2$; $n\in\mathbf{Z}^+$, $\alpha>-1$, then the solution of (2.19) is indicated by

$$\Psi(x)=U_n^{(\alpha)}(x)=(1-x^2)^{\frac{\alpha}{2}}P_n^{(\alpha,\alpha)}(x)\,, \tag{2.20}$$

where $P_n^{(\alpha,\alpha)}(x)$, known as Ultraspherical polynomials, is a special case of Jacobi polynomials [27, 70]

$$P_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}\binom{n+\alpha+\beta+k}{k}\binom{n+\alpha}{n-k}(\frac{x-1}{2})^k\,, \tag{2.21}$$

for $\alpha=\beta$. In this way, the solution (2.20) satisfies the orthogonality relation

$$\int_{-1}^{1}U_n^{(\alpha)}(x)U_m^{(\alpha)}(x)\,dx=(\int_{-1}^{1}(U_n^{(\alpha)}(x))^2\,dx)\,\delta_{n,m}=\frac{2^{2\alpha+1}\Gamma^2(n+\alpha+1)}{n!(2n+2\alpha+1)\Gamma(n+2\alpha+1)}\,\delta_{n,m}\,. \tag{2.22}$$

ii) If $\mu=n(n+1)$ and $\upsilon=m^2$; $m,n\in\mathbf{Z}^+$, (2.19) has a solution in the form (discovered by Legendre)

$$\Psi(x)=P_n^m(x)=(1-x^2)^{\frac{m}{2}}\frac{d^m(P_n(x))}{dx^m}\,, \tag{2.23}$$

where $P_n(x)=P_n^{(0,0)}(x)$ denotes the Legendre polynomials [56]. Moreover, according to [18], (2.23) satisfies the orthogonality relation

$$\int_{-1}^{1}P_i^m(x)\,P_j^m(x)\,dx=(\int_{-1}^{1}(P_i^m(x))^2\,dx)\,\delta_{i,j}=\frac{2\,(i+m)!}{(2i+1)(i-m)!}\,\delta_{i,j}\,. \tag{2.24}$$

iii) To obtain the third type of solution, let us first consider the Jacobi polynomials differential equation [27, 70]

$$(1-x^2)y''-\big((\alpha+\beta+2)x+(\alpha-\beta)\big)y'+n(n+\alpha+\beta+1)y=0\Leftrightarrow y=P_n^{(\alpha,\beta)}(x)\,, \tag{2.25}$$

and suppose $A_n(x) = (1-x)^{\alpha/2}(1+x)^{\beta/2} P_n^{(\alpha,\beta)}(x)$. After some computations in hand, the differential equation $A_n(x)$ is derived as

$$(1-x^2)^2 A_n''(x) - 2x(1-x^2)A_n'(x) - \frac{1}{4}[((\alpha+\beta)(\alpha+\beta+2) + 4n(n+\alpha+\beta+1))\,x^2 \\ + 2(\alpha+\beta)(\alpha-\beta)\,x - 2(\alpha+\beta) + (\alpha-\beta)^2 - 4n(n+\alpha+\beta+1)]\,A_n(x) = 0\ . \tag{2.26}$$

Now suppose $\alpha+\beta=0$ in (2.26) that reduces it to the differential equation

$$(1-x^2)A'' - 2xA' + (n(n+1) - \frac{\alpha^2}{1-x^2})A = 0\ . \tag{2.27}$$

The condition $-1<\alpha<1$ is necessary for the orthogonality property of the polynomials $P_n^{(\alpha,-\alpha)}(x)$, because we must have $\alpha+\beta=0$ and $\alpha,\beta>-1$. Therefore, by considering $\mu=n(n+1)$ and $\upsilon=\alpha^2$; $-1<\alpha<1$ in (2.19), the related solution takes the form

$$\Psi(x) = V_n^{(\alpha)}(x) = (\frac{1-x}{1+x})^{\frac{\alpha}{2}} P_n^{(\alpha,-\alpha)}(x)\,, \tag{2.28}$$

satisfying the orthogonality relation

$$\int_{-1}^{1} V_n^{(\alpha)}(x) V_m^{(\alpha)}(x)\,dx = (\int_{-1}^{1} (V_n^{(\alpha)}(x))^2\,dx)\,\delta_{n,m} = \frac{2\,\Gamma(n+1+\alpha)\Gamma(n+1-\alpha)}{(n!)^2(2n+1)}\ \delta_{n,m}\,. \tag{2.29}$$

To extend the associated Legendre functions, it is enough in (2.9) to choose

$$\begin{aligned} &A(x) = 1-x^2 = A(-x) &&; \quad B(x) = -2x = -B(-x)\,, \\ &C(x) = 1 = C(-x) &&; \quad D(x) = -\frac{\upsilon}{1-x^2} = D(-x)\,, \\ &\lambda_n = \mu_n &&; \quad E(x) = E(-x) \quad \text{Arbitrary}\,, \end{aligned} \tag{2.30}$$

and apply the theorem 1 to establish the orthogonality property on the same interval [-1,1]. To do this task, let $\Phi_n(x) = Q_n(x;\upsilon,E(x))$ be a known and predetermined solution of the differential equation

$$(1-x^2)\,\Phi_n''(x) - 2x\,\Phi_n'(x) + (\mu_n - \frac{\upsilon}{1-x^2} + \frac{1-(-1)^n}{2}E(x))\,\Phi_n(x) = 0\,. \tag{2.31}$$

According to the main theorem 1, we should have

$$\int_{-1}^{1} Q_n(x;\upsilon,E(x))Q_m(x;\upsilon,E(x))\,dx = (\int_{-1}^{1} (Q_n(x;\upsilon,E(x)))^2\,dx)\,\delta_{n,m}\,, \tag{2.32}$$

provided that the arbitrary function $E(x)$ is *even*.

The sequence $Q_n(x;\upsilon,E(x))$ (as a known solution of equation (2.31)) is now orthogonal under the pre-assigned condition $E(-x)=E(x)$. So it is applicable in the theory of the expansion of functions, as many boundary value problems of mathematical physics are solved by using the expansion of functions in terms of the eigenfunctions of a usual Sturm-Liouville problem. Hence, if we assume

$$f(x)=\sum_{n=0}^{\infty} q_n\, Q_n(x;\upsilon,E(x))\,, \tag{2.33}$$

then according to the property (2.32), the unknown coefficients q_n are found by

$$q_n=\int_{-1}^{1} f(x)Q_n(x;\upsilon,E(x))\,dx \bigg/ \int_{-1}^{1} (Q_n(x;\upsilon,E(x)))^2\,dx\,. \tag{2.34}$$

Clearly various options can be selected for $E(x)$, that are directly related to the orthogonal sequence $Q_n(x;\upsilon,E(x))$. For example, choosing the even function $E(x)=0$ gives one of the three cases of usual associated Legendre functions, stated before. A further special example is when $E(x)=E(-x)=-2/x^2$. Let us study this case in the next section in detail.

2.3.1.1. The special case $E(x)=-2/x^2$ in the generalized equation (2.31)

If $E(x)=-2/x^2$ in (2.31) it reduces to the equation

$$(1-x^2)\,\Phi''_n(x)-2x\,\Phi'_n(x)+(\mu_n-\frac{\upsilon}{1-x^2}+\frac{(-1)^n-1}{x^2})\,\Phi_n(x)=0\,. \tag{2.35}$$

To obtain the solution of this equation for specific values of μ_n and υ, one should refer to the generalized ultraspherical polynomials (GUP), which were first investigated by Chihara [27]. On the other hand, according to the main theorem 1 if in the general equation (2.9) one chooses

$$\begin{aligned}
&A(x)=x^2(1-x^2) &&; \quad \textit{An even function,}\\
&B(x)=-2x((p+q+1)x^2-q) &&; \quad \textit{An odd function,}\\
&C(x)=x^2 &&; \quad \textit{An even function,}\\
&D(x)=0 &&; \quad \textit{An even function,}\\
&E(x)=-p &&; \quad \textit{An even function,}\\
&\lambda_n=n(2p+2q+n+1), &&
\end{aligned} \tag{2.36}$$

then the symmetric differential equation

$$x^2(1-x^2)\Phi''_n(x)-2x((p+q+1)x^2-p)\Phi'_n(x) \\ +\Big(n(2p+2q+n+1)x^2+((-1)^n-1)p\Big)\Phi_n(x)=0, \tag{2.37}$$

is the generalized ultraspherical equation having the monic polynomial solution

$$\bar{U}_n^{(p,q)}(x)=\prod_{i=0}^{[n/2]-1}\frac{2i+2p+2-(-1)^n}{-2i-(2q+2p+2-(-1)^n+2[n/2])} \\ \times\sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\left(\prod_{i=0}^{[n/2]-(k+1)}\frac{-2i-(2q+2p+2-(-1)^n+2[n/2])}{2i+2p+2-(-1)^n}\right)x^{n-2k}. \tag{2.38}$$

The monic GUP are orthogonal with respect to the weight function $x^{2p}(1-x^2)^q$ on [-1,1] and satisfies the following orthogonality relation

$$\int_{-1}^{1}x^{2p}(1-x^2)^q\bar{U}_n^{(p,q)}(x)\bar{U}_m^{(p,q)}(x)\,dx \\ =\Big(\prod_{i=1}^{n}\frac{(i+(1-(-1)^i)p)(i+(1-(-1)^i)p+2q)}{(2i+2p+2q-1)(2i+2p+2q+1)}\Big)\frac{\Gamma(p+1/2)\Gamma(q+1)}{\Gamma(p+q+3/2)}\;\delta_{n,m}. \tag{2.39}$$

Now, by defining

$$G_n(x;a,b)=x^a(1-x^2)^{\frac{b}{2}}\bar{U}_n^{(a,b)}(x)\quad;\quad a>-\frac{1}{2}\,,\ b>-1\,, \tag{2.40}$$

and replacing it into equation (2.37), one gets (2.41)

$$(1-x^2)G''_n(x)-2xG'_n(x)+\left((n+a+b)(n+a+b+1)-\frac{b^2}{1-x^2}+\frac{a((-1)^n-a)}{x^2}\right)G_n(x)=0.$$

If the above equation is compared with (2.35), then it can be concluded that

$$Q_n(x;b^2,-\frac{2}{x^2})=G_n(x;1,b)\quad;\quad b>-1\,, \tag{2.42}$$

is a general solution of (2.31) for $\mu_n=(n+b+1)(n+b+2)$, $\upsilon=b^2$ and $E(x)=-2/x^2$. Moreover, substituting these values into (2.32) and noting (2.39) yields

$$\int_{-1}^{1}Q_n(x;b^2,-\frac{2}{x^2})Q_m(x;b^2,-\frac{2}{x^2})\,dx=\Big(\prod_{i=1}^{n}\frac{(i+1-(-1)^i)(i+1-(-1)^i+2b)}{(2i+2b+1)(2i+2b+3)}\Big)\frac{\sqrt{\pi}\,\Gamma(b+1)}{2\Gamma(b+5/2)}\;\delta_{n,m}. \tag{2.43}$$

Remark 1. Although some special functions such as Bessel functions [18], Fourier trigonometric sequences and so on are symmetric and satisfy a differential equation whose coefficients are alternatively even and odd, it is anyway important to note that they do not satisfy the conditions of the main theorem 1. For instance, if in the generic equation (2.9) we choose

$$\begin{aligned} A(x) &= x^2 \quad ; \quad B(x) = x \quad ; \quad C(x) = 1, \\ D(x) &= x^2 \quad ; \quad E(x) = 0 \quad ; \quad \lambda_n = -n^2, \end{aligned} \tag{2.44}$$

we get to the Bessel differential equation

$$x^2\,\Phi''_n(x) + x\,\Phi'_n(x) + (x^2 - n^2)\,\Phi_n(x) = 0\,, \tag{2.45}$$

with the symmetric solution as

$$(-1)^n J_n(-x) = J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k\,!(n+k)!}(\frac{x}{2})^{n+2k} \quad , \quad n \in \mathbf{Z}^+\,, \tag{2.46}$$

but the orthogonality interval of Bessel functions is not symmetric (i.e. $[0,1]$). Hence, theorem 1 cannot be applied for $J_n(x)$, unless there exists a specific even function $E(x)$ for equation (2.9) such that the corresponding solution has infinity zeros (see e.g. [18, 56] for more details), exactly similar to the case of usual Bessel functions of order n. Therefore the conclusion of this chapter can be summarized as follows: the usual Sturm-Liouville problem

$$\frac{d}{dx}\left(k(x)\frac{dy}{dx}\right) + (\lambda_n\,\rho(x) - q(x))y = 0 \quad ; \quad k(x) > 0\,, \quad \rho(x) > 0\,,$$

under the boundary conditions

$$\begin{cases} \alpha_1\, y(a) + \beta_1\, y'(a) = 0 \\ \alpha_2\, y(b) + \beta_2\, y'(b) = 0 \end{cases} \quad ; \quad a < x < b$$

can be extended to the following problem

$$\frac{d}{dx}\left(k(x)\frac{dy}{dx}\right) + (\lambda_n\,\rho(x) - q(x) + \frac{1-(-1)^n}{2}E(x))y = 0\,,$$

with the boundary condition $\alpha_3\, y(\theta) + \beta_3\, y'(\theta) = 0$; $-\theta < x < \theta$, provided that the solution of latter differential equation has the symmetry property, i.e. $y_n(-x) = (-1)^n\, y_n(x)$.

Chapter 3

A main class of symmetric orthogonal polynomials

3.1. Introduction

In the previous chapter, we determined how to extend the usual Sturm-Liouville problems with symmetric solutions. In this chapter, by using the extended Sturm-Liouville theorem for symmetric functions, we are going to introduce a main class of symmetric orthogonal polynomials (MCSOP) with four free parameters and obtain all its standard properties, such as a generic second order differential equation together with its explicit polynomial solution, a generic orthogonality relation, a generic three term recurrence relation and so on. Then, we show that essentially four main sequences of symmetric orthogonal polynomials can be extracted from the introduced class. They are respectively the generalized ultraspherical polynomials, generalized Hermite polynomials and two other sequences of symmetric polynomials, which are finitely orthogonal on the real line and can be expressed in terms of the mentioned class directly. In this way, we also introduce two half-trigonometric sequences of orthogonal polynomials as special sub-cases of MCSOP.

3.2. A Main Class of Symmetric Orthogonal Polynomials (MCSOP) using the extended Sturm-Liouville theorem [1]

By referring to the main theorem of chapter 2 and generic differential equation (2.9) choose the following options

$$\begin{aligned}
&A(x)=x^2(px^2+q) \quad &;\quad &An\ \ even\ \ function,\\
&B(x)=x(rx^2+s) \quad &;\quad &An\ \ odd\ \ function,\\
&C(x)=x^2 \quad &;\quad &An\ \ even\ \ function,\\
&D(x)=0 \quad &;\quad &An\ \ even\ \ function,\\
&E(x)=-s \quad &;\quad &An\ \ even\ \ function,
\end{aligned} \tag{3.1}$$

where p,q,r and s are real free parameters and $\lambda_n=-n\,(r+(n-1)p)$. Therefore, one deals with a second order differential equation of the form (3.2)

$$x^2(px^2+q)\,\Phi_n''(x)+x(rx^2+s)\,\Phi_n'(x)-\left(n(r+(n-1)p)x^2+(1-(-1)^n)\,s/2\right)\Phi_n(x)=0\,.$$

To obtain the polynomial solution of above equation, let us first suppose that $n=2m$, i.e.

$$x(px^2+q)\,\Phi''_{2m}(x)+(rx^2+s)\,\Phi'_{2m}(x)-2m\,(r+(2m-1)\,p)\,x\,\Phi_{2m}(x)=0\,. \tag{3.3}$$

After doing some calculations in hand, one can get the solution of equation (3.3) as

$$S_{2m}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)=\sum_{k=0}^{m}\binom{m}{k}\left(\prod_{j=0}^{m-(k+1)}\frac{(2j-1+2m)\,p+r}{(2j+1)\,q+s}\right)x^{2m-2k}\,, \tag{3.4}$$

where $\prod_{r=0}^{-1}a_r=1$. Similarly, if $n=2m+1$ is considered in (3.2), the simplified equation

$$x^2\,(px^2+q)\,\Phi''_{2m+1}(x)+x(rx^2+s)\,\Phi'_{2m+1}(x)-((2m+1)\,(r+2mp)\,x^2+s)\,\Phi_{2m+1}(x)=0\,, \tag{3.5}$$

has the polynomial solution

$$S_{2m+1}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)=\sum_{k=0}^{m}\binom{m}{k}\left(\prod_{j=0}^{m-(k+1)}\frac{(2j+1+2m)\,p+r}{(2j+3)\,q+s}\right)x^{2m+1-2k}\,. \tag{3.6}$$

Consequently, combining (3.4) with (3.6) gives

$$S_{n}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)=\sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\left(\prod_{i=0}^{[n/2]-(k+1)}\frac{\left(2i+(-1)^{n+1}+2[n/2]\right)p+r}{\left(2i+(-1)^{n+1}+2\right)q+s}\right)x^{n-2k}\,, \tag{3.7}$$

as the most common source of classical *symmetric* orthogonal polynomials with four free parameters p,q,r and s where neither both values *q* and *s* nor both values *p* and *r* can vanish together. Here let us point out that almost all known symmetric orthogonal polynomials, such as Legendre polynomials, first and second kind of Chebyshev polynomials, ultraspherical polynomials, generalized ultraspherical polynomials (GUP), Hermite polynomials and generalized Hermite polynomials (GHP) are special sub-cases of (3.7) and can be expressed in terms of it directly. Because of this matter, we call these polynomials, "The second kind of classical orthogonal polynomials". Furthermore, there are two other symmetric sequences of *finite* orthogonal polynomials that are special sub-cases of general representation (3.7) and we will introduce them in the sections 3.9 and 3.10.

The symbol $S_{n}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)$ is used only because of *Symmetry* property, however, to shorten this notation in the text, we will show it as $S_n(p,q,r,s\,;x)$. A straightforward result from (3.7) is that

$$S_{2n+1}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)=x\,S_{2n}\left(\begin{matrix} r+2p & s+2q \\ p & q \end{matrix}\middle| x\right). \tag{3.8}$$

Moreover, since the monic type of orthogonal polynomials (i.e. with leading coefficient 1) is often required, we define

$$\bar{S}_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) = \prod_{i=0}^{[n/2]-1} \frac{\left(2i+(-1)^{n+1}+2\right)q+s}{\left(2i+(-1)^{n+1}+2[n/2]\right)p+r}\ S_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right). \qquad (3.9)$$

For instance, if $n=0,...,5$ in (3.9) we have

$$\begin{aligned}
&\bar{S}_0\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)=1,\\
&\bar{S}_1\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)=x,\\
&\bar{S}_2\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)=x^2+\frac{q+s}{p+r},\\
&\bar{S}_3\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)=x^3+\frac{3q+s}{3p+r}x,\\
&\bar{S}_4\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)=x^4+2\frac{3q+s}{5p+r}x^2+\frac{(3q+s)(q+s)}{(5p+r)(3p+r)},\\
&\bar{S}_5\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)=x^5+2\frac{5q+s}{7p+r}x^3+\frac{(5q+s)(3q+s)}{(7p+r)(5p+r)}x.
\end{aligned} \qquad (3.9.1)$$

The explicit representation of (3.9) helps us now obtain a three-term recurrence relation for the polynomials. Hence, if we assume that they satisfy the relation

$$\bar{S}_{n+1}(x)=x\bar{S}_n(x)+C_n\begin{pmatrix} r & s \\ p & q \end{pmatrix}\bar{S}_{n-1}(x) \quad ; \quad \bar{S}_0(x)=1\ ,\ \bar{S}_1(x)=x\ ,\ n\in\mathbf{N}\ , \qquad (3.10)$$

then after doing a series of computations in hand, we obtain

$$C_n\begin{pmatrix} r & s \\ p & q \end{pmatrix}=\frac{pqn^2+\left((r-2p)q-(-1)^n ps\right)n+(r-2p)s(1-(-1)^n)/2}{(2pn+r-p)(2pn+r-3p)}, \qquad (3.10.1)$$

which reveals the explicit form of (3.10).

On the other hand, since the recurrence relation (3.10) has explicitly been specified, to determine the norm square value of the polynomials one can use Favard's theorem [27] by noting that there is orthogonality with respect to a weight function. According to this theorem, if $\{P_n(x)\}_{n=0}^{\infty}$ is defined by

$$xP_n(x)=A_n P_{n+1}(x)+B_nP_n(x)+C_nP_{n-1}(x) \quad ; \quad n=0,1,2,...\ , \qquad (3.11)$$

where $P_{-1}(x)=0$, $P_0(x)=1$, A_n, B_n, C_n *real* and $A_nC_{n+1}>0$ for $n=0,1,...$, then there exists a weight function $W(x)$ so that

$$\int_{-\infty}^{\infty} W(x)P_n(x)P_m(x)\,dx = \left(\prod_{i=0}^{n-1}\frac{C_{i+1}}{A_i}\int_{-\infty}^{\infty} W(x)\,dx\right)\delta_{n,m} \quad . \tag{3.12}$$

Moreover, if the positive condition $A_n C_{n+1} > 0$ only holds for $n = 0,1,...,N$ then the orthogonality relation (3.12) only holds for a finite number of *m, n*. The latter note will help us in the next sections to obtain two new sub-classes of $\bar{S}_n(p,q,r,s\,;x)$ that are *finitely* orthogonal on the interval $(-\infty,\infty)$. It is clear that the Favard theorem is also valid for the recurrence relation (3.10) in which $A_n = 1$, $B_n = 0$ and $C_n = -C_n(p,q,r,s)$. Furthermore, the condition $-C_{n+1}(p,q,r,s) > 0$ must always be satisfied if one demands to apply the Favard theorem for (3.10). By noting this subject and (3.12), the generic form of the orthogonality relation of MCSOP can be designed as

$$\int_{-\alpha}^{\alpha} W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle|\, x\right)\bar{S}_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle|\, x\right)\bar{S}_m\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle|\, x\right)dx = \left((-1)^n\prod_{i=1}^{i=n} C_i\left(\begin{matrix} r & s \\ p & q \end{matrix}\right)\int_{-\alpha}^{\alpha} W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle|\, x\right)dx\right)\delta_{n,m} \tag{3.13}$$

where the weight function, by referring to (2.16) and (3.1), is defined as

$$W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle|\, x\right) = x^2 \exp\left(\int \frac{(r-4p)x^2 + (s-2q)}{x(px^2+q)}dx\right) = \exp\left(\int \frac{(r-2p)x^2 + s}{x(px^2+q)}dx\right), \tag{3.14}$$

and α takes the standard values $1,\infty$. Note that the function $(px^2+q)W(p,q,r,s;x)$ must vanish at $x = \alpha$ in order to establish the main orthogonality relation (3.13).

3.3. An analogue of Pearson distributions family [1]

The positive function (3.14) can also be investigated from statistical point of view. In fact, this function is an analogue of Pearson distributions family having the general form

$$\rho\left(\begin{matrix} d & e & \\ a & b & c \end{matrix}\middle|\, x\right) = \exp\left(\int \frac{(d-2a)x + (e-b)}{ax^2+bx+c}dx\right), \tag{3.15}$$

and satisfying the first order differential equation

$$\frac{d}{dx}((ax^2+bx+c)\,\rho(x)) = (dx+e)\,\rho(x) \quad . \tag{3.15.1}$$

Therefore, we would like to point out that, similar to equation (3.15.1), the weight function (3.14) satisfies a first order differential equation as

$$x\frac{d}{dx}((px^2+q)W(x)) = (rx^2+s)W(x), \tag{3.16}$$

which is equivalent to

$$\frac{d}{dx}(x^2(px^2+q)W(x))=x(r^*x^2+s^*)W(x) \quad s.t. \quad \begin{cases} r^*=r+2p \\ s^*=s+2q \end{cases} \tag{3.16.1}$$

From (3.16) or (3.16.1) it can be deduced that $W(p,q,r,s;x)$ is an analytic integrable function and since it is also a positive function, its probability density function (pdf) must be available. In general, there are four main sub-classes of distributions family (3.14) (and consequently sub-solutions of equation (3.16)) whose explicit pdfs are respectively as follows

$$K_1 W\left(\begin{matrix} -2a-2b-2, & 2a \\ -1, & 1 \end{matrix}\middle| x\right)=\frac{\Gamma(a+b+3/2)}{\Gamma(a+1/2)\Gamma(b+1)}\, x^{2a}(1-x^2)^b \quad ; \quad -1\le x\le 1\,; \quad a+1/2>0\ ;\ b+1>0. \tag{3.17}$$

$$K_2 W\left(\begin{matrix} -2, & 2a \\ 0, & 1 \end{matrix}\middle| x\right)=\frac{1}{\Gamma(a+1/2)}\, x^{2a}e^{-x^2} \quad ; \quad -\infty<x<\infty\ ;\ a+1/2>0. \tag{3.18}$$

$$K_3 W\left(\begin{matrix} -2a-2b+2, & -2a \\ 1, & 1 \end{matrix}\middle| x\right)=\frac{\Gamma(b)}{\Gamma(b+a-1/2)\Gamma(-a+1/2)}\frac{x^{-2a}}{(1+x^2)^b} \quad ; \quad -\infty<x<\infty\,; \quad b+a>1/2\ ;\ a<1/2\ ;\ b>0. \tag{3.19}$$

$$K_4 W\left(\begin{matrix} -2a+2, & 2 \\ 1, & 0 \end{matrix}\middle| x\right)=\frac{1}{\Gamma(a-1/2)}\, x^{-2a}e^{-\frac{1}{x^2}} \quad ; \quad -\infty<x<\infty\ ;\ a>1/2. \tag{3.20}$$

The values $K_i\ ;\ i=1,2,3,4$ play the normalizing constant role in above distributions. Moreover, as it is observed, the value of distribution vanishes at $x=0$ in each four cases, i.e. $W(p,q,r,s;0)=0$ for $s\neq 0$. Hence, let us call the positive function (3.14) "The dual symmetric distributions family".

3.3.1. A generalization of dual symmetric distributions family

First it is important to remember that if $s=0$ in (3.16), the foresaid equation will be reduced to a special sub-case of Pearson differential equation (3.15.1). Hence, we hereafter suppose that $s\neq 0$. Since the explicit forms of $S_n(p,q,r,s;x)$ in (3.7), $C_n(p,q,r,s)$ in (3.10.1) and $W(p,q,r,s;x)$ in (3.14) are all known, a further main pdf can directly be defined by referring to the orthogonality relation (3.13), so that we have

$$D_m\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)=\frac{(-1)^m}{\prod_{i=1}^{i=m} C_i\begin{pmatrix} r & s \\ p & q \end{pmatrix}\int_{-\alpha}^{\alpha} W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)dx}\, W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)\times\left(\bar{S}_m\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right)\right)^2. \tag{3.21}$$

Clearly choosing $m=0$ in this definition gives the same as dual symmetric distributions family. Moreover, the Fisher information [36] and Boltzmann-Shannon information entropy [28] are two important factors in statistical estimation theory that can be investigated for the generalized distribution (3.21).

3.4. A direct relationship between first and second kind of classical orthogonal polynomials

One can verify that there is a direct relation between first kind of classical orthogonal polynomials (including Jacobi, Laguerre and Hermite polynomials as well as three finite classes of orthogonal polynomials, see chapter 4) and the explicit polynomials $S_n(p,q,r,s;x)$ indicated in (3.7). To find this relationship, we start with the following differential equation

$$(ax^2+bx+c)y_n''(x)+(dx+e)y_n'(x)-n((n-1)a+d)y_n(x)=0\ . \qquad (3.22)$$

According to [56], the monic polynomial solution of (3.22) can be shown by a Rodrigues-type formula as (3.23)

$$\bar{P}_n\left(\begin{matrix} d & e \\ a & b & c\end{matrix}\middle| x\right)=\frac{1}{(\prod_{k=1}^{n} d+(n+k-2)a)\,\rho\left(\begin{matrix} d & e \\ a & b & c\end{matrix}\middle| x\right)}\times\frac{d^n}{dx^n}((ax^2+bx+c)^n\,\rho\left(\begin{matrix} d & e \\ a & b & c\end{matrix}\middle| x\right)),$$

where $\rho(a,b,c,d,e;x)$ is defined as the same form as (3.15).

But, since $\bar{S}_{2n}(p,q,r,s;x)$ is generally an even function, taking $x=wt^2+v$ in (3.22) gives (3.24)

$$t^2(aw^2t^4+w(2av+b)t^2+av^2+bv+c)y_n''(t)+t((2d-a)w^2t^4+(2wv(d-a)+w(2e-b))t^2$$
$$-(av^2+bv+c))y_n'(t)-4w^2n(d+(n-1)a)t^4y_n(t)=0.$$

If (3.24) is equated with (3.3), we should have

$$av^2+bv+c=0 \quad or \quad v=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\ . \qquad (3.24.1)$$

The condition (3.24.1) simplifies the equation (3.24) to (3.24.2)

$$t(awt^2\pm\sqrt{b^2-4ac})y_n''(t)+((2d-a)wt^2+(\frac{d}{a}-1)(-b\pm\sqrt{b^2-4ac})+2e-b)y_n'(t)$$
$$-4wn(d+(n-1)a)t\,y_n(t)=0\Leftrightarrow y_n(t)=\bar{P}_n\left(\begin{matrix} d & e \\ a & b & c\end{matrix}\middle| wt^2+\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right).$$

The equation (3.24.2) is clearly a special case of (3.3). This means that (3.25)

$$P_n\left(\begin{matrix} d & e \\ a & b & c\end{matrix}\middle| wt^2+\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)=K\,S_{2n}\left(\begin{matrix} (2d-a)w, & (\frac{d}{a}-1)(-b\pm\sqrt{b^2-4ac})+2e-b \\ aw, & \pm\sqrt{b^2-4ac}\end{matrix}\middle| t\right)$$

where K is the leading coefficient of the left-hand side polynomial of relation (3.25) divided to leading coefficient of the right-hand side polynomial.
As we observe in the above relation, there exist 5 free parameters *a, b, c, d, e* in the left-hand side of (3.25). So, one of them must be pre-assigned in order that one can get the explicit form of $\bar{S}_{2n}(p,q,r,s;x)$ in terms of $\bar{P}_n(a,b,c,d,e;wt^2+v)$ similarly. For this purpose, if for instance $c=0$ is considered in (3.25), two following cases appear

$$\bar{S}_{2n}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| t\right) = w^{-n}\bar{P}_n\left(\begin{matrix} \frac{r+p}{2w}, & \frac{s+q}{2} \\ p/w, & q, \quad 0 \end{matrix}\middle| wt^2\right) \qquad ; \qquad w \neq 0, \tag{3.26}$$

$$\bar{S}_{2n}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| t\right) = w^{-n}\bar{P}_n\left(\begin{matrix} \frac{r+p}{2w}, & \frac{sp-rq}{2p} \\ p/w, & -q, \quad 0 \end{matrix}\middle| wt^2 + w\frac{q}{p}\right) \qquad ; \qquad p, w \neq 0. \tag{3.27}$$

Furthermore, if (3.8) is applied for two latter relations we respectively get

$$\bar{S}_{2n+1}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| t\right) = w^{-n}t\,\bar{P}_n\left(\begin{matrix} \frac{r+3p}{2w}, & \frac{s+3q}{2} \\ p/w, & q, \quad 0 \end{matrix}\middle| wt^2\right) \qquad ; \qquad w \neq 0 , \tag{3.28}$$

$$\bar{S}_{2n+1}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| t\right) = w^{-n}t\,\bar{P}_n\left(\begin{matrix} \frac{r+3p}{2w}, & \frac{sp-rq}{2p} \\ p/w, & -q, \quad 0 \end{matrix}\middle| wt^2 + w\frac{q}{p}\right) \qquad ; \qquad p, w \neq 0 . \tag{3.29}$$

3.5. Some further standard properties of MCSOP

The relations (3.26)-(3.29) are useful tool to get a generating function for MCSOP. Usually, a generating function for a system of polynomials $P_n(z)$ is defined by a function like $G(z,t)$ whose expansion in powers of *t* has, for sufficiently small $|t|$, the form

$$G(z,t) = \sum_{n=0}^{\infty} P_n(z)\frac{t^n}{n!} . \tag{3.30}$$

If $P_n(z)$ has the Rodrigues-type formula [56], Cauchy's integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\Omega} \frac{f(u)}{(u-z)^{n+1}}du , \tag{3.31}$$

where Ω is a closed contour surrounding the point $u=z$, is employed to obtain $G(z,t)$ explicitly, see e.g. [56, p. 27]. This means that by considering the Rodrigues-type representation (3.23) for $c=0$ and applying Cauchy's integral theorem on it we have

$$\sum_{n=0}^{\infty}\Big(\prod_{k=1}^{n} d+(n+k-2)a\Big)\,\bar{P}_n\left(\begin{matrix} d & e \\ a & b & 0\end{matrix}\,\middle|\, z\right)\frac{t^n}{n!}=\frac{\rho\left(\begin{matrix} d & e \\ a, & b, & 0\end{matrix}\,\middle|\, u\right)}{\rho\left(\begin{matrix} d & e \\ a, & b, & 0\end{matrix}\,\middle|\, z\right)}\times\frac{1}{\sqrt{(1-bt)^2-4atz}}\ . \qquad (3.32)$$

$$\text{where}: \quad u=\frac{1-bt+\sqrt{(1-bt)^2-4atz}}{2at}\ .$$

Now if $z\to z^2$, $t\to t^2$, $a=p$, $b=q$, $d=\dfrac{r+p}{2}$ and $e=\dfrac{s+q}{2}$ are substituted into (3.32), by noting the relations (3.26)-(3.29) we obtain (3.32.1)

$$\sum_{n=0}^{\infty}\Big(\prod_{k=1}^{n}\frac{r+p}{2}+(n+k-2)p\Big)\bar{S}_{2n}\left(\begin{matrix} r & s \\ p & q\end{matrix}\,\middle|\, z\right)\frac{t^{2n}}{n!}=\frac{\rho\left(\begin{matrix} (r+p)/2 & (s+q)/2 \\ p,\ q,\ 0 & \end{matrix}\,\middle|\, u^*\right)}{\rho\left(\begin{matrix} (r+p)/2 & (s+q)/2 \\ p,\ q,\ 0 & \end{matrix}\,\middle|\, z^2\right)}\frac{1}{\sqrt{(1-qt^2)^2-4pt^2z^2}}$$

$$\text{where}: \quad u^*=\frac{1-qt^2+\sqrt{(1-qt^2)^2-4pt^2z^2}}{2pt^2}\ .$$

Similarly by multiplying both sides of (3.32) by tz and using (3.28) we get (3.32.2)

$$\sum_{n=0}^{\infty}\Big(\prod_{k=1}^{n}\frac{r+3p}{2}+(n+k-2)p\Big)\bar{S}_{2n+1}\left(\begin{matrix} r & s \\ p & q\end{matrix}\,\middle|\, z\right)\frac{t^{2n+1}}{n!}==\frac{\rho\left(\begin{matrix} (r+3p)/2 & (s+3q)/2 \\ p,\ q,\ 0 & \end{matrix}\,\middle|\, u^*\right)}{\rho\left(\begin{matrix} (r+3p)/2 & (s+3q)/2 \\ p,\ q,\ 0 & \end{matrix}\,\middle|\, z^2\right)}\frac{t\,z}{\sqrt{(1-qt^2)^2-4pt^2z^2}}$$

Finally let us define

$$S_n^*\left(\begin{matrix} r & s \\ p & q\end{matrix}\,\middle|\, x\right)=\frac{\prod_{k=1}^{[n/2]} r/2+((2n-5+(-1)^{n+1})/4+k)p}{[n/2]!}\bar{S}_n\left(\begin{matrix} r & s \\ p & q\end{matrix}\,\middle|\, x\right)\ . \qquad (3.33)$$

Therefore, by noting (3.32.1) and (3.32.2) a kind of generating function for MCSOP is derived as (3.34)

$$\sum_{n=0}^{\infty}S_n^*\left(\begin{matrix} r & s \\ p & q\end{matrix}\,\middle|\, z\right)t^n=\frac{1}{\sqrt{(1-qt^2)^2-4pt^2z^2}}\left(\frac{\rho\left(\begin{matrix} (r+p)/2, & (s+q)/2 \\ p,\ q,\ 0 & \end{matrix}\,\middle|\, u^*\right)}{\rho\left(\begin{matrix} (r+p)/2, & (s+q)/2 \\ p,\ q,\ 0 & \end{matrix}\,\middle|\, z^2\right)}+\frac{(tz)\rho\left(\begin{matrix} (r+3p)/2, & (s+3q)/2 \\ p,\ q,\ 0 & \end{matrix}\,\middle|\, u^*\right)}{\rho\left(\begin{matrix} (r+3p)/2, & (s+3q)/2 \\ p,\ q,\ 0 & \end{matrix}\,\middle|\, z^2\right)}\right)$$

where u^* is the same form as (3.32.1).

The explicit form of polynomials (3.7) can also be applied to obtain a generic hypergeometric representation for the polynomials, because by using it one can easily indicate the coefficients of polynomials (3.7) in terms of the Pochhammer symbol: $(a)_m = a(a+1)...(a+m-1)$. For instance, we have

$$\prod_{i=0}^{[n/2]-(k+1)} (2i+(-1)^{n+1}+2[n/2])\,p+r = (p/2)^{[n/2]-k}\left(r/2p+[n/2]+(-1)^{n+1}/2\right)_{[\frac{n}{2}]-k} . \quad (3.35)$$

Hence, after some calculations, we eventually find that

$$\bar{S}_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) = x^n\, {}_2F_1\left(\begin{matrix} -[n/2], & (q-s)/2q-[(n+1)/2] \\ & -(r+(2n-3)p)/2p \end{matrix}\middle| -\frac{q}{px^2}\right), \quad (3.36)$$

where ${}_2F_1\left(\begin{matrix} \alpha & \beta \\ & \gamma \end{matrix}\middle| x\right) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k} \frac{x^k}{k!}$ denotes the hypergeometric function of order (2,1) [43]. Furthermore, since (3.37)

$${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt \Leftrightarrow \operatorname{Re} c > \operatorname{Re} b > 0 \;\; ; \;\; |z|<1,$$

the integral representation of MCSOP can easily be derived by referring to (3.36). In this way, by applying the Gauss identity [43]

$${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad (3.38)$$

one can also determine the value of polynomials $\bar{S}_n(p,q,r,s;x)$ at a specific boundary point, i.e. $x=\sqrt{-q/p}$. To reach this goal, it is sufficient to put $x=\sqrt{-q/p}$ in (3.36) and use (3.38) to get

$$\bar{S}_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| \sqrt{-\frac{q}{p}}\right) = \frac{(-\frac{q}{p})^{\frac{n}{2}}\,\Gamma(\frac{3}{2}-\frac{r}{2p})\Gamma(1+\frac{s}{2q}-\frac{r}{2p})}{(\frac{3}{2}-\frac{r}{2p}-n)_n\,\Gamma(\frac{3}{2}-\frac{r}{2p}+[\frac{n}{2}]-n)\,\Gamma(1+\frac{s}{2q}-\frac{r}{2p}-[\frac{n}{2}])} . \quad (3.39)$$

Note that to derive the above identity, the following equalities have been used

$$\begin{aligned} &[\frac{n+1}{2}]-[\frac{n}{2}] = \frac{1-(-1)^n}{2}, \qquad [\frac{n+1}{2}]+[\frac{n}{2}] = n, \\ &[\frac{n}{2}] = \frac{n}{2}-\frac{1-(-1)^n}{4}, \qquad (a)_n = (-1)^n(1-a-n)_n. \end{aligned} \quad (3.40)$$

The standard properties of MCSOP have been found now. So, it is a good position to introduce four special sub-cases of main polynomials (3.7) in detail.

3.6. First subclass, Generalized ultraspherical polynomials (GUP)

These polynomials were first investigated by Chihara in detail [see 27]. He obtained the main properties of GUP via a direct relation between them and Jacobi orthogonal polynomials. The asymptotic behaviors of foresaid polynomials were also studied by Konoplev [48]. On the other hand, Charris and Ismail in [24] (see also [25, 41]) introduced the Sieved random walk polynomials to show that the generalized ultraspherical polynomials are a special case of them. Of course, there are some other generalizations of ultraspherical polynomials. For instance, Askey in [19] introduced two classes of orthogonal polynomials as a limiting case of the q-Wilson polynomials on $[-1,1]$ with the weight functions

$$W_1(x,\alpha,\lambda)=(1-x^2)^{\alpha-\frac{1}{2}}\,|U_{k-1}(x)|^{2\alpha}|T_k(x)|^{\lambda}$$

and

$$W_2(x,\alpha,\lambda)=(1-x^2)^{\alpha+\frac{1}{2}}\,|U_{k-1}(x)|^{2\alpha}|T_k(x)|^{\lambda}$$

in which $k\in N$ is a fixed integer and $T_k(x)$ and $U_{k-1}(x)$ are respectively the Chebyshev polynomials of the first and second kind, to generalize GUP for $k=1$ and $\lambda=0$. For the case $\lambda=0$ the polynomials were introduced by Al-Salam, Allaway and Askey [15] as a limiting case of the q-Ultraspherical polynomials of Rogers [64].
Anyway, we intend in this section to show that GUP can directly be represented in terms of $\bar{S}_n(p,q,r,s;x)$ and consequently all its standard properties will be obtained. For this purpose, it is only enough to have the initial vector corresponding to these polynomials and replace it into the standard properties of MCSOP.

3.6.1. Definition

Choose the initial vector $(p,q,r,s)=(-1,1,-2a-2b-2,2a)$ and substitute it into (3.7) to get (3.41)

$$\bar{S}_n\left(\begin{matrix}-2a-2b-2, & 2a\\ -1, & 1\end{matrix}\,\middle|\,x\right)=\prod_{i=0}^{[n/2]-1}\frac{2i+2a+2-(-1)^n}{-2i-(2b+2a+2-(-1)^n+2[n/2])}$$
$$\times\sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\left(\prod_{i=0}^{[\frac{n}{2}]-(k+1)}\frac{-2i-(2b+2a+2-(-1)^n+2[n/2])}{2i+2a+2-(-1)^n}\right)x^{n-2k},$$

as the explicit form of monic GUP. Moreover, since the ultraspherical (Gegenbauer), Legendre, and Chebyshev polynomials of the first and second kind are all special cases of GUP, they can be expressed in terms of $\bar{S}_n(p,q,r,s;x)$ directly and we have

Ultraspherical polynomials:

$$C_n^a(x)=\frac{2^n(a)_n}{n!}\,\bar{S}_n\left(\begin{matrix}-2a-1 & 0\\ -1 & 1\end{matrix}\,\middle|\,x\right). \qquad (3.41.1)$$

Legendre polynomials:

$$P_n(x) = \frac{(2n)!}{(n!)^2 2^n}\ \bar{S}_n\left(\begin{array}{cc|c} -2 & 0 & \\ -1 & 1 & x \end{array}\right). \tag{3.41.2}$$

Chebyshev polynomials of the first kind:

$$T_n(x) = 2^{n-1}\ \bar{S}_n\left(\begin{array}{cc|c} -1 & 0 & \\ -1 & 1 & x \end{array}\right). \tag{3.41.3}$$

Chebyshev polynomials of the second kind:

$$U_n(x) = 2^n\ \bar{S}_n\left(\begin{array}{cc|c} -3 & 0 & \\ -1 & 1 & x \end{array}\right). \tag{3.41.4}$$

3.6.2. Recurrence relation of monic polynomials

By replacing the initial vector (3.41) into the explicit expression $C_n(p,q,r,s)$, given in (3.10.1), the recurrence relation of monic GUP takes the form (3.42)

$$\bar{S}_{n+1}(x) = x\bar{S}_n(x) + C_n\begin{pmatrix} -2a-2b-2, & 2a \\ -1, & 1 \end{pmatrix}\bar{S}_{n-1}(x) \quad ; \quad \bar{S}_0(x)=1\,,\ \bar{S}_1(x)=x\,,\ n\in\mathbf{N},$$

where according to (3.10.1) (3.42.1)

$$C_n\begin{pmatrix} -2a-2b-2, & 2a \\ -1, & 1 \end{pmatrix} = \frac{-n^2-(2b+2(1-(-1)^n)a)n-2a(a+b)(1-(-1)^n)}{(2n+2a+2b-1)(2n+2a+2b+1)}$$
$$= \frac{-(n+(1-(-1)^n)a)(n+(1-(-1)^n)a+2b)}{(2n+2a+2b-1)(2n+2a+2b+1)}.$$

3.6.3. Orthogonality relation

Clearly the weight function of GUP is the same distribution as (3.17) without considering its normalizing constant, i.e. $x^{2a}(1-x^2)^b$. Also, since this function must be even and positive, the condition $(-1)^{2a}=1$ is essential. Hence, the mentioned weight function can also be considered as $|x|^{2a}(1-x^2)^b$; $x\in[-1,1]$. By noting this comment and generic relation (3.13) for $\alpha=1$, the orthogonality relation of first sub-class takes the form (3.43)

$$\int_{-1}^{1} x^{2a}(1-x^2)^b\,\bar{S}_n\left(\begin{array}{cc|c} -2a-2b-2, & 2a & \\ -1, & 1 & x \end{array}\right)\bar{S}_m\left(\begin{array}{cc|c} -2a-2b-2, & 2a & \\ -1, & 1 & x \end{array}\right)dx$$
$$= \left((-1)^n\prod_{i=1}^{i=n} C_i\begin{pmatrix} -2a-2b-2, & 2a \\ -1, & 1 \end{pmatrix}\int_{-1}^{1} x^{2a}(1-x^2)^b\,dx\right)\delta_{n,m},$$

where

$$\int_{-1}^{1} x^{2a}(1-x^2)^b\,dx = B(a+\frac{1}{2},b+1) = \frac{\Gamma(a+1/2)\Gamma(b+1)}{\Gamma(a+b+3/2)}\ . \tag{3.43.1}$$

From (3.43.1) one can conclude that the constraints on the parameters *a* and *b* should be $a+1/2>0$, $(-1)^{2a}=1$ and $b+1>0$. Note that $B(\lambda_1,\lambda_2)$ in (3.43.1) denotes the Beta integral [18] having various definitions in the form

$$\begin{aligned} B(\lambda_1,\lambda_2) &= \int_0^1 x^{\lambda_1-1}(1-x)^{\lambda_2-1}dx = \int_{-1}^1 x^{2\lambda_1-1}(1-x^2)^{\lambda_2-1}dx = \int_0^\infty \frac{x^{\lambda_1-1}}{(1+x)^{\lambda_1+\lambda_2}}dx \\ &= 2\int_0^{\pi/2} \sin^{(2\lambda_1-1)}x\cos^{(2\lambda_2-1)}x\,dx = \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda_1+\lambda_2)} = B(\lambda_2,\lambda_1). \end{aligned} \tag{3.44}$$

3.6.4. Differential equation: $\Phi_n(x) = S_n(-1,1,-2a-2b-2,2a;x)$.

To derive the differential equation of GUP it is enough to substitute its initial vector into the main differential equation (3.2) to get to (3.45)

$$x^2(-x^2+1)\Phi_n''(x) - 2x((a+b+1)x^2-a)\Phi_n'(x) + \left(n(2a+2b+n+1)x^2+((-1)^n-1)a\right)\Phi_n(x) = 0$$

3.7. Fifth and Sixth kind of Chebyshev polynomials [1]

As we know, four kinds of trigonometric orthogonal polynomials, known as first, second, third and fourth kind of Chebyshev polynomials, have been investigated in the literature up to now, see e.g. [63, 37, 27, 70]. The explicit definitions of them are respectively as

$$T_n(x) = 2^{n-1}\prod_{k=1}^n (x-\cos\frac{(2k-1)\pi}{2n}) = \cos(n\theta) \quad ; \quad x=\cos\theta\ , \tag{3.46}$$

$$U_n(x) = 2^n\prod_{k=1}^n (x-\cos\frac{k\pi}{n+1}) = \frac{\sin((n+1)\theta)}{\sin\theta} \quad ; \quad x=\cos\theta\ , \tag{3.47}$$

$$V_n(x) = 2^n\prod_{k=1}^n (x-\cos\frac{(2k-1)\pi}{2n+1}) = \frac{\cos((2n+1)\theta)}{\cos(\theta)} \quad ; \quad x=\cos(2\theta)\ , \tag{3.48}$$

$$W_n(x) = 2^n\prod_{k=1}^n (x-\cos\frac{2k\pi}{2n+1}) = \frac{\sin((2n+1)\theta)}{\sin(\theta)} \quad ; \quad x=\cos(2\theta)\ . \tag{3.49}$$

Now, we would like to add here that there exist two further kinds of Half-trigonometric orthogonal polynomials, which are particular sub-cases of $\bar{S}_n(p,q,r,s;x)$. Since they are generated by using the first and second kind of Chebyshev polynomials and have the half-trigonometric forms, let us call them the fifth and sixth kind of Chebyshev polynomials.

To generate these two sequences, we should refer to the important relation (3.8). According to (3.41.3), the initial vector of first kind Chebyshev polynomials is: $(p,q,r,s)=(-1,1,-1,0)$. So, if this vector is replaced into (3.8) then we have

$$\bar{S}_{2n+1}\left(\begin{matrix}-1 & 0\\ -1 & 1\end{matrix}\middle|\, x\right)=\bar{T}_{2n+1}(x)=x\,\bar{S}_{2n}\left(\begin{matrix}-3 & 2\\ -1 & 1\end{matrix}\middle|\, x\right). \tag{3.50}$$

By means of (3.50), the secondary vector $(p,q,r,s)=(-1,1,-3,2)$, as a special case of the set of vectors (p,q,r,s), appears. By using this new vector first we define the half-trigonometric polynomials

$$X_n(x)=S_n\left(\begin{matrix}-3 & 2\\ -1 & 1\end{matrix}\middle|\, x\right)=\begin{cases}\dfrac{(-1)^{n/2}}{n+1}\dfrac{\cos((n+1)\theta)}{\cos\theta} & if \quad n=2m,\\ S_n(-1,1,-3,2\,;x) & if \quad n=2m+1,\end{cases} \tag{3.51}$$

where $\cos\theta=x$. According to (3.51) and (3.10), $\bar{X}_n(x)$ satisfies the recurrence relation

$$\bar{X}_{n+1}(x)=x\,\bar{X}_n(x)+C_n\begin{pmatrix}-3 & 2\\ -1 & 1\end{pmatrix}\bar{X}_{n-1}(x) \quad ; \quad \bar{X}_0(x)=1\,,\ \bar{X}_1(x)=x\,,\ n\in\mathbf{N}, \tag{3.52}$$

in which

$$C_n\begin{pmatrix}-3 & 2\\ -1 & 1\end{pmatrix}=\frac{-(n-(-1)^n)(n+1-(-1)^n)}{4n(n+1)}\ . \tag{3.52.1}$$

Consequently, substituting the secondary vector (-1,1,-3,2) into the generic relation (3.13) gives the orthogonality relation of the first kind of half-trigonometric orthogonal polynomials as

$$\int_{-1}^{1}W\left(\begin{matrix}-3 & 2\\ -1 & 1\end{matrix}\middle|\, x\right)\bar{X}_n(x)\bar{X}_m(x)dx=\left((-1)^n\prod_{i=1}^{n}C_i\begin{pmatrix}-3 & 2\\ -1 & 1\end{pmatrix}\int_{-1}^{1}W\left(\begin{matrix}-3 & 2\\ -1 & 1\end{matrix}\middle|\, x\right)dx\right)\delta_{n,m}. \tag{3.53}$$

On the other hand since

$$\int_{-1}^{1}W\left(\begin{matrix}-3 & 2\\ -1 & 1\end{matrix}\middle|\, x\right)dx=\int_{-1}^{1}\frac{x^2}{\sqrt{1-x^2}}dx=B(\frac{3}{2},\frac{1}{2})=\frac{\pi}{2}\ , \tag{3.54}$$

(3.54) is simplified to

$$\int_{-1}^{1}\frac{x^2}{\sqrt{1-x^2}}\bar{X}_n(x)\bar{X}_m(x)dx=\left((-1)^n\prod_{i=1}^{n}C_i\begin{pmatrix}-3 & 2\\ -1 & 1\end{pmatrix}\right)\frac{\pi}{2}\,\delta_{n,m}\ . \tag{3.55}$$

Clearly a half of polynomials $\bar{X}_n(x)$ is decomposable and we have

$$\bar{X}_{2n}(x)=\prod_{k=1}^{2n}(x-\cos\frac{(2k-1)\pi}{2(2n+1)}). \tag{3.56}$$

So, if one can find the roots of $\bar{X}_{2n+1}(x)$ too, these polynomials will find many applications in numerical analysis such as Gaussian quadrature rules [27, 70].

Similarly, the subject holds for the initial vector of the second kind Chebyshev polynomials, i.e. $(p,q,r,s)=(-1,1,-3,0)$. Again, if this vector is substituted into (3.8) then

$$\bar{S}_{2n+1}\left(\begin{matrix}-3 & 0\\ -1 & 1\end{matrix}\middle| x\right)=\bar{U}_{2n+1}(x)=x\,\bar{S}_{2n}\left(\begin{matrix}-5 & 2\\ -1 & 1\end{matrix}\middle| x\right), \qquad (3.57)$$

and subsequently the secondary vector is obtained in the form $(p,q,r,s)=(-1,1,-5,2)$. Now, by assuming that $\cos\theta = x$ let us define the polynomials

$$Y_n(x)=S_n\left(\begin{matrix}-5 & 2\\ -1 & 1\end{matrix}\middle| x\right)=\begin{cases}\dfrac{(-1)^{n/2}}{n+2}\dfrac{\sin((n+2)\theta)}{\cos\theta\sin\theta} & if \quad n=2m,\\ S_n(-1,1,-5,2;x) & if \quad n=2m+1,\end{cases} \qquad (3.58)$$

satisfying the recurrence relation

$$\bar{Y}_{n+1}(x)=x\bar{Y}_n(x)+C_n\begin{pmatrix}-5 & 2\\ -1 & 1\end{pmatrix}\bar{Y}_{n-1}(x) \quad ; \quad \bar{Y}_0(x)=1\,,\ \bar{Y}_1(x)=x\,,\ n\in\mathbf{N}, \qquad (3.59)$$

where

$$C_n\begin{pmatrix}-5 & 2\\ -1 & 1\end{pmatrix}=\frac{-(n+1-(-1)^n)(n+2-(-1)^n)}{4(n+1)(n+2)}\ , \qquad (3.59.1)$$

and having the orthogonality relation

$$\int_{-1}^{1} W\left(\begin{matrix}-5 & 2\\ -1 & 1\end{matrix}\middle| x\right)\bar{Y}_n(x)\bar{Y}_m(x)\,dx=\left((-1)^n\prod_{i=1}^{n}C_i\begin{pmatrix}-5 & 2\\ -1 & 1\end{pmatrix}\int_{-1}^{1}W\left(\begin{matrix}-5 & 2\\ -1 & 1\end{matrix}\middle| x\right)dx\right)\delta_{n,m}\ , \qquad (3.60)$$

where

$$\int_{-1}^{1} W\left(\begin{matrix}-5 & 2\\ -1 & 1\end{matrix}\middle| x\right)dx=\int_{-1}^{1}x^2\sqrt{1-x^2}\,dx=B\left(\frac{3}{2},\frac{3}{2}\right)=\frac{\pi}{8}\ . \qquad (3.61)$$

The relation (3.61) simplifies (3.60) as

$$\int_{-1}^{1}x^2\sqrt{1-x^2}\,\bar{Y}_n(x)\bar{Y}_m(x)dx=\left((-1)^n\prod_{i=1}^{n}C_i\begin{pmatrix}-5 & 2\\ -1 & 1\end{pmatrix}\right)\frac{\pi}{8}\,\delta_{n,m}\ . \qquad (3.62)$$

Similar to previous case, $\bar{Y}_{2n}(x)$ is decomposable as

$$\bar{Y}_{2n}(x)=\prod_{k=1}^{2n}\left(x-\cos\frac{k\pi}{2n+2}\right)\ . \qquad (3.63)$$

Here it should be added that there are two other sequences of half-trigonometric polynomials that are *not orthogonal*, but can be shown in terms of MCSOP. These sequences are defined as

$$\bar{S}_n\left(\begin{array}{cc|c} 1 & -2 & \\ -1 & 1 & x \end{array}\right) = \begin{cases} \bar{S}_n(-1,1,1,-2;x) & \text{if} \quad n=2m, \\ x\bar{T}_{n-1}(x) & \text{if} \quad n=2m+1. \end{cases} \tag{3.64}$$

$$\bar{S}_n\left(\begin{array}{cc|c} -1 & -2 & \\ -1 & 1 & x \end{array}\right) = \begin{cases} \bar{S}_n(-1,1,-1,-2;x) & \text{if} \quad n=2m, \\ x\bar{U}_{n-1}(x) & \text{if} \quad n=2m+1. \end{cases} \tag{3.65}$$

However since we have

$$\int_{-1}^{1} W\left(\begin{array}{cc|c} 1 & -2 & \\ -1 & 1 & x \end{array}\right) dx = \int_{-1}^{1} \frac{dx}{x^2\sqrt{1-x^2}} = +\infty \ , \tag{3.66}$$

$$\int_{-1}^{1} W\left(\begin{array}{cc|c} -1 & -2 & \\ -1 & 1 & x \end{array}\right) dx = \int_{-1}^{1} \frac{\sqrt{1-x^2}}{x^2} dx = +\infty \ , \tag{3.67}$$

they cannot fall into the half-trigonometric orthogonal polynomials category. Thus we can generally consider the following table showing some properties of the first kind to sixth kind of monic Chebyshev polynomials orthogonal on $[-1,1]$.

Table 1: Representations of Chebyshev polynomials by $\bar{S}_n(p,q,r,s;x)$

Type	Notation	Definition	Weight
First Kind	$\bar{T}_n(x)$	$\bar{S}_n\left(\begin{array}{cc\|c} -1 & 0 & \\ -1 & 1 & x \end{array}\right)$	$\frac{1}{\sqrt{1-x^2}}$
Second Kind	$\bar{U}_n(x)$	$\bar{S}_n\left(\begin{array}{cc\|c} -3 & 0 & \\ -1 & 1 & x \end{array}\right)$	$\sqrt{1-x^2}$
Third Kind	$\bar{V}_n(x)$	$2^n\bar{S}_{2n}\left(\begin{array}{cc\|c} -3 & 2 & \\ -1 & 1 & \sqrt{\frac{1+x}{2}} \end{array}\right)$	$\sqrt{\frac{1+x}{1-x}}$
Fourth Kind	$\bar{W}_n(x)$	$2^n\bar{S}_{2n}\left(\begin{array}{cc\|c} -3 & 2 & \\ -1 & 1 & \sqrt{\frac{1-x}{2}} \end{array}\right)$	$\sqrt{\frac{1-x}{1+x}}$
Fifth Kind	$\bar{X}_n(x)$	$\bar{S}_n\left(\begin{array}{cc\|c} -3 & 2 & \\ -1 & 1 & x \end{array}\right)$	$\frac{x^2}{\sqrt{1-x^2}}$
Sixth Kind	$\bar{Y}_n(x)$	$\bar{S}_n\left(\begin{array}{cc\|c} -5 & 2 & \\ -1 & 1 & x \end{array}\right)$	$x^2\sqrt{1-x^2}$

Remark 1. According to (3.41.4), the initial vector of the monic Chebyshev polynomials $\bar{U}_n(x)$ is $(p,q,r,s)=(-1,1,-3,0)$. If this vector is replaced in (3.10.1), a very simple case of three-term recurrence relation (3.10) with $C_n(-1,1,-3,0)=-1/4$ is derived. A system of monic orthogonal polynomials that satisfies the relation

$$\overline{P}_{n+1}(x)=(x-\alpha_n)\overline{P}_n(x)-\beta_n\overline{P}_{n-1}(x) \quad ; \quad \overline{P}_0(x)=1\,,\; n\in Z^+, \qquad (3.68)$$

and has the property

$$\lim_{n\to\infty}\alpha_n=a \quad \text{and} \quad \lim_{n\to\infty}\beta_n=\frac{b^2}{4}>0 \quad ; \quad a,b\in\mathbf{R}, \qquad (3.69)$$

is said to belong to the class $N(a,b)$. The monic polynomials corresponding to the conditions (3.69) are perturbations of $x\to b^n\overline{U}_n(\frac{x-a}{b})$. Now since

$$\lim_{n\to\infty}C_n\begin{pmatrix}-3 & 2\\ -1 & 1\end{pmatrix}=-\frac{1}{4} \quad \text{and} \quad \lim_{n\to\infty}C_n\begin{pmatrix}-5 & 2\\ -1 & 1\end{pmatrix}=-\frac{1}{4}\,, \qquad (3.70)$$

the defined polynomials $\overline{X}_n(x)$ and $\overline{Y}_n(x)$ belong to the class $N(0,1)$.

3.8. Second subclass, Generalized Hermite polynomials (GHP)

The GHP were first introduced by Szego who gave a second order differential equation for these polynomials [70, problem 25] as almost the same form as we will give in this section. These polynomials can be characterized by using a direct relationship between them and Laguerre orthogonal polynomials [27]. Of course, there are some other approaches for this matter, see e.g. [32]. Because of this, it is better to only point to the main properties of GHP in terms of the obtained properties of $\overline{S}_n(p,q,r,s;x)$.

3.8.1. Initial vector

$$(p,q,r,s)=(0,1,-2,2a)\;. \qquad (3.71)$$

3.8.2. Explicit form of monic GHP

$$\overline{S}_n\left(\begin{matrix}-2 & 2a\\ 0 & 1\end{matrix}\,\middle|\,x\right)=(-1)^{[\frac{n}{2}]}\left(a+1-\frac{(-1)^n}{2}\right)_{[\frac{n}{2}]}\sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\left(\prod_{i=0}^{[n/2]-(k+1)}\frac{-2}{2i+(-1)^{n+1}+2+2a}\right)x^{n-2k} \qquad (3.72)$$

3.8.3. Recurrence relation of monic GHP

$$\overline{S}_{n+1}(x)=x\,\overline{S}_n(x)+C_n\begin{pmatrix}-2 & 2a\\ 0 & 1\end{pmatrix}\overline{S}_{n-1}(x) \quad ; \quad \overline{S}_0(x)=1\,,\; \overline{S}_1(x)=x\,,\; n\in\mathbf{N}, \qquad (3.73)$$

where

$$C_n\begin{pmatrix}-2 & 2a\\ 0 & 1\end{pmatrix}=-\frac{1}{2}n-\frac{1-(-1)^n}{2}a\;. \qquad (3.73.1)$$

3.8.4. Orthogonality relation

$$\int_{-\infty}^{\infty}x^{2a}e^{-x^2}\,\overline{S}_n\left(\begin{matrix}-2 & 2a\\ 0 & 1\end{matrix}\,\middle|\,x\right)\overline{S}_m\left(\begin{matrix}-2 & 2a\\ 0 & 1\end{matrix}\,\middle|\,x\right)dx=\left(\frac{1}{2^n}\prod_{i=1}^{n}(1-(-1)^i)a+i\right)\Gamma(a+\frac{1}{2})\,\delta_{n,m}\,. \qquad (3.74)$$

The above relation shows that orthogonality is valid for $a+1/2>0$ and $(-1)^{2a}=1$.

3.8.5. Differential equation: $\Phi_n(x)=S_n(0,1,-2,2a\,;x)$.

$$x^2\Phi_n''(x)-2x(x^2-a)\,\Phi_n'(x)+\left(2n\,x^2+((-1)^n-1)a\right)\Phi_n(x)=0\ . \tag{3.75}$$

Finally note that since the leading coefficient of Hermite polynomials $H_n(x)$ is 2^n, the following equality holds

$$H_n(x)=2^n\,\bar{S}_n\left(\begin{matrix}-2 & 0\\ 0 & 1\end{matrix}\middle|\,x\right)\ . \tag{3.76}$$

3.9. Third subclass, A finite class of symmetric orthogonal polynomials with weight function $x^{-2a}(1+x^2)^{-b}$ on $(-\infty,\infty)$ [1]

According to Favard theorem, if the condition $-C_{n+1}(p,q,r,s)>0$ holds only for a finite number of positive integers, i.e. for $n=0,1,...,N$ then the related polynomials class would be finitely orthogonal. This note helps us obtain some new classes of finite symmetric orthogonal polynomials, which are special sub-cases of $\bar{S}_n(p,q,r,s;x)$ and can be indicated by it directly. To derive the first finite sub-class, we should first compute the logarithmic derivative of the weight function $W(x)=x^{-2a}(1+x^2)^{-b}$ as

$$\frac{W'(x)}{W(x)}=\frac{-2(a+b)x^2-2a}{x(1+x^2)}\ . \tag{3.77}$$

If the above fraction is compared with the logarithmic derivative of the main weight function $W(p,q,r,s;x)$ then we get

$$(p,q,r,s)=(1,1,-2a-2b+2,-2a)\ , \tag{3.78}$$

which is in fact the initial vector corresponding to the first finite sub-class of symmetric orthogonal polynomials. Hence, if (3.78) is replaced in (3.7) the explicit form of polynomials is derived as (3.79)

$$S_n\left(\begin{matrix}-2a-2b+2, & -2a\\ 1, & 1\end{matrix}\middle|\,x\right)=\sum_{k=0}^{[n/2]}\binom{[\frac{n}{2}]}{k}\left(\prod_{i=0}^{[\frac{n}{2}]-(k+1)}\frac{2i+2[n/2]+(-1)^{n+1}+2-2a-2b}{2i+(-1)^{n+1}+2-2a}\right)x^{n-2k}$$

Replacing (3.78) in the main recurrence relation (3.10) also gives (3.80)

$$\bar{S}_{n+1}(x)=x\,\bar{S}_n(x)+C_n\begin{pmatrix}-2a-2b+2, & -2a\\ 1, & 1\end{pmatrix}\bar{S}_{n-1}(x)\ \ ;\ \ \bar{S}_0(x)=1\,,\ \bar{S}_1(x)=x\ ,\ n\in\mathbf{N},$$

where

$$C_n\begin{pmatrix} -2a-2b+2, & -2a \\ 1, & 1 \end{pmatrix} = \frac{(n-(1-(-1)^n)a)(n-(1-(-1)^n)a-2b)}{(2n-2a-2b+1)(2n-2a-2b-1)}\ . \qquad (3.80.1)$$

Therefore, the orthogonality relation

$$\int_{-\infty}^{\infty} \frac{x^{-2a}}{(1+x^2)^b}\bar{S}_n\left(\begin{matrix} -2a-2b+2, & -2a \\ 1, & 1 \end{matrix}\middle| x\right)\bar{S}_m\left(\begin{matrix} -2a-2b+2, & -2a \\ 1, & 1 \end{matrix}\middle| x\right)dx = \left((-1)^n\prod_{i=1}^{n} C_i\begin{pmatrix} -2a-2b+2, & -2a \\ 1, & 1 \end{pmatrix}\right)\frac{\Gamma(b+a-1/2)\Gamma(-a+1/2)}{\Gamma(b)}\ \delta_{n,m}\ , \qquad (3.81)$$

is valid iff $-C_{n+1}\begin{pmatrix} -2a-2b+2 & -2a \\ 1 & 1 \end{pmatrix} > 0 \quad ; \quad \forall n \in Z^+ ;\ b+a > 1/2\ ;\ a < 1/2\ \text{ and }\ b > 0$ ·

Here is a good position to explain how we can determine the parameters conditions to be established the orthogonality property (3.81). For this purpose, there is an interesting technique. Let us consider the differential equation of polynomials (3.79) using the initial vector (3.78) and subsequently the main equation (3.2) as (3.82)

$$x^2(x^2+1)\Phi_n''(x) - 2x((a+b-1)x^2+a)\Phi_n'(x) + \left(n(2a+2b-(n+1))x^2 + (1-(-1)^n)a\right)\Phi_n(x) = 0$$

If the above equation is written in self-adjoint form, then according to theorem 1 the following term must vanish, i.e.

$$[x^{-2a}(1+x^2)^{-b+1}(\Phi_n'(x)\Phi_m(x) - \Phi_m'(x)\Phi_n(x))]_{-\infty}^{\infty} = 0\ . \qquad (3.83)$$

On the other hand, since $\Phi_n(x)$ is a polynomial of degree *n*, so

$$\max\ \deg(\Phi_n'(x)\Phi_m(x) - \Phi_m'(x)\Phi_n(x)) = n+m-1\ . \qquad (3.84)$$

Consequently from (3.83) and (3.84) we must have

$$-2a+2(-b+1)+n+m-1 \le 0\ , \qquad (3.85)$$

which gives the following result

$$\begin{cases} -2a-2b+n+m+1 \le 0, \\ a < 1/2,\ b > 0. \end{cases} \qquad (3.86)$$

In other words, (3.81) holds if and only if $m,n = 0,1,...,N \le a+b-1/2$ in which $N = \max\{m,n\}$, $a < 1/2$ and $b > 0$.

Corollary 1. The finite polynomial set $\{S_n(1,1,-2a-2b+2,-2a\,;x)\}_{n=0}^{n=N}$ is orthogonal with respect to the weight function $x^{-2a}(1+x^2)^{-b}$ on $(-\infty,\infty)$ if and only if $N \le a+b-1/2$, $a<1/2$ and $b>0$.

We add that the explained technique can similarly be applied for the first and second sub-classes of $S_n(p,q,r,s;x)$ i.e. GUP and GHP.

3.10. Fourth subclass, A finite class of symmetric orthogonal polynomials with weight function $x^{-2a}e^{-1/x^2}$ on $(-\infty,\infty)$ [1]

Similar to the first finite sub-class, one can compute the logarithmic derivative of the given weight function to get respectively

3.10.1. Initial vector

$$(p,q,r,s)=(1,0,-2a+2,2)\,. \tag{3.87}$$

3.10.2. Explicit form of polynomials

$$S_n\left(\left.\begin{matrix}-2a+2 & 2\\ 1 & 0\end{matrix}\right| x\right)=\sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\left(\prod_{i=0}^{[\frac{n}{2}]-(k+1)}\frac{2i+2[n/2]+(-1)^{n+1}+2-2a}{2}\right)x^{n-2k}\,. \tag{3.88}$$

3.10.3. Recurrence relation of monic polynomials

$$\bar{S}_{n+1}(x)=x\,\bar{S}_n(x)+C_n\begin{pmatrix}-2a+2 & 2\\ 1 & 0\end{pmatrix}\bar{S}_{n-1}(x)\quad ;\quad \bar{S}_0(x)=1\,,\ \bar{S}_1(x)=x\,,\ n\in\mathbf{N}, \tag{3.89}$$

where

$$C_n\begin{pmatrix}-2a+2 & 2\\ 1 & 0\end{pmatrix}=\frac{-2(-1)^n(n-a)-2a}{(2n-2a+1)(2n-2a-1)}\,. \tag{3.89.1}$$

3.10.4. Orthogonality relation

$$\begin{aligned}&\int_{-\infty}^{\infty}x^{-2a}e^{-\frac{1}{x^2}}\,\bar{S}_n\left(\left.\begin{matrix}-2a+2 & 2\\ 1 & 0\end{matrix}\right| x\right)\bar{S}_m\left(\left.\begin{matrix}-2a+2 & 2\\ 1 & 0\end{matrix}\right| x\right)dx\\ &\qquad=\left((-1)^n\prod_{i=1}^{n}C_i\begin{pmatrix}-2a+2 & 2\\ 1 & 0\end{pmatrix}\right)\Gamma(a-\tfrac{1}{2})\,\delta_{n,m}\,.\end{aligned} \tag{3.90}$$

But (3.90) is valid if

$$[x^{2-2a}\exp(-\frac{1}{x^2})(\Phi'_n(x)\Phi_m(x)-\Phi'_m(x)\Phi_n(x))]_{-\infty}^{\infty}=0\,, \tag{3.91}$$

or equivalently

$$2-2a+n+m-1\le 0 \Leftrightarrow N\le a-\frac{1}{2}\quad ;\quad N=\max\{m,n\}\,. \tag{3.92}$$

Corollary 2. The finite polynomial set $\{S_n(1,0,-2a+2,2;x)\}_{n=0}^{n=N}$ is orthogonal with respect to the weight function $x^{-2a}e^{-1/x^2}$ on $(-\infty,\infty)$ if and only if $N \le a-1/2$.

3.10.5. Differential equation: $\Phi_n(x) = S_n(1,0,-2a+2,2;x)$.

$$x^4\,\Phi_n''(x) + 2x((1-a)x^2+1)\,\Phi_n'(x) - \big(n(n+1-2a)\,x^2 + 1 - (-1)^n\big)\Phi_n(x) = 0\ . \qquad (3.93)$$

3.11. A unified approach for the classification of MCSOP

As we observed in the previous sections, each four introduced sub-classes of symmetric orthogonal polynomials were determined by $\bar{S}_n(p,q,r,s;x)$ directly and it was only sufficient to obtain the initial vector corresponding to them. On the other hand, it is clear that the orthogonality interval of the sub-classes, other than first one (GUP), are all infinite, i.e. $(-\infty,\infty)$. Hence, applying a linear transformation, say $x = wt+v$, preserves the orthogonality interval. For example, if $x = wt + v/w$ in (3.74), the orthogonality interval will not change and consequently a more extensive class with weight function $(w^2t+v)^{2a}\exp(-w^2t^2-2vt)$ will be derived on the interval $(-\infty,\infty)$. However, it is important to know that the latter weight corresponds to the class of orthogonal polynomials $S_n(p,q,r,s;wx+v/w)$. Therefore, only by having the initial vector we can have access to all other standard properties and design a unified approach for the cases that may occur. In other words, if one can obtain the parameters (p,q,r,s) by referring to the initial data, such as a given three term recurrence relation, a given weight function and so on, then all other properties will be derived straightforwardly. Here, we consider two special sub-cases of this approach.

3.11.1. How to find the parameters p,q,r,s if a special case of the main weight function $W(p,q,r,s;x)$ is given?

By referring to third and fourth orthogonal sub-classes, it is easy to find out that the best way for deriving p,q,r,s is to compute the logarithmic derivative $W'(x)/W(x)$ and then equate the pattern with (3.14). The following examples will clarify this matter.

Example 1. The weight functions

$i)\quad W_1(x) = -x^6+4x^4 \quad ; \quad -2 \le x \le 2$

$ii)\quad W_2(x) = (16x^2-8x+1)\exp(2x(1-2x)) \quad ; \quad -\infty < x < \infty$

$iii)\quad W_3(x) = (2x+1)^2(2x^2+2x+1)^{-5} \quad ; \quad -\infty < x < \infty$

and their orthogonality intervals are given. Find other standard properties such as explicit form of polynomials, orthogonality relation and ….

To solve the problem, it is only sufficient to find the initial vector corresponding to each given weight functions. For this purpose, if the logarithmic derivative of the first weight is computed, then we have

$$i)\quad \frac{W_1'(x)}{W_1(x)} = \frac{6x^2-16}{x(x^2-4)} = \frac{(r-2p)x^2+s}{x(px^2+q)} \Rightarrow (p,q,r,s)=(1,-4,8,-16)\,.$$

Hence the related monic polynomials are $\{\bar{S}_n(1,-4,8,-16;x)\}_{n=0}^{\infty}$. Note that these polynomials are orthogonal with respect to $x^4(4-x^2)$ on $[-2,2]$ for every n and it is *not* necessary to know that they are the same as shifted GUP on $[-2,2]$, because they can explicitly and independently be expressed by $\bar{S}_n(p,q,r,s;x)$.
But, for the weight function $W_2(x)$ it differs somewhat and we have

$$ii)\quad W_2(x) = 4e^{\frac{1}{4}}(2x-\frac{1}{2})^2\exp(-(2x-\frac{1}{2})^2) \Rightarrow \frac{1}{4}e^{-\frac{1}{4}}W_2(\frac{2t+1}{4}) = t^2e^{-t^2} = W_2^*(t)$$

$$\to \frac{W_2^{*\,\prime}(t)}{W_2^*(t)} = \frac{-2t^2+2}{t} \Rightarrow (p,q,r,s)=(0,1,-2,2)$$

Hence the related orthogonal polynomials are as $\left\{S_n(0{,}1{,}-2{,}2;\,2x-\frac{1}{2})\right\}_{n=0}^{\infty}$.

$$iii)\quad W_3(x) = 2^5\frac{(2x+1)^2}{(1+(2x+1)^2)^5} \Rightarrow 2^{-5}W_3(\frac{t-1}{2}) = \frac{t^2}{(1+t^2)^5} = W_3^*(t)$$

$$\to \frac{W_3^{*\,\prime}(t)}{W_3^*(t)} = \frac{-8t^2+2}{t(t^2+1)} \Rightarrow (p,q,r,s)=(1,1,-6,2)$$

Consequently, by noting the orthogonality relation of the third sub-class of MCSOP, the finite set $\{S_n(1,1,-6,2;\,2x+1)\}_{n=0}^{n=3}$ is orthogonal with respect to $W_3(x)$ on $(-\infty,\infty)$ and the upper bound of this set has been determined based on the condition $N \le a+b-1/2$ for $b=5$ and $a=1$.

3.11.2. How to find the parameters p,q,r,s if a special case of the main three-term recurrence equation (18) is given?

In general, there are two ways to determine the special case of $\bar{S}_n(p,q,r,s;x)$ corresponding to a given three-term recurrence equation. The first way is to directly compare the given recurrence equation with (3.10). This leads to a system of polynomial equations in terms of the four parameters p,q,r,s. In [46] a similar method is applied for the first kind of classical orthogonal polynomials. The second way is to equate the first four terms of each two recurrence equations together, which leads to a polynomial system with 4 equations and 4 unknowns p,q,r and s respectively. The following example will better illustrate these methods.
Example 2. If the recurrence equation

$$\bar{S}_{n+1}(x) = x\bar{S}_n(x) - 2\frac{6+(-1)^n(n-6)}{(2n-11)(2n-13)}\bar{S}_{n-1}(x) \quad ; \quad \bar{S}_0(x)=1 \quad \text{and} \quad \bar{S}_1(x)=x\,,$$

is given, then find its explicit polynomial solution, differential equation of polynomials, the related weight function and finally orthogonality relation of polynomials.

Solution. If the above recurrence equation is directly compared with the main equation (3.10) and subsequently (3.10.1), then one can obtain the values $(p,q,r,s)=(1,0,-10,2)$. Hence, the explicit solution of above recurrence equation is the polynomials $\bar{S}_n(1,0,-10,2;x)$ and therefore their differential equation is found as

$$x^4\,\Phi_n''(x)+x(-10\,x^2+2)\,\Phi_n'(x)-\left(n(n-11)\,x^2+1-(-1)^n\right)\Phi_n(x)=0\,.$$

Moreover, by replacing the initial vector in the main weight function (3.14) as

$$W\left(\begin{matrix}-10 & 2\\ 1 & 0\end{matrix}\middle|\,x\right)=\exp\left(\int\frac{-12x^2+2}{x^3}\,dx\right)=x^{-12}e^{-\frac{1}{x^2}}\,,$$

one can find out that the related polynomials are a particular case of the fourth introduced sub-class. Hence we have

$$\int_{-\infty}^{\infty}x^{-12}e^{-\frac{1}{x^2}}\,\bar{S}_n\left(\begin{matrix}-10 & 2\\ 1 & 0\end{matrix}\middle|\,x\right)\bar{S}_m\left(\begin{matrix}-10 & 2\\ 1 & 0\end{matrix}\middle|\,x\right)dx=\left((-1)^n\prod_{i=1}^{i=n}C_i\begin{pmatrix}-10 & 2\\ 1 & 0\end{pmatrix}\right)\Gamma\left(\frac{11}{2}\right)\delta_{n,m}\Leftrightarrow m,n\le 5$$

Second method. If the given recurrence relation is only expanded for n = 2, 3, 4, 5 and then equated with (3.9.1), the following system will be derived

$$\bar{S}_2(x)=x^2-\frac{2}{9}=x^2+\frac{q+s}{p+r}\;\Rightarrow\frac{q+s}{p+r}=-\frac{2}{9}\quad;\;\bar{S}_3(x)=x\;\;\bar{S}_2(x)-\frac{4}{63}\bar{S}_1(x)=x^3+\frac{3q+s}{3p+r}x$$

$$\bar{S}_4(x)=x\;\;\bar{S}_3(x)-\frac{18}{35}\bar{S}_2(x)=x^4+2\frac{3q+s}{5p+r}x^2+\frac{(3q+s)(q+s)}{(5p+r)(3p+r)}\;\text{ and }\;2\;\;\frac{5q+s}{7p+r}=-\frac{4}{3}\;.$$

Solving this system again results that $(p,q,r,s)=(1,0,-10,2)$.

In conclusion, by using the extended Sturm-Liouville theorem for symmetric functions explained in chapter 2, one can define a generic second order differential equation having a main polynomial solution with four free parameters. This solution satisfies a generic orthogonality relation whose weight function corresponds to an analogue of Pearson distributions. In other words, there are four special cases of the dual symmetric distributions family that can respectively be considered as the weight functions of four introduced sub-classes of MCSOP. In this way, the following table shows the explicit forms of the mentioned sub-classes in terms of $S_n(p,q,r,s;x)$ as well as their weight functions, kind of polynomials (finite or infinite), orthogonality interval and finally constraint on the parameters.

Table 2: Four special sub-cases of $S_n(p,q,r,s;x)$

Definition	Weight function	Interval & Kind	Parameters Constraint
$S_n\left(\left.\begin{matrix}-2a-2b-2, & 2a\\ -1, & 1\end{matrix}\right\vert x\right)$	$x^{2a}(1-x^2)^b$	$[-1,1]$, Infinite	$a>-1/2$ $b>-1$
$S_n\left(\left.\begin{matrix}-2, & 2a\\ 0, & 1\end{matrix}\right\vert x\right)$	$x^{2a}e^{-x^2}$	$(-\infty,\infty)$, Infinite	$a>-\frac{1}{2}$
$S_n\left(\left.\begin{matrix}-2a-2b+2, & -2a\\ 1, & 1\end{matrix}\right\vert x\right)$	$\frac{x^{-2a}}{(1+x^2)^b}$	$(-\infty,\infty)$, Finite	$\begin{cases}N\le a+b-1/2\\ a<1/2\ ,\ b>0\end{cases}$
$S_n\left(\left.\begin{matrix}-2a+2, & 2\\ 1, & 0\end{matrix}\right\vert x\right)$	$x^{-2a}e^{-\frac{1}{x^2}}$	$(-\infty,\infty)$, Finite	$N\le a-\frac{1}{2}$

Finally we repeat that since all weights in above table are even functions, the condition $(-1)^{2a}=1$ must always be satisfied by noting the constraint of parameters for each introduced weight functions. Therefore, they can also be considered in the forms $|x|^{2a}\,(1-x^2)^b$, $|x|^{2a}\,e^{-x^2}$, $|x|^{-2a}\,(1+x^2)^{-b}$ and $|x|^{-2a}\,e^{-1/x^2}$ respectively.

Chapter 4

Finite classical orthogonal polynomials

4.1. Introduction

It is well known that the classical orthogonal polynomials of Jacobi, Laguerre and Hermite are infinitely orthogonal and satisfy a second order differential equation of the form

$$(Ax^2+Bx+C)y_n''(x)+(Dx+E)y_n'(x)-n((n-1)A+D)y_n(x)=0$$

in which A,B,C,D and E are parameters independent of n. In this chapter, we intend to study three other sequences of hypergeometric polynomials in detail, which are special solutions of above equation and are finitely orthogonal with respect to three specific weight functions. These classes have respectively relation with the Jacobi and Laguerre polynomials. In particular, the second class is directly related to the generalized Bessel polynomials and consequently Laguerre polynomials. Let us start with the first finite case.

4.2. First finite class of hypergeometric orthogonal polynomials

Consider the differential equation of Sturm-Liouville type

$$\sigma(x)y_n''(x)+\tau(x)y_n'(x)-\lambda_n y_n(x)=0\ , \tag{4.1}$$

where $\sigma(x)=Ax^2+Bx+C$ and $\tau(x)=Dx+E$ are polynomials independent of n and $\lambda_n=n(n-1)A+nD$ is the eigenvalue parameter depending on $n=0,1,2,...$.
The Jacobi orthogonal polynomials for $\sigma(x)=1-x^2$, $\tau(x)=-(\alpha+\beta+2)x+(\beta-\alpha)$, Laguerre for $\sigma(x)=x$, $\tau(x)=\alpha+1-x$ and finally Hermit for $\sigma(x)=1$, $\tau(x)=-2x$ are three known types of polynomial solutions of equation (4.1). But, there are three other types of polynomial solutions that are finitely orthogonal. The first finite class is defined when $\sigma(x)=x^2+x$, $\tau(x)=(2-p)x+(1+q)$ in (4.1). So, substituting these values in (4.1) gives the following differential equation

$$(x^2+x)y_n''(x)+((2-p)x+q)y_n'(x)-n((n+1-p)y_n(x)=0\ . \tag{4.2}$$

By applying the Frobenius method, an explicit polynomial solution for the equation (4.2) will be derived as

$$M_n^{(p,q)}(x) = (-1)^n n! \sum_{k=0}^{k=n} \binom{p-(n+1)}{k}\binom{q+n}{n-k}(-x)^k \Leftrightarrow \binom{a}{k} = \frac{1}{k!}\prod_{i=0}^{k-1}(a-i) \ . \tag{4.3}$$

Moreover, one can prove that these polynomials are finitely orthogonal with respect to the weight function $W_1(x;p,q) = x^q(1+x)^{-(p+q)}$ on $[0,\infty)$ if and only if $p > 2\{\max n\}+1$ and $q > -1$. To prove this claim, one should first write the self-adjoint form of the equation (4.2) as

$$\begin{aligned}&(x^{1+q}(1+x)^{1-p-q}y_n'(x))' = n(n+1-p)x^q(1+x)^{-(p+q)}y_n(x),\\&(x^{1+q}(1+x)^{1-p-q}y_m'(x))' = m(m+1-p)x^q(1+x)^{-(p+q)}y_m(x); \quad y_n(x) = M_n^{(p,q)}(x).\end{aligned} \tag{4.4}$$

Then using the Sturm-Liouville theorem, one gets

$$\left[\frac{x^{q+1}}{(1+x)^{p+q-1}}(y_n'(x)y_m(x) - y_m'(x)y_n(x))\right]_0^\infty = (\lambda_n - \lambda_m)\int_0^\infty \frac{x^q}{(1+x)^{p+q}} M_n^{(p,q)}(x)M_m^{(p,q)}(x)\,dx \tag{4.5}$$

where $\lambda_n = n(n+1-p)$. As we applied in chapter 3, since

$$\max \deg\{y_n'(x)y_m(x) - y_m'(x)y_n(x)\} = m+n-1$$

so if $q > -1$, $p > 2N+1$, $N = \max\{m,n\}$, the left side of (4.5) tends to zero and we will have

$$\int_0^\infty \frac{x^q}{(1+x)^{p+q}} M_n^{(p,q)}(x)M_m^{(p,q)}(x)\,dx = 0 \Leftrightarrow \begin{cases} m \neq n,\ p > 2N+1,\ q > -1, \\ N = \max\{m,n\}. \end{cases} \tag{4.6}$$

Corollary 1. The finite set $\{M_n^{(p>2N+1,q>-1)}(x)\}_{n=0}^{N} = \{M_n^{(p,q)}(x)\}_{n=0}^{N<(p-1)/2}$ must be orthogonal with respect to the weight function $W_1(x;p,q) = x^q(1+x)^{-(p+q)}$ on $[0,\infty)$.

As the differential equation (4.2) shows, the polynomials (4.3) have a direct relation with the Jacobi polynomials. Hence, by referring to the Rodrigues representation of Jacobi polynomials [16], the Rodrigues formula of the defined polynomials (4.3) can be indicated as

$$M_n^{(p,q)}(x) = (-1)^n \frac{(1+x)^{p+q}}{x^q} \frac{d^n(x^{n+q}(1+x)^{n-p-q})}{dx^n} \ ; \quad n = 0,1,2,... \tag{4.7}$$

One of the advantages of above representation is to calculate the norm square value of the polynomials. For this purpose, if (4.7) is replaced in the norm 2 relation

$$\int_0^\infty \frac{x^q}{(1+x)^{p+q}}(M_n^{(p,q)}(x))^2\,dx = (-1)^n \int_0^\infty M_n^{(p,q)}(x)\frac{d^n(x^{n+q}(1+x)^{n-p-q})}{dx^n}\,dx\,, \tag{4.8}$$

then integration by parts yields

$$(-1)^n \int_0^\infty M_n^{(p,q)}(x) \frac{d^n(x^{n+q}(1+x)^{n-p-q})}{dx^n} dx = \frac{n!(p-(n+1))!}{(p-(2n+1))!} \int_0^\infty x^{n+q}(1+x)^{n-p-q}\, dx \,. \qquad (4.9)$$

On the other hand

$$\int_0^\infty x^{n+q}(1+x)^{n-p-q}\, dx = \frac{(p-(2n+2))!(q+n)!}{(p+q-(n+1))!} \,. \qquad (4.10)$$

So, we have

$$\int_0^\infty \frac{x^q}{(1+x)^{p+q}} (M_n^{(p,q)}(x))^2\, dx = \frac{n!(p-n-1)!(q+n)!}{(p-2n-1)(p+q-n-1)!} . \qquad (4.11)$$

The foresaid relation shows that $p > 2n+1$ is a necessary condition for the orthogonality of the polynomials $M_n^{(p,q)}(x)$. Therefore

Corollary 2 (Orthogonality relation)

$$\int_0^\infty \frac{x^q}{(1+x)^{p+q}} M_n^{(p,q)}(x) M_m^{(p,q)}(x)\, dx = \frac{n!(p-n-1)!(q+n)!}{(p-2n-1)(p+q-n-1)!} \delta_{n,m}$$

if and only if $m,n = 0,1,2,...,N < \dfrac{p-1}{2}$, $q > -1$.

For instance, the polynomial set $\{M_n^{(202,0)}(x)\}_{n=0}^{100} = \{(-1)^n n! \sum_{k=0}^{k=n} \binom{201-n}{k}\binom{n}{k}(-x)^k\}_{n=0}^{100}$ is finitely orthogonal with respect to the weight function $W_1(x,202,0) = (1+x)^{-202}$ on $[0,\infty)$ and

$$\int_0^\infty (1+x)^{-202} M_n^{(202,0)}(x) M_m^{(202,0)}(x) dx = \frac{(n!)^2}{201-2n} \delta_{n,m} \Leftrightarrow m,n \le 100 \,. \qquad (4.12)$$

4.3. Second finite class of hypergeometric orthogonal polynomials: If $\sigma(x) = x^2$ and $\tau(x) = (2-p)x + 1$ in (4.1)

The second case is directly related to the generalized Bessel polynomials. The Bessel polynomials for $\sigma(x) = x^2$, $\tau(x) = 2x+2$ were first studied by [51] in 1949. They established the complex orthogonality of Bessel polynomials on the unit circle (the real orthogonalizing weights of these polynomials have recently given in [35]). Then, in 1973, the generalized Bessel polynomials were reviewed by [38]. They are special solutions of equation (4.1) for $\sigma(x) = x^2$, $\tau(x) = (2+\alpha)x + 2$; $\alpha \ne -2,-3,...$ and are indicated by

$$\bar{B}_n^{(\alpha)}(x) = 2^n \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(n+k+\alpha+1)}{\Gamma(2n+\alpha+1)} (\frac{x}{2})^k \ , \qquad (4.13)$$

where $\bar{B}_n^{(\alpha)}(x)$ denotes the monic Bessel polynomial. Now, without loss of generality, let us consider the following differential equation

$$x^2 y_n''(x) + ((2-p)x+1)y_n'(x) - n((n+1-p)y_n(x) = 0 \ . \tag{4.14}$$

By applying the Frobenius method one can show that

$$N_n^{(p)}(x) = (-1)^n \sum_{k=0}^{n} k! \binom{p-(n+1)}{k} \binom{n}{n-k} (-x)^k , \tag{4.15}$$

is a polynomial solution of (4.14). Clearly these polynomials are related to the generalized Bessel polynomials as

$$N_n^{(p)}(x) = \frac{n!}{2^n} \binom{p-1-n}{n} \overline{B}_n^{(-p)}(2x) \ . \tag{4.16}$$

If the equation (4.14) is written in self-adjoint form

$$\begin{aligned} &(x^{-p+2}\exp(-1/x) y_n'(x))' = n(n+1-p)x^{-p}\exp(-1/x) y_n(x), \\ &(x^{-p+2}\exp(-1/x) y_m'(x))' = m(m+1-p)x^{-p}\exp(-1/x) y_m(x); \quad y_n(x) = M_n^{(p,q)}(x), \end{aligned} \tag{4.17}$$

then multiplying by $y_m(x)$, $y_n(x)$ in (4.17) respectively and subtracting them, we get

$$\left[x^{-p+2} e^{-\frac{1}{x}} (y_n'(x) y_m(x) - y_m'(x) y_n(x)) \right]_0^{\infty} = (\lambda_n - \lambda_m) \int_0^{\infty} x^{-p} e^{-\frac{1}{x}} N_n^{(p)}(x) N_m^{(p)}(x)\, dx \ , \tag{4.18}$$

where $\lambda_n = n(n+1-p)$. Again, since max $\deg\{y_n'(x)y_m(x) - y_m'(x)y_n(x)\} = m+n-1$ the condition $p > 2N+1$, $N = \max\{m,n\}$ causes the left side of (4.18) to tend to zero and therefore

$$\int_0^{\infty} x^{-p} e^{-\frac{1}{x}} N_n^{(p)}(x) N_m^{(p)}(x)\, dx = 0 \Leftrightarrow \begin{cases} m \neq n, \ p > 2N+1 \\ N = \max\{m,n\} \end{cases} \tag{4.19}$$

Corollary 3. The finite set $\{N_n^{(p>2N+1)}(x)\}_{n=0}^{N} = \{N_n^{(p)}(x)\}_{n=0}^{N<(p-1)/2}$ must be an orthogonal one with respect to the weight function $W_2(x;p) = x^{-p}\exp(-1/x)$ on $[0,\infty)$. For instance, the polynomial set $\{N_n^{(202)}(x)\}_{n=0}^{100}$ must be finitely orthogonal with respect to the weight function $W_2(x,202) = x^{-202}\exp(-1/x)$ on $[0,\infty)$.

But the Rodrigues representation of classical orthogonal polynomials can be denoted by $\frac{C_n}{W(x)} \frac{d^n(P^n(x)W(x))}{dx^n}$ where $W(x)$ is a weight function, C_n is a constant value and $P(x)$ is an appropriate polynomial. So, if $W(x) = x^{-p}\exp(-1/x)$, $C_n = (-1)^n$ and $P(x) = x^2$ are supposed in the mentioned formula, then

$$N_n^{(p)}(x) = (-1)^n x^p e^{\frac{1}{x}} \frac{d^n (x^{-p+2n} e^{-1/x})}{dx^n} \; ; \quad n = 0,1,2,... \tag{4.20}$$

To complete the orthogonality relation of $N_n^{(p)}(x)$, one can use the above formula and compute the norm square value of the polynomials. Accordingly, if (4.20) is replaced in the relation

$$\int_0^\infty x^{-p} e^{-\frac{1}{x}} (N_n^{(p)}(x))^2 \, dx = (-1)^n \int_0^\infty N_n^{(p)}(x) \frac{d^n (x^{-p+2n} e^{-\frac{1}{x}})}{dx^n} dx \; , \tag{4.21}$$

then integration by parts yields

$$(-1)^n \int_0^\infty N_n^{(p)}(x) \frac{d^n (x^{-p+2n} e^{-\frac{1}{x}})}{dx^n} dx = \frac{n!(p-(n+1))!}{(p-(2n+1))!} \int_0^\infty x^{-p+2n} e^{-\frac{1}{x}} dx \; . \tag{4.22}$$

Here one should note that if $p > 1$ then

$$\int_0^\infty W_2(x;p) dx = \int_0^\infty x^{-p} e^{-\frac{1}{x}} dx = \Gamma(p-1) \; . \tag{4.23}$$

Therefore, (4.22) is simplified as

$$\frac{n!(p-(n+1))!}{(p-(2n+1))!} \int_0^\infty x^{-p+2n} e^{-\frac{1}{x}} dx = \frac{n!(p-(n+1))!}{p-(2n+1)} \tag{4.24}$$

and we finally get

$$\int_0^\infty x^{-p} e^{-\frac{1}{x}} (N_n^{(p)}(x))^2 \, dx = \frac{n!(p-(n+1))!}{(p-1-2n)} . \tag{4.25}$$

Corollary 4. (Orthogonality Relation)

$$\int_0^\infty x^{-p} e^{-\frac{1}{x}} N_n^{(p)}(x) N_m^{(p)}(x) \, dx = \left(\frac{n!(p-(n+1))!}{(p-(2n+1)} \right) \delta_{n,m} \Leftrightarrow m,n = 0,1,2,...,N < \frac{p-1}{2}$$

4.3.1 A direct relationship between Bessel polynomials and Laguerre polynomials [8]

It is interesting to know that there is a direct relationship between Bessel polynomials and Laguerre polynomials. To find this relation, we first consider the generalized Bessel equation

$$x^2 y''(x) + (r_1 x + r_2) y'(x) + r_3 \, y(x) = 0 \quad , \quad r_2 \neq 0 \tag{4.26}$$

and suppose that $y(x) = x^r F(1/x)$. Therefore $F(x) = x^r y(1/x)$ and (4.26) is transformed to

$$x^2F''(x)+x(-r_2x+2-r_1-2r)F'(x)+(rr_2x+r^2+(r_1-1)r+r_3)\,F(x)=0\ . \tag{4.27}$$

Now if in (4.27)

$$r^2+(r_1-1)r+r_3=0\Rightarrow r=\frac{1-r_1\pm\sqrt{(r_1-1)^2-4r_3}}{2}\,, \tag{4.27.1}$$

then it changes to

$$xF''(x)+(1\mp\sqrt{(r_1-1)^2-4r_3}-r_2x)F'(x)+r_2\,\frac{1-r_1\pm\sqrt{(r_1-1)^2-4r_3}}{2}\,F(x)=0\ . \tag{4.27.2}$$

On the other hand, it is known that the general solution of

$$xg''(x)+(c-bx)g'(x)-ba\,g(x)=0\,, \tag{4.28}$$

can be indicated by the confluent hypergeometric functions [42]

$$g(x)={}_1F_1(a,c;bx)=\sum_{k=0}^{\infty}\frac{(a)_k}{(c)_k}\frac{(bx)^k}{k!}\ . \tag{4.28.1}$$

So, by comparing the equations (4.27.2) and (4.28) it is concluded that the function

$$y(x)=x^{\frac{1-r_1\pm\sqrt{(r_1-1)^2-4r_3}}{2}}\,{}_1F_1\Big(\frac{r_1-1\mp\sqrt{(r_1-1)^2-4r_3}}{2},1\mp\sqrt{(r_1-1)^2-4r_3}\,;\frac{r_2}{x}\Big) \tag{4.29}$$

satisfies the equation (4.26) if and only if $r_2\neq 0$. Subsequently the monic Bessel polynomials are representable in terms of the Laguerre polynomials and we have

$$\bar{B}_n^{(p)}(x)=\frac{(-1)^n n!(p+n)!}{(p+2n)!}\,x^n\,L_n^{(-p-2n-1)}\Big(\frac{2}{x}\Big)\ . \tag{4.30}$$

In this way, $N_n^{(p)}(x)$ can also be represented by the Laguerre polynomials so that

$$N_n^{(p)}(x)=n!\,x^n\,L_n^{(p-(2n+1))}\Big(\frac{1}{x}\Big)\,. \tag{4.31}$$

The latter relation is useful to generate a new definite integral for the Laguerre polynomials, because substituting (4.31) into (4.25) yields

$$\int_0^{\infty}x^{p-1}e^{-x}\left(L_n^{(p)}(x)\right)^2dx=\frac{1}{p}\frac{(n+p)!}{n!}\ . \tag{4.32}$$

4.4. Third finite class: Classical hypergeometric orthogonal polynomials with weight function $((ax+b)^2+(cx+d)^2)^{-p}\exp(q\arctan\frac{ax+b}{cx+d})$ on $(-\infty,\infty)$ [9]

As was expressed up to now, from the main equation

$$(Ax^2+Bx+C)y_n''(x)+(Dx+E)y_n'(x)-n((n-1)A+D)y_n(x)=0 \quad ; \; n\in\mathbf{Z}^+$$

one can extract six "infinite" and "finite" sequences of classical orthogonal polynomials. In this part, we study the last finite class of hypergeometric polynomials, which is "finitely" orthogonal with respect to the well-behaved weight function $((ax+b)^2+(cx+d)^2)^{-p}\exp(q\arctan(ax+b)/(cx+d))$ on $(-\infty,\infty)$. The foresaid function can be considered as an important statistical distribution too, because by having its explicit criterion, one can generalize the T sampling distribution and prove that it tends to the Normal distribution just like T- student distribution. Of course, the next chapter is devoted to this subject with more details. However, before studying the last finite case, we should introduce a special sub-case of the equation (4.1) for $\sigma(x)=1+x^2$ and $\tau(x)=(3-2p)x$ and show that the polynomials generated from this case are finitely orthogonal with respect to the positive measure $\rho(x,p)=(1+x^2)^{-(p-\frac{1}{2})}$ on the real interval $(-\infty,\infty)$. Hence, similar to previous cases, we see that the differential equation

$$(1+x^2)y_n''(x)+(3-2p)x\,y_n'(x)-n((n+2-2p)y_n(x)=0, \tag{4.33}$$

has a polynomial solution in the form [10]

$$I_n^{(p)}(x)=n!\sum_{k=0}^{[n/2]}(-1)^k\binom{p-1}{n-k}\binom{n-k}{k}(2x)^{n-2k} \quad . \tag{4.34}$$

By transforming (4.33) as a Sturm-Liouville equation and applying the technique that was applied for the first and second kind of finite classical orthogonal polynomials we arrive at

$$\int_{-\infty}^{\infty}(1+x^2)^{-(p-\frac{1}{2})}I_n^{(p)}(x)I_m^{(p)}(x)\,dx=0 \Leftrightarrow \begin{cases} m\neq n\ , \ \ p>N+1 \\ N=\max\{m,n\} \end{cases} \tag{4.35}$$

On the other hand, since (see [10])

$$I_n^{(p)}(x)=\frac{(-2)^n(p-n)_n}{(2p-2n-1)_n}(1+x^2)^{p-\frac{1}{2}}\frac{d^n((1+x^2)^{n-(p-\frac{1}{2})})}{dx^n} \quad ; \quad n=0,1,2,... \tag{4.36}$$

we finally get

Corollary 5. (Orthogonality Relation)

$$\int_{-\infty}^{\infty} \rho(x,p)\, I_n^{(p)}(x) I_m^{(p)}(x)\, dx = \left(\frac{n!\, 2^{2n-1}\sqrt{\pi}\,\Gamma^2(p)\Gamma(2p-2n)}{(p-n-1)\Gamma(p-n)\Gamma(p-n+1/2)\Gamma(2p-n-1)}\right) \delta_{n,m}$$

If and only if $m,n = 0,1,2,...,N < p-1$ and $\rho(x,p) = (1+x^2)^{-(p-1/2)}$.

It is important to point out that the weight function of above orthogonality relation corresponds to the usual T student distribution so that we have

$$T(x;n) = \left(\frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\,\Gamma(n/2)}\right) \rho\left(\frac{x}{\sqrt{n}}; \frac{n}{2}+1\right) \quad ; \quad -\infty < x < \infty \ , \ n \in \mathbf{N}. \tag{4.37}$$

Now, we here wish to say that it is not the end of the story of polynomials $I_n^{(p)}(x)$, rather, there is a much more extensive polynomial class which is finitely orthogonal on $(-\infty,\infty)$ and generalizes the sequence $I_n^{(p)}(x)$. For this purpose, first we consider the following polynomials

$$\begin{aligned} J_n^{(p,q)}(x;a,b,c,d) &= (-1)^n \left((ax+b)^2 + (cx+d)^2\right)^p \exp\left(-q \arctan\frac{ax+b}{cx+d}\right) \\ &\times \frac{d^n\left(\left((ax+b)^2+(cx+d)^2\right)^{n-p} \exp\left(q\arctan\frac{ax+b}{cx+d}\right)\right)}{dx^n}, \end{aligned} \tag{4.38}$$

such that $\det\begin{bmatrix} a & b \\ c & d\end{bmatrix} = ad-bc > 0$. After doing some computations on (4.38), the following differential equation will be derived for the polynomials (4.39)

$$\begin{aligned} ((ax+b)^2 + (cx+d)^2) y_n''(x) + \left(2(1-p)(a^2+c^2)x + q(ad-bc) + 2(1-p)(ab+cd)\right) y_n'(x) \\ - n(n+1-2p)(a^2+c^2) y_n(x) = 0. \end{aligned}$$

On the other hand, the equation (4.39) can be written in terms of the differential equation of hypergeometric function ${}_2F_1(a,b,c;x)$ [56]. So, after applying an appropriate change of variable one will reach this result that

$$\begin{aligned} J_n^{(p,q)}(x;a,b,c,d) &= (-1)^n ((ab+cd)+i(ad-bc))^n (n+1-2p)_n \times \\ \sum_{k=0}^{k=n} \binom{n}{k} &\left(\frac{a^2+c^2}{(ab+cd)+i(ad-bc)}\right)^k {}_2F_1\left(\left.\begin{matrix} k-n & p-n-iq/2 \\ & 2p-2n \end{matrix}\right| \frac{2(ad-bc)}{(ad-bc)-i(ab+cd)}\right) x^k \end{aligned} \tag{4.40}$$

in which $(a)_k = \Gamma(a+k)/\Gamma(a)$. This formula is an explicit representation for the defined polynomials (4.38).

Furthermore, the polynomials (4.40) have an important linear property. Using the Rodrigues representation (4.38) it can be obtained easily, because if $x = wt + v$ is considered in (4.38), then (4.41)

$$J_n^{(p,q)}(wt+v;a,b,c,d)=\left(-1/w\right)^n\left((awt+av+b)^2+(cwt+cv+d)^2\right)^p\exp(-q\arctan\frac{awt+av+b}{cwt+cv+d})$$
$$\times\frac{d^n(\left((awt+av+b)^2+(cwt+cv+d)^2\right)^{n-p}\exp(q\arctan\frac{awt+av+b}{cwt+cv+d}))}{dt^n}$$
$$=w^{-n}J_n^{(p,q)}(t;aw,av+b,cw,cv+d).$$

For instance, if $w=-1$ and $v=0$ in (4.41) then

$$J_n^{(p,q)}(-t;a,b,c,d)=(-1)^n J_n^{(p,q)}(t;-a,b,-c,d)\,. \tag{4.41.1}$$

Now, let us consider the differential equation (4.39) in the form of a self-adjoint equation

$$\begin{aligned}&(\left((ax+b)^2+(cx+d)^2\right)^{1-p}\exp(q\arctan\frac{ax+b}{cx+d})\ y_n'(x))'\\&=n(n+1-2p)(a^2+c^2)\left((ax+b)^2+(cx+d)^2\right)^{-p}\exp(q\arctan\frac{ax+b}{cx+d})y_n(x)\end{aligned} \tag{4.42}$$

and apply the Sturm-Liouville theorem for it on $(-\infty,\infty)$ to reach (4.43)

$$[\left((ax+b)^2+(cx+d)^2\right)^{1-p}\exp(q\arctan\frac{ax+b}{cx+d})(y_n'(x)y_m(x)-y_m'(x)y_n(x))]_{-\infty}^{\infty}=\left(\lambda_n-\lambda_m\right)$$
$$\times\int_{-\infty}^{\infty}\left((ax+b)^2+(cx+d)^2\right)^{-p}\exp(q\arctan\frac{ax+b}{cx+d})\,J_n^{(p,q)}(x;a,b,c,d)J_m^{(p,q)}(x;a,b,c,d)dx$$

where $\lambda_n=n(n+1-2p)$. Since $\max\deg\left(y_n'(x)y_m(x)-y_m'(x)y_n(x)\right)=n+m-1$ is valid in (4.43), if the conditions $p>N+1/2$, $N=\max\{m,n\}$ and $a,b,c,d,q\in\mathbf{R}$ hold, the left side of (4.43) tends to zero and consequently

$$\int_{-\infty}^{\infty}\left((ax+b)^2+(cx+d)^2\right)^{-p}\exp(q\arctan\frac{ax+b}{cx+d})\,J_n^{(p,q)}(x;a,b,c,d)J_m^{(p,q)}(x;a,b,c,d)dx=0$$
$$\Leftrightarrow\begin{cases}m\neq n,\ p>N+1/2,N=\max\{m,n\}\\ a,b,c,d,q\in\mathbf{R}\ ,\quad ad-bc>0\end{cases} \tag{4.44}$$

It just remains to compute the norm square value of the polynomials. To compute the norm 2, let us first replace the Rodrigues representation (4.38) in

$$\begin{aligned}&\int_{-\infty}^{\infty}\left((ax+b)^2+(cx+d)^2\right)^{n-p}\exp(q\arctan\frac{ax+b}{cx+d})(J_n^{(p,q)}(x;a,b,c,d))^2dx\\&=(-1)^n\int_{-\infty}^{\infty}J_n^{(p,q)}(x;a,b,c,d)\frac{d^n(\left((ax+b)^2+(cx+d)^2\right)^{n-p}\exp(q\arctan\frac{ax+b}{cx+d}))}{dx^n}.\end{aligned} \tag{4.45}$$

Then, by noting that $J_n^{(p,q)}(x;a,b,c,d)$ has the orthogonality property, integration by parts from (4.45) yields

$$\begin{aligned}&(-1)^n \int_{-\infty}^{\infty} J_n^{(p,q)}(x;a,b,c,d)\frac{d^n\left(\left((ax+b)^2+(cx+d)^2\right)^{n-p}\exp(q\arctan\frac{ax+b}{cx+d})\right)}{dx^n}\\&=\frac{n!(a^2+c^2)^n\Gamma(2p-n)}{\Gamma(2p-2n)}\int_{-\infty}^{\infty}\left((ax+b)^2+(cx+d)^2\right)^{n-p}\exp(q\arctan\frac{ax+b}{cx+d})\,dx.\end{aligned} \tag{4.46}$$

Now, suppose $\frac{ax+b}{cx+d}=t$ in the right hand side of (4.46). This simplifies it as (4.47)

$$\begin{aligned}&\int_{-\infty}^{\infty}\left((ax+b)^2+(cx+d)^2\right)^{n-p}\exp(q\arctan\frac{ax+b}{cx+d})=(ad-bc)^{2n+1-2p}\times\\&\int_{-\infty}^{\infty}(a-ct)^{2p-2n-2}(1+t^2)^{n-p}\exp(q\arctan)dt=(ad-bc)^{2n+1-2p}\int_{-\pi/2}^{\pi/2}(a\cos\theta-c\sin\theta)^{2p-2n-2}e^{q\theta}d\theta.\end{aligned}$$

Therefore, the orthogonality property of defined polynomials can be expressed as:

Corollary 6. If $W^{(p,q)}(x;a,b,c,d)=\left((ax+b)^2+(cx+d)^2\right)^{-p}\exp(q\arctan\frac{ax+b}{cx+d})$ is the weight function of the polynomials $J_n^{(p,q)}(x;a,b,c,d)$, then we have

$$\begin{aligned}&\int_{-\infty}^{\infty}W^{(p,q)}(x;a,b,c,d)\,J_n^{(p,q)}(x;a,b,c,d)J_m^{(p,q)}(x;a,b,c,d)dx\\&\qquad=\Big(\frac{n!(a^2+c^2)^n\Gamma(2p-n)}{(ad-bc)^{2p-2n-1}\Gamma(2p-2n)}\int_{-\pi/2}^{\pi/2}(a\cos\theta-c\sin\theta)^{2p-2n-2}e^{q\theta}d\theta\Big)\delta_{n,m}\end{aligned} \tag{4.48}$$

If and only if $m,n=0,1,2,...,N<p-1/2$, $q\in\mathbf{R}$ & $ad-bc>0$.

Note that (4.48) will be simplified more if one takes $2p$ as a natural number, because it is known that $\int(a\cos\theta-c\sin\theta)^m e^{q\theta}\,d\theta$ is analytically integrable if $m\in\mathbf{N}$. Hence, by knowing that

$$a\cos\theta-c\sin\theta=\frac{a+ic}{2}e^{i\theta}+\frac{a-ic}{2}e^{-i\theta}\quad \&\quad cis\theta=\cos\theta+i\sin\theta=e^{i\theta}, \tag{4.49}$$

we have (4.50)

$$\int_{-\pi/2}^{\pi/2}(a\cos\theta-c\sin\theta)^m e^{q\theta}d\theta=2^{-m}\sum_{k=0}^{k=m}\binom{m}{k}\frac{(a+ic)^{m-k}(a-ic)^k}{(m-2k)i+q}\left(e^{((m-2k)i+q)\frac{\pi}{2}}-e^{-((m-2k)i+q)\frac{\pi}{2}}\right)$$

The equality (4.50) implies that (4.50.1)

$$\int_{-\pi/2}^{\pi/2}(a\cos\theta-c\sin\theta)^{2m}e^{q\theta}d\theta=2^{-2m+1}(-1)^m\sinh(\frac{q\pi}{2})\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\frac{(a+ic)^{2m-k}(a-ic)^k}{(2m-2k)i+q}$$

and (4.50.2)

$$\int_{-\pi/2}^{\pi/2}(a\cos\theta-c\sin\theta)^{2m+1}e^{q\theta}d\theta=2^{-2m}i(-1)^m\cosh(\frac{q\pi}{2})\sum_{k=0}^{2m+1}(-1)^k\binom{2m+1}{k}\frac{(a+ic)^{2m+1-k}(a-ic)^k}{(2m+1-2k)i+q}$$

Thus, if $2p$ is a natural number in (4.48), the norm square value of the polynomials will explicitly be determined. For instance, set $c=0$, $d=1$, $p=\frac{m}{2}$ and $m\in\mathbf{N}$ to get

$$\int_{-\infty}^{\infty}(1+(ax+b)^2)^{\frac{-m}{2}}\exp(q\arctan(ax+b))(J_n^{(\frac{m}{2},q)}(x;a,b,0,1))^2\,dx$$

$$=\frac{a^{2n-1}n!(m-n-1)!(q-(1-(-1)^m)(q-1)/2)(e^{\frac{q\pi}{2}}-(-1)^m e^{\frac{-q\pi}{2}})}{(m-2n-1)(\prod_{k=0}^{[\frac{m}{2}]-(n+1)}q^2+(m-2n-2k-2)^2)}. \tag{4.51}$$

Here is a good position to propound a complete example of the above orthogonality property. Suppose the finite set $\{J_n^{(6,2)}(x;2,-1,0,1)\}_{n=0}^{n=5}$ is given. This set is orthogonal with respect to the measure $W^{(6,2)}(x;2,-1,0,1)=(2x^2-2x+1)^{-6}\exp(2\arctan(2x-1))$ on $(-\infty,\infty)$ and satisfies the orthogonality relation (4.51.1)

$$\int_{-\infty}^{\infty}\frac{\exp(2\arctan(2x-1))}{(2x^2-2x+1)^6}J_n^{(6,2)}(x;2,-1,0,1)J_m^{(6,2)}(x;2,-1,0,1)dx=\frac{2^{4n-5}n!(11-n)!(\sinh\pi)}{(11-2n)(\prod_{k=0}^{5-n}1+(5-n-k)^2)}\delta_{n,m}$$

if and only if $m,n\le 5$.

4.5. Application of defined polynomials in functions approximation and numerical integration

Usually, the finite sets of orthogonal polynomials are applied to the discrete orthogonal polynomials rather than continuous cases. In other words, there is a main difference equation in the form

$$\sigma(x)(\Delta\nabla y(x))+\tau(x)(\Delta y(x))-\lambda_n y(x)=0, \tag{4.52}$$

with $\Delta y(x)=y(x+1)-y(x)$; $\nabla y(x)=y(x)-y(x-1)$; $\sigma(x)=Ax^2+Bx+C$; $\tau(x)=Dx+E$ and $\lambda_n=n(n-1)A+nD$, that contains classical orthogonal polynomials of a discrete variable [57]. For instance, Hahn discrete polynomials [42]

$$Q_n(x,\alpha,\beta,N)={}_3F_2(-n,n+\alpha+\beta+1,-x,\alpha+1,-N;1),\qquad n=0,1,2,...,N \tag{4.53}$$

are finitely orthogonal with the known conditions $\alpha,\beta>-1$ or $N<-\alpha$, $N<-\beta$ and satisfy

$$\sum_{x=0}^{N}\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}Q_n(x)Q_m(x)=\frac{(-1)^n n!(\beta+1)_n(n+\alpha+\beta+1)_{N+1}}{(-N)_n N!(\alpha+1)_n(2n+\alpha+\beta+1)}\delta_{n,m}. \quad (4.53.1)$$

The Krawtchouk polynomials [57], Racah polynomials [57], Dual Hahn discrete polynomials [42] are some further cases that have been classified in the category of finite discrete hypergeometric orthogonal polynomials. According to orthogonal polynomials theory, it is obvious that any arbitrary function $f(x)$ can be expanded in terms of each mentioned polynomials, provided that $f(x)$ satisfies the Dirichlet conditions. Now, we would like to point out that the finite polynomial set $\{J_n^{(p,q)}(x;a,b,c,d)\}_{n=0}^{n=N}$; $N<p-1/2$ can similarly be applied for approximating the function $f(x)$. For this purpose, it is enough to consider the following finite approximation

$$f(x)\cong\sum_{n=0}^{N}C_n J_n^{(p,q)}(x;a,b,c,d) \quad ; \quad N<p-1/2\ ,\ q\in\mathbf{R}\ ,\ ad-bc>0\,. \quad (4.54)$$

According to Corollary 6

$$C_n=\frac{(ad-bc)^{2p-2n-1}\Gamma(2p-2n)}{n!(a^2+c^2)^n\Gamma(2p-n)\int_{-\pi/2}^{\pi/2}(a\cos\theta-c\sin\theta)^{2p-2n-2}\,e^{q\theta}d\theta} \quad (4.55)$$
$$\times\int_{-\infty}^{\infty}((ax+b)^2+(cx+d)^2)^{-p}\exp(q\arctan\frac{ax+b}{cx+d})J_n^{(p,q)}(x;a,b,c,d)\,f(x)\,dx.$$

Note that the integer $N<p-1/2$ is the maximum precision degree of the approximation (4.54). For example, in the polynomial set $\{J_n^{(4,1)}(x;1,0,0,1)\}$, the maximum precision degree is at most 3. The following example clarifies this matter by applying the Gram-Schmidt orthogonalization process [16].

Example 1. Let us compute $J_n^{(p,q)}(x;1,0,0,1)$ for $p=4$, $q=1$ and $n\le 3$. Therefore $S=\{J_n^{(4,1)}(x;1,0,0,1)\}_{n=0}^{n=3}=\{1,6x-1,20x^2-10x-3,24x^3-36x^2-12x+7\}$ is a finite orthogonal polynomial set with respect to the weight function $W^{(4,1)}(x;1,0,0,1)=(1+x^2)^{-4}\exp(Arctgx)$ on $(-\infty,\infty)$ and

$$\int_{-\infty}^{\infty}\frac{\exp(\arctan x)}{(1+x^2)^4}J_n^{(4,1)}(x;1,0,0,1)J_m^{(4,1)}(x;1,0,0,1)\,dx=\frac{n!(7-n)!(2\sinh\frac{\pi}{2})}{(7-2n)\prod_{k=0}^{3-n}(1+(6-2n-2k)^2)}\delta_{n,m}\Leftrightarrow m,n\le 3$$

For example by referring to the set S , $m=n=2$ in this relation gives

$$\int_{-\infty}^{\infty} \frac{\exp(\arctan x)}{(1+x^2)^4}(20x^2-10x-3)^2\,dx = 32\sinh\frac{\pi}{2}.$$

Now, by noting the above orthogonality relation and considering the members of set S, one can approximate a third degree polynomial for $f(x)$ as

$$f(x) \cong c_0 + c_1(6x-1) + c_2(20x^2-10x-3) + c_3(24x^3-36x^2-12x+7),$$

which eventually yields

$$\begin{aligned}(288\sinh\frac{\pi}{2})\, f(x) \cong {} & \left(629\int_{-\infty}^{\infty}(1+x^2)^{-4}\exp(\arctan x)\, f(x)dx\right) + \\ & \left(85\int_{-\infty}^{\infty}(1+x^2)^{-4}\exp(\arctan x)(6x-1)\, f(x)dx\right)(6x-1) + \\ & \left(9\int_{-\infty}^{\infty}(1+x^2)^{-4}\exp(\arctan x)(20x^2-10x-3)\, f(x)dx\right)(20x^2-10x-3) + \\ & \left(\int_{-\infty}^{\infty}(1+x^2)^{-4}\exp(\arctan x)(24x^3-36x^2-12x+7)\, f(x)dx\right)(24x^3-36x^2-12x+7).\end{aligned}$$

This approximation is exact for any arbitrary polynomial function of degree at most 3. Moreover, note that the monic polynomials set S can be obtained by applying the Gram-Schmidt orthogonalization process if the moments of weight function $W^{(4,1)}(x;1,0,0,1)$ exist on $(-\infty,\infty)$. In this case we have

$$\bar{P}_n(x) = (x-B_n)\,\bar{P}_{n-1}(x) - C_n\bar{P}_{n-2}(x) \quad \text{s.t. } \bar{P}_0(x)=1 \;;\; \bar{P}_1(x) = x - B_1 \quad \text{and}$$

$$B_n = \frac{\int_{-\infty}^{\infty} x(1+x^2)^{-4}\exp(\arctan x)(\bar{P}_{n-1}(x))^2\,dx}{\int_{-\infty}^{\infty}(1+x^2)^{-4}\exp(\arctan x)(\bar{P}_{n-1}(x))^2\,dx} \;;\; C_n = \frac{\int_{-\infty}^{\infty} x(1+x^2)^{-4}\exp(\arctan x)\bar{P}_{n-1}(x)\bar{P}_{n-2}(x)\,dx}{\int_{-\infty}^{\infty}(1+x^2)^{-4}\exp(\arctan x)(\bar{P}_{n-2}(x))^2\,dx}$$

After calculating the coefficients B_n, C_n for $n \le 3$, the monic orthogonal set S will be derived as

$$\bar{S} = \{1, x-\frac{1}{6}, x^2-\frac{1}{2}x-\frac{3}{20}, x^3-\frac{3}{2}x^2-\frac{1}{2}x+\frac{7}{24}\}\ .$$

One of the other advantages of defined polynomials is to estimate a type of definite integrals using Gauss integration theory. By employing the zeros of polynomials $J_n^{(p,q)}(x;a,b,c,d)$ as the interpolator points in the Lagrange interpolation, one can approximate the integrals $\int_{-\infty}^{\infty}((ax+b)^2+(cx+d)^2)^{-p}\exp(q\arctan\frac{ax+b}{cx+d})\, f(x)\,dx$ with the precision degree $N=2n-1$. This subject is clarified by the following example.

Example 2. By replacing the zeros of $J_2^{(4,1)}(x;1,0,0,1)=20x^2-10x-3$ in the Lagrange interpolation, a two-points approximation will be derived in the form

$$\int_{-\infty}^{\infty}\frac{\exp(\arctan x)}{(1+x^2)^4}f(x)\,dx\cong\frac{48}{10693}\sinh\frac{\pi}{2}\,((51+\sqrt{85})\,f(\frac{5-\sqrt{85}}{20})+(51-\sqrt{85})\,f(\frac{5+\sqrt{85}}{20}))$$

This approximation is precise for $f(x)=1,x,x^2,x^3$ and any linear combination of these elements. Anyway, it should be noted that the maximum precision degree of the numerical approximation $\int_{-\infty}^{\infty}((ax+b)^2+(cx+d)^2)^{-p}\exp(q\arctan\frac{ax+b}{cx+d})\,f(x)\,dx$ is: $N=2\{\max n\,|\,n<p-\frac{1}{2}\}-1$. For example if $p\in(7,9]$ then we have $N=5$. See also [6] and [7] in this regard. [6] is applied for the weight function of polynomials $I_n^{(p)}(x)$ and [7] for the weight function of $N_n^{(p)}(x)$.

4.6. A connection between infinite and finite classical orthogonal polynomials

The Rodrigues representation of the finite classical orthogonal polynomials are useful tool to find some limit relations between $J_n^{(p,q)}(x;a,b,c,d)$ and Hermite polynomials and also between $M_n^{(p,q)}(x)$ and Laguerre polynomials respectively. For this purpose, the following limits should first be considered

$$\lim_{p\to\infty}W^{(p,q)}(x;\frac{1}{\sqrt{p}},0,0,1)=\lim_{p\to\infty}(1+\frac{x^2}{p})^{-p}\exp(q\arctan\frac{x}{\sqrt{p}})=e^{-x^2},$$
$$\lim_{p\to\infty}(1+\frac{x^2}{p})^{n}=1. \tag{4.56}$$

Hence we have (4.57)

$$\lim_{p\to\infty}J_n^{(p,q)}(x;\frac{1}{\sqrt{p}},0,0,1)=\lim_{p\to\infty}\frac{(-1)^n}{W^{(p,q)}(x;p^{-1/2},0,0,1)}\times\frac{d^n((1+\frac{x^2}{p})^nW^{(p,q)}(x;\frac{1}{\sqrt{p}},0,0,1))}{dx^n}$$
$$=(-1)^ne^{+x^2}\frac{d^n(e^{-x^2})}{dx^n}=H_n(x).$$

Similarly, this technique can be applied to derive a limit relation between $M_n^{(p,q)}(x)$ and Laguerre polynomials. In this sense we have

$$M_n^{(p,q)}(\frac{t}{p})=(-1)^nt^{-q}(1+\frac{t}{p})^{p+q}\frac{d^n(t^{n+q}(1+t/p)^{n-p-q})}{dt^n}. \tag{4.58}$$

Now, taking limit as $p\to\infty$ yields

$$\lim_{p\to\infty}M_n^{(p,q)}(\frac{t}{p})=(-1)^nt^{-q}e^t\frac{d^n(t^{n+q}e^{-t})}{dt^n}=(-1)^n n!\,L_n^{(q)}(t). \tag{4.59}$$

At the end of this chapter we can summarize that generally there exist six sequences of classical orthogonal polynomials that are generated by the main differential equation (4.1). Also, the parameters A, B, C, D and E corresponding to each six equations of mentioned sequences determine their characteristics. The following table shows this matter.

Table(1)

Kind	Notation	A	B	C	D	E
1.Infinite	$P_n^{(\alpha,\beta)}(x)$	-1	0	1	$-\alpha-\beta-2$	$-\alpha+\beta$
2.Infinite	$L_n^{(\alpha)}(wx)$	0	1	0	$-w$	$\alpha+1$
3.Infinite	$H_n(wx+v/2w)$	0	0	1	$-2w^2$	$-v$
4. Finite	$M_n^{(p,q)}((w/v)x)$	w	v	0	$(-p+2)w$	$(q+1)v$
5. Finite	$N_n^{(p)}(wx)$	w	0	0	$(-p+2)w$	1
6. Finite	$J_n^{(p,q)}(x;a,b,c,d)$	a^*	b^*	c^*	d^*	e^*

where
$$a^* = a^2 + c^2 \ , \ \ b^* = 2(ab+cd) \ , \ \ c^* = b^2 + d^2 ,$$
$$d^* = 2(1-p)(a^2+c^2) \ , \ \ e^* = q(ad-bc) + 2(1-p)(ab+cd).$$

Moreover, the following table shows the general properties of these six classes such as shifted orthogonal polynomials and their weight function, kind of polynomials (Finite or Infinite), orthogonality interval, the distribution corresponding to the weight function and finally the conditions of parameters:

Table(2)

Shifted polynomials	Weight function	Kind, Interval, Distribution	Parameters conditions
1. $H_n(wx+\frac{v}{2w})$	$\exp(-x(v+w^2x))$	Infinite, $(-\infty,\infty)$ Normal	$\forall n, w\neq 0, v\in\mathbf{R}$
2*. $I_n^{(p)}(wx+v)$	$(1+(wx+v)^2)^{-(p-\frac{1}{2})}$	Finite, $(-\infty,\infty)$ T Sampling	$\max n < p-1$ $w\neq 0$, $v\in\mathbf{R}$,
2. $J_n^{(p,q)}(x;a,b,c,d)$	$W^{(p,q)}(x;a,b,c,d)$	Finite, $(-\infty,\infty)$ Generalized T	$\max n < (p-1)/2$ $w\neq 0, v\in\mathbf{R}, q\in\mathbf{R}$
3. $L_n^{(\alpha)}(wx)$	$x^\alpha e^{-wx}$	Infinite, $[0,\infty)$ Gamma	$\forall n, w>0$, $\alpha>-1$
4. $M_n^{(p,q)}(\frac{w}{v}x)$	$x^q(wx+v)^{-(p+q)}$	Finite, $[0,\infty)$ F sampling	$\max n < (p-1)/2$ $q>-1, w>0$, $v>0$
5. $N_n^{(p)}(wx)$	$x^{-p}e^{\frac{-1}{wx}}$	Finite, $[0,\infty)$ Inverse gamma	$\max n < (p-1)/2$ $w>0$,
6. $P_n^{(\alpha,\beta)}(x)$	$(1-x)^\alpha(1+x)^\beta$	Infinite, $[-1,1]$ Beta	$\forall n$, $\alpha>-1, \beta>-1$

Chapter 5

A Generalization of Student's *t*-distribution from the Viewpoint of Special Functions

5.1. Introduction

Student's *t*-distribution has found various applications in mathematical statistics. One of the main properties of the *t*-distribution is to converge to the normal distribution as the number of samples tends to infinity. In this chapter, by using a Cauchy integral we introduce a generalization of the *t*-distribution function with four free parameters and show that it converges to the normal distribution again. We provide a comprehensive treatment of mathematical properties of this new distribution. Moreover, since the Fisher *F*-distribution has a close relationship with the *t*-distribution, we also introduce a generalization of the *F*-distribution and prove that it converges to the chi-square distribution as the number of samples tends to infinity. Hence, we start our discussion again with the Pearson differential equation with a simpler form in comparison with (3.15.1), Sec. 3.3:

$$\frac{dW}{dx}=\frac{dx+e}{ax^2+bx+c}W(x)\ , \tag{5.1}$$

which is directly connected with classical orthogonal polynomials and defines their weight functions $W(x)$ [56]. The solution of equation (5.1) can be indicated as

$$W(x)=W\begin{pmatrix} d & & e \\ a & b & c \end{pmatrix}\Big|\, x\Bigg)=\exp\Big(\int\frac{dx+e}{ax^2+bx+c}dx\Big)\,, \tag{5.2}$$

where a,b,c,d,e are all real parameters. There are several special sub-cases of (5.2). One of them is the Beta distribution, which is usually represented by the integral

$$\int_C (L_1(t))^a (L_2(t))^b\, dt\,, \tag{5.3}$$

where $L_1(t)$ and $L_2(t)$ are linear functions, $a\,,b$ are *complex* numbers and C is an appropriate contour [61]. The Euler and Cauchy integrals [18] are two important sub-classes of Beta type integrals, which are often used in applied mathematics. The Euler integral is given by

$$\int_a^b (t-a)^{c-1}(t-b)^{d-1}dt=\frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)}\,(a+b)^{c+d-1}\quad (\mathrm{Re}c>0\ ,\ \mathrm{Re}d>0\ ,\ a>0\ ,\ b>0), \tag{5.4}$$

while the Cauchy integral is represented by the formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dt}{(a+it)^{c}(b-it)^{d}}=\frac{\Gamma(c+d-1)}{\Gamma(c)\Gamma(d)}(a+b)^{1-(c+d)}\ , \tag{5.5}$$

in which $i=\sqrt{-1}$, $\mathrm{Re}(c+d)>1$, $\mathrm{Re}\,a>0$ and $\mathrm{Re}\,b>0$. Note that in both relations (5.4) and (5.5) $\Gamma(a)=\int_0^{\infty}x^{a-1}e^{-x}dx$ denotes the Gamma function. The relation (5.5) is a suitable tool to compute some different looking definite integrals. For this purpose, we use the relation

$$\left(\frac{a-ib}{a+ib}\right)^{iq}=\exp(2q\arctan\frac{b}{a})\qquad (a,b,q\in\mathbf{R}), \tag{5.6}$$

which rewrites the complex left hand side in terms of the real right hand side. Consequently we have

$$(b-it)^{p+iq}(b+it)^{p-iq}=(b^2+t^2)^{p}\exp(2q\arctan\frac{t}{b}). \tag{5.7}$$

Now if (5.7) is substituted, then the integral (5.5) changes towards

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(b^2+t^2)^{p}\exp(2q\arctan\frac{t}{b})\,dt=\frac{\Gamma(-2p-1)}{\Gamma(-p+iq)\Gamma(-p-iq)}\ \ (2b)^{2p+1}. \tag{5.8}$$

The above integral plays a key role to introduce a generalization of the *t*-distribution.

5.2. A generalization of the *t*-distribution [4]

The Student *t*-distribution [69, 73] having the probability density function (pdf)

$$T\,(t\,,m)=\frac{\Gamma((m+1)/2)}{\sqrt{m\pi}\ \ \Gamma(m/2)}(1+\frac{t^{2}}{m})^{-(\frac{m+1}{2})}\qquad(-\infty<t<\infty\ ,\quad m\in\mathbf{N}) \tag{5.9}$$

is perhaps one of the most important distributions in the sampling problems of normal populations. According to a theorem in mathematical statistics, if $\overline{X}$ and S^2 are respectively the mean value and variance of a stochastic sample with the size m of a normal population having the expected value μ and variance σ^2, then the random variable $T=\dfrac{\overline{X}-\mu}{S/\sqrt{m}}$ has the probability density function (5.9) with $(m-1)$ degrees of freedom [73]. This theorem is used in the test of hypotheses and interval estimation theory when the size of the sample is small, for instance less than 30.

Now, by using (5.8) one can extend the pdf of the *t*-distribution. To do this task, we substitute $t \to \frac{t}{\sqrt{m}}$, $b=1$, $p=-\frac{m+1}{2}$ and $q \to \frac{q}{2}$ in (5.8) to get

$$\int_{-\infty}^{\infty} (1+\frac{t^2}{m})^{-(\frac{m+1}{2})} \exp(q \arctan \frac{t}{\sqrt{m}}) \ dt = \frac{\sqrt{m} \ \ 2^{1-m}\Gamma(m) \ \ \pi}{\Gamma(\frac{1+m+iq}{2})\Gamma(\frac{1+m-iq}{2})} \ . \tag{5.10}$$

Since the right hand side of (5.10) is an e*ven* function with respect to the variable q, we can take a linear combination and get accordingly (5.11)

$$\int_{-\infty}^{\infty} (1+\frac{t^2}{m})^{-(\frac{m+1}{2})} \left(\lambda_1 \exp(q \arctan \frac{t}{\sqrt{m}}) + \lambda_2 \exp(-q \arctan \frac{t}{\sqrt{m}}) \right) \ dt = \frac{(\lambda_1+\lambda_2)\sqrt{m} \ \ 2^{1-m}\Gamma(m) \ \ \pi}{\Gamma(\frac{1+m+iq}{2})\Gamma(\frac{1+m-iq}{2})} \ .$$

The above integral can be used to generalize (5.9) as (5.12)

$$T(t,m,q,\lambda_1,\lambda_2) = \frac{\Gamma(\frac{1+m+iq}{2})\Gamma(\frac{1+m-iq}{2})}{(\lambda_1+\lambda_2)\sqrt{m} \ \ 2^{1-m}\Gamma(m) \ \ \pi} (1+\frac{t^2}{m})^{-(\frac{m+1}{2})} \left(\lambda_1 \exp(q \arctan \frac{t}{\sqrt{m}}) + \lambda_2 \exp(-q \arctan \frac{t}{\sqrt{m}}) \right)$$

where $-\infty < t < \infty$, $m \in \mathbf{N}$, q is a complex number and $\lambda_1, \lambda_2 \geq 0$.
Note that $\lambda_1, \lambda_2 \geq 0$ is a necessary condition for (5.12), because the probability density function must always be positive. Also note that the normalizing constant

$$\Gamma(\frac{1+m+iq}{2})\Gamma(\frac{1+m-iq}{2}) / ((\lambda_1+\lambda_2)\sqrt{m} \ \ 2^{1-m}\,\Gamma(m) \ \pi)$$

of (5.12) is *real*, because the corresponding integrand is a *real* function on $(-\infty,\infty)$. It is clear that for $q=0$ in (5.12) the usual *t*-distribution is derived. Moreover, for $q=0$ the normalizing constant of distribution (5.12) is equal to the normalizing constant of the *t*-distribution. This fact can be proved by applying the Legendre duplication formula [18]

$$\Gamma(\frac{z}{2})\Gamma(\frac{z+1}{2}) = \frac{\sqrt{\pi}}{2^{z-1}}\Gamma(z) \, . \tag{5.13}$$

But, according to one of the basic theorems in sampling theory, $T(t,m)$ converges to the pdf of the standard normal distribution $N(t,0,1)$ as $m \to \infty$ [61, 73], that is

$$\lim_{m\to\infty} T(t,m) = N(t,0,1) \ . \tag{5.14}$$

Here we intend to show that this matter is also valid for the generalized distribution $T(t,m,q,\lambda_1,\lambda_2)$. To prove this claim, we use the dominated convergence theorem (DCT) [31] to the real sequence of functions

$$S_m^{(1)}(t,q,\lambda_1,\lambda_2)=(1+\frac{t^2}{m})^{-(\frac{m+1}{2})}\left(\lambda_1\exp(q\arctan\frac{t}{\sqrt{m}})+\lambda_2\exp(-q\arctan\frac{t}{\sqrt{m}})\right). \tag{5.15}$$

For every $m\in\mathbf{N}$ it is not difficult to see that

$$|S_m^{(1)}(t,q,\lambda_1,\lambda_2)|\le(\lambda_1+\lambda_2)\exp(|q|\frac{\pi}{2}) \qquad (t\in\mathbf{R}). \tag{5.16}$$

On the other hand, we have

$$\lim_{m\to\infty}(1+\frac{t^2}{m})^{-(\frac{m+1}{2})}(\lambda_1\exp(q\arctan\frac{t}{\sqrt{m}})+\lambda_2\exp(-q\arctan\frac{t}{\sqrt{m}}))=(\lambda_1+\lambda_2)\exp(-\frac{t^2}{2}). \tag{5.17}$$

Since the dominated convergence theorem states that if for a continuous and integrable function $g(x)$ we have $|f_m(x)|\le g(x)$, then

$$\lim_{m\to\infty}\int_a^b f_m(x)dx=\int_a^b\lim_{m\to\infty}f_m(x)dx\ , \tag{5.18}$$

by considering the limit relation (5.17) we obtain (5.19)

$$\lim_{m\to\infty}T(t,m,q,\lambda_1,\lambda_2)=\frac{\lim_{m\to\infty}(1+t^2/m)^{-(\frac{m+1}{2})}\left(\lambda_1\exp(q\arctan(t/\sqrt{m}))+\lambda_2\exp(-q\arctan(t/\sqrt{m}))\right)}{\int_{-\infty}^{\infty}\lim_{m\to\infty}(1+t^2/m)^{-(\frac{m+1}{2})}\left(\lambda_1\exp(q\arctan(t/\sqrt{m}))+\lambda_2\exp(-q\arctan(t/\sqrt{m}))\right)dt}$$
$$=\frac{(\lambda_1+\lambda_2)\exp(-t^2/2)}{\int_{-\infty}^{\infty}(\lambda_1+\lambda_2)\exp(-t^2/2)dt}=\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}t^2)=N(t,0,1)\ .$$

Remark 1. Taking the limit on both sides of (5.11) as $m\to\infty$, the following asymptotic relation is obtained for the Gamma function

$$\lim_{x\to\infty}\frac{\Gamma(x+iy)\Gamma(x-iy)}{2^{-(2x-1)}\sqrt{2x-1}\ \ \Gamma(2x-1)}=\frac{2}{\sqrt{2\pi}}. \tag{5.20}$$

To compute the expected value of the distribution given by the pdf (5.12) it is sufficient to consider the definite integral

$$\int_{-\infty}^{\infty} t\ (1+\frac{t^2}{m})^{-(\frac{m+1}{2})} \exp(q \arctan \frac{t}{\sqrt{m}})\ dt = \frac{\sqrt{m}\ \ 2^{1-m}\Gamma(m)\ \ \pi}{\Gamma(\frac{1+m+iq}{2})\Gamma(\frac{1+m-iq}{2})}(\frac{q\sqrt{m}}{m-1}), \tag{5.21}$$

which gives the expected value of (5.12) as

$$E\ [T] = (\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2})\frac{q\sqrt{m}}{m-1}\ . \tag{5.22}$$

On the other hand, since $E\ [1+T^2/m]$ can easily be computed, after some calculations, we get for the variance measure of (5.12)

$$Var\ [T] = E\ [T^2] - E^2[T] = \frac{m(q^2+m-1)}{(m-2)(m-1)} - \left(\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\right)^2 (\frac{mq^2}{(m-1)^2})\ . \tag{5.23}$$

It is valuable to point out that as expected $q = 0$ in (5.22) and (5.23) gives the expected value and variance of the usual *t*-distribution, respectively.
But it is known that the *t*-distribution has a close relationship with the Fisher *F*-distribution [56], defined by its pdf

$$F(x,m,k) = \frac{\Gamma((m+k)/2)(k/m)^{k/2}}{\Gamma(k/2)\ \ \Gamma(m/2)}\ x^{\frac{k}{2}-1}(1+\frac{k}{m}x)^{-(\frac{m+k}{2})}\quad (m,k \in \mathbf{N}\ ,\quad 0 < x < \infty), \tag{5.24}$$

where $x = t^2$ and $k = 1$ in (5.24). In other words we have

$$T(t,m) = F(t^2, m, 1)\ . \tag{5.25}$$

By referring to the above relation and the fact that the *t*-distribution was generalized by relation (5.12), it is now natural to generalize the pdf of the *F*-distribution (5.24) as follows

$$F(x,m,k,q,\lambda_1,\lambda_2) = Bx^{\frac{k}{2}-1}(1+\frac{k}{m}x)^{-(\frac{m+k}{2})}(\lambda_1 \exp(q \arctan\sqrt{\frac{k}{m}x}) + \lambda_2 \exp(-q \arctan\sqrt{\frac{k}{m}x})), \tag{5.26}$$

where

$$\frac{1}{B} = \int_0^{\infty} x^{\frac{k}{2}-1}(1+\frac{k}{m}x)^{-(\frac{m+k}{2})}(\lambda_1 \exp(q \arctan\sqrt{\frac{k}{m}x}) + \lambda_2 \exp(-q \arctan\sqrt{\frac{k}{m}x}))dx\ . \tag{5.26.1}$$

For $q = 0$, (5.26) is the usual *F*-distribution defined in (5.24).
According to the following theorem, the generalized function (5.26) converges to a special case of the Gamma distribution [73], defined by

$$G(x,\alpha,\beta) = \frac{\beta^{-\alpha}}{\Gamma(\alpha)} x^{\alpha-1} \exp(\frac{-x}{\beta})\qquad (\alpha,\beta > 0\ ,\quad 0 < x < \infty)\ . \tag{5.27}$$

Theorem 1. If the Gamma distribution is given by (5.27), then we have

$$\lim_{m\to\infty} F(x,m,k,q,\lambda_1,\lambda_2) = G(x,\alpha=\frac{k}{2},\beta=2) = \chi_k^2$$

where χ_k^2 denotes the pdf of the chi-square distribution.

Proof. Let us define the sequence

$$S_m^{(2)}(x,k,q,\lambda_1,\lambda_2) = x^{\frac{k}{2}-1}(1+\frac{k}{m}x)^{-(\frac{m+k}{2})}(\lambda_1 \exp(q \arctan\sqrt{\frac{k}{m}x}) + \lambda_2 \exp(-q \arctan\sqrt{\frac{k}{m}x})).$$

It is easy to show that

$$|S_m^{(2)}(x,k,q,\lambda_1,\lambda_2)| \le (\lambda_1+\lambda_2)\, x^{\frac{k}{2}-1} \exp(|q|\frac{\pi}{2}) \quad (x\in[0,\infty)\ ,\ \ k\in\mathbf{N})\,, \quad (5.28)$$

and

$$\lim_{m\to\infty} S_m^{(2)}(x,k,q,\lambda_1,\lambda_2) = (\lambda_1+\lambda_2)\, x^{\frac{k}{2}-1}\exp(-x/2)\,. \quad (5.29)$$

Therefore, according to the DCT we have (5.30)

$$\lim_{m\to\infty} F(x,m,k,q,\lambda_1,\lambda_2) = \frac{\lim_{m\to\infty} S_m^{(2)}(x,k,q,\lambda_1,\lambda_2)}{\int_0^\infty \lim_{m\to\infty} S_m^{(2)}(x,k,q,\lambda_1,\lambda_2)\,dx} = \frac{x^{(k/2)-1}\exp(-x/2)}{\int_0^\infty x^{(k/2)-1}\exp(-x/2)\,dx} = G(x,\frac{k}{2},2)\,.$$

Moreover, it is not difficult to show that

$$F(t^2,m,1,q,\lambda_1,\lambda_2) = T(t,m,q,\lambda_1,\lambda_2)\,. \quad (5.31)$$

5.3. Some particular sub-cases of the generalized *t* (and *F*) distribution

In this section, we intend to study some symmetric and asymmetric sub-cases of the generalized distributions (5.12) and (5.26).

5.3.1. A symmetric generalization of the *t*-distribution, the case $q = ib$ and $\lambda_1 = \lambda_2 = 1/2$

If the special case $q = ib$ and $\lambda_1 = \lambda_2 = 1/2$ is considered in (5.12), then (5.32)

$$T(t,m,ib,\frac{1}{2},\frac{1}{2}) = T_S(t,m,b) = \frac{\Gamma(\frac{1+m+b}{2})\Gamma(\frac{1+m-b}{2})}{\sqrt{m}\ \ 2^{1-m}\Gamma(m)\ \ \pi}\ (1+\frac{t^2}{m})^{-(\frac{m+1}{2})}\cos(b \arctan\frac{t}{\sqrt{m}})$$

is a symmetric generalization of the ordinary *t*-distribution in which $-1 \le b \le 1$.

The usual pdf of the *t*-distribution is obviously derived by $b=0$ in (5.32). Note that according to the Legendre duplication formula we will reach the normalizing constant of the *t*-distribution if $b=0$ is considered in (5.32). In other words, we have

$$b=0 \Rightarrow \frac{\Gamma^2((1+m)/2)}{\sqrt{m}\,2^{1-m}\Gamma(m)\pi}=\frac{\Gamma((1+m)/2)}{\sqrt{m\pi}\,\Gamma(m/2)}. \tag{5.33}$$

Also note that the parameter b in the generalized distribution (5.32) must belong to [-1,1], because the probability density function must always be positive and therefore we ought to have $\cos(b\arctan(t/\sqrt{m}))\geq 0$. On the other hand, since for $-\frac{\pi}{2}\leq\theta\leq\frac{\pi}{2}$ we have $\cos\theta\geq 0$, therefore to prove $\cos(b\arctan(t/\sqrt{m}))\geq 0$ it is sufficient to show that

$$-1\leq b\leq 1 \Leftrightarrow b\arctan\frac{t}{\sqrt{m}}\subseteq[-\frac{\pi}{2},\frac{\pi}{2}] \quad (t\in\mathbf{R}\ ,\quad m\in\mathbf{N})\,. \tag{5.34}$$

For this purpose, let us define the sequence $U_m(t)=\arctan\frac{t}{\sqrt{m}}$ to get (5.35)

$$U'_m(t)=(\frac{1}{\sqrt{m}})/(1+\frac{t^2}{m})>0 \Rightarrow [\min U_m(t),\max U_m(t)]=[U_m(-\infty),U_m(\infty)]=[-\frac{\pi}{2},\frac{\pi}{2}]\,.$$

Now if we demand the sequence $bU_m(t)=b\arctan\frac{t}{\sqrt{m}}$ to belong to $[-\frac{\pi}{2},\frac{\pi}{2}]$, it is clear that we must have $|b|\leq 1$, which proves (5.34). The following figures clarify this matter for $b\in[-1,1]$ and $b\notin[-1,1]$ in the interval (-10,10).

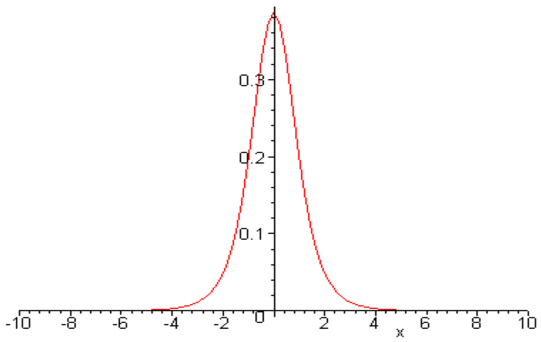

Figure 1: $b=1/2$, $m=4$

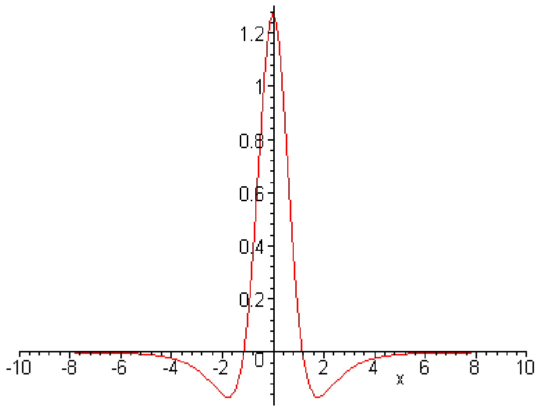

Figure 2: $b=3$, $m=4$

Fig. 1 shows the pdf $T_S(t,4,1/2)$ with normalizing constant $35\sqrt{2}/128$ and Fig. 2 shows the non-positive function $T_S(t,4,3)=\ (4/\pi)(1+t^2/4)^{-5/2}\cos(3\arctan(t/2))$ in the interval $(-10,10)$. As the above figures show, the generalized distribution (5.32) is symmetric, i.e.

$$T_S(-t,m,b)=T_S(t,m,b) \quad (t\in\mathbf{R})\,. \tag{5.36}$$

Moreover, according to (5.22) and (5.23) the expected value and variance of distribution (5.32) take the forms

$$E[t]=0 \quad , \quad Var[t]=\frac{m(m-1-b^2)}{(m-1)(m-2)} . \tag{5.37}$$

Clearly $b=0$ in these relations gives the expected value and variance of the *t*-distribution.

Theorem 2. $T_S(t,m,q)$ converges to $N(t,0,1)$ as $m\to\infty$.

Proof. If the sequence $S_m^{(3)}(t,b)=\cos(b\arctan\frac{t}{\sqrt{m}})(1+\frac{t^2}{m})^{-(\frac{m+1}{2})}$ is considered, then one can show that

$$|S_m^{(3)}(t,b)|=|\cos(b\arctan\frac{t}{\sqrt{m}})||(1+\frac{t^2}{m})^{-(\frac{m+1}{2})}|\le 1 \quad (t\in\mathbf{R}). \tag{5.38}$$

Consequently we have

$$\begin{aligned}\lim_{m\to\infty}T_S(t,m,b)&=\lim_{m\to\infty}\frac{\cos(b\arctan(t/\sqrt{m}))(1+t^2/m)^{-(\frac{m+1}{2})}}{\int_{-\infty}^{\infty}\cos(b\arctan(t/\sqrt{m}))(1+t^2/m)^{-(\frac{m+1}{2})}dt}\\ &=\frac{\lim_{m\to\infty}\cos(b\arctan(t/\sqrt{m}))(1+t^2/m)^{-(\frac{m+1}{2})}}{\int_{-\infty}^{\infty}\lim_{m\to\infty}\cos(b\arctan(t/\sqrt{m}))(1+t^2/m)^{-(\frac{m+1}{2})}dt}=\frac{\exp(-t^2/2)}{\int_{-\infty}^{\infty}\exp(-t^2/2)\;dt}=N(t,0,1) .\end{aligned} \tag{5.39}$$

By referring to (5.26), we can now define the generalized *F*-distribution corresponding to the first given sub-case as follows (5.40)

$$F(x,m,k,ib,\frac{1}{2},\frac{1}{2})=F_1(x,m,k,b)=B\,x^{\frac{k}{2}-1}(1+\frac{k}{m}x)^{-(\frac{m+k}{2})}\cos(b\arctan\sqrt{\frac{k}{m}x}) \quad (-1\le b\le 1)$$

where

$$\begin{aligned}\frac{1}{B}&=\int_0^{\infty}x^{\frac{k}{2}-1}(1+\frac{k}{m}x)^{-(\frac{m+k}{2})}\cos(b\arctan\sqrt{\frac{k}{m}x})\,dx\\ &=2(\frac{m}{k})^{k/2}\int_0^{\pi/2}\sin^{(k-1)}\theta\cos^{(m-1)}\theta\cos(b\theta)d\theta .\end{aligned} \tag{5.40.1}$$

Theorem 3. $\mathrm{F}_1(x,m,k,b)$ converges to the chi-square distribution as $m\to\infty$.

Proof. We define the sequence $S_m^{(4)}(x,k,q)=x^{\frac{k}{2}-1}(1+\frac{k}{m}x)^{-(\frac{m+k}{2})}\cos(b\arctan\sqrt{\frac{k}{m}x})$ to get

$$|S_m^{(4)}(x,k,b)|\le x^{\frac{k}{2}-1} \quad (x\in[0,\infty)) \;,\; k\in\mathbf{N} \;,\; |b|<1) . \tag{5.41}$$

Hence, according to DCT we find out that

$$\lim_{m\to\infty} F_1(x,m,k,b) = \lim_{m\to\infty} \frac{S_m^{(4)}(x,k,b)}{\int_0^\infty S_m^{(4)}(x,k,b)\ \ dx} = \frac{x^{(k/2)-1}\exp(-x/2)}{\int_0^\infty x^{(k/2)-1}\exp(-x/2)\ \ dx} = G(x,\frac{k}{2},2)\,. \quad (5.42)$$

It is not difficult to verify that the generalized distributions $T_S(t,m,b)$ and $F_1(x,m,k,b)$ are related to each other as follows

$$F_1(t^2,m,1,b) = T_S(t,m,b)\,. \qquad (5.43)$$

Remark 2. Here is a good position to return to previous chapter and remember that the weight function of orthogonal polynomials $M_n^{(p,q)}(x)$ corresponds to the ordinary F-distribution such that if $n=0$ is considered in (4.11), then an integral is derived that corresponds to the F distribution.

5.3.2. An asymmetric generalization of the *t*-distribution, the case $\lambda_2 = 0$

Again let us come back to the chapter 4 and consider the weight function of finite orthogonal polynomials $J_n^{(p,q)}(x\,;a,b,c,d)$, i.e. (5.44)

$$W_n^{(p,q)}(x;a,b,c,d) = ((ax+b)^2+(cx+d)^2)^{-p}\exp(q\arctan\frac{ax+b}{cx+d}) \qquad (-\infty < x < \infty)\,,$$

where *a, b, c, d, p, q* are all real parameters. This function is a sub-case of the Pearson distribution (5.2), because the logarithmic derivative of (5.44) is a rational function. For convenience, if $a = \frac{1}{\sqrt{m}}$, $b=0$, $c=0$, $d=1$ and $p = -\frac{m+1}{2}$ $(m\in\mathbf{N})$ is selected in (5.44) then

$$W^{(-\frac{m+1}{2},q)}(t\ \ ;\frac{1}{\sqrt{m}},0,0,1) = (1+\frac{t^2}{m})^{-(\frac{m+1}{2})}\exp(q\arctan\frac{t}{\sqrt{m}}) \quad (m\in\mathbf{N},\, q\in\mathbf{R})\,. \quad (5.45)$$

On the other hand since

$$\int_{-\infty}^{\infty}(1+\frac{t^2}{m})^{-(\frac{m+1}{2})}\exp(q\arctan\frac{t}{\sqrt{m}})\,dt = \sqrt{m}\int_{-\pi/2}^{\pi/2} e^{q\theta}\cos^{(m-1)}\theta\,d\theta\,, \qquad (5.46)$$

we have

$$\int_{-\pi/2}^{\pi/2} e^{q\theta}\cos^{(m-1)}\theta\ \ d\theta = \frac{(m-1)!(q-(\frac{1+(-1)^m}{2})(q-1))\ \ (e^{\frac{q\pi}{2}}+(-1)^m e^{\frac{-q\pi}{2}})}{\prod_{k=0}^{\lfloor (m-1)/2\rfloor}(q^2+(m-2k-1)^2)}\,. \qquad (5.47)$$

Therefore, an asymmetric generalization of the *t*-distribution may be defined as

$$T_A(t,m,q) = K\,(1+\frac{t^2}{m})^{-(\frac{m+1}{2})}\exp(q\arctan\frac{t}{\sqrt{m}}) \quad (-\infty < t < \infty,\ \ m \in \mathbf{N},\ \ q \in \mathbf{R}) \qquad (5.48)$$

where

$$K = \frac{\prod_{k=0}^{\lfloor (m-1)/2 \rfloor} (q^2 + (m-2k-1)^2)}{\sqrt{m}(m-1)!(q-(\frac{1+(-1)^m}{2})(q-1))\ \ (e^{\frac{q\pi}{2}} + (-1)^m e^{\frac{-q\pi}{2}})}. \qquad (5.48.1)$$

The distribution (5.48) with normalizing constant given by (5.48.1) was defined in [2] based on this particular approach. But we can still modify and simplify it. To do this task, we set $\lambda_2 = 0$ in (5.12) to get

$$T_A(t,m,q) = \frac{\Gamma(\frac{1+m+iq}{2})\Gamma(\frac{1+m-iq}{2})}{\sqrt{m}\ \ 2^{1-m}\,\Gamma(m)\ \ \pi}\ (1+\frac{t^2}{m})^{-(\frac{m+1}{2})}\exp(q\arctan\frac{t}{\sqrt{m}})\ , \qquad (5.49)$$

which is an explicit representation for the asymmetric distribution (5.48). For the latter distribution, we clearly have

$$T_A(-t,m,q) = T_A(t,m,-q)\ . \qquad (5.49.1)$$

The asymmetry of distribution (5.48) (or (5.49)) is shown by Fig. 3 and 4 for specific values of q and m .

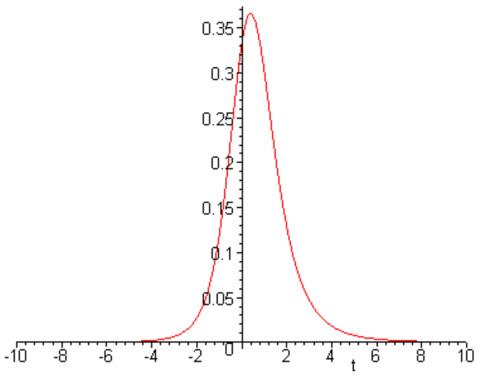

Figure 3: $q = 1$, $m = 4$

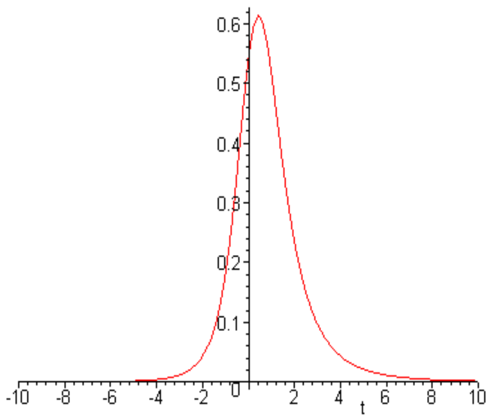

Figure 4: $q = 1$, $m = 3$

According to (5.48) and (5.48.1), the explicit definitions of the two mentioned figures have respectively the forms

$$\text{Fig. 3:}\quad T_A(t,4,1) = \frac{5}{6\ \ \cosh(\pi/2)}(1+\frac{t^2}{4})^{\frac{-5}{2}} e^{\arctan\frac{t}{2}}$$

$$\text{Fig. 4:}\quad T_A(t,3,1) = \frac{5\sqrt{3}}{12\ \ \sinh(\pi/2)}(1+\frac{t^2}{3})^{-2} e^{\arctan\frac{t}{\sqrt{3}}}$$

Now, the following statements (A1 to A5) collect the properties of the asymmetric distribution (5.48) (or (5.49)).

A1) The expected value and variance of (5.49) are respectively represented by

$$E[t]=\frac{q\sqrt{m}}{m-1} \quad , \qquad Var[t]=\frac{m(q^2+(m-1)^2)}{(m-2)(m-1)^2} \ . \tag{5.50}$$

$q=0$ in these relations gives the expected value and variance of the *t*-distribution.

A2) $T_A(t,m,q)$ converges to $N(t,0,1)$ as $m\to\infty$.
The proof is similar to the first case if one chooses $\lambda_2=0$ and $\lambda_1=1$ in the defined sequence $S_m^{(1)}(t,q,\lambda_1,\lambda_2)$.

A3) By the definition (5.26) and considering the case $\lambda_2=0$ we can define

$$F(x,m,k,q,\lambda_1,0)=F_2(x,m,k,q)=D\,x^{\frac{k}{2}-1}(1+\frac{k}{m}x)^{-(\frac{m+k}{2})}\exp(q\arctan\sqrt{\frac{k}{m}x}\,) \tag{5.51}$$

$$(q\in\mathbf{R}\ ,\ \ m,k\in\mathbf{N}\ ,\ \ 0<x<\infty)\ ,$$

where (5.51.1)

$$\frac{1}{D}=\int_0^\infty x^{\frac{k}{2}-1}(1+\frac{k}{m}x)^{-(\frac{m+k}{2})}\exp(q\arctan\sqrt{\frac{k}{m}x})\ dx=2(\frac{m}{k})^{k/2}\int_0^{\pi/2}\sin^{(k-1)}\theta\ \ \cos^{(m-1)}\theta\ \ e^{q\theta}d\theta\,.$$

A4) $F_2(x,m,k,q)$ converges to the chi-square distribution as $m\to\infty$.
The proof is similar to the proof of Theorem 1 when $\lambda_2=0$ and $\lambda_1=1$.

A5) The distributions $F_2(x,m,k,q)$ and $T_A(t,m,q)$ are related to each other by

$$F_2(t^2,m,1,q)=T_A(t,m,q)\,. \tag{5.52}$$

Remark 3. There is another symmetric generalization of the *t*-distribution when we set $\lambda_1=\lambda_2$ in (5.12). Its pdf is given as (5.53)

$$T(-t,m,q,\lambda_1,\lambda_1)=T(t,m,q,\lambda_1,\lambda_1)=\frac{\Gamma(\frac{1+m+iq}{2})\Gamma(\frac{1+m-iq}{2})}{\sqrt{m}\ \ 2^{1-m}\Gamma(m)\ \ \pi}(1+\frac{t^2}{m})^{-(\frac{m+1}{2})}\cosh(q\arctan\frac{t}{\sqrt{m}})\ .$$

Therefore, at the end of this chapter, we in fact considered the three following particular sub-cases of the general distribution (5.12):

a) $q=ib$ and $\lambda_1=\lambda_2=1/2$; symmetric case
b) $\lambda_2=0$; asymmetric case
c) $\lambda_1=\lambda_2$; symmetric case

Chapter 6

A generic polynomial solution for the differential equation of hypergeometric type and six sequences of orthogonal polynomials related to it

6.1. Introduction

In this chapter, we will present a generic formula for the polynomial solutions of the well-known differential equation of hypergeometric type

$$(ax^2 + bx + c)y_n''(x) + (dx + e)y_n'(x) - n(d + (n-1)a)y_n(x) = 0\,,$$

and show that all the three infinite classical orthogonal polynomial families as well as three finite orthogonal polynomial families, investigated in chapter 4, can be identified as special cases of this derived polynomial sequence [3]. We will also present some general properties of the mentioned sequence. For this purpose, we should reconsider the differential equation

$$\sigma(x)y_n''(x) + \tau(x)y_n'(x) - \lambda_n y_n(x) = 0\,, \qquad (6.1)$$

in which, as before, $\sigma(x) = ax^2 + bx + c$ is a polynomial of degree at most 2, $\tau(x) = dx + e$ is a polynomial of degree 1 and $\lambda_n = n(n-1)a + nd$ is the eigenvalue parameter depending on $n = 0,1,2,...$, and suppose that the polynomial solution of (6.1) is denoted by $P_n\left(\begin{matrix} & d & e \\ a & b & c \end{matrix}\middle| x\right)$.

So far extensive research has been done on equation (6.1) and its polynomial solutions. In 1929 Bochner [21] classified the polynomial solutions of (6.1) and showed that the only polynomial systems up to a linear change of variable arising as eigenfunctions of the differential equation (6.1) are (see also [14])

Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$, $\big(\alpha,\beta,\alpha+\beta+1 \notin \{-1,-2,...\}\big)$
Laguerre polynomials $\{L_n^{(\alpha)}(x)\}_{n=0}^{\infty}$, $\big(\alpha \notin \{-1,-2,...\}\big)$
Hermite polynomials $\{H_n(x)\}_{n=0}^{\infty}$
Bessel polynomials $\{B_n^{(\alpha,\beta)}(x)\}_{n=0}^{\infty}$, $\big(\alpha \notin \{0,-1,-2,...\}$ and $\beta \neq 0\big)$.

Then, in 1988, Nikiforov and Uvarov [56] gave some general properties of $P_n\left(\begin{matrix} & d & e \\ a & b & c \end{matrix}\middle| x\right)$, such as a generating function for the polynomials, a Cauchy integral representation and so on in terms of the given $\sigma(x)$ and $\tau(x)$. Of course their approach is based on the Rodrigues representation of the polynomials and is not expressed in an explicit form, the task that we will do in this chapter. Some other approaches in this regard are [14, 44, 72]. But before deriving a generic solution for (6.1), we should introduce an algebraic identity, which is easy to prove but important.

6.1.1. An algebraic identity

If a,b and C_k $(k=0,1,...,n)$ are real numbers then

$$\sum_{k=0}^{n} C_k (ax+b)^k = \sum_{k=0}^{n}\left(\sum_{i=0}^{n-k}\binom{n-i}{k} b^{n-i} C_{n-i}\right)\left(\frac{ax}{b}\right)^k = \sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{n-i}{k-i} b^{k-i} C_{n-i}\right)(ax)^{n-k}. \quad (6.2)$$

For instance, if we set $C_k = \dfrac{(-n)_k (p_2)_k (p_3)_k ...(p_m)_k}{(q_1)_k (q_2)_k ...(q_{m-1})_k \, k!}$ such that $(r)_k = \prod_{i=0}^{k-1} r+i$ then we get (6.3)

$$\begin{aligned} &{}_mF_{m-1}\left(\begin{matrix} -n, & p_2, ..., p_m \\ q_1, & q_2, ..., q_{m-1}\end{matrix} \,\middle|\, ax+b\right) \\ &= \frac{(-b)^n (p_2)_n ...(p_m)_n}{(q_1)_n (q_2)_n ...(q_{m-1})_n} \sum_{k=0}^{n}\binom{n}{k} {}_mF_{m-1}\left(\begin{matrix} k-n, & 1-q_1-n, ..., 1-q_{m-1}-n \\ 1-p_2-n, & 1-p_3-n, ..., 1-p_m-n \end{matrix} \,\middle|\, \frac{1}{b}\right)\left(\frac{ax}{b}\right)^k \\ &= \frac{(-ax)^n (p_2)_n ...(p_m)_n}{(q_1)_n (q_2)_n ...(q_{m-1})_n} \sum_{k=0}^{n}\binom{n}{k} {}_mF_{m-1}\left(\begin{matrix} -k, & 1-q_1-n, ..., 1-q_{m-1}-n \\ 1-p_2-n, & 1-p_3-n, ..., 1-p_m-n \end{matrix} \,\middle|\, \frac{1}{b}\right)\left(\frac{b}{ax}\right)^k \end{aligned}$$

where ${}_pF_q\left(\begin{matrix} a_1, & a_2, ..., a_p \\ b_1, & b_2, ..., b_q\end{matrix} \,\middle|\, x\right) = \sum_{k=0}^{\infty} \dfrac{(a_1)_k (a_2)_k ...(a_p)_k}{(b_1)_k (b_2)_k ...(b_q)_k} \dfrac{x^k}{k!}$ denotes the generalized hypergeometric function of order (*p*, *q*) (see e.g. [43], Chapter 2). Note that to compute the relations (6.3) we have generally used the two identities

$$\begin{cases} \Gamma(r+k) = \Gamma(r)(r)_k \\ \Gamma(r-k) = \dfrac{\Gamma(r)(-1)^k}{(1-r)_k} \end{cases} \Rightarrow \; (r)_{n-i} = \frac{(-1)^i (r)_n}{(1-r-n)_i} \quad ; \; r \in \mathbf{R} \; ; \; k,n,i \in \mathbf{Z}^+. \qquad (6.4)$$

An interesting case takes place for (6.3) when $m=2$ and $a=b=1$. In this case we have

$$\begin{aligned} {}_2F_1\left(\begin{matrix} -n & p_2 \\ & q_1 \end{matrix} \,\middle|\, x+1\right) &= \frac{(-1)^n (p_2)_n}{(q_1)_n} \sum_{k=0}^{n}\binom{n}{k} {}_2F_1\left(\begin{matrix} k-n & 1-q_1-n \\ & 1-p_2-n\end{matrix} \,\middle|\, 1\right) x^k \\ &= \frac{(-x)^n (p_2)_n}{(q_1)_n} \sum_{k=0}^{n}\binom{n}{k} {}_2F_1\left(\begin{matrix} -k & 1-q_1-n \\ & 1-p_2-n\end{matrix} \,\middle|\, 1\right) x^{-k} \end{aligned} \qquad (6.5)$$

which according to Gauss' identity (introduced in chapter 3, formula (3.38))

$$ {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \Big| 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \ , $$

simplifies to

$$ {}_2F_1\left(\begin{matrix} -n & p_2 \\ & q_1 \end{matrix} \ | \ x+1\right) = \frac{(q_1-p_2)_n}{(q_1)_n}\ {}_2F_1\left(\begin{matrix} -n & p_2 \\ 1-q_1+p_2-n & \end{matrix} \ | -x\right) $$

$$ = \frac{(-x)^n (p_2)_n}{(q_1)_n}\ {}_2F_1\left(\begin{matrix} -n & q_1-p_2 \\ 1-p_2-n & \end{matrix} \ | -\frac{1}{x}\right). \qquad (6.6) $$

The first identity of (6.6) can be found in [13], (15.3.6) and the second identity in (6.6) is a special case of [13], (15.3.7) for integer upper parameter.
Now it is a good position to propound the main theorem of finding a generic polynomial solution for equation (6.1).

6.2. The main Theorem. The monic polynomial solution of the differential equation

$$ (ax^2+bx+c)y_n''(x)+(dx+e)y_n'(x)-n((n-1)a+d)y_n(x)=0 \quad ; \quad n\in\mathbf{Z}^+ \qquad (6.7) $$

is given by the formula

$$ y_n(x) = \overline{P}_n\left(\begin{matrix} d & e & \\ a & b & c \end{matrix} \Big| \ x\right) = \sum_{k=0}^{n} \binom{n}{k} G_k^{(n)}(a,b,c,d,e)\, x^k \ , \qquad (6.8) $$

where

$$ G_k^{(n)} = \left(\frac{2a}{b+\sqrt{b^2-4ac}}\right)^{k-n} {}_2F_1\left(\begin{matrix} k-n & \dfrac{2ae-bd}{2a\sqrt{b^2-4ac}}+1-\dfrac{d}{2a}-n \\ 2-d/a-2n & \end{matrix} \Bigg| \ \frac{2\sqrt{b^2-4ac}}{b+\sqrt{b^2-4ac}}\right). $$

For $a=0$ the equality can be adapted by limit considerations and gives (6.8) in the form

$$ G_k^{(n)}(0,b,c,d,e) = \lim_{a\to 0} G_k^{(n)}(a,b,c,d,e) = \left(\frac{b}{c}\right)^{k-n} {}_2F_0\left(\begin{matrix} k-n, & \dfrac{cd-be}{b^2}+1-n \\ & - \end{matrix} \Bigg| \ \frac{b^2}{cd}\right), $$

which is valid for $c,d\neq 0$, leading to

$$ \overline{P}_n\left(\begin{matrix} d & e & \\ 0 & b & c \end{matrix} \Big| \ x\right) = \left(\frac{b}{d}\right)^n \left(\frac{eb-cd}{b^2}\right)_n \ {}_1F_1\left(\begin{matrix} -n \\ \dfrac{eb-cd}{b^2} \end{matrix} \Bigg| -\frac{d}{b}x-\frac{cd}{b^2}\right). $$

Finally for $a=b=0$ and $d\neq 0$ (6.8) is transformed to

$$ \overline{P}_n\left(\begin{matrix} d & e & \\ 0 & 0 & c \end{matrix} \Big| \ x\right) = \lim_{\substack{a\to 0 \\ b\to 0}} \overline{P}_n\left(\begin{matrix} d & e & \\ a & b & c \end{matrix} \Big| \ x\right) = \left(x+\frac{e}{d}\right)^n \ {}_2F_0\left(\begin{matrix} -\dfrac{n}{2}, & -\dfrac{n-1}{2} \\ & - \end{matrix} \Bigg| \ \frac{2cd}{(dx+e)^2}\right). $$

Proof. Consider the differential equation (6.7) and suppose that $x = pt + q$. Hence it will change to

$$(t^2 + \frac{2aq+b}{ap}t + \frac{aq^2+bq+c}{ap^2})\frac{d^2y}{dt^2} + (\frac{d}{a}t + \frac{e+dq}{pa})\frac{dy}{dt} - \frac{n}{a}(d+(n-1)a)y = 0. \qquad (6.9)$$

If $aq^2 + bq + c = 0$ and $(2aq+b)/ap = -1$ are assumed in (6.9), then

$$p = \mp\frac{\sqrt{b^2-4ac}}{a} \quad \text{and} \quad q = \frac{-b \pm \sqrt{b^2-4ac}}{2a}. \qquad (6.10)$$

Therefore (6.9) is simplified as

$$t(t-1)\frac{d^2y}{dt^2} + (\frac{d}{a}t + \frac{2ae - bd \pm d\sqrt{b^2-4ac}}{\mp 2a\sqrt{b^2-4ac}})\frac{dy}{dt} - \frac{n}{a}(d+(n-1)a)y = 0. \qquad (6.11)$$

On the other hand, equation (6.11) is a special case of the Gauss hypergeometric differential equation (see e.g. [43], p. 26):

$$t(t-1)\frac{d^2y}{dt^2} + ((\alpha+\beta+1)t - \gamma)\frac{dy}{dt} + \alpha\beta\, y = 0 \qquad (6.12)$$

for $\alpha = -n$, $\beta = n-1+d/a$ and $\gamma = \dfrac{2ae - bd \pm d\sqrt{b^2-4ac}}{\mp 2a\sqrt{b^2-4ac}}$ respectively. So, by considering $P_n\left(\begin{matrix} d & e \\ a\ b & c\end{matrix}\Big| x\right)$ as a pre-assigned solution of (6.7) and comparing the relations (6.11) and (6.12), we must have (6.13)

$$P_n\left(\begin{matrix} d & e \\ a\ b & c\end{matrix}\Bigg| \mp\frac{\sqrt{b^2-4ac}}{a}t + \frac{-b \pm \sqrt{b^2-4ac}}{2a}\right) = K\, {}_2F_1\left(\begin{matrix} -n \quad n-1+d/a \\ \dfrac{2ae - bd \pm d\sqrt{b^2-4ac}}{\pm 2a\sqrt{b^2-4ac}}\end{matrix}\Bigg|\, t\right).$$

(6.13) can also be written in terms of the variable *x* so that we have (6.14)

$$P_n\left(\begin{matrix} d & e \\ a\ b & c\end{matrix}\Big| x\right) = K\, {}_2F_1\left(\begin{matrix} -n \quad n-1+d/a \\ \dfrac{2ae - bd \pm d\sqrt{b^2-4ac}}{\pm 2a\sqrt{b^2-4ac}}\end{matrix}\Bigg| \mp(\frac{ax}{\sqrt{b^2-4ac}} + \frac{b \mp \sqrt{b^2-4ac}}{2\sqrt{b^2-4ac}})\right).$$

From (6.14) two following sub-cases are concluded (6.15)

$$\text{(i)} \quad \overline{P}_n\left(\begin{matrix} d & e \\ a\ b & c\end{matrix}\Big| x\right) = K^*\, {}_2F_1\left(\begin{matrix} -n \quad n-1+d/a \\ \dfrac{2ae - bd + d\sqrt{b^2-4ac}}{2a\sqrt{b^2-4ac}}\end{matrix}\Bigg| \frac{-ax}{\sqrt{b^2-4ac}} - \frac{b - \sqrt{b^2-4ac}}{2\sqrt{b^2-4ac}}\right),$$

$$\text{(ii)}\ \bar{P}_n\left(\begin{matrix} d & e \\ a\ \ b & c \end{matrix}\middle| x\right) = K^{**}\ {}_2F_1\left(\begin{matrix} -n \quad n-1+d/a \\ \dfrac{-(2ae-bd)+d\sqrt{b^2-4ac}}{2a\sqrt{b^2-4ac}} \end{matrix}\middle| \frac{ax}{\sqrt{b^2-4ac}}+\frac{b+\sqrt{b^2-4ac}}{2\sqrt{b^2-4ac}}\right).$$

Note that both above relations only differ by a minus sign in the argument of the second formula (ii), which does not affect on the differential equation (6.7). In other words, if in (i) we consider the case $\bar{P}_n\left(\begin{matrix} -d & -e \\ -a\ \ -b & -c \end{matrix}\middle| x\right)$, we will reach the second formula of (6.15). Therefore, only the formula (ii) must be considered as the main solution. To compute K^{**}, it is sufficient to obtain the leading coefficient of ${}_2F_1(\ldots|\ldots)$ corresponding to formula (ii), which is given by

$$K^{**} = \frac{n!\,(\sqrt{b^2-4ac})^n\,((bd-2ae+d\sqrt{b^2-4ac})/(2a\sqrt{b^2-4ac}))_n}{a^n(-n)_n(n-1+d/a)_n}. \tag{6.16}$$

But according to identity (6.3)

$${}_2F_1\left(\begin{matrix} -n & p \\ & q \end{matrix}\ \middle|\ rx+s\right) = \frac{(-1)^n(p)_n}{(q)_n}\sum_{k=0}^{n}\binom{n}{k}s^{n-k}\ {}_2F_1\left(\begin{matrix} k-n & 1-q-n \\ & 1-p-n \end{matrix}\ \middle|\ \frac{1}{s}\right)(rx)^k. \tag{6.17}$$

So, by considering the main solution (6.15) and assuming

$$\begin{cases} p = n-1+\dfrac{d}{a} \quad , \quad q = \dfrac{bd-2ae+d\sqrt{b^2-4ac}}{2a\sqrt{b^2-4ac}} \\ r = \dfrac{a}{\sqrt{b^2-4ac}} \quad , \quad s = \dfrac{b+\sqrt{b^2-4ac}}{2\sqrt{b^2-4ac}} \end{cases} \tag{6.18}$$

(6.17) is changed to (6.19)

$${}_2F_1\left(\begin{matrix} -n \quad n-1+d/a \\ \dfrac{bd-2ae+d\sqrt{b^2-4ac}}{2a\sqrt{b^2-4ac}} \end{matrix}\middle| \frac{ax}{\sqrt{b^2-4ac}}+\frac{b+\sqrt{b^2-4ac}}{2\sqrt{b^2-4ac}}\right) = \frac{(-1)^n(\frac{d}{a}+n-1)_n}{(\frac{bd-2ae+d\sqrt{b^2-4ac}}{2a\sqrt{b^2-4ac}})_n}$$

$$\times\sum_{k=0}^{n}\binom{n}{k}\left(\frac{b+\sqrt{b^2-4ac}}{2\sqrt{b^2-4ac}}\right)^{n-k}\left(\frac{a}{\sqrt{b^2-4ac}}\right)^k \times$$

$${}_2F_1\left(\begin{matrix} k-n & \dfrac{2ae-bd-(d+(2n-2)a)\sqrt{b^2-4ac}}{2a\sqrt{b^2-4ac}} \\ & -d/a-2n+2 \end{matrix}\ \middle|\ \frac{2\sqrt{b^2-4ac}}{b+\sqrt{b^2-4ac}}\right)x^k.$$

Simplifying this relation and substituting K^{**} by (6.16) finally gives the monic polynomial solution of equation (6.7) in the form (6.8). Hence the first part of the theorem is proved. To deduce the limiting case when $a \to 0$, one should use the limit relation $\lim_{a\to 0} a^r (-\frac{d}{a}+2-2n)_r = (-d)^r$ and the following identity

$$ {}_1F_1\begin{pmatrix} -n \\ p \end{pmatrix}| rx+s \Bigg) = \frac{\Gamma(1-p-n)}{\Gamma(1-p)} \sum_{k=0}^{n} \binom{n}{k} {}_2F_0\begin{pmatrix} k-n & 1-p-n \\ & - \end{pmatrix}| -\frac{1}{s} \Bigg) (\frac{r}{s}x)^k , \tag{6.20} $$

which is a special case of identity (6.2) for $C_k = \frac{(-n)_k}{(p)_k\, k!}$.

Here we would like to point out that the general formula $G_k^{(n)}(a,b,c,d,e)$ is a suitable tool to compute the coefficients of x^k for any fixed degree *k* and arbitrary *a*. If for example the coefficient of x^{n-1} is needed in the generic polynomial $\bar{P}_n\begin{pmatrix} d & e \\ a\ b\ c \end{pmatrix}| x \Big)$ then it is enough to calculate the term

$$ \begin{aligned} G_{n-1}^{(n)}(a,b,c,d,e) &= (\frac{2a}{b+\Delta})^{-1}\, {}_2F_1\left(\begin{matrix} -1 & \frac{2ae-bd-(d+(2n-2)a)\Delta}{2a\Delta} \\ & -(\frac{d+(2n-2)a}{a}) \end{matrix} \;\Bigg|\; \frac{2\Delta}{b+\Delta} \right) \\ &= (\frac{b+\Delta}{2a})(1+\frac{2ae-bd-(d+(2n-2)a)\Delta}{2a\Delta} \times \frac{2\Delta}{b+\Delta} \times \frac{a}{d+(2n-2)a}) = \frac{e+(n-1)b}{d+(2n-2)a} \end{aligned} \tag{6.21} $$

in which $\Delta = \sqrt{b^2-4ac}$. Note that in the above simplified relation, all parameters a,b,c,d and e are free and can adopt any value including zero since it is easy to find out that neither both values *a* and *d* nor both values *b* and *e* in (6.8) can vanish together. After simplifying $G_k^{(n)}(a,b,c,d,e)$ for $k = n-1, n-2, ...$ we eventually get (6.22)

$$ \begin{aligned} \bar{P}_n\begin{pmatrix} d & e \\ a\ b\ c \end{pmatrix}| x \Bigg) &= x^n + \binom{n}{1}\frac{e+(n-1)b}{d+(2n-2)a}x^{n-1} + \binom{n}{2}\frac{(e+(n-1)b)(e+(n-2)b)+c(d+(2n-2)a)}{(d+(2n-2)a)(d+(2n-3)a)}x^{n-2} \\ &+ ... + \binom{n}{n}(\frac{b+\sqrt{b^2-4ac}}{2a})^n\, {}_2F_1\left(\begin{matrix} -n & \frac{2ae-bd-(d+(2n-2)a)\sqrt{b^2-4ac}}{2a\sqrt{b^2-4ac}} \\ & -(\frac{d+(2n-2)a}{a}) \end{matrix} \;\Bigg|\; \frac{2\sqrt{b^2-4ac}}{b+\sqrt{b^2-4ac}} \right). \end{aligned} $$

The above relation implies that (6.22.1)

$$ \bar{P}_n\begin{pmatrix} d & e \\ a\ b\ c \end{pmatrix}| 0 \Bigg) = (\frac{b+\sqrt{b^2-4ac}}{2a})^n\, {}_2F_1\left(\begin{matrix} -n & \frac{2ae-bd-(d+(2n-2)a)\sqrt{b^2-4ac}}{2a\sqrt{b^2-4ac}} \\ & -(\frac{d+(2n-2)a}{a}) \end{matrix} \;\Bigg|\; \frac{2\sqrt{b^2-4ac}}{b+\sqrt{b^2-4ac}} \right). $$

Moreover, (6.22) shows, for example, that if *n* = 0,1,2,3 then (6.22.2)

$$\bar{P}_0\begin{pmatrix} d & e \\ a\ b\ c \end{pmatrix} x = 1,$$

$$\bar{P}_1\begin{pmatrix} d & e \\ a\ b\ c \end{pmatrix} x = x + \frac{e}{d},$$

$$\bar{P}_2\begin{pmatrix} d & e \\ a\ b\ c \end{pmatrix} x = x^2 + 2\frac{e+b}{d+2a}x + \frac{c(d+2a)+e(e+b)}{(d+2a)(d+a)},$$

$$\bar{P}_3\begin{pmatrix} d & e \\ a\ b\ c \end{pmatrix} x = x^3 + 3\frac{e+2b}{d+4a}x^2 + 3\frac{c(d+4a)+(e+b)(e+2b)}{(d+4a)(d+3a)}x + \frac{2c(d+3a)(e+2b)+ce(d+4a)+e(e+b)(e+2b)}{(d+4a)(d+3a)(d+2a)}.$$

6.3. A special case of the generic polynomials (6.8)

In the sequel, let us apply the Gauss identity again and assume that $\frac{2\sqrt{b^2-4ac}}{b+\sqrt{b^2-4ac}} = 1$ in (6.8). This assumption implies that $ac = 0$. If $c = 0$, the following special case for the generic polynomial (6.8) is derived (6.23)

$$\bar{P}_n\begin{pmatrix} d & e \\ a\ b\ 0 \end{pmatrix} x = \sum_{k=0}^{n}\binom{n}{k}\left(\frac{a}{b}\right)^{k-n} {}_2F_1\left(\begin{matrix} k-n & \frac{2ae-bd}{2ab}+1-\frac{d}{2a}-n \\ 2-d/a-2n & \end{matrix}\Big| 1\right) x^k$$

$$= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{a}{b}\right)^{k-n}\frac{\Gamma(2-2n-d/a)\Gamma(1-k-e/b)}{\Gamma(2-n-k-d/a)\Gamma(1-n-e/b)}x^k = \frac{b^n\Gamma(2-2n-d/a)\Gamma(1-e/b)}{a^n\Gamma(2-n-d/a)\Gamma(1-n-e/b)} \times {}_2F_1\left(\begin{matrix} -n & n-1+d/a \\ & e/b \end{matrix}\Big| -\frac{a}{b}x\right).$$

Furthermore, if one comes back to the Nikiforov and Uvarov approach and consideres the differential equation (6.7) as a self-adjoint form, then according to (3.23), chapter 3, the Rodrigues representation of $\bar{P}_n\begin{pmatrix} d & e \\ a\ b\ c \end{pmatrix} x$ is given by (6.24)

$$\bar{P}_n\begin{pmatrix} d & e \\ a\ b\ c \end{pmatrix} x = \frac{1}{\left(\prod_{k=1}^{n} d+(n+k-2)a\right)\rho\begin{pmatrix} d & e \\ a\ b\ c \end{pmatrix} x} \times \frac{d^n\left((ax^2+bx+c)^n\rho\begin{pmatrix} d & e \\ a\ b\ c \end{pmatrix} x\right)}{dx^n},$$

where $\rho\begin{pmatrix} d & e \\ a\ b\ c \end{pmatrix} x = \exp\left(\int\frac{(d-2a)x+(e-b)}{ax^2+bx+c}dx\right)$.

Now if we suppose $c = 0$ in this relation and refer to (6.23), we get (6.24.1)

$$\frac{\exp\left(-\int \frac{(d-2a)x+(e-b)}{ax^2+bx}dx\right)}{\left(\prod_{k=1}^{n} d+(n+k-2)a\right)} \times \frac{d^n\left((ax^2+bx)^n \exp\left(\int \frac{(d-2a)x+(e-b)}{ax^2+bx}dx\right)\right)}{dx^n}$$

$$= \frac{\Gamma(2-2n-d/a)}{\Gamma(1-n-e/b)} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{a}{b}\right)^{k-n} \frac{\Gamma(1-k-e/b)}{\Gamma(2-n-k-d/a)} x^k .$$

6.4. Some further properties of the main polynomials (6.8)

6.4.1. A linear change of variables

Using the representation (6.24) one can derive a linear change of variables for the monic polynomials $\bar{P}_n\begin{pmatrix} d & e & \\ a & b & c \end{pmatrix}\!\!\left| x \right.$ explicitly. Assuming $x = wt + v$, the mentioned relation changes to (6.25)

$$\bar{P}_n\left(\begin{matrix} d & e & \\ a & b & c \end{matrix} \,\middle|\, wt+v\right) = \frac{w^n\, d^n\left((aw^2t^2+(2awv+bw)t+(av^2+bv+c))^n\, \rho\left(\begin{matrix} d & e & \\ a & b & c \end{matrix} \,\middle|\, wt+v\right)\right)}{\left(\prod_{k=1}^{n} d+(n+k-2)a\right) \rho\left(\begin{matrix} d & e & \\ a & b & c \end{matrix} \,\middle|\, wt+v\right) dt^n}.$$

On the other hand $\rho\left(\begin{matrix} d & e & \\ a & b & c \end{matrix} \,\middle|\, wt+v\right)$ can be simplified as

$$\rho\left(\begin{matrix} d & e & \\ a & b & c \end{matrix} \,\middle|\, wt+v\right) = \exp\left(\int \frac{(d-2a)(wt+v)+(e-b)}{aw^2t^2+(2av+b)wt+av^2+bv+c}\, w\,dt\right)$$
$$= \rho\left(\begin{matrix} dw^2, & (dv+e)w & \\ aw^2, & (2av+b)w, & av^2+bv+c \end{matrix} \,\middle|\, t\right). \qquad (6.26)$$

Therefore (6.25) is transformed to

$$\bar{P}_n\left(\begin{matrix} d & e & \\ a & b & c \end{matrix} \,\middle|\, wt+v\right) = w^n\, \bar{P}_n\left(\begin{matrix} dw^2, & (dv+e)w & \\ aw^2, & (2av+b)w, & av^2+bv+c \end{matrix} \,\middle|\, t\right), \qquad (6.27)$$

which shows the effect of a linear change of variables on the polynomials (6.8). For instance, if $w=-1$ and $v=0$ in (6.27), then we have

$$\bar{P}_n\left(\begin{matrix} d & e & \\ a & b & c \end{matrix} \,\middle|\, -t\right) = (-1)^n\, \bar{P}_n\left(\begin{matrix} d, & -e & \\ a, & -b, & c \end{matrix} \,\middle|\, t\right). \qquad (6.28)$$

6.4.2. A generic three-term recurrence equation

The second formula of (6.15) is a suitable relation to compute the recurrence equation of the generic polynomials (6.8). In other words, it can be applied along with various identities of the Gauss hypergeometric function for generating a recurrence relation. For example, the following identity holds for the function ${}_2F_1(p,q,r\,;\,t)$ [18], (6.29)

$$(p-q)\big((p-q-1)(p-q+1)t+2pq+r(1-p-q)\big)\,{}_2F_1(p,q,r\,;\,t)+$$
$$q(r-p)(p-q+1)\,{}_2F_1(p-1,q+1,r\,;\,t)+p(r-q)(p-q-1)\,{}_2F_1(p+1,q-1,r\,;\,t)=0\,.$$

By using the formula (ii) in (6.15) and its coefficient in (6.16) if one assumes in (6.29) that (6.29.1)

$$p=-n\,,\;\; q=\frac{d}{a}+n-1\,,\;\; r=\frac{bd-2ae+d\sqrt{b^2-4ac}}{2a\sqrt{b^2-4ac}} \quad \text{and} \quad t=\frac{a}{\sqrt{b^2-4ac}}\,x+\frac{b+\sqrt{b^2-4ac}}{2\sqrt{b^2-4ac}}\,,$$

then after some computations, one finally gets (6.30)

$$\bar{P}_{n+1}(x)=\left(x+\frac{2n(n+1)ab+(d-2a)(e+2nb)}{(d+2na)(d+(2n-2)a)}\right)\bar{P}_n(x)$$
$$+\frac{n(d+(n-2)a)\big(c(d+(2n-2)a)^2-nb^2(d+(n-2)a)+(e-b)(a(e+b)-bd)\big)}{(d+(2n-3)a)(d+(2n-2)a)^2(d+(2n-1)a)}\bar{P}_{n-1}(x)$$

in which $\bar{P}_n(x)$ denotes the monic polynomials of (6.8) and the initial values $\bar{P}_0(x)=1$ and $\bar{P}_1(x)=x+\dfrac{e}{d}$ are given. For other approaches to extract (6.30), see [52] and [45].

6.4.3. A generic formula for the norm square value of the polynomials

Let $[L,U]$ be a predetermined orthogonality interval which (besides for finite families) of course consists of the zeros of $\sigma(x)=ax^2+bx+c$ or $\pm\infty$. By using the Rodrigues representation of the polynomials (6.8) we have

$$\left\|\bar{P}_n\right\|^2=\int_L^U \bar{P}_n^{\,2}\begin{pmatrix} d & e \\ a\;\; b & c\end{pmatrix}\!\!\Big|\,x\Big)\rho\begin{pmatrix} d & e \\ a\;\; b & c\end{pmatrix}\!\!\Big|\,x\Big)dx=\frac{1}{\prod_{k=1}^{n} d+(n+k-2)a}$$
$$\times\int_L^U \bar{P}_n\begin{pmatrix} d & e \\ a\;\; b & c\end{pmatrix}\!\!\Big|\,x\Big)\left(d^n((ax^2+bx+c)^n\rho\begin{pmatrix} d & e \\ a\;\; b & c\end{pmatrix}\!\!\Big|\,x\Big)\right)/dx^n)\,dx\,. \tag{6.31}$$

Consequently integrating by parts from right hand side of (6.31) yields

$$\left\|\bar{P}_n\right\|^2=\frac{n!(-1)^n}{\prod_{k=1}^{n} d+(n+k-2)a}\int_L^U (ax^2+bx+c)^n(\exp\int\frac{(d-2a)x+(e-b)}{ax^2+bx+c}dx)\,dx\,. \tag{6.32}$$

6.5. Six special cases of the generic polynomials (6.8) as classical orthogonal polynomials

As we mentioned in chapter 4, from the main equation (6.1) one can extract six special sequences of orthogonal polynomials on the real line. Jacobi, Laguerre and Hermite polynomials are three of them, which are infinitely orthogonal and three other ones are finitely orthogonal for some restricted values of *n*. In this section we intend to use the previous generic formulas to redetect the properties of each of these six sequences.

6.5.1. Jacobi orthogonal polynomials

If $a=-1$, $b=0$, $c=1$, $d=-\alpha-\beta-2$ and $e=-\alpha+\beta$ are selected in (6.8) then

$$\begin{aligned} P_n^{(\alpha,\beta)}(x) &= \frac{(n+\alpha+\beta+1)_n}{2^n n!}\,\bar{P}_n\begin{pmatrix} -\alpha-\beta-2, & -\alpha+\beta \\ -1,\ 0,\ 1 & \end{pmatrix}\Big|\, x \Big) \\ &= \frac{(n+\alpha+\beta+1)_n}{2^n n!}\sum_{k=0}^{n}(-1)^{k-n}\binom{n}{k}\,{}_2F_1\left(\begin{matrix} k-n,-n-\alpha \\ -2n-\alpha-\beta \end{matrix}\Big|\, 2\right)x^k , \end{aligned} \tag{6.33}$$

are the Jacobi orthogonal polynomials with weight function

$$\rho\left(\begin{matrix} -\alpha-\beta-2, & -\alpha+\beta \\ -1,\ 0,\ 1 & \end{matrix}\Big|\, x\right) = \exp\left(\int \frac{-(\alpha+\beta)x+\beta-\alpha}{-x^2+1}dx\right) = (1-x)^{\alpha}(1+x)^{\beta}, \tag{6.34}$$

and orthogonality relation (6.35)

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)dx = \frac{2^{\alpha+\beta+1}}{n!(2n+\alpha+\beta+1)}\,\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)}\,\delta_{n,m}.$$

These polynomials can also be represented as

$$\bar{P}_n\left(\begin{matrix} -\alpha-\beta-2, & -\alpha+\beta \\ -1,\ 0,\ 1 & \end{matrix}\Big|\, x\right) = (-1)^n(1-x)^n\,{}_2F_1\left(\begin{matrix} -n,-\alpha-n \\ -\alpha-\beta-2n \end{matrix}\Big|\, \frac{2}{1-x}\right), \tag{6.36}$$

which is one of the known hypergeometric representations for the Jacobi polynomials (see e.g. [13], [70]).

Since the Gegenbauer (ultraspherical), Legendre and Chebyshev polynomials of the first and second kind are all special sub-cases of the Jacobi polynomials, the following representations are straightforwardly concluded for them.

Gegenbauer polynomials: (6.37)

$$C_n^{(\lambda)}(x) = \frac{2^n(\lambda)_n}{n!}\,\bar{P}_n\left(\begin{matrix} -2\lambda-1 & 0 \\ -1 & 0 & 1 \end{matrix}\Big|\, x\right) = \frac{2^n(\lambda)_n}{n!}(-1)^n(1-x)^n\,{}_2F_1\left(\begin{matrix} -n,1/2-\lambda-n \\ 1-2\lambda-2n \end{matrix}\Big|\, \frac{2}{1-x}\right),$$

Legendre polynomials:

$$P_n(x) = \frac{(2n)!}{(n!)^2 2^n} \bar{P}_n\begin{pmatrix} -2 & 0 & \\ -1 & 0 & 1 \end{pmatrix} x \Bigg) = \frac{(2n)!}{(n!)^2 2^n} (-1)^n (1-x)^n {}_2F_1\left(\begin{matrix} -n, -n \\ -2n \end{matrix} \middle| \frac{2}{1-x} \right), \qquad (6.38)$$

Chebyshev polynomials of first kind:

$$T_n(x) = 2^{n-1} \bar{P}_n\begin{pmatrix} -1 & 0 & \\ -1 & 0 & 1 \end{pmatrix} x \Bigg) = 2^{n-1} (-1)^n (1-x)^n {}_2F_1\left(\begin{matrix} -n, 1/2-n \\ 1-2n \end{matrix} \middle| \frac{2}{1-x} \right), \qquad (6.39)$$

Chebyshev polynomials of second kind:

$$U_n(x) = 2^n \bar{P}_n\begin{pmatrix} -3 & 0 & \\ -1 & 0 & 1 \end{pmatrix} x \Bigg) = 2^n (-1)^n (1-x)^n {}_2F_1\left(\begin{matrix} -n, -1/2-n \\ -1-2n \end{matrix} \middle| \frac{2}{1-x} \right). \qquad (6.40)$$

6.5.2. Laguerre orthogonal polynomials

If one puts $a = 0$, $b = 1$, $c = 0$, $d = -1$, $e = \alpha + 1$ in (6.8) then

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} \bar{P}_n\begin{pmatrix} -1 & \alpha+1 & \\ 0 & 1 & 0 \end{pmatrix} x \Bigg) = \frac{(\alpha+1)_n}{n!} {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix} \middle| x \right), \qquad (6.41)$$

are the Laguerre orthogonal polynomials with weight function

$$\rho\begin{pmatrix} -1, & \alpha+1 & \\ 0, & 1, & 0 \end{pmatrix} x \Bigg) = \exp\left(\int \frac{-x+\alpha}{x} dx\right) = x^{\alpha} e^{-x}, \qquad (6.42)$$

and orthogonality relation

$$\int_0^{\infty} x^{\alpha} e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)\, dx = \frac{(n+\alpha)!}{n!} \delta_{n,m} \ . \qquad (6.43)$$

6.5.3. Hermite orthogonal polynomials

If $a = 0$, $b = 0$, $c = 1$, $d = -2$, $e = 0$ are selected in (6.8) then

$$H_n(x) = 2^n \bar{P}_n\begin{pmatrix} -2 & 0 & \\ 0 & 0 & 1 \end{pmatrix} x \Bigg) = (2x)^n {}_2F_0\left(\begin{matrix} -\frac{n}{2}, -\frac{n-1}{2} \\ - \end{matrix} \middle| -\frac{1}{x^2} \right), \qquad (6.44)$$

are the Hermite orthogonal polynomials with weight function

$$\rho\begin{pmatrix} -2, & 0 & \\ 0, & 0, & 1 \end{pmatrix} x \Bigg) = \exp\left(\int (-2x)\, dx\right) = \exp(-x^2), \qquad (6.45)$$

and orthogonality relation

$$\int_{-\infty}^{\infty} \exp(-x^2) H_n(x) H_m(x)\, dx = n! 2^n \sqrt{\pi}\ \delta_{n,m} \ . \qquad (6.46)$$

6.5.4. Finite classical orthogonal polynomials with weight function $W_1(x,p,q)=x^q(1+x)^{-(p+q)}$ on $[0,\infty)$

According to the approach explained in chapter 3, section 3.11.1, if one computes the logarithmic derivative of the given weight as $\frac{W_1'(x)}{W_1(x)}=\frac{-px+q}{x^2+x}$, then by referring to the Pearson's differential equation one respectively gets

$$a=1\ ,\ b=1\ ,\ c=0\ ,\ d=-p+2\ ,\ e=q+1\ . \tag{6.47}$$

In [53] the family of this type of polynomials are called Romanovski-Jacobi polynomials, see also [65, 54]. In [10] the related polynomials are denoted by $M_n^{(p,q)}(x)$, for which we get (6.48)

$$M_n^{(p,q)}(x)=(-1)^n(n+1-p)_n\,\bar{P}_n\left(\begin{matrix}-p+2 & q+1\\ 1\ \ 1 & 0\end{matrix}\Big|\,x\right)=(-1)^n(q+1)_n\ {}_2F_1\left(\left.\begin{matrix}-n,n+1-p\\ q+1\end{matrix}\right|-x\right).$$

Also, as it was shown in chapter 4, section 4.2, the finite set $\{M_n^{(p,q)}(x)\}_{n=0}^{n=N}$ is orthogonal with respect to the weight function $W_1(x,p,q)$ on $[0,\infty)$ if and only if $q>-1$ and $p>2N+1$. Let us add that to compute the norm square value of the polynomials one can also use the general relation (6.32). According to the foresaid relation we have

$$\int_0^\infty \frac{x^q}{(1+x)^{p+q}}\,\bar{P}_n^{\,2}\left(\begin{matrix}-p+2 & q+1\\ 1\ \ 1 & 0\end{matrix}\Big|\,x\right)dx=\frac{n!(-1)^n}{\prod_{k=1}^{n}(-p+n+k)}\int_0^\infty (x^2+x)^n\,\frac{x^q}{(1+x)^{p+q}}\,dx. \tag{6.49}$$

Thus, noting (6.48) yields

$$\int_0^\infty \frac{x^q}{(1+x)^{p+q}}\,(M_n^{(p,q)}(x))^2\,dx=\frac{n!(p-n-1)!(q+n)!}{(p-2n-1)(p+q-n-1)!}. \tag{6.50}$$

6.5.5. Finite classical orthogonal polynomials with weight function $W_2(x,p)=x^{-p}e^{-1/x}$ on $[0,\infty)$

If the fraction $\frac{W_2'(x)}{W_2(x)}=\frac{-px+1}{x^2}$ is generated then the main parameters are derived as

$$a=1\ ,\ b=0\ ,\ c=0\ ,\ d=-p+2\ ,\ e=1\ . \tag{6.51}$$

In [53], the polynomials of this type are called Romanovski-Bessel polynomials. In [10] these polynomials are denoted by $N_n^{(p)}(x)$, for which we have

$$N_n^{(p)}(x) = (-1)^n (n+1-p)_n \, \overline{P}_n\left(\begin{matrix} -p+2 & 1 \\ 1 \quad 0 & 0 \end{matrix}\,\middle|\, x\right) = (-1)^n \, {}_2F_0\left(\begin{matrix} -n, 1-p+n \\ - \end{matrix}\,\middle|\, -x\right). \tag{6.52}$$

Moreover, in chapter 4, section 4.3, it was shown that the finite set $\{N_n^{(p)}(x)\}_{n=0}^{n=N}$ is orthogonal with respect to the weight function $W_2(x,p)$ on $[0,\infty)$ if and only if $p > 2N+1$. Nevertheless, using the general relation (6.32) helps us obtain the norm square value of these polynomials as follows

$$\int_0^\infty x^{-p} e^{-\frac{1}{x}} \overline{P}_n^{\,2}\left(\begin{matrix} -p+2 & 1 \\ 1 \quad 0 & 0 \end{matrix}\,\middle|\, x\right) dx = \frac{n!(-1)^n}{\prod_{k=1}^{n}(-p+n+k)} \int_0^\infty x^{2n} x^{-p} e^{-\frac{1}{x}} \, dx. \tag{6.53}$$

Consequently, the complete orthogonality relation takes the form (6.54)

$$\int_0^\infty x^{-p} e^{\frac{-1}{x}} N_n^{(p)}(x) N_m^{(p)}(x)\, dx = \left(\frac{n!(p-(n+1))!}{p-(2n+1)}\right)\delta_{n,m} \quad \text{for} \quad m,n = 0,1,2,\ldots, N < \frac{p-1}{2}.$$

6.5.6. Finite classical orthogonal polynomials with weight function $W_3^{(p,q)}(x;A,B,C,D) = ((Ax+B)^2 + (Cx+D)^2)^{-p} \exp(q \arctan \frac{Ax+B}{Cx+D})$ on $(-\infty,\infty)$

Similar to the two previous cases, by computing $\dfrac{W_3'(x)}{W_3(x)}$ we get the parameters

$$\begin{aligned} &a = A^2 + C^2 \ , \ \ b = 2(AB+CD) \ \ , \ \ c = B^2 + D^2, \\ &d = 2(1-p)(A^2+C^2) \ \ , \ \ e = q(AD-BC) + 2(1-p)(AB+CD). \end{aligned} \tag{6.55}$$

In [53], the polynomials of this type are called Romanovski-Pseudo-Jacobi polynomials. In [9] the related polynomials are denoted by $J_n^{(p,q)}(x;A,B,C,D)$, for which we have

$$\begin{aligned} &J_n^{(p,q)}(x;A,B,C,D) = (-1)^n (n+1-2p)_n (A^2+C^2)^n \\ &\qquad \times \overline{P}_n\left(\begin{matrix} 2(1-p)(A^2+C^2), & q(AD-BC)+2(1-p)(AB+CD) \\ A^2+C^2, & 2(AB+CD), \quad B^2+D^2 \end{matrix}\,\middle|\, x\right) = \\ &(-1)^n (n+1-2p)_n \big(AB+CD+(AD-BC)i + x(A^2+C^2)\big)^n \\ &\qquad \times {}_2\mathrm{F}_1\left(\begin{matrix} -n, -n+p-\frac{q}{2}i \\ 2p-2n \end{matrix}\,\middle|\, \frac{2i(AD-BC)}{(A-Ci)(B+Di+x(A+Ci))}\right). \end{aligned} \tag{6.56}$$

6.6. How to find the parameters if a special case of the main weight function is given?

Similar to the chapter 3, section 3.11.1, it is easy to find out that the best way for finding a,b,c,d,e is to compute the logarithmic derivative $W'(x)/W(x)$ and match the pattern with $\dfrac{\rho'(x)}{\rho(x)}=\dfrac{(d-2a)x+(e-b)}{ax^2+bx+c}$. Let us clarify the subject by some examples given below.

Example 1. Consider the weight function $W(x)=(-x^2+3x-2)^{10}$; $1<x<2$. If the logarithmic derivative of this weight function is computed, then we get

$$\frac{W'(x)}{W(x)}=\frac{-20x+30}{-x^2+3x-2}=\frac{(d-2a)x+(e-b)}{ax^2+bx+c}\Rightarrow a=-1\ ,\ b=3\ ,\ c=-2\ ,\ d=-22\ ,\ e=33.$$

Consequently the related monic orthogonal polynomials are $\bar{P}_n\left(\begin{matrix}-22, & 33 & \\ -1, & 3, & -2\end{matrix}\,\Big|\,x\right)$. Note that these polynomials are orthogonal with respect to the given weight function on [1,2] for every value *n*. Therefore it is *not* necessary to know that these polynomials are the shifted Jacobi polynomials on the interval [1,2] since they can explicitly be expressed by the generic polynomials (6.8).

Example 2. The weight function $W(x)=(2x^2+2x+1)^{-10}$; $-\infty<x<\infty$ is given. Thus

$$\frac{W'(x)}{W(x)}=\frac{-40x-20}{2x^2+2x+1}\Rightarrow(a,b,c,d,e)=(2,2,1,-38,-18)\,,$$

and the related monic orthogonal polynomials are $\bar{P}_n\left(\begin{matrix}-38, & -18 & \\ 2, & 2, & 1\end{matrix}\,\Big|\,x\right)$. These polynomials are finitely orthogonal for $n\le 9$, because according to (4.44) in chapter 4 we must have $N=\max\{n\}<10-(1/2)$. Hence, the finite set $\{\bar{P}_n(2,2,1,-38,-18;x)\}_{n=0}^{n=9}$ is orthogonal with respect to the weight function $(2x^2+2x+1)^{-10}$ on $(-\infty,\infty)$.

Example 3. Consider the weight function $W(x)=\exp(x(\theta-x))$; $-\infty<x<\infty,\ \theta\in\mathbf{R}$. Then we have

$$\frac{W'(x)}{W(x)}=-2x+\theta\Rightarrow(a,b,c,d,e)=(0,0,1,-2,\theta)\,,$$

which gives the related monic orthogonal polynomials as $\bar{P}_n\left(\begin{matrix}-2, & \theta & \\ 0, & 0, & 1\end{matrix}\,\Big|\,x\right)$ for every value *n*.

Example 4. For the weight function $W(x)=\dfrac{\sqrt{x}}{(x+2)^8}$; $0<x<\infty$, which is a special case of the second kind of the Beta distribution, we have

$$\frac{W'(x)}{W(x)}=\frac{-15x+2}{2x^2+4x}\Rightarrow(a,b,c,d,e)=(2,4,0,-11,6)\,.$$

Therefore, the monic orthogonal polynomials are $\bar{P}_n\left(\begin{matrix} -11, & 6 \\ 2, & 4, & 0 \end{matrix} \Big| \, x\right)$ for $n \le 3$, because according to (4.6) in chapter 4, $2n+1<8-(1/2)$. This means that the finite set $\{\bar{P}_n(2,4,0,-11,6;x)\}_{n=0}^{n=3}$ is orthogonal with respect to the weight function $\sqrt{x}/(x+2)^8$ on $[0,\infty)$.

6.7. A generic formula for the values at the boundary points of monic classical orthogonal polynomials [2]

In the previous sections of this chapter, we found a generic formula for the polynomial solution families of the well-known differential equation (6.1). Now, in the section (6.8) we intend to obtain another such formula, which enables us to present a generic formula for the values of monic classical orthogonal polynomials at their boundary points of definition. For this goal, we should again use the general form of the Rodrigues representation of the polynomials in (6.24) and recall its corresponding weight function, i.e.

$$\rho\left(\begin{matrix} & d & e \\ a & b & c \end{matrix} \Big| \, x\right) = \exp\left(\int \frac{(d-2a)x+(e-b)}{ax^2+bx+c}dx\right).$$

Without loss of generality, let us suppose that $ax^2+bx+c=a(x+\theta_1)(x+\theta_2)$ in which

$$\theta_1 = \frac{b-\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad \theta_2 = \frac{b+\sqrt{b^2-4ac}}{2a}. \tag{6.57}$$

In the general case, $-\theta_1$ and $-\theta_2$ in (6.57) are the boundary points of the underlying interval for the corresponding classical orthogonal polynomials. This means that if θ_1 and θ_2 are finite and equal, the polynomials are of the Bessel type and if both θ_1 and θ_2 are finite but different from each other, the polynomials are of the Jacobi type, whereas if one of these values tends to $\pm\infty$, the polynomials are of the Laguerre type, and finally if both values are $\pm\infty$, then the polynomials are of the Hermite type.
The relation (6.57) implies that the main weight function is simplified as

$$\rho\left(\begin{matrix} & d & e \\ a & b & c \end{matrix} \Big| \, x\right) = R(x+\theta_1)^A(x+\theta_2)^B, \tag{6.58}$$

where R is a constant and

$$A = \frac{d}{2a} - 1 + \frac{2ae-bd}{2a\sqrt{b^2-4ac}} \quad \text{and} \quad B = \frac{d}{2a} - 1 - \frac{2ae-bd}{2a\sqrt{b^2-4ac}}. \tag{6.59}$$

Note that the relation (6.58) follows because the logarithmic derivative of the function $W^*(x)=(x+\theta_1)^A(x+\theta_2)^B$ equals the logarithmic derivative of the main weight function, and since

$$\frac{u'(x)}{u(x)}=\frac{v'(x)}{v(x)}\Leftrightarrow u(x)=R\,v(x)\,, \tag{6.60}$$

so (6.58) is valid.

On the other hand, according to (6.24) the Rodrigues representation of $\bar{P}_n\begin{pmatrix} d & e & \\ a & b & c \end{pmatrix}\Big| x$ is

$$\bar{P}_n\left(\begin{matrix} d & e & \\ a & b & c \end{matrix}\Big| x\right)=\frac{1}{\left(\prod_{k=1}^{k=n} d+(n+k-2)a\right)W\left(\begin{matrix} d & e & \\ a & b & c \end{matrix}\Big| x\right)}\frac{d^n}{dx^n}\left((ax^2+bx+c)^n W\left(\begin{matrix} d & e & \\ a & b & c \end{matrix}\Big| x\right)\right).$$

So, if (6.58) is replaced into the above representation, then

$$\begin{aligned}\bar{P}_n\left(\begin{matrix} d & e & \\ a & b & c \end{matrix}\Big| x\right)&=\frac{\frac{d^n}{dx^n}\left(R\quad a^n(x+\theta_1)^n(x+\theta_2)^n(x+\theta_1)^A(x+\theta_2)^B\right)}{R\left(\prod_{k=1}^{k=n} d+(n+k-2)a\right)(x+\theta_1)^A(x+\theta_2)^B}\\ &=\frac{1}{(n-1+d/a)_n(x+\theta_1)^A(x+\theta_2)^B}\frac{d^n}{dx^n}\left((x+\theta_1)^{n+A}(x+\theta_2)^{n+B}\right)\,.\end{aligned} \tag{6.61}$$

But according to the Leibniz rule

$$\frac{d^n(f(x)g(x))}{dx^n}=\sum_{k=0}^{k=n}\binom{n}{k}f^{(k)}(x)g^{(n-k)}(x)\,, \tag{6.62}$$

we have (6.63)

$$\frac{d^n((x+\theta_1)^{n+A}(x+\theta_2)^{n+B})}{dx^n}=(-1)^n\sum_{k=0}^{k=n}\binom{n}{k}(-n-A)_k(-n-B)_{n-k}(x+\theta_1)^{n+A-k}(x+\theta_2)^{B+k}\,.$$

Hence, (6.61) is simplified as (6.64)

$$\begin{aligned}\bar{P}_n\left(\begin{matrix} d & e & \\ a & b & c \end{matrix}\Big| x\right)&=\frac{1}{(2-2n-d/a)_n}\times\\ &\sum_{k=0}^{k=n}\binom{n}{k}\left(-n-\frac{d}{2a}+1-\frac{2ae-bd}{2a\sqrt{b^2-4ac}}\right)_k\left(-n-\frac{d}{2a}+1+\frac{2ae-bd}{2a\sqrt{b^2-4ac}}\right)_{n-k}\\ &\times\left(x+\frac{b-\sqrt{b^2-4ac}}{2a}\right)^{n-k}\left(x+\frac{b+\sqrt{b^2-4ac}}{2a}\right)^k.\end{aligned}$$

This is in fact another general representation for the polynomial solution of equation (6.7). Combining (6.8) and (6.64), we get straightforwardly (6.65)

$$\sum_{k=0}^{k=n}\binom{n}{k}(-n-\frac{d}{2a}+1-\frac{2ae-bd}{2a\Delta})_k(-n-\frac{d}{2a}+1+\frac{2ae-bd}{2a\Delta})_{n-k}(x+\frac{b-\Delta}{2a})^{n-k}(x+\frac{b+\Delta}{2a})^k$$
$$=(2-2n-d/a)_n\sum_{k=0}^{k=n}\binom{n}{k}\left(\frac{2a}{b+\Delta}\right)^{k-n}{}_2F_1\left(\begin{matrix}k-n & \frac{2ae-bd}{2a\Delta}+1-\frac{d}{2a}-n\\ & 2-d/a-2n\end{matrix}\,\middle|\,\frac{2\Delta}{b+\Delta}\right)x^k.$$

where again $\Delta=\sqrt{b^2-4ac}$.

Relation (6.64) can also be represented in terms of the hypergeometric form (6.66)

$$\bar{P}_n\left(\begin{matrix}d & e\\ a\ b & c\end{matrix}\,\middle|\,x\right)=\frac{(-n-\frac{d}{2a}+1+\frac{2ae-bd}{2a\Delta})_n(x+\theta_1)^n}{(2-2n-d/a)_n}\,{}_2F_1\left(\begin{matrix}-n & -n-\frac{d}{2a}+1-\frac{2ae-bd}{2a\Delta}\\ \frac{d}{2a}-\frac{2ae-bd}{2a\Delta}\end{matrix}\,\middle|\,\frac{x+\theta_2}{x+\theta_1}\right)$$

where θ_1 and θ_2 are defined by (6.57).

Of course, this hypergeometric representation can still be simplified. To simplify (6.66), we use the hypergeometric identity

$${}_2F_1\left(\begin{matrix}-n & p\\ q\end{matrix}\,\middle|\,\frac{r}{t}+s\right)=\frac{(-1)^n(p)_n}{(q)_n t^n}\sum_{k=0}^{n}\binom{n}{k}r^{n-k}\,{}_2F_1\left(\begin{matrix}-k & 1-q-n\\ 1-p-n\end{matrix}\,\middle|\,\frac{1}{s}\right)(st)^k, \tag{6.67}$$

which was used already in (6.17) with a rather different form. If we choose in particular (6.68)

$$p=-n-\frac{d}{2a}+1-\frac{2ae-bd}{2a\sqrt{b^2-4ac}},\ q=\frac{d}{2a}-\frac{2ae-bd}{2a\sqrt{b^2-4ac}},\ r=\theta_2-\theta_1=\frac{\sqrt{b^2-4ac}}{a},$$
$$s=1\ \ and\ \ t=x+\frac{b-\sqrt{b^2-4ac}}{2a},$$

then by (6.67) relation (6.66) reads as (6.69)

$$\bar{P}_n\left(\begin{matrix}d & e\\ a\ \ b & c\end{matrix}\,\middle|\,x\right)=\frac{(-1)^n(-n-B)_n(-n-A)_n}{(2-2n-d/a)_n(1+B)_n}\sum_{k=0}^{k=n}\binom{n}{k}\ (\frac{\Delta}{a})^{n-k}\,{}_2F_1\left(\begin{matrix}-k & -n-B\\ 1+A\end{matrix}\,\middle|\,1\right)(x+\theta_1)^k.$$

On the other hand, using Gauss's identity (i.e. ${}_2F_1(a,b,c;1)=...$) (6.69) can be further simplified as

$$\bar{P}_n\left(\begin{matrix}d & e\\ a\ \ b & c\end{matrix}\,\middle|\,x\right)=\frac{(\sqrt{b^2-4ac})^n\ \ (\frac{d}{2a}+\frac{2ae-bd}{2a\sqrt{b^2-4ac}})_n}{a^n(2-2n-d/a)_n}\times$$
$${}_2F_1\left(\begin{matrix}-n & n-1+d/a\\ \frac{d}{2a}+\frac{2ae-bd}{2a\sqrt{b^2-4ac}}\end{matrix}\,\middle|\,\frac{-ax}{\sqrt{b^2-4ac}}+\frac{-b+\sqrt{b^2-4ac}}{2\sqrt{b^2-4ac}}\right), \tag{6.70}$$

which is the same form as the first formula of (6.15). Furthermore, since the identity

$$\bar{P}_n\left(\begin{matrix} \lambda d & \lambda e \\ \lambda a \;\; \lambda b & \lambda c \end{matrix}\Big| x\right) = \bar{P}_n\left(\begin{matrix} d & e \\ a \;\; b & c \end{matrix}\Big| x\right) \qquad \forall\ \lambda \neq 0\ , \tag{6.71}$$

is also valid for $\lambda = -1$, the relation (6.70) can be brought in the form of the second formula of (6.15) too.

6.7.1. Values of the classical orthogonal polynomials at the boundary points

Using the explicit representations for the monic classical orthogonal polynomials, we can now compute the generic value of the polynomials at their boundary points of definition, $-\theta_1$ and $-\theta_2$, respectively. If we set in the second formula of (6.15)

$$\frac{ax}{\sqrt{b^2-4ac}} + \frac{b+\sqrt{b^2-4ac}}{2\sqrt{b^2-4ac}} = \begin{cases} 0 \\ 1 \end{cases} , \tag{6.72}$$

then

$$x = \frac{-b-\sqrt{b^2-4ac}}{2a} = -\theta_2 \qquad \text{and} \qquad x = \frac{-b+\sqrt{b^2-4ac}}{2a} = -\theta_1\ , \tag{6.73}$$

respectively. Therefore we get

$$\bar{P}_n\left(\begin{matrix} d & e \\ a \;\; b & c \end{matrix}\Big| -\theta_2\right) = \frac{(\sqrt{b^2-4ac})^n\ \left((d/2a)-(2ae-bd)/(2a\sqrt{b^2-4ac})\right)_n}{(-a)^n (n-1+d/a)_n}\ , \tag{6.74}$$

and

$$\bar{P}_n\left(\begin{matrix} d & e \\ a \;\; b & c \end{matrix}\Big| -\theta_1\right) = \frac{(\sqrt{b^2-4ac})^n\ \left((d/2a)+(2ae-bd)/(2a\sqrt{b^2-4ac})\right)_n}{a^n (n-1+d/a)_n}\ . \tag{6.75}$$

For example, by noting the section 6.5.1 for the monic Jacobi orthogonal polynomials $\bar{P}_n^{(\alpha,\beta)}(x)$ we have $(a,b,c,d,e) = (-1,0,1,-\alpha-\beta-2,-\alpha+\beta)$. Consequently, (6.74) and (6.75) yield

$$\bar{P}_n^{(\alpha,\beta)}(+1) = 2^n\,\frac{(\alpha+1)_n}{(n+1+\alpha+\beta)_n} = 2^n\,\frac{\Gamma(n+1+\alpha)\Gamma(n+1+\alpha+\beta)}{\Gamma(\alpha+1)\Gamma(2n+1+\alpha+\beta)}\ , \tag{6.76}$$

$$\bar{P}_n^{(\alpha,\beta)}(-1) = (-2)^n\,\frac{(\beta+1)_n}{(n+1+\alpha+\beta)_n} = (-2)^n\,\frac{\Gamma(n+1+\beta)\Gamma(n+1+\alpha+\beta)}{\Gamma(\beta+1)\Gamma(2n+1+\alpha+\beta)}\ . \tag{6.77}$$

Moreover, by noting the section 6.5.2 for the monic Laguerre polynomials $\bar{L}_n^{(\alpha)}(x)$ with $(a,b,c,d,e) = (0,1,0,-1,\alpha+1)$ we have $ax^2+bx+c = x$. Therefore just one root i.e. $\theta_1 = \theta_2 = 0$ is derived and by computing the corresponding limit one gets

$$\overline{L}_n^{(\alpha)}(0) = (-1)^n (1+\alpha)_n \ . \tag{6.78}$$

Furthermore, since the Hermite polynomials can be written in terms of the Laguerre polynomials (see e.g. [13], relation 22.5.40), we can also conclude that

$$\overline{H}_n(0) = \frac{n!}{2^{n+1}(n/2)!}(1+(-1)^n) \tag{6.79}$$

Similarly, for the Bessel polynomials $\overline{B}_n^{(\alpha)}(x)$ with $(a,b,c,d,e) = (1,0,0,\alpha+2,2)$ we have $ax^2+bx+c = x^2$. So, after computing the corresponding limit one can obtain that

$$\overline{B}_n^{(\alpha)}(0) = \frac{2^n}{(n+1+\alpha)_n} \ . \tag{6.80}$$

Chapter 7

Application of rational classical orthogonal polynomials for explicit computation of inverse Laplace transforms

7.1. Introduction

In the chapters 4 and 6, it was shown that from the hypergeometric differential equation, six finite and infinite classes of orthogonal polynomials could be extracted. In this chapter, we first apply the Mobius transform $x = pz^{-1} + q$, $p \neq 0$, $q \in \mathbf{R}$ for the mentioned equation to generate the classical orthogonal polynomials with negative powers. Then we show that the generated rational orthogonal polynomials are a very suitable tool to compute the inverse Laplace transform directly, with no additional calculation for finding their roots. In this way, by applying infinite and finite rational classical orthogonal polynomials, we present (for example) three basic expansions of six ones to explicitly obtain the inverse Laplace transform. To do this task, we should again reconsider the hypergeometric differential equation

$$(ax^2 + bx + c)y_n''(x) + (dx + e)y_n'(x) - n(d + (n-1)a)y_n(x) = 0\,, \qquad (7.1)$$

and suppose that $x = pz^{-1} + q$, $p \neq 0$, $q \in \mathbf{R}$ .Therefore, the equation (7.1) eventually changes to

$$x^2(l_2x^2 + l_1x + l_0)y'' + x(2l_2x^2 + l_3x + l_4)y' - n((n+1)l_0 - l_4)y = 0\,, \qquad (7.2)$$

where l_4, l_3, l_2, l_1, l_0 are real parameters and n is a positive integer.
Obviously one of the main solutions of equation (7.2) is a rational (negative power) polynomial like $P_{-n}\left(\begin{matrix} l_4 & & l_3 \\ l_2 & l_1 & l_0 \end{matrix} \,\middle|\, x\right) = \sum_{k=0}^{n} r_k\, x^{-k}$ that depends on the parameters l_4, l_3, l_2, l_1, l_0 respectively. According to the Sturm-Liouville theorem, one can find a weight function corresponding to the differential equation (7.2) as

$$W\left(\begin{matrix} l_4 & & l_3 \\ l_2 & l_1 & l_0 \end{matrix} \,\middle|\, x\right) = \exp\left(\int \frac{-2l_2x^2 + (l_3 - 3l_1)x + (l_4 - 2l_0)}{x(l_2x^2 + l_1x + l_0)}\,dx\right). \qquad (7.3)$$

Now assume that $[L, U]$ is a predetermined orthogonality interval. So, we should have

$$\int_L^U W\begin{pmatrix} l_4 & l_3 \\ l_2 \ l_1 & l_0 \end{pmatrix} | \, x \Bigg) P_{-n}\begin{pmatrix} l_4 & l_3 \\ l_2 \ l_1 & l_0 \end{pmatrix} | \, x \Bigg) P_{-m}\begin{pmatrix} l_4 & l_3 \\ l_2 \ l_1 & l_0 \end{pmatrix} | \, x \Bigg) dx = A_n \, \delta_{n,m} \,, \tag{7.4}$$

where

$$A_n = \int_L^U W\begin{pmatrix} l_4 & l_3 \\ l_2 \ l_1 & l_0 \end{pmatrix} | \, x \Bigg) (P_{-n}\begin{pmatrix} l_4 & l_3 \\ l_2 \ l_1 & l_0 \end{pmatrix} | \, x \Bigg))^2 \, dx \,, \tag{7.5}$$

denotes the norm square value. There exist six cases, corresponding to the main equation (7.2), that are orthogonal for some specific values of l_4, l_3, l_2, l_1, l_0.

As it is seen, the connection between equations (7.1) and (7.2) is a Mobius transform as $x = pz^{-1} + q$ for different values of p and q. For example, if $p = -2$ and $q = 1$ are considered then the rational Jacobi polynomials can be defined as

$$P_{-n}^{(\alpha,\beta)}(x) = P_n^{(\alpha-\beta-2,\beta)}(-2x^{-1}+1) = \sum_{k=0}^{k=n} (-1)^k \binom{n+\alpha-2+k}{k} \binom{n+\alpha-2-\beta}{n-k} x^{-k} \,. \tag{7.6}$$

For n = 0,1,2 this definition respectively gives

$$\begin{cases} P_0^{(\alpha,\beta)}(x) = 1 \\ P_{-1}^{(\alpha,\beta)}(x) = -\alpha\, x^{-1} + (\alpha - \beta - 1) \\ P_{-2}^{(\alpha,\beta)}(x) = \dfrac{1}{2}(\alpha+2)(\alpha+1)\, x^{-2} - (\alpha+1)(\alpha-\beta)\, x^{-1} + \dfrac{1}{2}(\alpha-\beta)(\alpha-\beta-1) \end{cases} \tag{7.7}$$

Since the orthogonality relation of Jacobi polynomials is known, the orthogonality relation of $P_{-n}^{(\alpha,\beta)}(x)$ will also be known as (7.8)

$$\int_1^\infty x^{-\alpha}(x-1)^\beta \, P_{-n}^{(\alpha,\beta)}(x) P_{-m}^{(\alpha,\beta)}(x) dx = \frac{(n+\alpha-\beta-2)!(n+\beta)!}{(2n+\alpha-1)n!(n+\alpha-2)!} \, \delta_{n,m} \Leftrightarrow \alpha > 0 \text{ and } \beta > -1.$$

Moreover, the differential equation of $y = P_{-n}^{(\alpha,\beta)}(x)$ is a special case of the main equation (7.2) for $l_4 = \alpha - 2$, $l_3 = \beta - \alpha + 3$, $l_2 = 0$, $l_1 = 1$, $l_0 = -1$ and we have

$$x^2(x-1)y'' + x((\beta - \alpha + 3)x + \alpha - 2)y' + n(n + \alpha + 1)y = 0 \,. \tag{7.9}$$

Similarly one can obtain the Mobius transforms of other five classes of rational classical orthogonal polynomials. Hence, it is better not to enter in details of them rather just we note that all differential equations of these transforms must however be the special cases of (7.2). For instance, the rational Laguerre orthogonal polynomials $L_{-n}^{(\alpha)}(x) = L_n^{(\alpha)}(x^{-1} - 1)$ satisfies

$$x^3(1-x)y'' - x(2x^2 + \alpha x - 1)y' + n\,y = 0\,, \tag{7.10}$$

as well as the rational Hermite orthogonal polynomials $H_{-n}(x) = H_n(\alpha\, x^{-1} + \beta)$ satisfies

$$x^4 y'' + 2x(x^2 + \alpha\beta\, x + \alpha^2)y' + 2n\alpha^2 y = 0\,. \tag{7.11}$$

7.2. Evaluation of Inverse Laplace Transform using rational classical orthogonal polynomials [8]

It is well known that the Laplace transform provides a powerful method for analyzing the linear systems. However, many physical problems lead to Laplace transforms whose inverses are not readily expressed in terms of tabulated functions. Because of this problem, extensive researches have been done on this matter and its applications up to now.

For example, Chandran and Pallath [23] have computed inverse Laplace transforms of a class of non-rational fractional functions. Evans and Chung in [34] have obtained Laplace transform inversions using optimal contours in the complex plane, see also [17]. Iqbal in [40] has stated a classroom note regarding the Fourier method for computation of Laplace transform inversion. The problem of inverse two-sided Laplace transform for probability density functions has been stated by Tagliani in 1998 [71]. Furthermore, the problem of numerical inversion of Laplace transform has been studied by several authors. For example, Cunha and Viloche in [29] have presented an iterative method for the numerical inversion of Laplace transform. In [26], Dong has introduced a regularization method for this purpose. In [30], Crump has used Fourier series approximation (see also [39]) while Miller and Guy in [55] have used Jacobi polynomials and Sidi [68] has applied a window function for Laplace transform inversion. Finally Piessens' work [60] is a good bibliography in this regard that one can refer to it. But, in cases where the inverse Laplace transform is required for many values of the independent variable, it is convenient to obtain the inverse as a series expansion in terms of a set of linearly independent functions. Procedure based on this idea can be calculated by solving a system of equations, which can be reduced to a triangular system if one chooses to use "orthogonal polynomials". Such a method, using orthogonal polynomials, gives an approximate evaluation of the inversion integral using "Gauss quadrature" in the complex plane [66, 58, 59, 67]. Of course, the chief disadvantage of this method is the necessity of finding *all roots*, real and complex, of a polynomial of high degree, and of the calculation of a set of complex Christoffel numbers [74, p. 419]. Hence, we wish here to insist that the orthogonal polynomials with negative powers are in turn suitable tool to compute the inverse Laplace transform without any effort for finding the roots of orthogonal polynomials. To achieve this goal, we should use the orthogonality properties of rational classical orthogonal polynomials introduced in section 7.1. In this way, we present three basic expansions for explicit computation of Laplace transform inversion.

7.2.1. Inverse Laplace transform using rational Jacobi orthogonal polynomials $P_{-n}^{(\alpha,\beta)}(x)$.

Let us consider the Laplace transform together with its inverse as (7.12)

$$F(s)=L[f(x)]=\int_0^{\infty}e^{-sx}f(x)\,dx \Leftrightarrow f(x)=L^{-1}[F(s)]=\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}e^{sx}F(s)\,ds \quad ; \quad \forall s>0\,.$$

By referring to the previous sections we can find an explicit solution of the above integral equation provided that $F(s)$ is known and expandable. First by noting (7.8), we clearly have

$$\int_0^{\infty}(t+1)^{-\alpha}t^{\beta}\,P_{-n}^{(\alpha,\beta)}(t+1)P_{-m}^{(\alpha,\beta)}(t+1)dt=\frac{(n+\alpha-\beta-2)!(n+\beta)!}{(2n+\alpha-1)n!(n+\alpha-2)!}\,\delta_{n,m}\quad . \qquad (7.13)$$

Now, let $F(s)$ satisfy the Dirikhlet conditions and

$$F(s)=\sum_{n=0}^{\infty}C_nP_{-n}^{(\alpha,\beta)}(s+1)=\sum_{n=0}^{\infty}\frac{A_n}{(s+1)^n}\quad . \qquad (7.14)$$

By applying the property (7.13) in the expansion (7.14), the coefficients C_n are found as

$$C_n=\frac{(2n+\alpha-1)n!(n+\alpha-2)!}{(n+\alpha-\beta-2)!(n+\beta)!}\int_0^{\infty}(s+1)^{-\alpha}s^{\beta}\,P_{-n}^{(\alpha,\beta)}(s+1)\,F(s)\,ds\,. \qquad (7.15)$$

On the other hand, taking the inverse Laplace transform from the equality (7.14) yields

$$f(x)=L^{-1}[F(s)]=\sum_{n=0}^{\infty}C_nL^{-1}[P_{-n}^{(\alpha,\beta)}(s+1)]\,. \qquad (7.16)$$

Moreover, according to the definition (7.6) we have

$$\begin{aligned}L^{-1}[P_{-n}^{(\alpha,\beta)}(s+1)]&=\sum_{k=0}^{k=n}(-1)^k\binom{n+\alpha-2+k}{k}\binom{n+\alpha-2-\beta}{n-k}L^{-1}[(s+1)^{-k}]\\&=\binom{n+\alpha-2-\beta}{\alpha-2-\beta}\delta(x)-e^{-x}\sum_{k=1}^{k=n}\binom{n+\alpha-2+k}{k}\binom{n+\alpha-2-\beta}{n-k}\frac{(-x)^{k-1}}{(k-1)!}\,,\end{aligned} \qquad (7.17)$$

in which $\delta(x)=\operatorname*{Lim}_{\varepsilon\to 0}f_{\varepsilon}(x)$ such that $f_{\varepsilon}(x)=\begin{cases}1/\varepsilon & 0\le x\le\varepsilon\\ 0 & x>\varepsilon\end{cases}$ is the Dirac function. Consequently the special series (7.18)

$$\begin{aligned}f(x)=&\left(\frac{(\alpha-1)!}{(\alpha-\beta-2)!\beta!}\int_0^{\infty}(s+1)^{-\alpha}s^{\beta}\,F(s)\,ds\right)\delta(x)\\&+\sum_{n=1}^{\infty}\left(\frac{(2n+\alpha-1)n!(n+\alpha-2)!}{(n+\alpha-\beta-2)!(n+\beta)!}\int_0^{\infty}(s+1)^{-\alpha}s^{\beta}\,P_{-n}^{(\alpha,\beta)}(s+1)\,F(s)\,ds\right)\\&\qquad\times\left(\binom{n+\alpha-2-\beta}{\alpha-2-\beta}\delta(x)-e^{-x}\sum_{k=1}^{k=n}\binom{n+\alpha-2+k}{k}\binom{n+\alpha-2-\beta}{n-k}\frac{(-x)^{k-1}}{(k-1)!}\right)\end{aligned}$$

is an expanded solution for the integral equation (7.12). Note that the foresaid solution is valid if and only if its definite integrals are convergent and $F(s)$ in (7.14) is an expandable function under the Dirikhlet conditions. The function $(s+1)^{-\alpha}s^{\beta}$ in the right hand side of (7.18) plays in fact a weighted distribution role for computation of definite integrals (7.18) on $[0,\infty)$. Hence, if this distribution changes, another expanded solution will appear. The next section will specify this subject.

7.2.2. Inverse Laplace transform using rational Laguerre orthogonal polynomials $L_{-n}^{(\alpha)}(x)$.

First, let us define the sequence

$$L_{-n}^{(\alpha)}(s) = L_n^{(\alpha)}(\frac{1}{s}) = \sum_{k=0}^{k=n} \frac{(-1)^k}{k!}\binom{n+\alpha}{n-k} s^{-k} , \tag{7.19}$$

that satisfies the orthogonality relation

$$\int_0^\infty s^{-(\alpha+2)} e^{\frac{-1}{s}} L_{-n}^{(\alpha)}(s) L_{-m}^{(\alpha)}(s)\, ds = \frac{(n+\alpha)!}{n!}\,\delta_{n,m} . \tag{7.20}$$

If the expansion

$$F(s) = \sum_{n=0}^\infty C_n L_{-n}^{(\alpha)}(s) = \sum_{n=0}^\infty \frac{A_n}{s^n} , \tag{7.21}$$

is considered, then by applying (7.20) in (7.21), the coefficients C_n will be derived as

$$C_n = \frac{n!}{(n+\alpha)!}\int_0^\infty s^{-(\alpha+2)} e^{\frac{-1}{s}} L_{-n}^{(\alpha)}(s) F(s)\, ds . \tag{7.22}$$

Therefore, we similarly have

$$f(x) = L^{-1}[F(s)] = \sum_{n=0}^\infty C_n L^{-1}[L_{-n}^{(\alpha)}(s)] . \tag{7.23}$$

On the other hand, since

$$L^{-1}[L_{-n}^{(\alpha)}(s)] = \sum_{k=0}^{k=n} \frac{(-1)^k}{k!}\binom{n+\alpha}{n-k} L^{-1}[\frac{1}{s^k}] , \tag{7.24}$$

so, eventually the special series

$$\begin{aligned} f(x) &= \left(\frac{1}{\alpha!}\int_0^\infty s^{-(\alpha+2)} e^{\frac{-1}{s}} F(s)\, ds\right)\delta(x) \\ &+ \sum_{n=1}^\infty \left(\frac{n!}{(n+\alpha)!}\int_0^\infty s^{-(\alpha+2)} e^{\frac{-1}{s}} L_{-n}^{(\alpha)}(s) F(s)\, ds\right)\left(\binom{n+\alpha}{n}\delta(x) - \sum_{k=1}^{k=n}\binom{n+\alpha}{n-k}\frac{(-x)^{k-1}}{k!(k-1)!}\right) \end{aligned} \tag{7.25}$$

is a series solution for the Laplace transform inversion. Again, we mention that (7.25) is valid only if its definite integrals are convergent and the function $F(s)$ is expandable under the Dirikhlet conditions. In relation (7.25), the function $s^{-(\alpha+2)}e^{-1/s}$ is in fact a weighted distribution on $[0,\infty)$.

But, by noting the section 7.1, finite rational classical orthogonal polynomials can also be applied for approximate computation of inverse Laplace transform, because the function $F(s)$ can be expanded by them finitely, and only some limit conditions are imposed on their parameters. For example, if we define the sequence

$$M_{-n}^{(p,q)}(s)=M_{n}^{(p+2,q)}(\frac{1}{s})=(-1)^n n!\sum_{k=0}^{k=n}(-1)^k\binom{p+1-n}{k}\binom{q+n}{n-k}s^{-k}\ , \tag{7.26}$$

that satisfies the orthogonality relation

$$\int_0^\infty \frac{s^p}{(1+s)^{p+q+2}}M_{-n}^{(p,q)}(s)M_{-m}^{(p,q)}(s)\,ds=\frac{n!(p+1-n)!(q+n)!}{(p+1-2n)(p+q+1-n)!}\delta_{n,m}\ , \tag{7.27}$$

with $q>-1,\ p>2N-1,\ N=Max\{m,n\}$, then by considering the approximation

$$F(s)\cong\sum_{n=0}^{N}C_nM_{-n}^{(p,q)}(s)=\sum_{n=0}^{N}\frac{A_n}{s^n}\quad;\quad N<\frac{p+1}{2}\ , \tag{7.28}$$

and applying (7.27) on (7.28) we get

$$C_n=\frac{(p+1-2n)(p+q+1-n)!}{n!(p+1-n)!(q+n)!}\int_0^\infty\frac{s^p}{(1+s)^{p+q+2}}M_{-n}^{(p,q)}(s)F(s)\,ds\ . \tag{7.29}$$

Therefore

$$f(x)=L^{-1}[F(s)]\cong\sum_{n=0}^{N}C_nL^{-1}[M_{-n}^{(p,q)}(s)]\ , \tag{7.30}$$

in which

$$L^{-1}[M_{-n}^{(p,q)}(s)]=(-1)^n n!\sum_{k=0}^{k=n}(-1)^k\binom{p+1-n}{k}\binom{q+n}{n-k}L^{-1}[\frac{1}{s^k}]. \tag{7.31}$$

Hence (7.32)

$$\begin{aligned} f(x)\cong&\left(\frac{(p+q+1)!}{p!q!}\int_0^\infty\frac{s^p}{(1+s)^{p+q+2}}F(s)\,ds\right)\delta(x)\\ &+\sum_{n=1}^{N}\left(\frac{(p+1-2n)(p+q+1-n)!}{n!(p+1-n)!(q+n)!}\int_0^\infty\frac{s^p}{(1+s)^{p+q+2}}M_{-n}^{(p,q)}(s)F(s)\,ds\right)\\ &\qquad\times\left(\binom{q+n}{n}\delta(x)-(-1)^n n!\sum_{k=1}^{k=n}\binom{p+1-n}{k}\binom{q+n}{n-k}\frac{(-x)^{k-1}}{(k-1)!}\right) \end{aligned}$$

is an approximate solution for the integral equation $L[f(x)]=F(s)$. Here we add that the inversion problem can also be propounded for the negative power polynomials

$N_{-n}^{(p)}(x), H_{-n}(x)$ and $J_{-n}^{(p,q)}(x;a,b,c,d)$ similarly. It is now a good position to present two practical examples in this way.

7.3. Special examples of section 7.2

Example 1. Let us consider a special case of rational Jacobi polynomials for $\alpha = 1$, $\beta = -1/2$. This case is defined by

$$P_{-n}^{(1,-\frac{1}{2})}(x+1) = T_n(\frac{x-1}{x+1}) = \cos(n \arccos \frac{x-1}{x+1}), \tag{7.33}$$

and called the rational Chebyshev polynomials of the first kind [70]. By noting the expansion (7.18) let us also suppose that for instance $F(s) = \frac{1}{1+s}$. This implies that the integrals of (7.18) are simplified as

$$\int_0^\infty \frac{s^{-1/2}}{(1+s)^2} ds = \frac{\pi}{2} \quad \text{and} \quad \int_0^\infty \frac{s^{-1/2}}{(1+s)^2} P_{-n}^{(1,-\frac{1}{2})}(s) ds =$$
$$\sum_{k=0}^{k=n} \binom{n+k-1}{k} \binom{n-1/2}{n-k} \int_0^\infty \frac{s^{-1/2}}{(1+s)^{k+2}} ds = \frac{\sqrt{\pi}\, \Gamma(n+1/2)}{2\, \Gamma(n)} \sum_{k=0}^{k=n} \frac{(-1)^k (n+k-1)!(2k+1)}{k!(n-k)!(k+1)!}$$

Therefore the inverse Laplace transform for the given $F(s)$ takes the form

$$L^{-1}(\frac{1}{1+s}) = e^{-x} = \frac{1}{2}\delta(x) +$$
$$\sum_{n=1}^{\infty}\left(\frac{n\sqrt{\pi}\Gamma(n+1)}{2\Gamma(n)} \sum_{i=0}^{i=n} \frac{(-1)^i (n+i-1)!(2i+1)}{i!(n-i)!(i+1)!}\right)\left(\frac{\Gamma(n+1/2)}{\sqrt{\pi}\, n!}\delta(x) - e^{-x} \sum_{k=1}^{k=n} \binom{n+k-1}{k}\binom{n-1/2}{n-k}\frac{(-x)^{k-1}}{(k-1)!}\right)$$

Clearly the above relation is valid if it is expanded.

Example 2. For this example, we use the analytic expansion (7.25) and consider the equation $L[f(x)] = \frac{\sqrt{\pi}}{2s\sqrt{s}}$. To derive the solution of this integral equation, first we have to evaluate the definite integrals corresponding to (7.25). Hence we have

$$F(s) = \frac{\sqrt{\pi}}{2s\sqrt{s}} \Rightarrow \int_0^\infty s^{-(\alpha+2)} e^{\frac{-1}{s}} F(s) ds = \frac{\sqrt{\pi}}{2}\Gamma(\alpha + \frac{5}{2}) \quad \text{and} \quad \int_0^\infty s^{-(\alpha+2)} e^{\frac{-1}{s}} L_{-n}^{(\alpha)}(s) F(s) ds =$$
$$\frac{\sqrt{\pi}}{2} \sum_{k=0}^{k=n} \frac{(-1)^k}{k!} \binom{n+\alpha}{n-k} \int_0^\infty s^{-(\alpha+\frac{7}{2}+k)} e^{\frac{-1}{s}} ds = \frac{\sqrt{\pi}\, \Gamma(n+1+\alpha)\Gamma(\alpha+5/2)}{\Gamma(n+1)\Gamma(\alpha+1)} {}_2F_1\left(\begin{matrix} -n & \alpha+5/2 \\ & \alpha+1 \end{matrix} \Big| 1\right)$$
$$= \frac{3}{8} \frac{\Gamma(-n-3/2)\Gamma(\alpha+1+n)\Gamma(\alpha+5/2)}{\Gamma(n+1)\Gamma(\alpha+1-n)}.$$

Note that for the latter integral we have used again the well-known Gauss identity. Consequently the expansion (7.25) is transformed to

$$L^{-1}\left(\frac{\sqrt{\pi}}{2s\sqrt{s}}\right)=\sqrt{x}=\frac{\sqrt{\pi}}{2}\frac{\Gamma(\alpha+5/2)}{\Gamma(\alpha+1)}\delta(x)+$$
$$\sum_{n=1}^{\infty}\left(\frac{3}{8}\frac{\Gamma(-n-3/2)\Gamma(\alpha+5/2)}{\Gamma(\alpha+1-n)}\right)\left(\binom{n+\alpha}{n}\delta(x)-\sum_{k=1}^{k=n}\binom{n+\alpha}{n-k}\frac{(-x)^{k-1}}{k!(k-1)!}\right).$$

The above relation holds if and only if $\alpha>-1$ and $\alpha+1-n\notin \mathbf{Z}^{-}$.

Chapter 8

Some further special functions and their applications

8.1. Introduction

In this chapter, we will introduce some new classes of special functions and study their applications in the classical equations of applied physics.

8.2. Application of zero eigenvalue for solving the potential, heat and wave equations using a sequence of special functions [5]

In the solution of boundary value problems, usually zero eigenvalue is ignored. This case also happens in calculating the eigenvalues of matrices, so that we would often like to find the nonzero solutions of the linear system $AX = \lambda X$ when $\lambda \neq 0$. But on the other hand $\lambda = 0$ implies that $\det A = 0$ for $X \neq 0$ and then the rank of matrix A is reduced at least one degree. This approach can be stated for the boundary value problems similarly. In other words, if at least one of the eigens of equations related to the main problem is considered zero, then one of the solutions will be specified in advance. By using this note, we can introduce a class of special functions and apply for the potential, heat and wave equations in spherical coordinate. Hence let us first define the following sequences

$$\begin{aligned} C_n(z;a(z)) &= \frac{(a(z))^n + (a(z))^{-n}}{2} = \cosh(n\ln(a(z))), \\ S_n(z;a(z)) &= \frac{(a(z))^n - (a(z))^{-n}}{2} = \sinh(n\ln(a(z))), \end{aligned} \tag{8.1}$$

where $a(z)$ can be a complex (or real) function and n is a positive integer number.
It is not difficult to verify that both of defined sequences satisfy a unique second order differential equation in the form

$$a^2(z)a'(z)y'' + \left(a(z)(a'(z))^2 - a^2(z)a''(z)\right)y' - n^2(a'(z))^3\,y = 0\,, \tag{8.2}$$

provided that $a^2(z)a'(z) \neq 0$. Consequently, we deal with only one class of special functions, which is in fact the solution of equation (8.2). The functions $C_n(z;a(z))$ and $S_n(z;a(z))$ have several important sub-cases that are useful to study. First sub-case is the Chebyshev polynomials if one chooses $a(z) = \exp(i\arccos z)$ in (8.1) and uses the well-known Euler identity. In this case, the following sequences will appear

$$C_n(z;\exp(i\arccos z)) = \cos(n\arccos z) = T_n(z),$$
$$S_n(z;\exp(i\arccos z)) = i\sin(nArc\cos z) = i\sqrt{1-z^2}\,U_{n-1}(z), \tag{8.3}$$

where $T_n(z)$ & $U_n(z)$ denote the same as first and second kind of Chebyshev polynomials. Moreover, if the selected $a(z)$ is replaced in (8.2), the differential equation of the first kind of Chebyshev polynomials is derived as

$$(1-z^2)y'' - z\,y' + n^2 y = 0. \tag{8.4}$$

The second sample is the rational Chebyshev functions that can be generated by $a(z) = \exp(i\,\mathrm{arccot}\,z)$. Thus, for this selected case we have

$$C_n(z;\exp(i\mathrm{arccot}\,z)) = \cos(n\,\mathrm{arccot}\,z),$$
$$S_n(z;\exp(i\mathrm{arccot}\,z)) = i\sin(n\mathrm{arccot}\,z). \tag{8.5}$$

In this way, replacing the related $a(z)$ in (8.2) yields

$$a^2(z)a'(z) = \frac{-i\exp(3i\,\mathrm{arccot}\,z)}{1+z^2}, \quad a(z)(a'(z))^2 = \frac{-\exp(3i\,\mathrm{arccot}\,z)}{(1+z^2)^2},$$
$$a^2(z)a''(z) = \frac{(2iz-1)\exp(3i\,\mathrm{arccot}\,z)}{(1+z^2)^2}, \quad (a'(z))^3 = \frac{i\exp(3i\,\mathrm{arccot}\,z)}{(1+z^2)^3}. \tag{8.6}$$

Therefore the functions (8.5) eventually satisfy

$$(1+z^2)^2 y'' + 2z(1+z^2)\,y' + n^2 y = 0. \tag{8.7}$$

It should be noted that the explicit forms of the real functions $C_n(z;\exp(i\mathrm{arccot}\,z))$ and $-i\,S_n(z;\exp(i\mathrm{arccot}\,z))$ in (8.5) could be derived by the Moivre's formula. In other words, let us substitute $\theta = \mathrm{arccot}\,x$ in the Moivre formula to get

$$\frac{(x+i)^n}{(\sqrt{1+x^2})^n} = \cos(n\mathrm{arccot}\,x) + i\sin(n\mathrm{arccot}\,x). \tag{8.8}$$

Consequently we have

$$C_n(x;\exp(i\mathrm{arccot}\,x)) = \Big(\sum_{k=0}^{[n/2]}(-1)^k\binom{n}{2k}x^{n-2k}\Big)/(\sqrt{1+x^2})^n = T_n^*(x),$$
$$-i\,S_{n+1}(x;\exp(i\mathrm{arccot}\,x)) = \Big(\sum_{k=0}^{[n/2]}(-1)^k\binom{n+1}{2k+1}x^{n-2k}\Big)/(\sqrt{1+x^2})^{n+1} = U_n^*(x). \tag{8.9}$$

It can be shown that the above rational Chebyshev functions $T_n^*(x)$ and $U_n^*(x)$ are orthogonal with respect to the weight function $W(x)=\dfrac{1}{1+x^2}$ on $(-\infty,\infty)$ and satisfy the following orthogonality properties respectively

$$\int_{-\infty}^{\infty}\frac{T_n^*(x)T_n^*(x)}{1+x^2}dx=\begin{cases}0 & if \quad m\neq n,\\ \pi & if \quad m=n,\\ 2\pi & if \quad m=n=0.\end{cases} \tag{8.10}$$

$$\int_{-\infty}^{\infty}\frac{U_n^*(x)U_n^*(x)}{1+x^2}dx=\begin{cases}0 & if \quad m\neq n,\\ \pi & if \quad m=n.\end{cases}$$

Here is worthy to point out that J.P. Boyd in 1987 [22] applied the rational functions $T_n^*((x^{1/2}-x^{-1/2})/2)$ on the interval $[0,\infty)$ in spectral methods, and we should here mention that his functions could be derived only by replacing $a(z)=\exp(2i\,\mathrm{arccot}\sqrt{z})$ in (8.1).
But, so far it has been investigated that the Legendre (or Associated Legendre) differential equation (introduced in chapter 2, equation (2.19))

$$(1-x^2)y''(x)-2xy'(x)+(p-\frac{q}{1-x^2})y(x)=0\,, \tag{8.11}$$

has three solutions in Cartesian coordinate as follows

(a) $p\neq 0$, $q\neq 0$ that generates the associated Legendre functions;
(b) $p\neq 0$, $q=0$ that generates the Legendre polynomials;
(c) $p=0$, $q=0$ that is reduced to the simple equation $(1-x^2)y''(x)-2xy'(x)=0$, which has the solution $y(x)=c_1\ln\dfrac{1+x}{1-x}+c_2$.
So, a fourth case $p=0$, $q\neq 0$ remains, which is different from above mentioned ones and should be solved. To find the solution of fourth case, we substitute $a(z)=(\dfrac{1-z}{1+z})^{\frac{1}{2}}$ in (8.2) to arrive at the differential equation

$$(1-z^2)y''-2z\,y'-\frac{n^2}{1-z^2}\,y=0\,. \tag{8.12}$$

Clearly (8.12) is a specific case of (8.11) for $p=0$, $q=n^2$. According to (8.1), the solutions of this equation are respectively

$$\begin{aligned}C_n(z;(\frac{1-z}{1+z})^{\frac{1}{2}})&=\frac{1}{2}((\frac{1-z}{1+z})^{\frac{n}{2}}+(\frac{1-z}{1+z})^{\frac{-n}{2}}),\\ S_n(z;(\frac{1-z}{1+z})^{\frac{1}{2}})&=\frac{1}{2}((\frac{1-z}{1+z})^{\frac{n}{2}}-(\frac{1-z}{1+z})^{\frac{-n}{2}}).\end{aligned} \tag{8.13}$$

Here let us claim that the mentioned functions are very applied in the Helmholtz equation in spherical coordinate (see chapter 2, section 2.3.1). In other words, if the equation $\nabla^2 U(r,\theta,\Phi) = k^2 U(r,\theta,\Phi)$ is separated to several ordinary equations, then one of the separate equations takes the form

$$\frac{1}{\sin\theta}\frac{d}{d\theta}(\sin\theta\frac{dy}{d\theta}) + (m(m+1) - \frac{n^2}{\sin^2\theta})y(\theta) = 0\,, \tag{8.14}$$

which is equivalent to equation (8.11) for $x = \cos\theta$. Hence, if $z = \cos\theta$ is considered in (8.12) or equivalently $a(z) = \tan\frac{z}{2}$ in (8.2), the special case of (8.14) for $m = 0$, i.e.

$$y'' + (\cot z)y' - \frac{n^2}{\sin^2 z}y = 0\,, \tag{8.15}$$

has the following solutions

$$\begin{aligned} C_n(z;\tan\frac{z}{2}) &= \frac{1}{2}((\tan\frac{z}{2})^n + (\tan\frac{z}{2})^{-n})\,, \\ S_n(z;\tan\frac{z}{2}) &= \frac{1}{2}((\tan\frac{z}{2})^n - (\tan\frac{z}{2})^{-n})\,. \end{aligned} \tag{8.16}$$

These sequences will frequently appear in the given problems of the next section.

8.2.1. Application of defined functions (8.16) in the solution of potential, heat and wave equations in spherical coordinate

Usually most of the boundary value problems related to the wave, heat and potential equations in spherical coordinate are reduced to the Helmholtz partial differential equation $\nabla^2 U(r,\theta,\Phi) = k^2 U(r,\theta,\Phi)$. But, for the special case of k = 0 in this relation we have

$$\frac{\partial^2 U}{\partial r^2} + \frac{2}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial U}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\Phi^2} = 0\,, \tag{8.17}$$

which is known as the potential (Laplace) equation in spherical coordinate. Now, if the related variables in equation (8.17) are separated as $U(r,\theta,\Phi) = R(r)A(\theta)B(\Phi)$, then the following ordinary differential equations will be derived

$$\begin{aligned} &r^2 R'' + 2rR' - \lambda_1 R = 0\,, \\ &B'' - \lambda_2 B = 0\,, \\ &A'' + (\cot\theta)A' + (\lambda_1 + \frac{\lambda_2}{\sin^2\theta})A = 0\,. \end{aligned} \tag{8.18}$$

As we said, the solution of Laplace (potential) equation is generally determined when the boundary conditions are known, nevertheless, if in (8.18) $\lambda_1 = 0$ and $\lambda_2 = -k^2 \neq 0$

are assumed, then for the variable r we must have $R(r) = -c_1^2 \frac{1}{r} + c_2$ (c_1 and c_2 are constant) and for the third equation the same form as equation (8.15). Therefore, the general solution corresponding to the third equation would be

$$A(\theta) = A_1 C_n(\theta; \tan\frac{\theta}{2}) + A_2 S_n(\theta; \tan\frac{\theta}{2}) = a_1 \tan^k(\frac{\theta}{2}) + a_2 \tan^{-k}(\frac{\theta}{2}), \tag{8.19}$$

where A_1 and A_2 & a_1 and a_2 are all constant values. Accordingly, (8.19) implies to have the general solution of the potential equation $\nabla^2 U(r,\theta,\Phi) = 0$ with the pre-assigned condition $R(r) = -c_1^{\ 2} \frac{1}{r} + c_2$ as

$$U(r,\theta,\Phi) = (\frac{-c_1^{\ 2}}{r} + c_2)\,(b_1 \cos k\Phi + b_2 \sin k\Phi)(a_1 \tan^k(\frac{\theta}{2}) + a_2 \tan^{-k}(\frac{\theta}{2})). \tag{8.20}$$

It is interesting to know that the solution (8.20) shows the sensitivity of potential equation with respect to the variable r, so that we have $\lim_{r \to 0} U(r,\theta,\Phi) = \infty$.

As an example, here let us consider the Laplace equation $\nabla^2 U(r,\theta,\Phi) = 0 \;\; ; \; 0 < r < a$ (in spherical coordinate) when the variable r takes the pre-assigned form $R(r) = \frac{-c_1^{\ 2}}{r} + c_2$ and the following initial and boundary conditions are given

$$\lim_{r \to 0} U(r,\theta,\Phi) = \infty\,,$$
$$U(\frac{a}{2},\theta,\Phi) = 0\,,$$
$$U(r,\frac{\pi}{2},\Phi) = 0\,,$$
$$U(a,\frac{\pi}{3},\Phi) = \Phi\,.$$

The general solution of this problem, according to the given conditions and assuming $A_k = a_1 b_1 c_2$, $B_k = a_2 b_1 c_2$ would be finally as

$$U(r,\theta,\Phi) = \sum_k U_k(r,\theta,\Phi)\,,$$
$$U_k(r,\theta,\Phi) = (1 - \frac{a}{2r})(\tan^k(\frac{\theta}{2}) - \tan^{-k}(\frac{\theta}{2}))(A_k \cos k\Phi + B_k \sin k\Phi).$$

On the other hand, putting the last condition in the above general solution yields

$$2\Phi = \sum_k (\sqrt{3}^{-k} - \sqrt{3}^{+k})(A_k \cos k\Phi + B_k \sin k\Phi)\,,$$

where A_k and B_k are calculated by

$$(\sqrt{3}^{-k}-\sqrt{3}^{+k})A_k=\frac{4}{\pi}\int_0^{\pi}\Phi\cos k\Phi\,d\Phi=\frac{4((-1)^k-1)}{\pi k^2},$$
$$(\sqrt{3}^{-k}-\sqrt{3}^{+k})B_k=\frac{4}{\pi}\int_0^{\pi}\Phi\sin k\Phi\,d\Phi=\frac{4(-1)^k}{k}.$$

Consequently, the particular solution of potential equation under the given conditions is

$$U(r,\theta,\Phi)=(1-\frac{a}{2r})\sum_{k=1}^{\infty}\left(\frac{4((-1)^k-1)k^{-2}}{\pi(3^{-k/2}-3^{k/2})}\cos k\Phi+\frac{4(-1)^k k^{-1}}{(3^{-k/2}-3^{k/2})}\sin k\Phi\right)(\tan^k\frac{\theta}{2}-\tan^{-k}\frac{\theta}{2}).$$

As we see, a special case of the functions (8.1) has appeared in the above solution.
Similarly application of zero eigenvalue can be propounded for the heat and wave equations respectively. For instance, if the classical heat equation $\nabla^2 U=\frac{\partial U}{\partial t}$ is considered, then separating the variables as $U(r,\theta,\Phi,t)=S(r,\theta,\Phi)T(t)$, where $S(r,\theta,\Phi)=R(r)A(\theta)B(\Phi)$, yields

$$\begin{cases}\nabla^2 U=\nabla^2(ST)=T\nabla^2 S\\ \dfrac{\partial U}{\partial t}=\dfrac{\partial(ST)}{\partial t}=S\dfrac{\partial T}{\partial t}\end{cases}\Rightarrow T\nabla^2 S=T'(t)S\Rightarrow\begin{cases}\nabla^2 S-\alpha S=0\\ T'-\alpha T=0\end{cases}\tag{8.21}$$

So, the following ordinary differential equations are derived

$$\begin{aligned}&T'-\alpha\,T=0,\\ &r^2R''+2rR'-(\alpha\,r^2+\lambda_1)R=0,\\ &B''-\lambda_2B=0,\\ &A''+(\cot\theta)\,A'+(\lambda_1+\frac{\lambda_2}{\sin^2\theta})A=0.\end{aligned}\tag{8.22}$$

Again, if $\lambda_1=0$, $\lambda_2=-n^2\neq 0$ and $\alpha=-k^2\neq 0$ are assumed in (8.22), then the general solution, when $R(r)=\frac{c_1J_{1/2}(kr)+c_2J_{-1/2}(kr)}{\sqrt{kr}}$ is pre-assigned, takes the form

$$\begin{aligned}U(r,\theta,\Phi,t)=&\frac{e^{-k^2t}}{\sqrt{kr}}(c_1J_{1/2}(kr)+c_2J_{-1/2}(kr))(b_1\tan^n(\frac{\theta}{2})+b_2\tan^{-n}(\frac{\theta}{2}))\\ &\times(a_1\cos n\Phi+a_2\sin n\Phi),\end{aligned}\tag{8.23}$$

in which $J_{1/2}(x)$ and $J_{-1/2}(x)$ are two particular cases of the Bessel functions $J_p(x)$. For example, let us consider the heat equation $\nabla^2 U(r,\theta,\Phi,t) = \frac{\partial U}{\partial t}$; $0 < r < a$ (in spherical coordinate) when the variable r takes the pre-assigned form $R(r) = \frac{c_1 J_{1/2}(kr) + c_2 J_{-1/2}(kr)}{\sqrt{kr}}$ and the following conditions hold

$$\lim_{r\to 0} U(r,\theta,\Phi,t) < M,$$

$$U(r,\frac{\pi}{2},\Phi,t) = 0,$$

$$U(r,\theta,0,t) = 0,$$

$$U(r,\frac{\pi}{3},\Phi,0) = f(r,\Phi) \quad ; \quad \text{arbitrary}$$

By referring to the general solution (8.23) and using the given conditions we get $c_2 = a_1 = 0$ and $b_1 + b_2 = 0$. So, if $A_{k,n} = a_2 b_1 c_1$, then

$$U(r,\theta,\Phi,t) = \sum_k \sum_n U_{k,n}(r,\theta,\Phi,t),$$

$$U_{k,n}(r,\theta,\Phi,t) = A_{k,n} \frac{e^{-k^2 t}}{\sqrt{kr}} J_{1/2}(kr)(\tan^n \frac{\theta}{2} - \tan^{-n} \frac{\theta}{2}) \sin n\Phi,$$

is the general solution of this specific example. On the other hand, since the orthogonality relation of Bessel functions $J_p(x)$ is denoted by [18]

$$\int_0^a x J_p(Z_{(p,m)} \frac{x}{a}) J_p(Z_{(p,n)} \frac{x}{a})\, dx = \frac{a^2}{2} J_{p+1}^2(Z_{(p,m)})\, \delta_{n,m}, \tag{8.24}$$

where $Z_{(p,m)}$ is mth zero of $J_p(x)$ (i.e. $J_p(Z_{(p,m)}) = 0$), it is better for the eigenvalues k to be considered as $k = \frac{Z_{(1/2,m)}}{a}$; $p = \frac{1}{2}$. Therefore, the general solution of the problem is simplified as

$$U(r,\theta,\Phi,t) = \sum_n \sum_m A_{n,m}^* \frac{\exp(-(\frac{Z_{(1/2,m)}}{a})^2 t)}{\sqrt{Z_{(1/2,m)} r/a}} J_{\frac{1}{2}}(Z_{(1/2,m)} r/a)(\tan^n \frac{\theta}{2} - \tan^{-n} \frac{\theta}{2}) \sin n\Phi$$

in which $A_{n,m}^* = A_{n,\frac{Z_{(1/2,m)}}{a}}$. Now, it is sufficient to compute the coefficients $A_{n,m}^*$. To do this, substituting the last condition of the problem in above relation yields

$$\sqrt{\frac{r}{a}}\, f(r,\Phi) = \sum_n \sum_m \frac{A_{n,m}^* (3^{\frac{-n}{2}} - 3^{\frac{n}{2}})}{\sqrt{Z_{(1/2,m)}}} J_{1/2}(Z_{(1/2,m)} \frac{r}{a}) \sin(n\Phi).$$

Therefore, by applying the orthogonality relation of Bessel functions and using the orthogonality property of the sequence $\{\sin(n\Phi)\}_{n=1}^{\infty}$ on $[0,\pi]$, $A^{*}_{n,m}$ are found as

$$A^{*}_{n,m}=\frac{4\sqrt{Z_{(1/2,m)}}\int_0^a\int_0^\pi f(r,\Phi)J_{1/2}(Z_{(1/2,m)}\frac{r}{a})\sin(n\Phi)r^{\frac{3}{2}}\,drd\Phi}{\pi(3^{\frac{-n}{2}}-3^{\frac{n}{2}})a^{\frac{3}{2}}J^{2}_{3/2}(Z_{(1/2,m)})}.$$

Finally, the problem can be stated for the wave equation $\nabla^2 U=\dfrac{\partial^2 U}{\partial t^2}$ according to the following stages. First we have

$$\begin{cases}\nabla^2 U=\nabla^2(ST)=T\nabla^2 S\\ \dfrac{\partial^2 U}{\partial t^2}=\dfrac{\partial^2(ST)}{\partial t^2}=S\dfrac{\partial^2 T}{\partial t^2}\end{cases}\Rightarrow T\nabla^2 S=T''(t)S\Rightarrow\begin{cases}\nabla^2 S-\alpha S=0\\ T''-\alpha T=0\end{cases}\tag{8.25}$$

which results the ordinary equations

$$\begin{aligned}&T''-\alpha\,T=0,\\ &r^2R''+2rR'-(\alpha\,r^2+\lambda_1)R=0,\\ &B''-\lambda_2 B=0,\\ &A''+(\cot\theta)\,A'+(\lambda_1+\frac{\lambda_2}{\sin^2\theta})A=0.\end{aligned}\tag{8.26}$$

Now, if $\lambda_1=0,\lambda_2=-k^2\neq 0$ and $\alpha=-n^2\neq 0$ are assumed in (8.26), then the general solution of the classical wave equation when $R(r)=\dfrac{c_1J_{1/2}(nr)+c_2J_{-1/2}(nr)}{\sqrt{nr}}$ would be as follows

$$\begin{aligned}U(r,\theta,\Phi,t)&=(d_1\cos nt+d_2\sin nt)(b_1\cos k\Phi+b_2\sin k\Phi).\\ &\times(a_1\tan^{k}\frac{\theta}{2}+a_2\tan^{-k}\frac{\theta}{2})(\frac{c_1J_{1/2}(nr)+c_2J_{-1/2}(nr)}{\sqrt{nr}}).\end{aligned}\tag{8.27}$$

Here let us consider a specific problem regarding the wave equation in spherical coordinate when the variable r has the form $R(r)=\dfrac{c_1J_{1/2}(nr)+c_2J_{-1/2}(nr)}{\sqrt{nr}}$ and the conditions

$$\begin{cases}1.\ \lim\limits_{r\to 0}U(r,\theta,\Phi,t)<M\quad , \quad 2.\ U(r,\frac{\pi}{2},\Phi,t)=0\\ 3.\ U(r,\theta,0,t)=0\quad , \quad 4.\ U(r,\theta,\Phi,0)=0\\ 5.\ U(r,\frac{\pi}{3},\Phi,q)=g(r,\Phi)\quad ; \quad \text{arbitrary}\end{cases}$$

are given. To solve the problem, replacing the given conditions in the general solution (8.27) gives $c_2 = b_1 = d_1 = 0$ and $a_1 + a_2 = 0$. If $B_{k,n} = a_1 b_2 c_1 d_2$ is supposed, then (8.27) becomes

$$U(r,\theta,\Phi,t) = \sum_k \sum_n U_{k,n}(r,\theta,\Phi,t),$$

$$U_{k,n}(r,\theta,\Phi,t) = B_{k,n} \sin(nt) \frac{J_{1/2}(nr)}{\sqrt{nr}} (\tan^k \frac{\theta}{2} - \tan^{-k} \frac{\theta}{2}) \sin k\Phi .$$

But similar to the previous problem, if $n = \dfrac{Z_{(1/2,m)}}{a}$ is taken, then

$$U(r,\theta,\Phi,t) = \sum_k \sum_m B^*_{k,m} \sin(Z_{(1/2,m)} \frac{t}{a}) \frac{J_{1/2}(Z_{(1/2,m)} r/a)}{\sqrt{Z_{(1/2,m)} r/a}} (\tan^k \frac{\theta}{2} - \tan^{-k} \frac{\theta}{2}) \sin k\Phi ,$$

where $B^*_{k,m} = B_{k,\frac{Z_{(1/2,m)}}{a}}$. By substituting the last condition of the problem in the above relation, i.e.

$$\sqrt{\frac{r}{a}}\, g(r,\Phi) = \sum_k \sum_m \frac{B^*_{k,m} (3^{\frac{-k}{2}} - 3^{\frac{k}{2}}) \sin(Z_{(1/2,m)} \frac{q}{a})}{\sqrt{Z_{(1/2,m)}}} J_{1/2}(Z_{(1/2,m)} \frac{r}{a}) \sin(k\Phi) ,$$

and using the orthogonality relation of Bessel functions $J_{1/2}(Z_{(1/2,m)} \frac{r}{a})$ on [0,a], the unknown coefficients $B^*_{k,m}$ will be derived as

$$B^*_{k,m} = \frac{4\sqrt{Z_{(1/2,m)}} \int_0^a \int_0^\pi g(r,\Phi) J_{1/2}(Z_{(1/2,m)} \frac{r}{a}) \sin(k\Phi) r^{\frac{3}{2}}\, dr d\Phi}{\pi \sin(Z_{(1/2,m)} \frac{q}{a}) (3^{\frac{-k}{2}} - 3^{\frac{k}{2}}) a^{\frac{3}{2}} J^2_{3/2}(Z_{(1/2,m)})} ,$$

which determines the final solution of the given problem straightforwardly.

8.3. Two classes of special functions using Fourier transforms of finite classical orthogonal polynomials [11]

Some orthogonal polynomial systems are mapped onto each other by the Fourier transform or by another ones such as the Mellin or Hankel transforms, see [33]. The best-known examples of this type are the Hermite functions, i.e. the Hermite polynomials $H_n(x)$ multiplied by $\exp(-x^2/2)$, which are eigenfunctions of the Fourier transform. More examples of this type are found in [49, 50] and [47]. The latter author showed that the Jacobi and continuous Hahn polynomials can be mapped onto each other in such a way, and the orthogonality relations for the continuous Hahn polynomials then follow from the orthogonality relations of the Jacobi polynomials and the Parseval formula. Now, we intend to introduce two new examples of finite systems of this type in this section and obtain their orthogonality relations. We then estimate a complicated integral and propose a conjecture for a further example of finite orthogonal sequences. For this purpose, we should come back to the chapter 4 and recall the orthogonality property of the finite classical orthogonal polynomials. Hence, let us here recall the polynomials

$$M_n^{(p,q)}(x) = (-1)^n n! \sum_{k=0}^{n} \binom{p-(n+1)}{k} \binom{q+n}{n-k} (-x)^k ,$$

that satisfy the orthogonality relation

$$\int_0^\infty \frac{x^q}{(1+x)^{p+q}} M_n^{(p,q)}(x) M_m^{(p,q)}(x)\, dx = \left(\frac{n!(p-(n+1))!(q+n)!}{(p-(2n+1))(p+q-(n+1))!} \right) \delta_{n,m}$$

for $m,n = 0,1,2,...,N < \dfrac{p-1}{2}$ and $q > -1$. To derive the Fourier transform of the polynomials $M_n^{(p,q)}(x)$ we should first refer to the definition of the Fourier transform of a function, say $g(x)$, as

$$\mathfrak{F}(s) = \mathfrak{F}(g(x)) = \int_{-\infty}^{\infty} e^{-isx} g(x)dx \ , \tag{8.28}$$

and consider its inverse transform as

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} \mathfrak{F}(s) ds \ . \tag{8.29}$$

In this sense, the Parseval identity of Fourier theory should also be considered as

$$\int_{-\infty}^{\infty} g(x)h(x)dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathfrak{F}(g(x)) \overline{\mathfrak{F}(h(x))} ds \ , \tag{8.30}$$

for $g, h \in L^2(\mathbf{R})$. Now, to obtain the Fourier transform of $M_n^{(p,q)}(x)$ first define the following functions

$$g(x) = \frac{e^{qx}}{(1+e^x)^{p+q}} M_n^{(r,u)}(e^x) \quad \& \quad h(x) = \frac{e^{ax}}{(1+e^x)^{a+b}} M_m^{(c,d)}(e^x) \ . \tag{8.31}$$

It is clear that the Fourier transform exists for both above functions. However, for example, for the function $g(x)$ defined in (8.31) we have

$$\mathfrak{F}(g(x)) = \int_{-\infty}^{\infty} e^{-isx} \frac{e^{qx}}{(1+e^x)^{p+q}} M_n^{(r,u)}(e^x) dx = \int_0^\infty t^{-is-1} \frac{t^q}{(1+t)^{p+q}} M_n^{(r,u)}(t) dt$$

$$= (-1)^n n! \binom{u+n}{n} \sum_{k=0}^{n} \frac{(-1)^k (-n)_k (n+1-r)_k}{(u+1)_k \, k!} \left(\int_0^\infty \frac{t^{q-is-1+k}}{(1+t)^{p+q}} \, dt \right)$$

$$
\begin{aligned}
&= (-1)^n \frac{(u+n)!}{u!} \sum_{k=0}^{n} \frac{(-1)^k (-n)_k (n+1-r)_k}{(u+1)_k \, k!} \frac{\Gamma(q-is+k)\Gamma(p+is-k)}{\Gamma(p+q)} \\
&= \frac{(-1)^n \Gamma(u+n+1)\Gamma(q-is)\Gamma(p+is)}{\Gamma(u+1)\Gamma(p+q)} \, {}_3F_2\left(\begin{matrix} -n & n+1-r & q-is \\ u+1 & -p+1-is & \end{matrix} \Big| \, 1 \right)
\end{aligned} \tag{8.32}
$$

where ${}_3F_2(...)$ is a special case of the generalized hypergeometric function defined by

$$
{}_pF_q\left(\begin{matrix} a_1, & a_2, & .... & ,a_p \\ b_1, & b_2, & .... & ,b_q \end{matrix} \Big| \, x \right) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k ...(a_p)_k}{(b_1)_k (b_2)_k ...(b_q)_k} \frac{x^k}{k!} \ ,
$$

for $(p,q)=(3,2)$ and again $(r)_k = r(r+1)...(r+k-1)$. Note that to derive (8.32) we have used the two identities

$$
\Gamma(a+k) = \Gamma(a)(a)_k \quad \text{and} \quad \Gamma(a-k) = \frac{(-1)^k \Gamma(a)}{(1-a)_k} \ . \tag{8.33}
$$

Now, by substituting (8.32) in Parseval's identity we get

$$
\begin{aligned}
&2\pi \int_{-\infty}^{\infty} \frac{e^{(q+a)x}}{(1+e^x)^{(p+q+a+b)}} M_n^{(r,u)}(e^x) M_m^{(c,d)}(e^x) dx = 2\pi \int_0^{\infty} \frac{t^{q+a-1}}{(1+t)^{p+q+a+b}} M_n^{(r,u)}(t) M_m^{(c,d)}(t) dt \\
&= \frac{(-1)^{n+m} \Gamma(u+n+1)\Gamma(d+m+1)}{\Gamma(u+1)\Gamma(p+q)\Gamma(d+1)\Gamma(a+b)} \times \\
&\qquad \int_{-\infty}^{\infty} \Gamma(q-is)\Gamma(p+is)\overline{\Gamma(a-is)\Gamma(b+is)} \, {}_3F_2\left(\begin{matrix} -n, & n+1-r, & q-is \\ u+1, & -p+1-is & \end{matrix} \Big| \, 1 \right) \\
&\qquad \times \overline{{}_3F_2\left(\begin{matrix} -m, & m+1-c, & a-is \\ d+1, & -b+1-is & \end{matrix} \Big| \, 1 \right)} \, ds
\end{aligned} \tag{8.34}
$$

On the other hand, if in the left hand side of (8.34) we take

$$
u = d = q+a-1 \quad \text{and} \quad r = c = p+b+1 \ , \tag{8.35}
$$

then according to the orthogonality relation of polynomials $M_n^{(p,q)}(x)$ given above, relation (8.34) reads as

$$
\begin{aligned}
&\frac{(2\pi) \ n!(p+b-n)!(q+a-1+n)!}{(p+b-2n)(p+q+a+b-n-1)!} \frac{\Gamma^2(q+a)\Gamma(p+q)\Gamma(a+b)}{(-1)^{n+m}\Gamma(q+a+n)\Gamma(q+a+m)} \delta_{n,m} = \\
&\int_{-\infty}^{\infty} \Gamma(q-is)\Gamma(p+is)\overline{\Gamma(a-is)\Gamma(b+is)} \, {}_3F_2\left(\begin{matrix} -n, & n-p-b, & q-is \\ q+a, & -p+1-is & \end{matrix} \Big| \, 1 \right) \\
&\qquad \times \overline{{}_3F_2\left(\begin{matrix} -m, & m-p-b, & a-is \\ q+a, & -b+1-is & \end{matrix} \Big| \, 1 \right)} \, ds \ .
\end{aligned} \tag{8.36}
$$

Theorem 1. The special function

$$A_n(x;a,b,c,d)=\frac{\Gamma(a+d+n)}{\Gamma(a+d)}\ {}_3F_2\left(\begin{matrix}-n, & n-b-c, & d-x\\ a+d, & -c+1-x\end{matrix}\middle|\,1\right) \quad (8.37)$$

has a finite orthogonality property as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(a+ix)\Gamma(b-ix)\Gamma(c+ix)\Gamma(d-ix)A_n(ix;a,b,c,d)A_m(-ix;d,c,b,a)\ dx$$
$$=\frac{n!}{(b+c-2n)}\frac{\Gamma(a+d+n)\Gamma(b+c+1-n)\Gamma(c+d)\Gamma(a+b)}{\Gamma(a+b+c+d-n)}\ \delta_{n,m} \quad (8.38)$$

where $a+d>-n,\quad b+c>2n,\quad a+b>0$ and $c+d>0$.

Remark 1. (i) Replacing $n = m = 0$ in (8.38) gives the Barnes' first lemma from 1908, see e.g. Bailey [20] or Whittaker and Watson [76] for the original proof by Barnes.
(ii) The weight function of the orthogonality relation (8.38) is positive for $a=d$, $b=c$ or $a=b$, $c=d$.
Similarly, the mentioned approach can be applied to the finite orthogonal polynomials

$$N_n^{(p)}(x)=(-1)^n\sum_{k=0}^{n}k!\binom{p-(n+1)}{k}\binom{n}{n-k}(-x)^k,$$

that satisfy the orthogonality relation

$$\int_0^{\infty}x^{-p}e^{\frac{-1}{x}}N_n^{(p)}(x)N_m^{(p)}(x)\,dx=\left(\frac{n!\,(p-(n+1))!}{p-(2n+1)}\right)\delta_{n,m}\quad\text{for}\quad m,n=0,1,2,...,N<\frac{p-1}{2}\ .$$

Thus, if we define the sequences

$$u(x)=\exp(-px-\frac{1}{2}e^{-x})N_n^{(q)}(e^x)\quad\text{and}\quad v(x)=\exp(-rx-\frac{1}{2}e^{-x})N_m^{(u)}(e^x), \quad (8.39)$$

and take the Fourier transform from $u(x)$, then we get

$$\begin{aligned}\mathfrak{F}(u(x))&=\int_{-\infty}^{\infty}e^{-isx}e^{-(px+\frac{1}{2}e^{-x})}N_n^{(q)}(e^x)dx=\int_0^{\infty}t^{-is-1-p}e^{\frac{-1}{2t}}\ N_n^{(q)}(t)dt\\ &=(-1)^n n!(q-n-1)!\ \sum_{k=0}^{n}\frac{(-1)^k}{(q-n-1-k)!k!(n-k)!}\left(\int_0^{\infty}t^{-is-1-p+k}e^{\frac{-1}{2t}}dt\right)\\ &=(-1)^n2^{p+is}\Gamma(p+is)\sum_{k=0}^{n}\frac{(-n)_k(n+1-q)_k}{(1-p-is)_k}\frac{(2^{-1})^k}{k!}\\ &=(-1)^n2^{p+is}\Gamma(p+is)\ \ {}_2F_1\left(\begin{matrix}-n, & n+1-q\\ 1-p-is\end{matrix}\middle|\,\frac{1}{2}\right).\end{aligned} \quad (8.40)$$

In relation (8.40) the following definite integral has been used

$$\int_0^\infty t^{-is-1-p+k} e^{\frac{-1}{2t}} dt = 2^{p+is-k}\,\Gamma(p+is-k)\ . \tag{8.41}$$

Now, according to definitions (8.39) apply the Parseval's identity again to get (8.42)

$$2\pi\int_{-\infty}^{\infty} e^{-(p+r)x} e^{-e^{-x}} N_n^{(q)}(e^x) N_m^{(u)}(e^x)dx = 2\pi\int_0^\infty t^{-(p+r+1)} e^{\frac{-1}{t}}\ N_n^{(q)}(t)\ N_m^{(u)}(t)dt =$$

$$(-1)^{n+m} 2^{p+r}\int_{-\infty}^{\infty}\Gamma(p+is)\overline{\Gamma(r+is)}\,{}_2F_1\left(\begin{matrix}-n, & n+1-q\\ & 1-p-is\end{matrix}\middle|\ \frac{1}{2}\right)\overline{{}_2F_1\left(\begin{matrix}-m, & m+1-u\\ & 1-r-is\end{matrix}\middle|\ \frac{1}{2}\right)}\ ds\ .$$

By assuming

$$q = u = p + r + 1 \tag{8.43}$$

in (8.42) and noting the orthogonality relation of $N_n^{(p)}(x)$ given above we can finally deduce the following theorem.

Theorem 2. The special function

$$B_n(x;a,b) = {}_2F_1\left(\begin{matrix}-n, & n-a-b\\ & -a+1-x\end{matrix}\middle|\ \frac{1}{2}\right) \tag{8.44}$$

has a finite orthogonality relation as

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(a+ix)\Gamma(b-ix)\,B_n(ix;a,b)B_m(-ix;b,a)\,dx = \frac{n!\,\Gamma(a+b+1-n)}{(a+b-2n)\,2^{a+b}}\,\delta_{n,m} \tag{8.45}$$

if $a + b > 2n$.

Remark 2. (i) If we put $n = m = 0$ in (8.45) then we obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(a+ix)\Gamma(b-ix)\ dx = \frac{\Gamma(a+b)}{2^{a+b}}\ . \tag{8.46}$$

(ii) The weight function of (8.46) is positive if $a = b$.

8.3.1. Evaluating a complicated integral and a conjecture

By applying the Ramanujan integral [62] (8.47)

$$\int_{-\infty}^{\infty}\frac{dx}{\Gamma(a+x)\Gamma(b+x)\Gamma(c-x)\Gamma(d-x)} = \frac{\Gamma(a+b+c+d-3)}{\Gamma(a+c-1)\Gamma(a+d-1)\Gamma(b+c-1)\Gamma(b+d-1)}$$

we can obtain the explicit value of the following definite integral (8.48)

$$I_n(m)=\int_{-\infty}^{\infty} {}_3F_2\left(\begin{matrix}-n, & n-b-c, & a-x \\ a+d, & -c+1-x & \end{matrix}\ \middle|\ 1\right)\frac{(d-x)_m}{(1-b-x)_m}$$
$$\times\frac{dx}{\Gamma(1-a+x)\Gamma(1-d+x)\Gamma(1-c-x)\Gamma(1-b-x)}\ .$$

For this purpose, we have

(8.49)

$$I_n(m)=\sum_{k=0}^{n}\frac{(-n)_k(n-b-c)_k}{(a+d)_k\,k!}\times$$
$$\times\int_{-\infty}^{\infty}\frac{(a-x)_k(d-x)_m}{(1-c-x)_k(1-b-x)_m}\frac{dx}{\Gamma(1-a+x)\Gamma(1-d+x)\Gamma(1-c-x)\Gamma(1-b-x)}$$

and since

$$\frac{(a-x)_k}{\Gamma(1-a+x)}=\frac{(-1)^k}{\Gamma(1-a+x-k)}\quad\text{and}\quad\frac{(d-x)_m}{\Gamma(1-d+x)}=\frac{(-1)^m}{\Gamma(1-d+x-m)} \tag{8.50}$$

according to Ramanujan's integral defined in (8.47) $I_n(m)$ can be simplified towards

(8.51)

$$I_n(m)=\sum_{k=0}^{n}\frac{(-n)_k(n-b-c)_k}{(a+d)_k\,k!}\ (-1)^{k+m}\times$$
$$\times\int_{-\infty}^{\infty}\frac{dx}{\Gamma(1-a+x-k)\Gamma(1-d+x-m)\Gamma(1-c-x+k)\Gamma(1-b-x+m)}$$
$$=\sum_{k=0}^{n}\frac{(-1)^{k+m}(-n)_k(n-b-c)_k\,\Gamma(1-a-b-c-d)}{(a+d)_k\,k!\,\Gamma(1-a-c)\Gamma(1-b-d)\Gamma(1-a-b+m-k)\Gamma(1-c-d-m+k)}$$
$$=\frac{(-1)^m\Gamma(1-a-b-c-d)\ \ {}_3F_2\left(\begin{matrix}-n, & n-b-c, & a+b-m \\ a+d, & 1-c-d-m & \end{matrix}\ \middle|\ 1\right)}{\Gamma(1-a-c)\Gamma(1-b-d)\Gamma(1-a-b+m)\Gamma(1-c-d-m)}$$

On the other hand, the Gosper-Saalschütz identity [75] implies that if

$$e=a+b+c+1-d$$

then

$${}_3F_2\left(\begin{matrix}a, & b, & c \\ d, & & e\end{matrix}\ \middle|\ 1\right)=\frac{\pi^2}{\cos(d\pi)\cos(e\pi)+\cos(a\pi)\cos(b\pi)\cos(c\pi)}\times$$
$$\times\frac{\Gamma(d)}{\Gamma(d-a)\Gamma(d-b)\Gamma(d-c)}\frac{\Gamma(e)}{\Gamma(e-a)\Gamma(e-b)\Gamma(e-c)}\ . \tag{8.52}$$

Therefore, the final value of the definite integral $I_n(m)$ is obtained as

$$I_n(m)=\frac{\pi^2\Gamma(a+d)}{\cos((a+b)\pi)\cos((b+c)\pi)-\cos((a+d)\pi)\cos((c+d)\pi)\Gamma(1-a-c)\Gamma(1-b-d)}\times$$
$$\frac{1}{\Gamma(a+d+n)\Gamma(a+b+c+d-n)\Gamma(d-b+m)\Gamma(1-c-d-m+n)\Gamma(1+b-d-m-n)\Gamma(1-a-b+m)} \tag{8.53}$$

But we evaluated the complicated integral (8.48) to be able to claim that the function $A_n(x;a,b,c,d)$ defined in Theorem 1 might essentially be orthogonal with respect to the Ramanujan integral. In other words, we conjecture that

$$\begin{aligned}&\int_{-\infty}^{\infty} {}_3F_2\left(\begin{matrix}-n, & n-b-c, & d-x\\ a+d, & -b+1-x & \end{matrix}\ \Big|\ 1\right){}_3F_2\left(\begin{matrix}-m, & m-b-c, & a-x\\ a+d, & -c+1-x & \end{matrix}\ \Big|\ 1\right)\\ &\qquad\times\frac{dx}{\Gamma(1-a+x)\Gamma(1-d+x)\Gamma(1-c-x)\Gamma(1-b-x)}=K_n\delta_{n,m}\ .\end{aligned} \tag{8.54}$$

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