A new type of Euler polynomials and numbers

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Abstract

By defining two specific exponential generating functions, we introduce a kind of Euler polynomials and study its basic properties in detail. As an application of the introduced polynomials, we use them in computing some new series of Taylor type that contain the associated Euler numbers $E_n(0)$ where $E_n(x)$ is the Euler polynomial.

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1 Introduction

The Appell polynomials $A_n(x)$ defined by

$$f(t)\mathrm{e}^{xt} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}, \tag{1}$$

where f is a formal power series in t, have found remarkable applications in different branches of mathematics, theoretical physics and chemistry [1, 2]. One of the special cases of Appell polynomials are Euler polynomials $E_n(x)$ where $E_n = 2^n E_n(\frac{1}{2})$ are usually known as Euler numbers. These numbers and polynomials have a close relationship with Bernoulli numbers and polynomials [5].

The Euler numbers are defined by an exponential generating function as [3, 7]

$$\frac{1}{\cosh t} = \frac{2\mathrm{e}^t}{\mathrm{e}^{2t}+1} = \sum_{n=0}^{\infty} E_n\frac{t^n}{n!} \qquad (|t| < \frac{\pi}{2}), \tag{2}$$

where, for instance, we have

$E_0 = 1,\ E_2 = -1,\ E_4 = 5,\ E_6 = -61,\ E_8 = 1385,\ \cdots$ and in general $E_{2n-1} = 0\ (n \in \mathbb{N})$.

Computing the finite sums of powers of integers, such as $\sum_{k=1}^{n} k$ and $\sum_{k=1}^{n} k^2$, was one of the main interests for mathematicians in the $17th$ century. Although the closed forms of these sums were known, the sum $1^m + 2^m + \cdots + n^m$ was not known in the general case. It was Bernoulli who could solve this problem and introduced Bernoulli numbers to evaluate the sum

$$S_m(n) = \sum_{k=1}^{n} k^m = 1^m + 2^m + \cdots + n^m.$$

In the sequel, Euler introduced his numbers to evaluate the alternating sum

$$A_m(n) = \sum_{k=1}^{n} (-1)^{n-k} k^m = n^m - (n-1)^m + \cdots + (-1)^{n-1}.$$

As a special case of Appell polynomials, if $f(t) = \frac{2}{\mathrm{e}^t+1}$ in (1), the Euler polynomials are generated by

$$\frac{2\mathrm{e}^{xt}}{\mathrm{e}^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad (|t| < \pi). \tag{3}$$

For $x = 0$ in (3), the associated Euler numbers $E_n(0)$ are given by the formula

$$\frac{2}{\mathrm{e}^t + 1} = \sum_{n=0}^{\infty} E_n(0) \frac{t^n}{n!} \quad (|t| < \pi). \tag{4}$$

Differentiating both sides of (3) with respect to x yields

$$\frac{\mathrm{d}}{\mathrm{d}x} E_n(x) = n E_{n-1}(x) \quad \text{and} \quad \deg\, E_n(x) = n.$$

Consequently we have

$$\int_a^b E_n(x)\ \mathrm{d}x = \frac{E_{n+1}(b) - E_{n+1}(a)}{n+1}.$$

Recently in [3], the authors have introduced a generalization of poly-Euler polynomials with three parameters and established some of their properties. They have also introduced a more general form of multi poly-Euler polynomials and obtained some identities similar to those of the generalized poly-Euler polynomials.
In [4], poly-Euler numbers with negative index are treated and their parity is shown as the main theorem. The divisibility of these numbers is also discussed therein.

This paper is organized as follows: In the next section, we introduce two kinds of Euler polynomials and define their exponential generating functions. We also study their basic properties and prove them. Finally in section 3, an application of these polynomials are given to compute some new series of Taylor type involving the associated Euler numbers $E_n(0)$.

2 A new type of Euler polynomials

We begin our treatment with the defining (binomial) convolution of two sequences. If a_n and b_n are two sequences with the following exponential generating functions

$$A(t)=\sum_{n=0}^{\infty}a_n\frac{t^n}{n!},\quad \text{and}\quad B(t)=\sum_{n=0}^{\infty}b_n\frac{t^n}{n!},$$

then their convolution is defined as

$$c_n=a_n*b_n=\sum_{k=0}^{n}\binom{n}{k}a_kb_{n-k}.$$

Hence, the corresponding exponential generating function takes the form

$$C(t)=A(t)B(t)=\sum_{n=0}^{\infty}c_n\frac{t^n}{n!}.$$

Now, let us define two bivariate polynomials as follows:
If $p,q\in\mathbb{R}$, it is known that the Taylor expansion of the two functions $\mathrm{e}^{pt}\cos qt$ and $\mathrm{e}^{pt}\sin qt$ are, respectively, as follows [6]

$$\mathrm{e}^{pt}\cos qt=\sum_{n=0}^{\infty}C_n(p,q)\frac{t^n}{n!}, \tag{5}$$

and

$$\mathrm{e}^{pt}\sin qt=\sum_{n=0}^{\infty}S_n(p,q)\frac{t^n}{n!}, \tag{6}$$

where

$$C_n(p,q)=\sum_{k=0}^{[\frac{n}{2}]}(-1)^k\binom{n}{2k}p^{n-2k}q^{2k}, \tag{7}$$

and

$$S_n(p,q)=\sum_{k=0}^{[\frac{n-1}{2}]}(-1)^k\binom{n}{2k+1}p^{n-2k-1}q^{2k+1}. \tag{8}$$

By considering $C_n(p,q)$, $S_n(p,q)$ and the associated Euler numbers $E_n(0)$ in (4), one can introduce two types of Euler polynomials as

$$E_n^{(c)}(p,q)=E_n(0)*C_n(p,q), \tag{9}$$

and

$$E_n^{(s)}(p,q) = E_n(0) * S_n(p,q), \tag{10}$$

whose exponential generating functions are, respectively, given by

$$\frac{2\mathrm{e}^{pt}}{\mathrm{e}^t+1}\cos qt = \sum_{n=0}^{\infty} E_n^{(c)}(p,q)\frac{t^n}{n!} \quad (|t|<\pi), \tag{11}$$

and

$$\frac{2\mathrm{e}^{pt}}{\mathrm{e}^t+1}\sin qt = \sum_{n=0}^{\infty} E_n^{(s)}(p,q)\frac{t^n}{n!} \quad (|t|<\pi). \tag{12}$$

Hence, we can be represent these polynomials as follows

$$E_n^{(c)}(p,q) = \sum_{k=0}^{n} \binom{n}{k} E_k(0) C_{n-k}(p,q), \tag{13}$$

and

$$E_n^{(s)}(p,q) = \sum_{k=0}^{n} \binom{n}{k} E_k(0) S_{n-k}(p,q). \tag{14}$$

Note that $E_n^{(c)}(p,0) = E_n(p)$. For instance, we have

$$E_0^{(c)}(p,q) = 1,$$
$$E_1^{(c)}(p,q) = p - \frac{1}{2},$$
$$E_2^{(c)}(p,q) = p^2 - p - q^2,$$
$$E_3^{(c)}(p,q) = p^3 - \frac{3}{2}p^2 - 3q^2p + \frac{3}{2}q^2 + \frac{1}{4},$$
$$E_4^{(c)}(p,q) = p^4 - 2p^3 - 6q^2p^2 + (6q^2+1)p + q^4,$$
$$E_5^{(c)}(p,q) = p^5 - \frac{5}{2}p^4 - 10q^2p^3 + (15q^2 + \frac{5}{2})p^2 + 5q^4p - \frac{5}{2}q^4 - \frac{5}{2}q^2 - \frac{1}{2},$$

and

$$E_0^{(s)}(p,q) = 0,$$
$$E_1^{(s)}(p,q) = q,$$
$$E_2^{(s)}(p,q) = 2qp - q,$$
$$E_3^{(s)}(p,q) = 3qp^2 - 3qp - q^3,$$
$$E_4^{(s)}(p,q) = 4qp^3 - 6qp^2 - 4q^3p + 2q^3 + q,$$
$$E_5^{(s)}(p,q) = 5qp^4 - 10qp^3 - 10q^3p^2 + (10q^3+5q)p + q^5,$$

Proposition 2.1. *For every* $n \in \mathbb{Z}^+$ *we have*

$$E_n^{(c)}(1-p,q) = (-1)^n E_n^{(c)}(p,q), \tag{15}$$

and

$$E_n^{(s)}(1-p,q) = (-1)^{n+1} E_n^{(s)}(p,q). \tag{16}$$

Proof. Applying the exponential generating function (11) gives

$$\sum_{n=0}^{\infty} E_n^{(c)}(1-p,q)\frac{t^n}{n!} = \frac{2\mathrm{e}^{(1-p)t}}{\mathrm{e}^t+1}\cos qt,$$

as well as

$$\sum_{n=0}^{\infty}(-1)^n E_n^{(c)}(p,q)\frac{t^n}{n!} = \frac{2\mathrm{e}^{-pt}}{\mathrm{e}^{-t}+1}\cos(-qt) = \frac{2\mathrm{e}^{(1-p)t}}{\mathrm{e}^t+1}\cos qt.$$

The property (16) can be similarly proved. □

Corollary 2.1. *Relations* (15) *and* (16) *imply that*

$$E_{2n+1}^{(c)}\left(\frac{1}{2},q\right) = 0,$$

$$E_{2n}^{(s)}\left(\frac{1}{2},q\right) = 0,$$

$$\int_0^1 E_{2n+1}^{(c)}(p,q)\ \mathrm{d}p = 0,$$

and

$$\int_0^1 E_{2n}^{(s)}(p,q)\ \mathrm{d}p = 0.$$

Proposition 2.2. *For every* $n \in \mathbb{Z}^+$ *, the following identities hold*

$$E_n^{(c)}(1+p,q) + E_n^{(c)}(p,q) = 2C_n(p,q), \tag{17}$$

and

$$E_n^{(s)}(1+p,q) + E_n^{(s)}(p,q) = 2S_n(p,q). \tag{18}$$

Proof. We have

$$\sum_{n=0}^{\infty} E_n^{(c)}(1+p,q)\frac{t^n}{n!} = \frac{2\mathrm{e}^{pt}(\mathrm{e}^t+1-1)}{\mathrm{e}^t+1}\cos qt = 2\mathrm{e}^{pt}\cos qt - \frac{2\mathrm{e}^{pt}}{\mathrm{e}^t+1}\cos qt$$
$$= \sum_{n=0}^{\infty} 2C_n(p,q)\frac{t^n}{n!} - \sum_{n=0}^{\infty} E_n^{(c)}(p,q)\frac{t^n}{n!},$$

which proves (17), and the proof of (18) is similar. □

Corollary 2.2. *Relations* (17) *and* (18) *first imply that*

$$E_{2n}^{(c)}(1,q) + E_{2n}^{(c)}(0,q) = 2(-1)^n q^{2n},$$

and

$$E_{2n+1}^{(s)}(1,q) + E_{2n+1}^{(s)}(0,q) = 2(-1)^n q^{2n+1}.$$

Hence, by applying Proposition 2.1 we obtain

$$E_{2n}^{(c)}(0,q) = E_{2n}^{(c)}(1,q) = (-1)^n q^{2n},$$

and

$$E_{2n+1}^{(s)}(0,q) = E_{2n+1}^{(s)}(1,q) = (-1)^n q^{2n+1}.$$

Proposition 2.3. *For every* $n \in \mathbb{Z}^+$ *, the following identities hold*

$$E_n^{(c)}(p+r,q) = \sum_{k=0}^{n} \binom{n}{k} E_k^{(c)}(p,q) r^{n-k}, \tag{19}$$

and

$$E_n^{(s)}(p+r,q) = \sum_{k=0}^{n} \binom{n}{k} E_k^{(s)}(p,q) r^{n-k}. \tag{20}$$

Proof. It is enough to apply (11) to obtain

$$\sum_{n=0}^{\infty} E_n^{(c)}(p+r,q)\frac{t^n}{n!} = \left(\frac{2\mathrm{e}^{pt}}{\mathrm{e}^t+1}\cos qt\right)\mathrm{e}^{rt} = \left(\sum_{n=0}^{\infty} E_n^{(c)}(p,q)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} r^n\frac{t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} E_k^{(c)}(p,q) r^{n-k}\right)\frac{t^n}{n!},$$

which proves (19). The result (20) can be similarly proved. □

Corollary 2.3. *For every* $n \in \mathbb{Z}^+$ *we have*

$$E_n^{(c)}(p,q) = \begin{cases} \sum_{k=0}^{m} \binom{2m}{2k} E_{2m-2k}^{(c)}\left(\frac{1}{2}, q\right) \left(p - \frac{1}{2}\right)^{2k} & n = 2m, \\ \sum_{k=0}^{m} \binom{2m+1}{2k+1} E_{2m-2k}^{(c)}\left(\frac{1}{2}, q\right) \left(p - \frac{1}{2}\right)^{2k+1} & n = 2m+1, \end{cases}$$

and

$$E_n^{(s)}(p,q) = \begin{cases} \sum_{k=0}^{m-1} \binom{2m}{2k+1} E_{2m-1-2k}^{(s)}\left(\frac{1}{2}, q\right) \left(p - \frac{1}{2}\right)^{2k+1} & n = 2m, \\ \sum_{k=0}^{m} \binom{2m+1}{2k} E_{2m+1-2k}^{(s)}\left(\frac{1}{2}, q\right) \left(p - \frac{1}{2}\right)^{2k} & n = 2m+1. \end{cases}$$

Corollary 2.4. *We have*

$$E_n^{(c)}(p,q) + \sum_{k=0}^{n} \binom{n}{k} E_k^{(c)}(p,q) = 2C_n(p,q), \tag{21}$$

and

$$E_n^{(s)}(p,q) + \sum_{k=0}^{n} \binom{n}{k} E_k^{(s)}(p,q) = 2S_n(p,q). \tag{22}$$

The result (21) *follows from* (17) *and* (19), *and to obtain* (22), *it is enough to combine* (18) *and* (20).

Corollary 2.5. *Relations* (21) *and* (22) *imply that*

$$E_n^{(c)}(0,q) + \sum_{k=0}^{n} \binom{n}{k} E_k^{(c)}(0,q) = 2q^n \cos\frac{n\pi}{2} = \begin{cases} 2(-1)^m q^{2m} & n = 2m, \\ 0 & n = 2m+1, \end{cases}$$

and

$$E_n^{(s)}(0,q) + \sum_{k=0}^{n} \binom{n}{k} E_k^{(s)}(0,q) = 2q^n \sin\frac{n\pi}{2} = \begin{cases} 0 & n = 2m, \\ 2(-1)^m q^{2m+1} & n = 2m+1. \end{cases}$$

Proposition 2.4. *For every* $n \in \mathbb{N}$, *the following partial differential equations hold*

$$\frac{\partial}{\partial p} E_n^{(c)}(p,q) = nE_{n-1}^{(c)}(p,q), \tag{23}$$

$$\frac{\partial}{\partial q} E_n^{(c)}(p,q) = -nE_{n-1}^{(s)}(p,q), \tag{24}$$

$$\frac{\partial}{\partial p} E_n^{(s)}(p,q) = nE_{n-1}^{(s)}(p,q), \tag{25}$$

and

$$\frac{\partial}{\partial q}E_n^{(s)}(p,q)=nE_{n-1}^{(c)}(p,q). \tag{26}$$

Proof. Let us just prove (23), as the other equations (24), (25) and (26) can be similarly derived. To prove (23), apply relation (11) to get

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\partial E_n^{(c)}(p,q)}{\partial p}\frac{t^n}{n!}&=\frac{2t\mathrm{e}^{pt}}{\mathrm{e}^t+1}\cos qt=\sum_{n=0}^{\infty}E_n^{(c)}(p,q)\frac{t^{n+1}}{n!}\\&=\sum_{n=1}^{\infty}E_{n-1}^{(c)}(p,q)\frac{t^n}{(n-1)!}=\sum_{n=1}^{\infty}nE_{n-1}^{(c)}(p,q)\frac{t^n}{n!}.\end{aligned}$$

□

Proposition 2.5. *If* $E_n^{(c)}(p,q)$ *and* $E_n^{(s)}(p,q)$ *are sorted in terms of the variable* p *, then they are polynomials of degree* n *and* $n-1$ *, respectively, such that we have*

$$E_n^{(c)}(p,q)=p^n-\frac{n}{2}p^{n-1}+\cdots, \tag{27}$$

and

$$E_n^{(s)}(p,q)=nqp^{n-1}-\binom{n}{2}qp^{n-2}+\cdots. \tag{28}$$

Conversely, if they are sorted in terms of the variable q *, then*

$$E_n^{(c)}(p,q)=\begin{cases}(-1)^{\frac{n-1}{2}}n\left(p-\frac{1}{2}\right)q^{n-1}+(-1)^{\frac{n+1}{2}}\binom{n}{3}\left(p^3-\frac{3}{2}p^2+\frac{1}{4}\right)q^{n-3}+\cdots\ (n\ \textit{odd}),\\[2ex](-1)^{\frac{n}{2}}q^n+(-1)^{\frac{n+2}{2}}\binom{n}{2}(p^2-p)q^{n-2}+\cdots\ (n\ \textit{even}),\end{cases} \tag{29}$$

and

$$E_n^{(s)}(p,q)=\begin{cases}(-1)^{\frac{n+2}{2}}n\left(p-\frac{1}{2}\right)q^{n-1}+(-1)^{\frac{n}{2}}\binom{n}{3}\left(p^3-\frac{3}{2}p^2+\frac{1}{4}\right)q^{n-3}+\cdots\ (n\ \textit{even}),\\[2ex](-1)^{\frac{n-1}{2}}q^n+(-1)^{\frac{n+1}{2}}\binom{n}{2}(p^2-p)q^{n-2}+\cdots\ (n\ \textit{odd}).\end{cases} \tag{30}$$

Proof. We first prove (27) by induction. It is known from (21) that

$$E_0^{(c)}(p,q)=1,\ E_1^{(c)}(p,q)=p-\frac{1}{2}\quad\text{and}\quad E_2^{(c)}(p,q)=p^2-p-q^2.$$

Therefore (27) holds for $n = 0, 1, 2$. Now, assume that it is valid for $n-1$. By noting (23), we have

$$\frac{\partial}{\partial p} E_n^{(c)}(p,q) = np^{n-1} - \frac{n(n-1)}{2} p^{n-2} + \cdots .$$

To complete the proof, it is enough to integrate both sides of the above equation with respect to the variable p to get the result (27). By referring to relation (26), the result (28) can be similarly derived.
To prove (29), suppose that it first holds for $0, 1, \ldots, n-1$. If $n = 2m$, then from (21) we have

$$E_{2m}^{(c)}(p,q) = -\frac{1}{2} \sum_{k=0}^{2m-1} \binom{2m}{k} E_k^{(c)}(p,q) + \sum_{k=0}^{m} (-1)^k \binom{2m}{2k} p^{2m-2k} q^{2k}. \tag{31}$$

Hence, the coefficient of q^{2m} on the right hand side of (31) is equal to

$$(-1)^m \binom{2m}{2m} p^{2m-2m} = (-1)^m,$$

and the coefficient of q^{2m-2} is equal to

$$-\frac{1}{2}\left(\binom{2m}{2m-1}(-1)^{m-1}(2m-1)\left(p-\frac{1}{2}\right) + \binom{2m}{2m-2}(-1)^{m-1}\right) \\ + (-1)^{m-1}\binom{2m}{2m-2} p^2 = (-1)^{m+1}\binom{2m}{2}(p^2-p).$$

So, (29) is true for $n = 2m$. In the second case, taking $n = 2m+1$ in (21) gives

$$E_{2m+1}^{(c)}(p,q) = -\frac{1}{2} \sum_{k=0}^{2m} \binom{2m+1}{k} E_k^{(c)}(p,q) + \sum_{k=0}^{m} (-1)^k \binom{2m+1}{2k} p^{2m+1-2k} q^{2k}. \tag{32}$$

Hence, the coefficient of q^{2m} in the right hand side of (32) is equal to

$$\frac{-1}{2m+2}\binom{2m+2}{2m}(-1)^m + (-1)^m \binom{2m+1}{2m} p = (-1)^m (2m+1)\left(p - \frac{1}{2}\right),$$

and the coefficient of q^{2m-2} is equal to

$$-\frac{1}{2}\left(\binom{2m+1}{2m}(-1)^{m+1}\binom{2m}{2}(p^2-p) + \binom{2m+1}{2m-1}(-1)^{m-1}(2m-1)\left(p-\frac{1}{2}\right)\right. \\ \left. + \binom{2m+1}{2m-2}(-1)^{m-1}\right) + (-1)^{m-1}\binom{2m+1}{2m-2}p^3 = (-1)^{m+1}\binom{2m+1}{3}\left(p^3 - \frac{3}{2}p^2 + \frac{1}{4}\right),$$

which completes the proof of (29). By combining (26) and (29), we can also obtain the result (30). □

Proposition 2.6. *The following identities hold*

$$E_n^{(c)}(p,q)=\sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{n}{2k}E_{n-2k}^{(c)}(p,0)q^{2k}, \tag{33}$$

and

$$E_n^{(s)}(p,q)=\sum_{k=0}^{\left[\frac{n-1}{2}\right]}(-1)^k\binom{n}{2k+1}E_{n-2k-1}^{(c)}(p,0)q^{2k+1}, \tag{34}$$

in which $E_{n-2k}^{(c)}(p,0)=E_{n-2k}(p)$ *and* $E_{n-2k-1}^{(c)}(p,0)=E_{n-2k-1}(p)$ *are Euler polynomials.*

Proof. According to (24) and (26), first we have

$$\frac{\partial^{2k}}{\partial q^{2k}}E_n^{(c)}(p,q)=(-1)^k\frac{n!}{(n-2k)!}E_{n-2k}^{(c)}(p,q) \quad \text{for} \quad k=0,1,\ldots,\left[\frac{n}{2}\right],$$

and

$$\frac{\partial^{2k+1}}{\partial q^{2k+1}}E_n^{(c)}(p,q)=(-1)^{k+1}\frac{n!}{(n-2k-1)!}E_{n-2k-1}^{(s)}(p,q) \quad \text{for} \quad k=0,1,\ldots,\left[\frac{n-2}{2}\right],$$

because $E_n^{(c)}(p,q)$ is a polynomial of degree n for even n and of degree $n-1$ for odd n in terms of the variable q according to Proposition 2.5. The Taylor expansion of $E_n^{(c)}(p,q)$ gives

$$E_n^{(c)}(p,q+h)=\sum_{k=0}^{n}\frac{1}{k!}\frac{\partial^k}{\partial q^k}E_n^{(c)}(p,q)h^k,$$

in which $h\in\mathbb{R}$. Since $E_n^{(s)}(p,0)=0$ for every $n\in\mathbb{Z}^+$, By replacing $q=0$ and $h=q$, we obtain the relation (33). In a similar way, equality (34) can be derived. □

Proposition 2.7. *If* m *is an odd number and* $n\in\mathbb{Z}^+$, *then*

$$E_n^{(c)}(mp,q)=m^n\sum_{k=0}^{m-1}(-1)^kE_n^{(c)}\left(p+\frac{k}{m},\frac{q}{m}\right), \tag{35}$$

and

$$E_n^{(s)}(mp,q)=m^n\sum_{k=0}^{m-1}(-1)^kE_n^{(s)}\left(p+\frac{k}{m},\frac{q}{m}\right). \tag{36}$$

Proof. First, it is known that

$$\sum_{n=0}^{\infty} E_n^{(c)}\left(p+\frac{k}{m},\frac{q}{m}\right)\frac{t^n}{n!}=\frac{2\mathrm{e}^{\left(p+\frac{k}{m}\right)t}}{\mathrm{e}^t+1}\cos\left(\frac{q}{m}t\right).$$

If we multiply both sides of the above relation by $(-1)^k$ and take a sum over k from 0 to $m-1$, then we obtain

$$\sum_{k=0}^{m-1}(-1)^k\left(\sum_{n=0}^{\infty} E_n^{(c)}\left(p+\frac{k}{m},\frac{q}{m}\right)\frac{t^n}{n!}\right)=\frac{2\mathrm{e}^{pt}}{\mathrm{e}^t+1}\cos\left(\frac{q}{m}t\right)\sum_{k=0}^{m-1}\left(-\mathrm{e}^{\frac{t}{m}}\right)^k$$
$$=\frac{2\mathrm{e}^{mp\frac{t}{m}}}{\mathrm{e}^{\frac{t}{m}}+1}\cos\left(q\frac{t}{m}\right)\frac{1-(-1)^m\mathrm{e}^t}{\mathrm{e}^t+1}.$$

Since m is an odd number, relation (35) is true, and in a similar way, equality (36) can be proved. □

For instance, for $m=3$, relations (35) and (36) read as

$$E_n^{(c)}(1,3q)=3^n\left(E_n^{(c)}\left(\frac{1}{3},q\right)-E_n^{(c)}\left(\frac{2}{3},q\right)+E_n^{(c)}(1,q)\right),$$

and

$$E_n^{(s)}(1,3q)=3^n\left(E_n^{(s)}\left(\frac{1}{3},q\right)-E_n^{(s)}\left(\frac{2}{3},q\right)+E_n^{(s)}(1,q)\right).$$

Proposition 2.8. *For every $n\in\mathbb{N}$ and $q\in\mathbb{R}$, the two following Propositions are valid:*

$\mathcal{P}_n$: *The function $p\mapsto(-1)^nE_{2n-1}^{(c)}(p,q)$ is positive on $\left(0,\frac{1}{2}\right)$ and negative on $\left(\frac{1}{2},1\right)$. Moreover, $p=\frac{1}{2}$ is a unique simple root on $(0,1)$, i.e. the aforesaid function has no zero in the intervals $\left(0,\frac{1}{2}\right)$ and $\left(\frac{1}{2},1\right)$.*

$\mathcal{Q}_n$: *The function $p\mapsto(-1)^nE_{2n}^{(c)}(p,q)$ is strictly increasing on $\left[0,\frac{1}{2}\right]$ and strictly decreasing on $\left[\frac{1}{2},1\right]$ and always takes a positive value at $p=\frac{1}{2}$.*

Proof. The Proposition $\mathcal{P}_1$ is clear, because $-E_1^{(c)}(p,q)=-\left(p-\frac{1}{2}\right)=-p+\frac{1}{2}$. Now define $f(p)=(-1)^nE_{2n}^{(c)}(p,q)$ to get $f'(p)=2n(-1)^nE_{2n-1}^{(c)}(p,q)$. By noting $\mathcal{P}_n$, we see that f is strictly increasing on $[0,\frac{1}{2}]$ and strictly decreasing on $\left[\frac{1}{2},1\right]$. Moreover, since $f(0)=q^{2n}\geq 0$ (by Corollary 2.2) and f has a maximum in $p=\frac{1}{2}$, then one can conclude that $f\left(\frac{1}{2}\right)>0$.

Finally define $g(p)=(-1)^{n+1}E_{2n+1}^{(c)}(p,q)$ to get $g'(p)=-(2n+1)(-1)^nE_{2n}^{(c)}(p,q)$. Since $g'(0)=g'(1)=-(2n+1)q^{2n}\leq 0$, and $E_{2n}^{(c)}(1-p,q)=E_{2n}^{(c)}(p,q)$, so by noting $\mathcal{Q}_n$ we have $\forall p\in(0,1):\ g'(p)<0$. Therefore, g takes the following table of variations
As $g\left(\frac{1}{2}\right)=0$ (by Corollary 2.1) and $g'\left(\frac{1}{2}\right)<0$, then $p=\frac{1}{2}$ is a simple root of g. So the proof of $\mathcal{P}_{n+1}$ is complete. □

p	0		$\frac{1}{2}$		1
$g'(p)$		$-$	$-$	$-$	
$g(p)$		$\searrow$	0	$\searrow$	

Proposition 2.9. *For every $n \in \mathbb{Z}^+$ and $q \in \mathbb{R}$ we have*

$$\sup_{p\in[0,1]} \left|E_{2n}^{(c)}(p,q)\right| = \max\left\{\left|E_{2n}^{(c)}(0,q)\right|,\ \left|E_{2n}^{(c)}\left(\frac{1}{2},q\right)\right|\right\}, \tag{37}$$

and

$$\sup_{p\in[0,1]} \left|E_{2n+1}^{(c)}(p,q)\right| \leq \frac{2n+1}{2}\max\left\{\left|E_{2n}^{(c)}(0,q)\right|,\ \left|E_{2n}^{(c)}\left(\frac{1}{2},q\right)\right|\right\}. \tag{38}$$

Proof. The result (37) follows from Propositions 2.1 and 2.8. To prove (38), if $p \in \left[0,\frac{1}{2}\right]$ then we have

$$E_{2n+1}^{(c)}(p,q) = E_{2n+1}^{(c)}(p,q) - E_{2n+1}^{(c)}\left(\frac{1}{2},q\right) = (2n+1)\int_{\frac{1}{2}}^{p} E_{2n}^{(c)}(t,q)\ \mathrm{d}t.$$

Therefore

$$\begin{aligned}\left|E_{2n+1}^{(c)}(p,q)\right| &\leq (2n+1)\int_{p}^{\frac{1}{2}} \left|E_{2n}^{(c)}(t,q)\right|\ \mathrm{d}t \leq (2n+1)\left(\frac{1}{2}-p\right)\sup_{t\in\left[p,\frac{1}{2}\right]}\left|E_{2n}^{(c)}(t,q)\right| \\ &\leq (2n+1)\left(\frac{1}{2}-p\right)\max\left\{\left|E_{2n}^{(c)}(0,q)\right|,\ \left|E_{2n}^{(c)}(\frac{1}{2},q)\right|\right\}.\end{aligned}$$

So we have

$$\sup_{p\in\left[0,\frac{1}{2}\right]} \left|E_{2n+1}^{(c)}(p,q)\right| \leq \frac{2n+1}{2}\max\left\{\left|E_{2n}^{(c)}(0,q)\right|,\ \left|E_{2n}^{(c)}(\frac{1}{2},q)\right|\right\}.$$

On the other hand, $E_{2n+1}^{(c)}(1-p,q) = -E_{2n+1}^{(c)}(p,q)$ completes the proof of (38). □

Proposition 2.10. *For every $n \in \mathbb{N}$ and $q > 0$, the two following Propositions are valid:*

$\mathcal{P}_n$ *: The function $p \mapsto (-1)^n E_{2n}^{(s)}(p,q)$ is positive on $\left[0,\frac{1}{2}\right)$ and negative on $\left(\frac{1}{2},1\right]$. Moreover, $p=\frac{1}{2}$ is a unique simple root on $[0,1]$, i.e. the aforesaid function has no zero in the intervals $[0,\frac{1}{2})$ and $(\frac{1}{2},1]$.*

$\mathcal{Q}_n$ *: The function $p \mapsto (-1)^n E_{2n+1}^{(s)}(p,q)$ is strictly increasing on $\left[0,\frac{1}{2}\right]$ and strictly decreasing on $\left[\frac{1}{2},1\right]$ and always takes a positive value at $p=\frac{1}{2}$.*

Proof. The Proposition $\mathcal{P}_1$ is clear, because $-E_2^{(s)}(p,q) = -q(2p-1) = q(1-2p)$. Now define $f(p) = (-1)^n E_{2n+1}^{(s)}(p,q)$ to get $f'(p) = (2n+1)(-1)^n E_{2n}^{(s)}(p,q)$. By noting $\mathcal{P}_n$, we see that f is strictly increasing on $\left[0, \frac{1}{2}\right]$ and decreasing on $\left[\frac{1}{2}, 1\right]$. Moreover, since $f(0) = q^{2n+1} > 0$ (by Corollary 2.2), one can conclude that $f\left(\frac{1}{2}\right) > 0$.

Finally define $g(p) = (-1)^{n+1} E_{2n+2}^{(s)}(p,q)$ to get $g'(p) = -(2n+2)(-1)^n E_{2n+1}^{(s)}(p,q)$. Since $g'(0) = g'(1) = -(2n+2)q^{2n+1} > 0$, and $E_{2n+1}^{(s)}(1-p,q) = E_{2n+1}^{(s)}(p,q)$, so by noting $\mathcal{Q}_n$ we have $\forall p \in (0,1) : \ g'(p) < 0$. Therefore, g takes the following table of variations

p	0		$\frac{1}{2}$		1
$g'(p)$		$-$	$-$	$-$	
$g(p)$		$\searrow$	0	$\searrow$	

As $g\left(\frac{1}{2}\right) = 0$ (by Corollary 2.1) and $g'\left(\frac{1}{2}\right) < 0$, then $p = \frac{1}{2}$ is a simple root of function g. So the proof of $\mathcal{P}_{n+1}$ is complete. $\square$

Corollary 2.6. *For every $n \in \mathbb{N}$ and $q \in \mathbb{R}$ we have*

$$\sup_{p\in[0,1]} \left| E_{2n+1}^{(s)}(p,q) \right| = \max\left\{ \left| E_{2n+1}^{(s)}(0,q) \right|, \ \left| E_{2n+1}^{(s)}\left(\frac{1}{2}, q\right) \right| \right\},$$

and

$$\sup_{p\in[0,1]} \left| E_{2n}^{(s)}(p,q) \right| \le n \max\left\{ \left| E_{2n-1}^{(s)}(0,q) \right|, \ \left| E_{2n-1}^{(s)}\left(\frac{1}{2}, q\right) \right| \right\}.$$

Proposition 2.11. *Let m and n be two positive integers and*

$$I^{(c)} = \int_0^1 E_m^{(c)}(p,q) E_n^{(c)}(p,q) \ \mathrm{d}p.$$

If $m+n$ is odd then $I^{(c)} = 0$ and if it is even then

$$I^{(c)} = \sum_{k=0}^{m+n} \frac{1}{(k+1)!} \left(\sum_{j=A}^{B} \binom{k}{j} \frac{n!m!}{(n-j)!(m-k+j)!} E_{n-j}^{(c)}(0,q) E_{m-k+j}^{(c)}(0,q) \right),$$

where $A = \max\{0, k-m\}$ and $B = \min\{n,k\}$.

Proof. First, suppose that $m+n$ is odd. By using (15) we have

$$I^{(c)} = \int_0^1 E_m^{(c)}(1-p,q) E_n^{(c)}(1-p,q) \ \mathrm{d}p = (-1)^{m+n} \int_0^1 E_m^{(c)}(p,q) E_n^{(c)}(p,q) \ \mathrm{d}p = -I^{(c)}.$$

Now, assume that $m+n$ is even. Since $\deg_p \left(E_m^{(c)} E_n^{(c)}\right) = m+n$ (from Proposition 2.5), using (23) we obtain

$$
\begin{aligned}
E_m^{(c)}(p,q)E_n^{(c)}(p,q) &= \sum_{k=0}^{m+n} \left(\frac{\partial^k}{\partial p^k} \left(E_m^{(c)}(p,q)E_n^{(c)}(p,q)\right)\right)\Bigg|_{p=0} \frac{p^k}{k!} \\
&= \sum_{k=0}^{m+n} \left(\sum_{j=0}^{k} \binom{k}{j} \left(\frac{\partial^j}{\partial p^j} E_n^{(c)}(p,q) \frac{\partial^{k-j}}{\partial p^{k-j}} E_m^{(c)}(p,q)\right)\Bigg|_{p=0}\right) \frac{p^k}{k!} \\
&= \sum_{k=0}^{m+n} \left(\sum_{j=A}^{E} \binom{k}{j} \frac{n!m!}{(n-j)!(m-k+j)!} E_{n-j}^{(c)}(0,q)E_{m-k+j}^{(c)}(0,q)\right) \frac{p^k}{k!},
\end{aligned}
$$

which leads to the second result. □

Corollary 2.7. *Let m and n be two positive integers and*

$$
I^{(s)} = \int_0^1 E_m^{(s)}(p,q)E_n^{(s)}(p,q)\ \mathrm{d}p.
$$

If $m+n$ is odd then $I^{(s)} = 0$ and if $m+n$ is even then

$$
I^{(s)} = \sum_{k=0}^{m+n-2} \frac{1}{(k+1)!} \left(\sum_{j=A}^{B} \binom{k}{j} \frac{n!m!}{(n-j)!(m-k+j)!} E_{n-j}^{(s)}(0,q)E_{m-k+j}^{(s)}(0,q)\right),
$$

where $A = \max\{0, k-m\}$ and $B = \min\{n,k\}$.

3 Some new series of Taylor type involving associated Euler numbers $E_n(0)$

One of the applications of relations (11) and (12) is that they can be considered as the Taylor expansion of two special functions at $t=0$ involving associated Euler numbers $E_n(0)$. In other words, substituting the relations (13) and (14) in, respectively, (11) and (12) yield

$$
\begin{aligned}
f_c(t;p,q) = \frac{2\mathrm{e}^{pt}}{\mathrm{e}^t+1}\cos qt &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} E_k(0) C_{n-k}(p,q)\right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} E_{n-k}(0) C_k(p,q)\right) \frac{t^n}{n!}, \qquad (39)
\end{aligned}
$$

and

$$f_s(t;p,q)=\frac{2\mathrm{e}^{pt}}{\mathrm{e}^t+1}\sin qt=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}E_k(0)S_{n-k}(p,q)\right)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)S_k(p,q)\right)\frac{t^n}{n!},\tag{40}$$

where $C_k(p,q)$ and $S_k(p,q)$ are defined in (7) and (8). In order to evaluate the two functions f_c and f_s at some specific parameters, first let us prove the following identities

$$C_k(p,p)=2^{\frac{k}{2}}p^k\cos\frac{k\pi}{4},\tag{41}$$

$$S_k(p,p)=2^{\frac{k}{2}}p^k\sin\frac{k\pi}{4},\tag{42}$$

$$C_k(0,q)=q^k\cos\frac{k\pi}{2},\tag{43}$$

$$S_k(0,q)=q^k\sin\frac{k\pi}{2},\tag{44}$$

and

$$C_k(p,0)=p^k,\qquad S_k(p,0)=0.\tag{45}$$

It is easy to find out that

$$\cos k\theta+\mathrm{i}\sin k\theta=(\cos\theta+\mathrm{i}\sin\theta)^k$$
$$=\sum_{j=0}^{[\frac{k}{2}]}(-1)^j\binom{k}{2j}\sin^{2j}\theta\cos^{k-2j}\theta+\mathrm{i}\sum_{j=0}^{[\frac{k-1}{2}]}(-1)^j\binom{k}{2j+1}\sin^{2j+1}\theta\cos^{k-2j-1}\theta.$$

By replacing $\theta=\frac{\pi}{4}$ in the above relation, we obtain

$$\cos\frac{k\pi}{4}+\mathrm{i}\sin\frac{k\pi}{4}=2^{-\frac{k}{2}}\sum_{j=0}^{[\frac{k}{2}]}(-1)^j\binom{k}{2j}+\mathrm{i}\;2^{-\frac{k}{2}}\sum_{j=0}^{[\frac{k-1}{2}]}(-1)^j\binom{k}{2j+1},$$

which leads to relations (41) and (42), respectively. Relations (43), (44) and (45) are also clear by noting relations (7) and (8).

Now, we can consider some particular examples.

Example 1. As the hyperbolic secant function is even, so by referring to (2) we have

$$\text{sech }t=\frac{2\mathrm{e}^t}{\mathrm{e}^{2t}+1}=\sum_{n=0}^{\infty}\frac{E_{2n}}{(2n)!}t^{2n},\quad |t|<\frac{\pi}{2}.\tag{46}$$

On the other hand, replacing $t \to 2t$, $p = \frac{1}{2}$ and $q = 0$ in (39) gives

$$f_c\left(2t; \frac{1}{2}, 0\right) = \operatorname{sech} t = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}E_k(0)C_{n-k}\left(\frac{1}{2}, 0\right)\right)\frac{2^n t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}E_k(0)2^{k-n}\right)\frac{2^n t^n}{n!}.$$

This means that we have the following relationship between the Euler numbers E_{2n} and associated Euler numbers $E_k(0)$ where $k = 0, 1, \ldots, 2n$:

$$E_{2n} = \sum_{k=0}^{2n}\binom{2n}{k}2^k E_k(0). \tag{47}$$

Remark: From equation (4), we observe that

$$E_{2k}(0) = 0 \ \ \forall k \in \mathbb{N} \quad \text{and} \quad E_0(0) = 1.$$

Hence, by the above relations, formula (47) is simplified as

$$\sum_{k=1}^{n}\binom{2n}{2k-1}2^{2k-1}E_{2k-1}(0) = E_{2n} - 1,$$

which can be written as the matrix form

$$\begin{pmatrix} u_{1,1} & 0 & 0 & \cdots & 0 \\ u_{2,1} & u_{2,2} & 0 & \cdots & 0 \\ \vdots & & & & \\ u_{n,1} & u_{n,2} & u_{n,3} & \cdots & u_{n,n} \end{pmatrix}\begin{pmatrix} E_1(0) \\ E_3(0) \\ \vdots \\ E_{2n-1}(0) \end{pmatrix} = \begin{pmatrix} E_2 - 1 \\ E_4 - 1 \\ \vdots \\ E_{2n} - 1 \end{pmatrix}$$

in which

$$u_{n,k} = \binom{2n}{2k-1}2^{2k-1}.$$

The above system is triangular and can be therefore solved explicitly. This means that we will eventually have

$$E_{2n-1}(0) = \sum_{k=1}^{n} v_{n,k}E_{2k}. \tag{48}$$

Hence, the standard Euler numbers in (48) can be used throughout the paper instead of the associated Euler numbers.

Example 2. Let $f(t) = \dfrac{\cos t}{\mathrm{e}^t + 1}$ and $g(t) = \dfrac{\sin t}{\mathrm{e}^t + 1}$. If in (39) we take $p = 0$ and $q = 1$, then by noting (43) we obtain

$$\begin{aligned} f_c(t;0,1) = \frac{2}{\mathrm{e}^t+1}\cos t &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)\cos\frac{k\pi}{2}\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[\frac{n}{2}]}\binom{n}{2k}E_{n-2k}(0)(-1)^k\right)\frac{t^n}{n!}. \end{aligned}$$

Therefore we have

$$\frac{\cos t}{\mathrm{e}^t+1} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k}{2}\binom{n}{2k}E_{n-2k}(0)\right)\frac{t^n}{n!},$$

and in a similar way

$$\frac{\sin t}{\mathrm{e}^t+1} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{[\frac{n-1}{2}]}\frac{(-1)^k}{2}\binom{n}{2k+1}E_{n-2k-1}(0)\right)\frac{t^n}{n!}.$$

Example 3. Let $f(t) = \dfrac{\mathrm{e}^{\mathrm{t}}}{\mathrm{e}^t+1}\cos t$ and $g(t) = \dfrac{\mathrm{e}^{\mathrm{t}}}{\mathrm{e}^t+1}\sin t$. replacing $p = q = 1$ in (39) gives

$$\begin{aligned} \frac{2\mathrm{e}^t}{\mathrm{e}^t+1}\cos t &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)C_k(1,1)\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)2^{\frac{k}{2}}\cos\frac{k\pi}{4}\right)\frac{t^n}{n!}. \end{aligned}$$

Similarly we obtain

$$\begin{aligned} \frac{2\mathrm{e}^t}{\mathrm{e}^t+1}\sin t &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)S_k(1,1)\right)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)2^{\frac{k}{2}}\sin\frac{k\pi}{4}\right)\frac{t^n}{n!}. \end{aligned}$$

Therefore, we have

$$\frac{\mathrm{e}^t}{\mathrm{e}^t+1}\cos t = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)2^{\frac{k}{2}-1}\cos\frac{k\pi}{4}\right)\frac{t^n}{n!},$$

and

$$\frac{\mathrm{e}^t}{\mathrm{e}^t+1}\sin t=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}E_{n-k}(0)2^{\frac{k}{2}-1}\sin\frac{k\pi}{4}\right)\frac{t^n}{n!}.$$

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