Bivariate Chromatic Polynomials in Computer Algebra

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Abstract

The computation of the chromatic polynomial of a graph, which was introduced by (BIR), is an NP-complete problem. Consequently, this is also valid for the bivariate generalization of the chromatic polynomial by (DPT). A recursion formula, which was presented by (AGM), has exponential complexity. Hence, the aim of the current article is to find efficient algorithms or formulas for the calculation of the bivariate chromatic polynomial for special types of graphs. The following results will be presented:

We found efficient formulas for complete graphs, from which the edges of stars with pairwise different vertices are deleted, for complete partite graphs and for special split graphs.

Furthermore, in a section about separators, we show that the special case of separating complete subgraphs, which is very simple in the univariate case, requires rather complex methods in the bivariate case.

Finally, we establish a connection between the bivariate chromatic polynomial and the matching polynomial for complete graphs from which the edges of stars with pairwise different vertices are removed, as well as for bi-cliques.

Some algorithms are implemented in *Mathematica*, and examples are presented.

Key words: Graph Theory, Algorithms

1. Introduction

The bivariate chromatic polynomial by (DPT) is a generalization of the chromatic polynomial. It can be generalized to the trivariate chromatic polynomial, which is a special case of the multivariate chromatic polynomial by (WHI). Before the multivariate chromatic polynomial was known, (AGM) introduced the edge-elimination-polynomial, which is the most general polynomial that satisfies a special recurrence relation. In general, this polynomial is P-hard to compute, but for graphs of tree-width at most k it is polynomial time computable. As (TRI) showed, the edge-elimination-polynomial is

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equivalent to the trivariate chromatic polynomial.
Chromatic polynomials have many applications in graph theory, topology, theoretical physics and in planning mobile networks. Nevertheless, there are only comparatively few publications on the bivariate chromatic polynomial. In their work (DPT) introduced explicit representations for the bivariate chromatic polynomial of complete graphs, complete bipartite graphs, paths and cycles. They also showed that the polynomial can be evaluated in polynomial time for trees and graphs of restricted pathwidth. In this text, we will introduce more formulas for further classes of graphs, show the complexity of computing the bivariate chromatic polynomial for graphs with complete separators in general and show a connection between the bivariate chromatic polynomial and the matching polynomial for special classes of graphs.

In the following, we will only consider simple and finite graphs $G = (V, E)$. For a subgraph H of G we write $H \subseteq G$, and for $X \subseteq V$, the graph $G[X]$ is the subgraph induced by the set X. If $G = (V, E)$ is a graph with $U \subset V$, $G - U$ denotes the graph $G[V \setminus U]$. Furthermore, $G - v = G - \{v\}$ with $v \in V$ is the graph with the vertex set $V \setminus \{v\}$ and the edge set $E \setminus \{e \in E | v \in e\}$ (*deletion* of v).
For terminology see (BOL).
A proper coloring of a graph G is a map from V to a set of colors Y with $|Y| = y$ without adjacent vertices having the same color. The *chromatic polynomial* $P(G; y)$ of a graph G, which was introduced by (BIR), denotes the number of all such maps. For this polynomial the following relation holds.

Theorem 1 ((BIR), page 42)**.** *The number $P(G; y)$ of all proper colorings of $G = (V, E)$ in y colors is a polynomial in y of degree $n = |V|$, which can be computed as follows: Let m_i for $i = 1, \ldots, n$ be the number of all proper colorings of G using exactly i colors modulo permutations of the colors. Then, counting also all permutations, we have $m_i y\,(y-1)\ldots(y-i+1)$ possibilities to color the vertices of G with exactly i colors. That means we also distinguish colorings, which only differ by permutation. Altogether, we get*

$$P(G; y) = \sum_{i=1}^{n} m_i y^{\underline{i}} ,$$

where $y^{\underline{i}} = y\,(y-1)\cdots(y-i+1)$ denotes the falling factorial.

Thus, the chromatic polynomial has only integral coefficients, *i.e.* $P(G; y) \in \mathbb{Z}[y]$.
Since the computation of the chromatic polynomial of a graph is an NP-complete problem (see (SKI), section 5.5.1) for complicated graphs it is difficult to compute the chromatic polynomial using for example the package `GraphTheory` contained in *Maple* (see (FKKM)) or the packages `Combinatorica` (see (SKI)) and `GraphUtilities` (see (WOL)) contained in *Mathematica*. Later we will compute some examples with *Mathematica* using our package `BivariatePolynomials`. We will present the package `BivariatePolynomials` in the appendix.
Now we give a generalization of the univariate chromatic polynomial which was introduced by (DPT) and which also allows colors that may be distributed arbitrarily to the vertices.

Definition 2 ((DPT))**.** Let $G = (V, E)$ be a graph and $X = Y \cup Z$ with $Y \cap Z = \emptyset$ the set of all available colors. The colors from Y are called *proper* and the colors from Z *improper.* We set $|X| = x$ and $|Y| = y$. A *generalized proper coloring* of G is a map $\phi : V \to X$ such that for each $e = \{v_1, v_2\} \in E$ if $\phi(v_1) \in Y$ and $\phi(v_2) \in Y$ then $\phi(v_1) \neq \phi(v_2)$. The number of all generalized proper colorings of G in x colors is called $P(G; x, y)$.

Theorem 3 ((DPT))**.** *In the setting of Definition 2 $P(G; x, y)$ can be written as*

$$P(G; x, y) = \sum_{X \subseteq V} (x - y)^{|X|} \, P(G - X; y) \; .$$

Because $P(G; x, y)$ can easily be expressed using the univariate chromatic polynomial, $P(G; x, y)$ is obviously a polynomial too, namely the *generalized bivariate chromatic polynomial.* As a result also $P(G; x, y) \in \mathbb{Z}[x, y]$ is true.
A generalized proper vertex coloring of a graph $G = (V, E)$ corresponds to a partition of V, such that there exists a bijection from the set of these subsets of V to $Y \cup \{a\}$, where a is chosen from Z arbitrarily and the pre-image of every member of Y is an independent set. The following definition will prepare this presentation of $P(G; x, y)$.

Definition 4. Let $G = (V, E)$ be a graph. We call a partition of V a *partition of G.* $\Pi(G)$ denotes the set of all partitions of G. Each member of a partition is called a *block.*

In this work we will only be interested in partitions which are defined as follows.

Definition 5. Non adjacent vertices and edges which don't have any vertex in common are called *independent of each other.* A set $X \subseteq V$ or $Y \subseteq E$ is said to be *independent* if its members are pairwise independent of each other.
An *independent partition* π of $G = (V, E)$ denotes, as defined in (DPT), section 3, a partition of G, such that each block $A \in \pi$ is independent. We denote the set of all independent partitions by $\Pi_I(G)$.

2. The recursion formula for $P(G; x, y)$ by Averbouch, Godlin and Makowsky

(AGM) proved a recursive relation for the generalized bivariate chromatic polynomial $P(G; x, y)$ of a graph $G = (V, E)$; it is a special case of a recursion formula for the edge elimination polynomial, that they also proved.
To prepare the recursion representation of $P(G; x, y)$, we define for an edge set $F \subseteq E$ the graph $G - F = (V, E \setminus F)$ and especially for one single edge $e \in E$ the graph $G - e = (V, E \setminus \{e\})$ (*deletion* of e). Moreover, for an edge $e = \{v_1, v_2\} \in E$ we construct a new graph G/e from G by removing e, identifying v_1 and v_2 as one new single vertex and replacing all possibly arising multiple edges by single edges (*contraction* of e).
For $P(G; x, y)$ the following relation holds.

Theorem 6 ((AGM))**.** *Let $G = (V, E)$ be a graph and $e = \{v_1, v_2\} \in E$. Then*

$$P(G; x, y) = P(G - e; x, y) - P(G/e; x, y) + (x - y)\, P(G - \{v_1, v_2\}\, ; x, y) \qquad \text{(AGM1)}$$

with the initial conditions $P(K_1;x,y) = x$ *and* $P(\emptyset;x,y) = 1$*, where* K_n *for* $n \geq 1$ *denotes the complete graph with* n *vertices and* $\emptyset$ *the zero graph without any vertices. For graphs* G_1, G_2 *with disjoint vertex sets also*

$$P(G_1 \cup G_2; x, y) = P(G_1; x, y)P(G_2; x, y)$$

holds.

The equation in Theorem 6 can also be solved for $P(G - e, x, y)$ and can be used to compute this polynomial using the other three graphs.
Because we will often analyze the complexity concerning our results, it is important to compare it with the complexity of the formula in Theorem 6. As it can be seen easily, the complexity of the method in Theorem 6 in general turns out to be exponential with base 3. Thus, our aim is to find more efficient ways for certain special graphs. In the following, we will assume that numerical calculations are much less complex than graph operations.

3. More general representations by using independent partitions

In this section we introduce two general representations of the bivariate chromatic polynomial $P(G; x, y)$ of an arbitrary graph G. For this purpose we return to the representation which was shown in Theorem 1. It can be written as follows.

Theorem 7. *For the univariate case* $Z = \emptyset$ *the equation in Theorem 1 can also be written as*

$$P(G; y) = \sum_{\pi \in \Pi_I(G)} y^{\underline{|\pi|}}\ .$$

From this we get the following result, which we will often use as an efficient method to prove more special cases.

Theorem 8. *Let* M *be a set and let* $\mathcal{P}(M) = \{L | L \subseteq M\}$ *denote the power set of* M*. For a natural number* $k \geq 1$ *let* $\mathcal{P}_k(M) = \{L \in \mathcal{P}(M) | \, |L| = k\}$*. Furthermore, let* $G = (V, E)$ *be a graph,* $\Pi_I(G[X])$ *the set of all independent partitions of* $X \subseteq V$ *and* $n = |V|$*. Then*

$$\begin{aligned} P(G; x, y) &= \sum_{W \subseteq V} (x - y)^{|W|} \sum_{\pi \in \Pi_I(G[V \setminus W])} y^{\underline{|\pi|}} \\ &= \sum_{i=0}^{n} (x - y)^{n-i} \sum_{X \in \mathcal{P}_i(V)} \sum_{\pi \in \Pi_I(G[X])} y^{\underline{|\pi|}}\ . \end{aligned}$$

Proof. By Theorem 3

$$P(G;x,y) = \sum_{W \subseteq V} (x-y)^{|W|} \, P(G-W;y) \; .$$

Together with Theorem 7 we get the first equation. The second equation follows from

$$\begin{aligned}
\sum_{W \subseteq V} (x-y)^{|W|} \sum_{\pi \in \Pi_I(G[V \setminus W])} y^{\underline{|\pi|}} &= \sum_{i=0}^{n} \sum_{W \in \mathcal{P}_{n-i}(V)} (x-y)^{|W|} \sum_{\pi \in \Pi_I(G[V \setminus W])} y^{\underline{|\pi|}} \\
&= \sum_{i=0}^{n} (x-y)^{n-i} \sum_{X \in \mathcal{P}_i(V)} \sum_{\pi \in \Pi_I(G[X])} y^{\underline{|\pi|}} \; .
\end{aligned}$$

□

Now we will show a further method of representing the bivariate chromatic polynomial by using independent partitions. Here we form the outer sum by using all independent partitions. For some proofs this method will turn out to be more advantageous.

Theorem 9. *Let G be a graph, $\Pi_I(G)$ the set of all independent partitions of G. Then*

$$P(G;x,y) = \sum_{\pi \in \Pi_I(G)} \sum_{i=0}^{|\pi \cap \mathcal{P}_1(V)|} \binom{|\pi \cap \mathcal{P}_1(V)|}{i} (x-y)^i \, y^{\underline{|\pi|-i}}$$

is an alternative representation of Theorem 6 and Corollary 7 in (DPT).

Proof. First, for each independent partition $\pi \in \Pi_I(G)$ we color the different blocks from π differently using proper colors. Thus, we get $y^{\underline{|\pi|}}$ possibilities as in the univariate case. This corresponds with the case $i = 0$. To use the colors from Z we choose i blocks with $i \geq 1$, but we only include all $X \in \pi$ with $|X| = 1$ to avoid double colorings. This problem could occur as follows: Refering to the definition of improper colors, we also may distribute the same improper color to different blocks from π. If we color a block $X \in \pi$ with $|X| \geq 2$ improper, we get the same coloring also by using each refinement π' of π, such that exactly the block $X \in \pi$ is replaced by blocks $X'_1, \ldots, X'_k \in \pi'$ with $2 \leq k \leq |X|$. For this purpose we respectively color $X'_1, \ldots, X'_k$ in the same improper color we have already chosen for X and transfer for all other, not changed blocks the same color as before.
To color i blocks of all $|\{X \in \pi | \, |X| = 1\}|$ blocks of cardinality one in an improper color, there are $\binom{|\{X \in \pi | |X|=1\}|}{i}$ possibilities. □

4. Deletion of stars in complete graphs

The representations introduced in the previous section yield efficient methods of proving the bivariate chromatic polynomial for graphs belonging to certain special graph classes. We start with a family of graphs which have comparatively many edges: they are constructed from a complete graph $K_n = (V, E)$ with $n = V$ by removing the edges of certain pairwise disjoint subgraphs.

Definition 10. A *star* is a bipartite graph $K_{1,r}$ with blocks of cardinalities 1 and r for a natural number $r \geq 1$.
Let $N, t_1 \ldots, t_N$ be positive integers. A graph obtained from the complete graph K_n by removing the edge sets of $\sum_{m=1}^{N} t_m$ vertex-disjoint stars is denoted by
$K_n - \sum\limits_{m=1}^{N} t_m K_{1,m}$, where we suppose that there are exactly t_m stars of order $m+1$ removed, $m = 1, \ldots, N$.

For this graph type we obtain a recursion-free method of computing the bivariate chromatic polynomial as follows.

Theorem 11. *Let n, N be positive integers and $t_1, t_2, \ldots, t_N \geq 0$ integers such that $n \geq \sum_{m=1}^{N} t_m (m+1)$. Define $G = K_n - \sum\limits_{m=1}^{N} t_m K_{1,m}$. Then*

$$P(G; x, y) = \sum_{b_N=0}^{t_N} \binom{t_N}{b_N} N^{b_N} \sum_{b_{N-1}=0}^{t_{N-1}} \binom{t_{N-1}}{b_{N-1}} (N-1)^{b_{N-1}} \ldots \sum_{b_2=0}^{t_2} \binom{t_2}{b_2} 2^{b_2} \sum_{b_1=0}^{t_1} \binom{t_1}{b_1}$$
$$\sum_{i=0}^{n-2b_N-\ldots-2b_1} \binom{n-2b_N-\ldots-2b_1}{i} (x-y)^i \, y^{\underline{n-b_N-\ldots-b_1-i}} \, .$$

Proof. For our proof we use Theorem 9:
The first sum in Theorem 9 is formed over all independent partitions of G. To generate these partitions of G, each block only may consist of either one vertex or two vertices, where the edge between them has been deleted. There are $\prod_{m=1}^{N} \sum_{b_m=0}^{t_m} \binom{t_m}{b_m} m^{b_m}$ ways of choosing blocks X_2 with $|X_2| = 2$. The remaining blocks X_1 are of cardinality $|X_1| = 1$. So we get $|\Pi_I(G)| = \prod_{m=1}^{N} \sum_{b_m=0}^{t_m} \binom{t_m}{b_m} m^{b_m}$, which leads to

$$\sum_{b_N=0}^{t_N} \binom{t_N}{b_N} N^{b_N} \sum_{b_{N-1}=0}^{t_{N-1}} \binom{t_{N-1}}{b_{N-1}} (N-1)^{b_{N-1}} \ldots \sum_{b_2=0}^{t_2} \binom{t_2}{b_2} 2^{b_2} \sum_{b_1=0}^{t_1} \binom{t_1}{b_1}$$

possibilities to choose a set of blocks X_2 with $|X_2| = 2$.
From the remaining $n - 2b_N - \ldots - 2b_1$ vertices we choose i vertices which are colored improper. For this, there are

$$\sum_{i=0}^{n-2b_N-\ldots-2b_1} \binom{n-2b_N-\ldots-2b_1}{i} (x-y)^i$$

possibilities.
Because we want to color both vertices of X_2 in the same proper color, we count each pair of vertices in X_2 as only one vertex. So, in the end, we need $n - b_N - \ldots - b_1 - i$ different proper colors and have $y^{\underline{n-b_N-\ldots-b_1-i}}$ possibilities to choose them. $\square$

As already mentioned before, the method in Theorem 6 has exponential cost because of the three graph operations in every recursion step. In contrast, the formula in Theorem 11 does not need any graph operations and recursion steps. There are only computation steps, which are given by a product of sums. Hence, the computation complexity must be polynomial bounded.

We want to show an example using the *Mathematica* packages `Combinatorica` ((SKI)), `GraphUtilities` ((WOL)) and our package `BivariatePolynomials`. We denote the recursion formula of Theorem 6 by `BivariatePolynomialAGM1` in our package `BivariatePolynomials`. If we solve this equation for $P(G-e;x,y)$, we denote it by `BivariatePolynomialAGM2`.
Now, we consider the special case $K_9-K_{1,3}-K_{1,2}-K_{1,1}$ from Theorem 11. To compute its bivariate chromatic polynomial we use the implementations `BivariatePolynomialAGM1` and `BivariatePolynomialAGM2` and compare the results with the result of the equation in Theorem 11, which is implemented directly.

```
G1=DeleteEdges[Combinatorica`CompleteGraph[9],{{1,2},{1,3},{1,4},{5,6},{5,
7},{8,9}}];

Timing[BivariatePolynomialAGM1[G1,x,y]]
```

$$\{127.515, x^9+17280y-21288xy+13332x^2y-5688x^3y+1874x^4y-515x^5y$$
$$+126x^6y-30x^7y-27648y^2+31030xy^2-17218x^2y^2+6212x^3y^2-1580x^4y^2$$
$$+263x^5y^2+11484y^3-10225xy^3+3976x^2y^3-734x^3y^3-1114y^4+462xy^4\}$$

```
Timing[BivariatePolynomialAGM2[G1,x,y]]
```

$$\{0.280802, x^9+17280y-21288xy+13332x^2y-5688x^3y+1874x^4y-515x^5y$$
$$+126x^6y-30x^7y-27648y^2+31030xy^2-17218x^2y^2+6212x^3y^2-1580x^4y^2$$
$$+263x^5y^2+11484y^3-10225xy^3+3976x^2y^3-734x^3y^3-1114y^4+462xy^4\}$$

```
Timing[Expand[
  Sum[Binomial[1, c]*3^c*
    Sum[Binomial[1, b]*2^b*
      Sum[Binomial[1, a]*1^a*
        Sum[Binomial[9 - 2*c - 2*b - 2*a, i]*(x - y)^i*
          FunctionExpand[FactorialPower[y, 9 - c - b - a - i]],
       {i, 0, 9 - 2*c - 2*b - 2*a}], {a, 0, 1}], {b, 0, 1}], {c, 0, 1}]]]
```

$$\{0.0156001, x^9+17280y-21288xy+13332x^2y-5688x^3y+1874x^4y-515x^5y$$
$$+126x^6y-30x^7y-27648y^2+31030xy^2-17218x^2y^2+6212x^3y^2-1580x^4y^2$$
$$+263x^5y^2+11484y^3-10225xy^3+3976x^2y^3-734x^3y^3-1114y^4+462xy^4\}$$

For the graph $K_{40}-K_{1,5}-2\cdot K_{1,4}-K_{1,3}-4\cdot K_{1,1}$ we use only the second implementation `BivariatePolynomialAGM2`, which stopped after 1 hour computing time while the direct computation using Theorem 11 leads to the result in 1.63801 seconds.

5. Complete k-partite graphs

A well-studied family of graphs are the complete k-partite graphs with $k \geq 2$. Let $K_{n_1,n_2,\ldots,n_k}$ be the complete k-partite graph with k blocks of cardinality n_i for $i = 1,\ldots,k$. For the special case with an arbitrary $k \geq 2$ and with $n_i = 2$ for all $i \in \{1,\ldots,k\}$ the computation of the bivariate chromatic polynomial is very simple and does not need any recursion.

Theorem 12. *Let G be the complete k-partite graph $K_{\underbrace{2,\ldots,2}_{k\ times}}$. Then we get*

$$P(G;x,y) = \sum_{i=0}^{k} \binom{k}{i} \sum_{l=0}^{2k-2i} \binom{2k-2i}{l} (x-y)^{2k-2i-l}\, y^{\underline{i+l}}\,.$$

Proof. Observe that $K_{2,\ldots,2} \cong K_{2k} - kK_{1,1}$. Now we apply Theorem 11 with $n = 2k$, $N = 1$ and $t_1 = k$.

$$\begin{aligned} P(K_{2k} - kK_{1,1};x,y) &= \sum_{b_1=0}^{k} \binom{k}{b_1} \sum_{l=0}^{2k-2b_1} \binom{2k-2b_1}{l} (x-y)^{b_1+l}\, y^{\underline{2k-b_1-l}} \\ &= \sum_{b_1=0}^{k} \binom{k}{b_1} \sum_{l=0}^{2k-2b_1} \binom{2k-2b_1}{l} (x-y)^{2k-2b_1-l}\, y^{\underline{b_1+l}}\,. \end{aligned}$$

□

We look at the complexity of the formula in Theorem 12. This method does not need any graph operations and recursion steps, and we have again a product of sums of computation steps. So, in opposition to the recursion formula in Theorem 6, the complexity of our equation in Theorem 12 is polynomial bounded.

Now, we compute the bivariate chromatic polynomial of the graph $K_{2,2,2,2}$ and use again the implementations `BivariatePolynomialAGM1` and `BivariatePolynomialAGM2`. We compare them with the directly implemented equation from Theorem 12.

```
G2=CompleteKPartiteGraph[2,2,2,2];
```

```
Timing[BivariatePolynomialAGM1[G2,x,y]]
```

$$\begin{gathered}\{16.3333, x^8 - 2790y + 3408xy - 2128x^2y + 912x^3y - 306x^4y + 88x^5y - 24x^6y \\ +4059y^2 - 4392xy^2 + 2320x^2y^2 - 768x^3y^2 + 156x^4y^2 - 1332y^3 + 1008xy^3 \\ -272x^2y^3 + 60y^4\}\end{gathered}$$

```
Timing[BivariatePolynomialAGM2[G2,x,y]]
```

$$\begin{gathered}\{0.156001, x^8 - 2790y + 3408xy - 2128x^2y + 912x^3y - 306x^4y + 88x^5y - 24x^6y + 4059y^2 \\ -4392xy^2 + 2320x^2y^2 - 768x^3y^2 + 156x^4y^2 - 1332y^3 + 1008xy^3 - 272x^2y^3 + 60y^4\}\end{gathered}$$

```
Timing[Expand[
  Sum[Binomial[4, i]*
    Sum[Binomial[8 - 2*i, l]*(x - y)^{8 - 2*i - l}*
      FunctionExpand[FactorialPower[y, i + l]], {l, 0, 8 - 2*i}],
   {i, 0, 4}]]]
```

$$\{0., x^8 - 2790y + 3408xy - 2128x^2y + 912x^3y - 306x^4y + 88x^5y - 24x^6y + 4059y^2$$
$$-4392xy^2 + 2320x^2y^2 - 768x^3y^2 + 156x^4y^2 - 1332y^3 + 1008xy^3 - 272x^2y^3 + 60y^4\}$$

For the graph $K_{\underbrace{2,\ldots,2}_{30\text{ times}}}$ we use only the implementation `BivariatePolynomialAGM2` and compute the equation from Theorem 12 again directly. While the implementation `BivariatePolynomialAGM2` stopped after 1 hour, the second computation came to a result after 1.48201 seconds.

Now we assume again an arbitrary $k \geq 2$, but instead of $n_i = 2$ we set $n_i = 3$ for all $i \in \{1, \ldots, k\}$. At this, we also get a simple recursion-free equation for the bivariate chromatic polynomial.

Theorem 13. *Let G be the complete k-partite graph $K_{\underbrace{3,\ldots,3}_{k\text{ times}}}$. Then we get*

$$P(G; x, y) = \sum_{i=0}^{k} \binom{k}{i} \sum_{t=0}^{i} \binom{i}{t} 3^{i-t} \sum_{l=0}^{3k-2i-t} \binom{3k-2i-t}{l} (x-y)^{3k-2i-t-l}\, y^{\underline{i+l}} \, .$$

Proof. We apply Theorem 9 to $K_{3,\ldots,3}$.

$$|\Pi_I(K_{3,\ldots,3})| = \sum_{i=0}^{k} \binom{k}{i} \sum_{t=0}^{i} \binom{i}{t} 3^{i-t}$$

and for each $\pi_{i,t} \in \Pi_I(K_{3,\ldots,3})$ because of

$$\begin{aligned} |\pi_{i,t}| &= 3k - 2i - t + (i - t) + t \\ &= 3k - i - t \end{aligned}$$

also

$$|\{X \in \pi_{i,t} \,|\, |X| = 1\}| = 3k - 2i - t \, .$$

Thus, we have

$$
\begin{aligned}
P(K_{3,\dots,3};x,y) &= \sum_{i=0}^{k}\binom{k}{i}\sum_{t=0}^{i}\binom{i}{t}3^{i-t}\sum_{l=0}^{3k-2i-t}\binom{3k-2i-t}{l}(x-y)^{l}\,y^{\underline{3k-i-t-l}}\\
&= \sum_{i=0}^{k}\binom{k}{i}\sum_{t=0}^{i}\binom{i}{t}3^{i-t}\sum_{l=0}^{3k-2i-t}\binom{3k-2i-t}{l}(x-y)^{3k-2i-t-l}\,y^{\underline{i+l}}\,.
\end{aligned}
$$

□

Similar to the situation in Theorem 12 the complexity of the formula in Theorem 13 is polynomial bounded. Hence, we found an efficient method.

To compute the bivariate chromatic polynomial for the graph $K_{3,3,3}$, we use again the implementations `BivariatePolynomialAGM1` and `BivariatePolynomialAGM2` together with a direct computation using Theorem 13.

```
G3=CompleteKPartiteGraph[3,3,3];

Timing[BivariatePolynomialAGM1[G3,x,y]]
```

$$
\{90.2934,,x^9+11828y-15048xy+9738x^2y-4305x^3y+1476x^4y-423x^5y
+108x^6y-27x^7y-18918y^2+21942xy^2-12582x^2y^2+4698x^3y^2-1242x^4y^2
+216x^5y^2+7848y^3-7236xy^3+2916x^2y^3-558x^3y^3-756y^4+324xy^4\}
$$

```
Timing[BivariatePolynomialAGM2[G3,x,y]]
```

$$
\{1.46641,x^9+11828y-15048xy+9738x^2y-4305x^3y+1476x^4y-423x^5y
+108x^6y-27x^7y-18918y^2+21942xy^2-12582x^2y^2+4698x^3y^2-1242x^4y^2
+216x^5y^2+7848y^3-7236xy^3+2916x^2y^3-558x^3y^3-756y^4+324xy^4\}
$$

```
Timing[Expand[
  Sum[Binomial[3, i]*
    Sum[Binomial[i, t]*3^(i - t)*
      Sum[Binomial[9 - 2 i - t, l]*(x - y)^(9 - 2 i - t - l)*
        FunctionExpand[FactorialPower[y, i + l]],
     {l, 0, 9 - 2 i - t}], {t, 0, i}], {i, 0, 3}]]]
```

$$
\{0.,x^9+11828y-15048xy+9738x^2y-4305x^3y+1476x^4y-423x^5y
+108x^6y-27x^7y-18918y^2+21942xy^2-12582x^2y^2+4698x^3y^2-1242x^4y^2
+216x^5y^2+7848y^3-7236xy^3+2916x^2y^3-558x^3y^3-756y^4+324xy^4\}
$$

We look at the running time result for the graph $K_{\underbrace{3,\dots,3}_{20\text{ times}}}$ using the implementation `BivariatePolynomialAGM2`. It stopped after 1 hour running time while a direct computing using Theorem 13 comes to a result in 7.28525 seconds.

More generally, we will now introduce a way of representing the bivariate chromatic polynomial of an arbitrary complete k-partite graph. We will not be able to avoid recursion, but it will still turn out to be efficient. The *complementary graph* $\overline{G}=(V,F)$ to a graph $G=(V,E)$, where $F=\{\{v_1,v_2\}\in\mathcal{P}_2(V)|\,\{v_1,v_2\}\notin E\}$, will be important. Let $S(r,k)$ be the Stirling number of the second kind for natural numbers r and k.

Theorem 14. *We assume $t \geq 1$ and $n_i \geq 1$ for all $i \in \{1, 2, \ldots, t\}$. Then, analogous to the bipartite case in (DPT), section 5.2, for the graph $K_{n_1,n_2,\ldots,n_t}$ we have the recursive representation*

$$P(K_{n_1,n_2,\ldots,n_t}; x, y) = \sum_{i=0}^{n_t} \binom{n_t}{i} (x-y)^i \sum_{j=0}^{n_t - i} S(n_{t-i}, j)\, y^{\underline{j}}\, P(K_{n_1,n_2,\ldots,n_{t-1}}; x-j, y-j)$$

with the initial condition

$$P(\overline{K_{n_1}}; x, y) = x^{n_1} \; .$$

First, we introduce a definition for the proof.

Definition 15. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ with $V_1 \cap V_2 = \emptyset$. Then $G_1 * G_2 = (W, F)$ denotes the *graph join* of G_1 and G_2 with vertex set $W = V_1 \cup V_2$ and edge set $F = E_1 \cup E_2 \cup \{\{v_1, v_2\} \,|v_1 \in V_1 \wedge v_2 \in V_2\}$.

Proof. Obviously, the graph $K_{n_1,n_2,\ldots,n_t}$ can be identified as follows:

$$P(K_{n_1,n_2,\ldots,n_t}; x, y) = P(K_{n_1,n_2,\ldots,n_{t-1}} * \overline{K_{n_t}}; x, y) \; .$$

From all n_t vertices of $\overline{K_{n_t}}$ we choose i vertices for an improper coloring, where $i = 0, \ldots, n_t$. For each i the number of all possible choices of i vertices from n_t vertices is $\binom{n_t}{i}$. The remaining $n_t - i$ vertices are colored properly. Because of their pairwise independence we may partition them arbitrarily to assign pairwise different colors to the blocks. As possible numbers of blocks we get all numbers from 0 to $n_t - i$. Because each vertex is adjacent to all vertices in $K_{n_1,n_2,\ldots,n_{t-1}}$, we may not use any of the proper colors already chosen for the graph $K_{n_1,n_2,\ldots,n_{t-1}}$. Finally, we get

$$P(K_{n_1,n_2,\ldots,n_t}; x, y) = \sum_{i=0}^{n_t} \binom{n_t}{i} (x-y)^i \sum_{j=0}^{n_t - i} S(n_{t-i}, j)\, y^{\underline{j}}\, P(K_{n_1,n_2,\ldots,n_{t-1}}; x-j, y-j) \; .$$

□

The Stirling number $S(n, k)$ can be calculated in $O(n^2)$ time. Theorem 14 shows that we only need one graph operation in every recursion step multiplied with a product of two sums of computation steps and a Stirling number. Hence, the complexity of our method must be polynomial bounded.

For an example in *Mathematica* we consider the special case $n_1 = n_2 = \ldots = n_t = r$ and return to the graph $K_{3,3,3}$. We use the implementations `BivariatePolynomialAGM1` and `BivariatePolynomialAGM2`. An implementation of the recursion formula in Theorem 14 in our package `BivariatePolynomials` is denoted by `BivariatePolynomialGerling`.

```
G4=CompleteKPartiteGraph[3,3,3];

Timing[BivariatePolynomialAGM1[G4,x,y]]
```

$$\{91.1358, x^9 + 11828y - 15048xy + 9738x^2y - 4305x^3y + 1476x^4y - 423x^5y$$
$$+108x^6y - 27x^7y - 18918y^2 + 21942xy^2 - 12582x^2y^2 + 4698x^3y^2 - 1242x^4y^2$$
$$+216x^5y^2 + 7848y^3 - 7236xy^3 + 2916x^2y^3 - 558x^3y^3 - 756y^4 + 324xy^4\}$$

```
Timing[BivariatePolynomialAGM2[G4,x,y]]
```

$$\{1.46641, x^9 + 11828y - 15048xy + 9738x^2y - 4305x^3y + 1476x^4y - 423x^5y$$
$$+108x^6y - 27x^7y - 18918y^2 + 21942xy^2 - 12582x^2y^2 + 4698x^3y^2 - 1242x^4y^2$$
$$+216x^5y^2 + 7848y^3 - 7236xy^3 + 2916x^2y^3 - 558x^3y^3 - 756y^4 + 324xy^4\}$$

```
Timing[Expand[BivariatePolynomialGerling[3,3,x,y]]]
```

$$\{0.0312002, x^9 + 11828y - 15048xy + 9738x^2y - 4305x^3y + 1476x^4y - 423x^5y$$
$$+108x^6y - 27x^7y - 18918y^2 + 21942xy^2 - 12582x^2y^2 + 4698x^3y^2 - 1242x^4y^2$$
$$+216x^5y^2 + 7848y^3 - 7236xy^3 + 2916x^2y^3 - 558x^3y^3 - 756y^4 + 324xy^4\}$$

For the graph $K_{10,10,10,10}$ we use the implementations `BivariatePolynomialAGM1`, `BivariatePolynomialAGM2` and `BivariatePolynomialGerling`. The first and second one stopped after 1 hour running time while the implementation `BivariatePolynomialGerling` leads to the result after 77.7821 seconds.

6. Split graphs

Another important graph class is the class of *split graphs*. These graphs are definable as graphs whose vertex set is partioned into two disjoint subsets, such that one of them is independent and the other one induces a complete subgraph. For the general case we consider the following result.

Theorem 16. *Let $G = (V, E)$ be a split graph with $G_1 = K_k$ and $G_2 = \overline{K_s}$, whose vertex set is given by $V = V(G_1) \cup V(G_2)$ with $V(G_1) \cap V(G_2) = \emptyset$. For $v \in V$ let $\Gamma(v)$ be the set of all independent sets of V containing v, and for $\{v_1, v_2, \ldots, v_i\} \subseteq V(G_1)$ with $i = 0, \ldots, k$ let*
$\alpha = \{(I_1, \ldots, I_i) \in \Gamma(v_1) \times \ldots \times \Gamma(v_i) \mid I_l \cap I_m = \emptyset$ for all $l, m \in \{1, \ldots, i\}$ with $l \neq m\}$. Then

$$P(G; x, y) = \sum_{i=0}^{k} (x-y)^{k-i} \, y^{\underline{i}} \sum_{\{v_1, v_2, \ldots, v_i\} \subseteq V(G_1)} \sum_{\alpha} (x-i)^{s-|I_1|-|I_2|-\ldots-|I_i|+i}$$

holds.

Proof. We apply Theorem 8 to G_1. Within G_1 each choice $\{v_1, v_2, \ldots, v_i\}$ allows only the trivial independent partition $\{\{v_1\}, \{v_2\}, \ldots, \{v_i\}\}$. Thus, all proper colors in G_1 must be pairwise different. Hence, we get $(x-y)^{k-i}\, y^{\underline{i}}$ possibilities for G_1 to color the chosen vertices properly and the remaining vertices in G_1 improperly.
We now count the possibilities of coloring the graph G_2 and consider all possibilities of coloring vertices from $V(G_2)$ respectively in the same color, such as the above chosen vertices v_r with $r = 1, 2, \ldots, i$. The vertices v_r are colored pairwise different and properly, hence also the sets I_r. Thus, $I_l \cap I_m = \emptyset$ for all $l, m \in \{1, 2, \ldots, i\}$ with $l \neq m$ has to be true. Now we use none of the already employed colors from Y for the remaining vertices in G_2 to exclude forbidden colorings, because adjacencies between vertices from $V(G_1)$ and $V(G_2)$ in general exist. In total, all allowed cases were already considered in the choice of our sets $(I_1, I_2, \ldots, I_i)$. $\square$

Furthermore, we also observe the following alternative representation.

Theorem 17. *We consider a split graph $G = (V, E)$ like in Theorem 16 with a vertex set $V = V(K_k) \cup V(\overline{K_s})$, such that $V(K_k) \cap V(\overline{K_s}) = \emptyset$. Furthermore, we define $V(\overline{K_s}) = \{v_i \in V | i = 1, \ldots, s\}$ and for $W \subseteq V(K_k)$ the set $N_{G[V(K_k)\setminus W]}(v_i) = \{w \in V(K_k) \setminus W | \{v_i, w\} \in E(G)\}$. This leads to*

$$P(G; x, y) = \sum_{W \subseteq V(K_k)} (x-y)^{|W|}\, y^{\underline{k-|W|}} \prod_{i=1}^{s} \left(x - \left|N_{G[V(K_k)\setminus W]}\,(v_i)\right|\right) .$$

Proof. Applying Theorem 2.1 in (TAHU) we get

$$P(G; y) = y^{\underline{k}} \prod_{i=1}^{s} \left(y - |N(v_i)|\right)$$

with $N(v_i) = \{v \in V | \{v_i, v\} \in E\}$. For the subgraph K_k there are $y^{\underline{k}}$ possibilities to color the vertices, because they are pairwise adjacent, and for each vertex v_i in the subgraph $\overline{K_s}$ we may exactly choose the colors which do not occur in $N(v_i)$. Together with the equality

$$P(G; x, y) = \sum_{W \subseteq V} (x-y)^{|W|}\, P(G - W; y)$$

from Theorem 1 in (DPT) the proof is finished. $\square$

The sums in Theorem 16 and Theorem 17 obviously differ in their complexities significantly. Hence, the method in Theorem 17 is much faster than the method in Theorem 16.

7. Separators in graphs

For the univariate case it was shown (see (REA), Theorem 3) that seperators in graphs are helpful to simplify the calculation of the chromatic polynomial. They are defined as follows: If $G = (V, E)$ is a connected graph and for $W \subseteq V$ or $W \subseteq E$ the graph $G - W$ is not connected, then W *separates* the graph. We call W a *separator* in G.
Now we consider the following case: Let G be a graph with $G = G_1 \cup G_2$ for two graphs G_1 and G_2 satisfying $G_1 \cap G_2 = K_r$ for an $r \geq 1$. According to (REA), Theorem 3, we have

$$P(G; y) = \frac{P(G_1; y)P(G_2; y)}{y^{\underline{r}}} ,$$

where

$$y^{\underline{r}} = P(K_r; y) .$$

The corresponding result in the bivariate case turns out to be much more complicated.

Theorem 18. *Let the graph* $G = (V, E) = \bigcup_{i=1}^{r} G_i$ *with* $G_i \cap G_j = K_s$ *for all* $i, j \in \{1, \dots, r\}$ *with* $i \neq j$ *and* $\bigcap_{i=1}^{r} G_i = K_s$*, i.e.* K_s *is a separator in* G*. Let* $\Gamma(v)$ *for* $v \in V$ *be defined as in Theorem 16 and*
$\alpha = \{(I_1, \dots, I_k) \in \Gamma(v_1) \times \ldots \times \Gamma(v_k) \mid I_l \cap I_m = \emptyset$ *for all* $l, m \in \{1, \dots, k\}$ *with* $l \neq m\}$*.*
Then we get

$$P(G; x, y) = \sum_{k=0}^{s} (x-y)^{s-k}\, y^{\underline{k}} \sum_{\{v_1,\dots,v_k\} \subseteq V\left(\bigcap_{i=1}^{r} G_i\right)} \sum_{\alpha} \prod_{i=1}^{r} P\Big(G_i - \bigcap_{j=1}^{r} G_j - \bigcup_{t=1}^{k} I_t; x-k, y-k\Big) .$$

Here, the summand for $k = 0$ *is defined as* $(x-y)^s \prod_{i=0}^{r} P\big(G_i - \bigcap_{j=1}^{r} G_j; x, y\big)$*.*

Proof. By using Theorem 8 the claim follows. □

As we will see now, the observation can be generalized to separators that are not complete in general. But here, we have to accept a constraint to the bivariate chromatic polynomial as well as to construct a new graph. This one is defined as follows.

Definition 19 ((TIT))**.** Let G be a graph and H a subgraph of G. For each $\pi \in \Pi_I(H)$ a new graph G_π can be defined as follows: Each block $X \in \pi$ can be replaced by a vertex. We add an edge between each pair of vertices if there was no edge between the corresponding blocks. Besides, there is an adjacency between each pair of vertices $v \in V(H)$ and $w \in V(G - H)$ if and only if w in G was adjacent to a vertex from $X \in \pi$ with $v \in X$.

This leads to the following result.

Theorem 20. *Let $G = (V, E) = G^1 \cup G^2$ with $G^1 \cap G^2 = H$ and $W \subseteq V(H)$. Assuming that all vertices from $(H - W)_\pi$ are colored properly, we define the number of all generalized proper colorings of $G^i - W$ as $P^{H-W}\Big((G^i - W)_\pi; x, y\Big)$. For each $W \subseteq V(K_{|\pi|})$ the number $P^{K_{|\pi|}-W}(G^i_\pi; x, y)$ is defined analogous. Then the following holds:*

$$P(G;x,y) = \sum_{W\subseteq V} (x-y)^{|W|} \sum_{\pi} \frac{P\Big((G^1-W)_\pi;y\Big)\, P\Big((G^2-W)_\pi;y\Big)}{y^{\underline{|\pi|}}}$$

$$= \sum_{W\subseteq V(H)} (x-y)^{|W|} \sum_{\pi} \frac{P^{H-W}\Big((G^1-W)_\pi;x,y\Big)\, P^{H-W}\Big((G^2-W)_\pi;x,y\Big)}{y^{\underline{|\pi|}}}$$

where the inner sum is over $\pi \in \Pi_I(G[V(H) \setminus W])$. In the case $x \neq y$ we also get

$$P(G;x,y) = \sum_{\pi\in\Pi_I(G[V(H)])} \; \sum_{W\subseteq V(K_{|\pi|})} \frac{P^{K_{|\pi|}-W}(G^1_\pi;x,y) P^{K_{|\pi|}-W}(G^2_\pi;x,y)}{(x-y)^{|W|}\, y^{\underline{|\pi|-|W|}}}$$

and for $x = y$ by (TIT) in equation (6.2) in section 6.3.3

$$P(G;y) = \sum_{\pi\in\Pi_I(G[V(H)])} \frac{1}{y^{\underline{|\pi|}}} P(G^1_\pi;y)P(G^2_\pi;y) \ .$$

Proof. The first equation is given by a combination of the following two equations: From Theorem 1 in (DPT) we use

$$P(G;x,y) = \sum_{W\subseteq V} (x-y)^{|W|}\, P(G-W;y)$$

and from (TIT), equation (6.2),

$$P(G;y) = \sum_{\pi\in\Pi_I(G[V(H)])} \frac{1}{y^{\underline{|\pi|}}} P(G^1_\pi;y)P(G^2_\pi;y) \ .$$

The last relation can be found as follows: We first choose G^1 and G^2 as above. After that, we construct a complete subgraph of $G = G^1 \cup G^2$ from each independent partition of H as in Definition 19. Thus, we can apply Theorem 3 in (REA) for a graph $G = G_1 \cup G_2$ with $G^1 \cap G^2 = K_r$ and $r \geq 1$ to obtain

$$P(G;y) = \frac{P(G_1;y)P(G_2;y)}{P(K_r;y)} \ .$$

For the second equation we sum over all $W \subseteq V(H)$ instead of summing over all $W \subseteq V$. To consider the improper colors in $V(G - H)$, we have to use the bivariate chromatic

polynomials of $G_i - W$ for $i \in \{1,2\}$, but on condition of using only proper colors for the vertices of $(H-W)_\pi$.
Finally, we change the order of both sums as well as the parts of W and π. $\square$

Here, a simple special case arises.

Theorem 21. *Given the situation in Theorem 20 for the case $H \cong K_r$ with $r = \left|V(G^1) \cap V(G^2)\right|$, we get*

$$P(G;x,y) = \sum_{W \subseteq V(K_r)} \frac{P^{K_r - W}(G^1;x,y) P^{K_r - W}(G^2;x,y)}{(x-y)^{|W|}\, y^{\underline{r-|W|}}} \,.$$

Proof. Since $\Pi_I(K_r) = \Big\{ \big\{ \{v_1\}, \ldots, \{v_r\} \big\} \Big\}$ with $V(K_r) = \{v_1, \ldots, v_r\}$ the claim follows. $\square$

To avoid the inconvenient constraints to the chromatic polynomials in the relation in Theorem 20, we use the following method, which contains the inclusion-exclusion principle. Assumed, Möbius inversion is known, the inclusion-exclusion principle can be proved easily.

Theorem 22. *In the situation of Theorem 20 we get*

$$\begin{aligned} P(G;x,y) = &\sum_{W \subseteq V(H)} (x-y)^{|W|} \\ &\sum_{\pi \in \Pi_I(G[V(H)\setminus W])} \frac{\prod_{i=1}^{2} \sum_{\sigma \subseteq \pi} (-1)^{|\sigma|} (x-y)^{|\sigma|} P\Big(\big(G^i - W - \sigma\big)_{\pi - \sigma}; x, y\Big)}{y^{\underline{|\pi|}}} \,. \end{aligned}$$

Proof. Using the inclusion-exclusion principle, we show for $i = 1,2$ the equality

$$P^{H-W}\Big((G^i - W)_\pi; x, y\Big) = \sum_{\sigma \subseteq \pi} (-1)^{|\sigma|} (x-y)^{|\sigma|} P\Big((G^i - W - \sigma)_{\pi - \sigma}; x, y\Big) \,.$$

Definition 19 gives $\pi = V\Big((H-W)_\pi\Big)$. Let $\sigma \subseteq \pi$, such that exactly all blocks in $\pi - \sigma$, *i.e.* all vertices in $V\Big((H-W-\sigma)_\pi\Big)$, are colored proper. Let $P_{\leq}^{H-W}\Big(\big(G^i - W\big)_\pi; x, y\Big)$ be the number of all generalized proper vertex colorings, such that in $V\Big((H-W)_\pi\Big)$ also improper colors may occur. Möbius inversion gives

$$P_{\leq}^{H-W}\Big((G^i - W)_\pi; x, y\Big) = \sum_{\sigma \subseteq \pi} P^{H-W}\Big(\big(G^i - W\big)_{\pi - \sigma}; x, y\Big) (x-y)^{|\sigma|}$$

$\Leftrightarrow$

$$P^{H-W}\Big((G^i - W)_\pi; x, y\Big) = \sum_{\sigma\subseteq\pi} (-1)^{|\pi|-(|\pi|-|\sigma|)} P_{\leq}^{H-W}\Big(\left(G^i - W\right)_{\pi-\sigma}; x, y\Big)$$
$$= \sum_{\sigma\subseteq\pi} (-1)^{|\sigma|} (x-y)^{|\sigma|} P^{H-W}\Big(\left(G^i - W - \sigma\right)_{\pi-\sigma}; x, y\Big) .$$

□

8. The bivariate chromatic polynomial and the matching polynomial

Some special graphs have independent vertex partitions which consist only of blocks with cardinalities one or two. Hence, each of them induces a vertex or an edge in the complementary graph. Thus, each independent partition of such a graph implies a so-called matching in the complementary graph.

Definition 23. Let $G = (V, E)$ be a graph and $M \subseteq E$. M is called a *matching* if no two edges of M have a vertex in common. If M consists of exactly k edges, M is a *k-matching* of G.

In the literature different definitions of matching polynomials occur. We will use the following definition introduced by (LOPL), section 8.5.

Definition 24 ((LOPL))**.** Let $G = (V, E)$ be a graph and let $n := |V|$ and a_k be the number of k-matchings in G for $k = 1, \ldots, \lfloor n/2 \rfloor$. The *matching generating polynomial* is defined as

$$M(G; x) = \sum_{k=0}^{\lfloor n/2\rfloor} a_k x^k .$$

Now we return to the graph type in Definition 10, whose independent partitions fulfill the property mentioned above.

Theorem 25. *For natural numbers $t_1, \ldots, t_N$ we call a graph $G = (V, E)$ according to Definition 10 a $K_n - \sum\limits_{m=1}^{N} t_m K_{1,m}$, if we can delete the edges of respectively t_m stars $K_{1,m}$ from a complete graph K_n, such that no two stars have any vertices in common. Furthermore, let $H = \sum_{m=1}^{N} t_m K_{1,m}$, $\overline{n} := |V(H)|$ and $M(H; x) = \sum_{k=0}^{\lfloor \overline{n}/2\rfloor} a_k x^k$. Then we get*

$$P(G; x, y) = \sum_{k=0}^{\lfloor \overline{n}/2\rfloor} a_k \sum_{i=0}^{n-2k} \binom{n-2k}{i} (x-y)^i \, y^{\underline{n-k-i}} .$$

Hence, k is the number of all blocks $X \in \pi \in \Pi_I(G)$ with $|X| > 1$ while a_k is the number of all such π with k blocks of the form X. Observe, that in general the relation $a_k = 0$ holds starting with a certain k.

Proof. In each independent partition of V there are only blocks which only consist of just one vertex or one deleted edge, *i.e.* an edge of $E(H)$. To achieve this, no two edges of H may have a vertex in common. So we are looking for all possible matchings of H. Hence, the number a_k of all k-matchings in H is equivalent to the number of all partitions π of V with exactly k blocks X with $|X| > 1$.
□

Theorem 26. *For $k \leq \sum_{i=1}^{N} t_i$ Theorem 11 and Definition 24 lead to*

$$a_k = \sum_{b_N=0}^{k} \binom{t_N}{b_N} N^{b_N} \sum_{b_{N-1}=0}^{k-b_N} \binom{t_{N-1}}{b_{N-1}} (N-1)^{b_{N-1}} \cdots \sum_{b_2=0}^{k-b_N-\ldots-b_3} \binom{t_2}{b_2} 2^{b_2} \binom{t_1}{k-b_N-\ldots-b_2}$$

and for $b_N, \ldots, b_1$ we have also

$$k = \sum_{i=0}^{N-1} b_{N-i} \ .$$

□

Furthermore, there is also a connection between the bivariate chromatic polynomial of a biclique and the matching polynomial of its complementary graph. This relation was already shown in (BOH), section 2, for the univariate chromatic polynomial. Before we present a more general form for the bivariate case we need a Definition by (BOH), section 1.

Definition 27 ((BOH))**.** A (m, n)*-biclique* is a graph G, which is the complementary graph of a bipartite graph with partition sets of cardinalities m and n with $n \geq m$.

Now we generalize the result by (BOH), section 2, mentioned above in the following Theorem.

Theorem 28. *Let G be an (m, n)-biclique with $n \geq m$ and*

$$M_{\overline{G}} = \left\{F \subseteq E(G) | F \text{ is a matching in } \overline{G}\right\} \ .$$

In addition, let $a_{\overline{G}}^{i}$ be the number of all i-matchings in $\overline{G}$. Analogous to the univariate case in (BOH), section 2, we get

$$\begin{aligned} P(G; x, y) &= \sum_{X \in M_{\overline{G}}} \sum_{W \subseteq V(\overline{G}) \setminus V(X)} (x-y)^{\underline{|W|}} \, y^{\underline{m+n-|W|-|E(X)|}} \\ &= \sum_{H \subseteq \overline{G}} (x-y)^{\underline{\overline{G}-H}} \sum_{X \in M_{\overline{H}}} y^{\underline{m+n-|W|-|E(X)|}} \\ &= \sum_{i=0}^{m} a_{\overline{G}}^{i} \sum_{j=0}^{m+n-2i} (x-y)^{\underline{j}} \binom{m+n-2i}{j} y^{\underline{m+n-j-i}} \ . \end{aligned}$$

Proof. Equal proper colors may only be assigned to pairwise non-adjacent vertices. As already mentioned in (BOH), section 2, the only possibility to do this is given by vertices which belong to a matching in $\overline{G}$. Improper colors are exclusively assigned to vertices which do not belong to the currently considered matching. To choose exactly j vertices not belonging to the currently considered matching, we have $\binom{m+n-2i}{j}$ possibilities. All remaining $m+n-j$ vertices are colored pairwise different and properly. Here, the vertices of all i edges of each i-matching are respectively colored in the same color. $\square$

9. Conclusion

In this text we first introduced two general representations of the bivariate chromatic polynomial for arbitrary graphs. These representations are very helpful to prove many formulas for special graph classes, which are much more efficient than the recursion formula by (AGM). Furthermore, we showed methods of computing the bivariate chromatic polynomial by using separators of graphs and could see that this is very complex in the bivariate case even for complete seperators. Finally, we could show a connection between the bivariate chromatic polynomial and the matching polynomial of two special graph classes.

A very important case is the class of graphs with restricted tree-width. As already mentioned in the introduction, the edge-elemination-polynomial is P-hard to compute in general, but for graphs of restricted tree-width it is computable in polynomial time. In particular, the bivariate chromatic polynomial of graphs of tree-width at most k is polynomial time computable. Hence, it should be interesting to find algorithms to compute the bivariate chromatic polynomial for graphs of restricted tree-width.

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Appendix: Mathematica package `BivariatePolynomials`

Our *Mathematica* package `BivariatePolynomials` contains some new implementations.

The first one is `BivariatePolynomialAGM1[graph_,x_,y_]`. It computes the bivariate chromatic polynomial for a graph $G = (V; E)$ and an edge $e = \{v_1, v_2\} \in E$ using the recursion formula

$$P(G;x,y) = P(G-e;x,y) - P(G/e;x,y) + (x-y)\,P(G-\{v_1,v_2\}\,;x,y) \qquad \text{(AGM1)}$$

with the initial conditions $P(K_1; x, y) = x$ and $P(\emptyset; x, y) = 1$ by (AGM) in Theorem 6.

The implementation `BivariatePolynomialAGM2[graph_,x_,y_]` solves the equation above for $P(G-e,x,y)$ using the other three graphs.

Our last implementation `BivariatePolynomialGerling[r_,k_,x_,y_]` computes the bivariate chromatic polynomial for a complete partite graph $K_{n_1,n_2\dots,n_t}$ with $t \geq 1$ and $n_i \geq 1$ for all $i \in \{1, 2, \dots, t\}$ in Theorem 14.

The *Mathematica* package `BivariatePolynomials` and a *Mathematica* notebook containing all examples is available at `http://www.mathematik.uni-kassel.de/~koepf/Publikationen/` under Software Downloads.

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