On Gould's Identity No. 1.45

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Abstract

A reliability property of a particular n-block system leads to the formulation of a known identity via a specified definite integral. A second proof is given by showing that a certain (order 1) recurrence equation is satisfied by each side of the identity.

Introduction

<u>**Identity No. 1.45**</u> For $n \geq 1$,

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} = \sum_{k=1}^{n} \frac{1}{k}.$$

In H.W. Gould's familiar *Combinatorial Identities* (Rev. Ed., University of West Virginia, U.S.A., 1972), the result is listed on page 6. In this short communication we first offer a formulation of the above, whose origin lies in some elementary reliability analysis conducted recently by the author E.J.F. Such a context is briefly outlined and the proof, which we hope readers will find interesting, follows. A first order recurrence equation is then shown to be satisfied by each side of the identity, providing an alternative proof on which a computational approach is remarked.

Proof I

Context

Consider a system of $n \geq 1$ independent blocks—all identical and working in parallel (as opposed to series)—and suppose that the system operates so long as at least one block works, repair being needed only at the point of failure of the final nth block. Associated with each block is a reliability function $e^{-\lambda t} \in (0, 1]$, where $\lambda > 0$ is its failure rate. Denoting a probability by $\Pr\{\cdot\}$, the System Reliability Function $R(t)$, say, is defined for $t \geq 0$ as

$$\begin{aligned}
R(t) &= \Pr\{\text{System is operational at time } t\} \\
&= 1 - \Pr\{\text{System is not operational at time } t\} \\
&= 1 - \Pr\{\text{All } n \text{ blocks have failed at time } t\} \\
&= 1 - \Pr{}^n\{\text{A single block has failed at time } t\} \\
&= 1 - (1 - \Pr\{\text{A single block is operational at time } t\})^n \\
&= 1 - (1 - e^{-\lambda t})^n. \qquad (1)
\end{aligned}$$

Thus, the so called Mean Time to Failure for the system is

$$I(n, \lambda) = \int_0^\infty R(t) \; dt = \int_0^\infty [1 - (1 - e^{-\lambda t})^n] \; dt. \qquad (2)$$

We derive the identity in question by treating the integral $I(n, \lambda)$ in two different ways.

Proof

Firstly, the substitution $x(t) = 1 - e^{-\lambda t}$ applied to (2) gives rise to a transformed integral

$$I(n,\lambda) = \frac{1}{\lambda}\int_0^1 \frac{1-x^n}{1-x}\ dx. \tag{I1}$$

Noting that, for $n \geq 1$,

$$(x-1)(1+x+x^2+x^3+\cdots+x^{n-2}+x^{n-1}) = x^n - 1, \tag{I2}$$

then

$$\begin{aligned} I(n,\lambda) &= \frac{1}{\lambda}\int_0^1 (1+x+x^2+x^3+\cdots+x^{n-2}+x^{n-1})\ dx \\ &= \frac{1}{\lambda}\left[1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n-1}+\frac{1}{n}\right] \\ &= \frac{1}{\lambda}\sum_{k=1}^{n}\frac{1}{k}. \end{aligned} \tag{I3}$$

We now expand binomially that part of the integrand

$$\begin{aligned} (1-e^{-\lambda t})^n &= \sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix}(-e^{-\lambda t})^k \\ &= 1+\sum_{k=1}^{n}(-1)^k\begin{pmatrix} n \\ k \end{pmatrix}e^{-\lambda tk}, \end{aligned} \tag{I4}$$

so that, by definition (2),

$$\begin{aligned} I(n,\lambda) &= -\sum_{k=1}^{n}(-1)^k\begin{pmatrix} n \\ k \end{pmatrix}\int_0^\infty e^{(-\lambda k)t}\ dt \\ &= -\sum_{k=1}^{n}(-1)^k\begin{pmatrix} n \\ k \end{pmatrix}\cdot\frac{1}{\lambda k} \\ &= \frac{1}{\lambda}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}\begin{pmatrix} n \\ k \end{pmatrix}; \end{aligned} \tag{I5}$$

the identity is immediate upon equating (I3) and (I5).□

Proof II

Consider the sum $\sum_{k=1}^{n}\frac{1}{k} = S(n)$, say. It is clearly the unique solution of the recurrence equation $S(n+1) - S(n) = \frac{1}{n+1}$ (with initial condition

$S(1) = 1$). Hence, writing

$$T(n) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k}, \tag{II1}$$

then noting that $T(1) = 1$ we need to show that $T(n+1) - T(n) = \frac{1}{n+1}$ likewise in order to establish the identity for $n \geq 1$. This is a straightforward procedure, since

$$\begin{aligned}
&T(n+1) - T(n) \\
&\quad= \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} \binom{n+1}{k} - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} \\
&\quad= \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} \left[\binom{n}{k} + \binom{n}{k-1} \right] - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} \\
&\quad= \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} \binom{n}{k} + \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} \binom{n}{k-1} \\
&\qquad\qquad - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} \\
&\quad= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} + \sum_{k=1}^{n+1} \frac{(-1)^{k-1}}{k} \binom{n}{k-1} \\
&\qquad\qquad - \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \binom{n}{k} \\
&\quad= \sum_{k=0}^{n} \frac{(-1)^{k}}{k+1} \binom{n}{k} \\
&\quad= \frac{1}{n+1}
\end{aligned} \tag{II2}$$

as desired, the last step a consequence of having set $x = 1$ in the result

$$\sum_{k=0}^{n} \frac{(-1)^{k}}{k+x} \binom{n}{k} = \frac{1}{x \binom{n+x}{n}} \tag{II3}$$

(true for $n \geq 0$, $x \neq 0, -1, -2, \ldots, -n$) which appears as Identity No. 1.41 in the aforesaid work of Gould (it is also set on p.71 as Exercise 48 (Section 1.2.6) of Volume 1 of Knuth's *The Art of Computer Programming* (Addison-Wesley, Reading, U.S.A., 1968)); by way of verification, the reader is invited to confirm that (II3) readily lends itself to proof by induction (although various derivations are possible). This finishes the second proof, about which some pertinent observations are now made.□

Remark 1 For completeness we remark that, since the summand of $T(n)$ (II1) is a hypergeometric term (*i.e.*, its ratio of successive terms is a rational function in k), Zeilberger's algorithm invoked through modern computer algebra tools (see, for instance, Koepf's text *Hypergeometric Summation: An Algorithmic Approach to Summation and Special Function Identities* (Vieweg, Wiesbaden, Germany, 1998), and the accompanying software package "hsum6.mpl" accessible at **http://www.mathematik.uni-kassel.de/∼koepf/Publikationen**) gives the required recursion in T instantly. The final step of Proof II can equally be effected computationally using Gosper's algorithm (also available as part of the package hsum6.mpl).

Remark 2 Consider a hypergeometric term $f(k;n)$, say, which is summed over k. Many functions of this type are zero outside a certain set of values $k \in [k_l, k_u]$—the binomial coefficient $\binom{n}{k}$, for example, has $k_l = 0$ and $k_u = n$. The interval given by the largest value of k_l and the smallest value of k_u can be regarded as describing the 'natural bounds' for the sum, and Zeilberger's algorithm will generate an inhomogeneous (rather than a homogeneous) recurrence for any sum of hypergeometric terms whose bounds are not the natural ones. This is the case here for $T(n)$ (II1), for which no such bounds exist.

Remark 3 At the time of writing (January 2002) it has recently been brought to our attention that J. Riordan shows that $T(n) - T(n-1) = \frac{1}{n}$ in a similar way to us on p.5 of his 1968 text *Combinatorial Identities* (Wiley, New York, U.S.A.), where he notes that, given $T(1) = 1$, $T(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ is immediate. It is felt instructive to retain Proof II here, however, especially in view of Remarks 1,2 which bring it up to date.

To end, it is perhaps worth mentioning that Identity No. 1.45 can itself be established *directly* from a few other results in Gould's listing. Setting, for instance, $a = 1$ in Identity No. 1.134 (p.17) yields it as a special case, as does Identity No. 7.17 (p.60) for $m = n$. In addition, also appearing in this document is the interesting Identity No. Z.7 (p.82), which states that

$$\sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} \frac{f(x-k)}{k} = f(x) \sum_{k=1}^{n} \frac{1}{k} - \frac{df}{dx} \tag{3}$$

and holds for any polynomial $f(x)$ in x of degree $\leq n$. Choosing $f(x) = 1$ (of degree $0 < n$ for $n \geq 1$), the result follows trivially.

Summary

We have given two different proofs of Gould's Identity No. 1.45, with comments as appropriate. Professor Gould (in a private communication to P.J.L.) has indicated that it possesses a long history, a topic about which he is writing currently. This work ("Differences of the Harmonic Series, Stirling Numbers, and q-Analogs", submitted)—which is much more comprehensive than ours—includes several of his own proofs of the result, whose natural generalisation is

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^p} \binom{n}{k} = \sum_{1 \le s_1 \le s_2 \le \cdots \le s_p \le n} \frac{1}{s_1 s_2 \cdots s_p}, \qquad p \ge 1, \tag{4}$$

that is addressed also.