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WEINSTEIN'S FUNCTIONS AND THE ASKEY-GASPER IDENTITY

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In his 1984 proof of the Bieberbach and Milin conjectures de Branges used a positivity result of special functions which follows from an identity about Jacobi polynomial sums that was found by Askey and Gasper in 1973, published in 1976.
In 1991 Weinstein presented another proof of the Bieberbach and Milin conjectures, also using a special function system which (by Todorov and Wilf) was realized to be the same as de Branges'.
In this article, we show how a variant of the Askey-Gasper identity can be deduced by a straightforward examination of Weinstein's functions which are intimately related with a Löwner chain of the Koebe function, and therefore with univalent functions.

KEY WORDS: Bieberbach conjecture, Weinstein's functions, Askey-Gasper identity

MSC (1991): 30C50, 33C45

1. INTRODUCTION

Let S denote the family of analytic and univalent functions $f(z) = z + a_2 z^2 + \dots$ of the unit disk $\mathbb{D}$. S is compact with respect to the topology of locally uniform convergence so that $k_n := \max\limits_{f \in S} |a_n(f)|$ exists. In 1916 Bieberbach [3] proved that $k_2 = 2$, with equality if and only if f is a rotation of the *Koebe function*

$$K(z) := \frac{z}{(1-z)^2} = \frac{1}{4}\left(\left(\frac{1+z}{1-z}\right)^2 - 1\right) = \sum_{n=1}^{\infty} n z^n \,, \tag{1}$$

and in a footnote he mentioned "Vielleicht ist überhaupt $k_n = n$.". This statement is known as the *Bieberbach conjecture*.

In 1923 Löwner [13] proved the Bieberbach conjecture for $n = 3$. His method was to embed a univalent function $f(z)$ into a *Löwner chain*, i.e. a family $\{f(z,t) \mid t \geq 0\}$

of univalent functions of the form

$$f(z,t) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n, \quad (z \in \mathbb{D}, t \geq 0, a_n(t) \in \mathbb{C}\ (n \geq 2))$$

which start with f

$$f(z,0) = f(z) \,,$$

and for which the relation

$$\operatorname{Re} p(z,t) = \operatorname{Re} \left(\frac{\dot{f}(z,t)}{z f'(z,t)} \right) > 0 \qquad (z \in \mathbb{D}) \tag{2}$$

is satisfied. Here $'$ and $\dot{}$ denote the partial derivatives with respect to z and t, respectively. Equation (2) is referred to as the *Löwner differential equation*, and geometrically it states that the image domains of f_t expand as t increases.

The history of the Bieberbach conjecture showed that it was easier to obtain results about the *logarithmic coefficients* of a univalent function f, i.e. the coefficients d_n of the expansion

$$\varphi(z) = \ln \frac{f(z)}{z} =: \sum_{n=1}^{\infty} d_n z^n$$

rather than for the coefficients a_n of f itself. So Lebedev and Milin [12] in the mid sixties developed methods to exponentiate such information. They proved that if for $f \in S$ the *Milin conjecture*

$$\sum_{k=1}^{n} (n+1-k) \left(k|d_k|^2 - \frac{4}{k} \right) \leq 0$$

on its logarithmic coefficients is satisfied for some $n \in \mathbb{N}$, then the Bieberbach conjecture for the index $n+1$ follows.

In 1984 de Branges [4] verified the Milin, and therefore the Bieberbach conjecture, and in 1991, Weinstein [18] gave a different proof. A reference other than [4] concerning de Branges' proof is [5], and a German language summary of the history of the Bieberbach conjecture and its proofs was given in [10].

Both proofs use the positivity of special function systems, and independently Todorov [16] and Wilf [19] showed that both de Branges' and Weinstein's functions essentially are the same (see also [11]),

$$\dot{\tau}_k^n(t) = -k \Lambda_k^n(t) \,, \tag{3}$$

$\tau_k^n(t)$ denoting the de Branges functions and $\Lambda_k^n(t)$ denoting the Weinstein functions, respectively.

Whereas de Branges applied an identity of Askey and Gasper [2] to his function system, Weinstein applied an addition theorem for Legendre polynomials to his function system to deduce the positivity result needed.

The identity of Askey and Gasper used by de Branges was stated in ([2], (1.16)) in the form

$$\sum_{j=0}^{n} P_j^{(\alpha,0)}(x) = \sum_{j=0}^{[n/2]} \frac{(1/2)_j \left(\frac{\alpha+2}{2}\right)_{n-j} \left(\frac{\alpha+3}{2}\right)_{n-2j} (n-2j)!}{j! \left(\frac{\alpha+3}{2}\right)_{n-j} \left(\frac{\alpha+1}{2}\right)_{n-2j} (\alpha+1)_{n-2j}} \times \left(C_{n-2j}^{(\alpha+1)/2} \left(\sqrt{\frac{1+x}{2}} \right) \right)^2 , \tag{4}$$

where $C_n^\lambda(x)$ denote the Gegenbauer polynomials, $P_j^{(\alpha,\beta)}(x)$ denote the Jacobi polynomials (see e.g. [1], § 22), and

$$(a)_j := a(a+1)\cdots(a+j-1) = \frac{\Gamma(a+j)}{\Gamma(a)}$$

denotes the shifted factorial (or Pochhammer symbol).

In this article, we show how a variant of the Askey-Gasper identity can be deduced by a straightforward examination of Weinstein's functions which are intimately related with the bounded Löwner chain of the Koebe function.

The application of an addition theorem for the Gegenbauer polynomials quite naturally arises in this context. We present a simple proof of this result so that this article is self-contained.

2. THE LÖWNER CHAIN OF THE KOEBE FUNCTION AND THE WEINSTEIN FUNCTIONS

We consider the Löwner chain

$$w(z,t) := K^{-1}\left(e^{-t}K(z)\right) \quad (z \in \mathbb{D}, t \geq 0) \tag{5}$$

of bounded univalent functions in the unit disk $\mathbb{D}$ which is defined in terms of the Koebe function (1). Since K maps the unit disk onto the entire plane slit along the negative x-axis in the interval $(-\infty, 1/4]$, the image $w(\mathbb{D},t)$ is the unit disk with a radial slit on the negative x-axis increasing with t.

Weinstein [18] used the Löwner chain (5), and showed the validity of Milin's conjecture if for all $n \geq 2$ the *Weinstein functions* $\Lambda_k^n : \mathbb{R}^+ \to \mathbb{R}$ $(k = 0, \ldots, n)$ defined by

$$\frac{e^t w(z,t)^{k+1}}{1 - w^2(z,t)} =: \sum_{n=k}^{\infty} \Lambda_k^n(t) z^{n+1} = W_k(z,t) , \tag{6}$$

satisfy the relations

$$\Lambda_k^n(t) \geq 0 \qquad (t \in \mathbb{R}^+, \quad 0 \leq k \leq n) . \tag{7}$$

Weinstein did not identify the functions $\Lambda_k^n(t)$, but was able to prove (7) without an explicit representation.

In this section we apply Weinstein's following interesting observation to show that $\Lambda_k^n(t)$ are the Fourier coefficients of a function that is connected with the Gegenbauer and Chebyshev polynomials.

The range of the function $w = K^{-1}(e^{-t}K)$ is the unit disk with a slit on the negative real axis. Since for all $\gamma \in \mathbb{R}, \gamma \neq 0 \pmod{\pi}$ the mapping

$$h_\gamma(z) := \frac{z}{1 - 2\cos\gamma \cdot z + z^2}$$

maps the unit disk onto the unit disk with two slits on the real axis, we can interpret w as composition $w = h_\theta^{-1}(e^{-t}h_\gamma)$ for a suitable pair (θ, γ), and a simple calculation shows that the relation

$$\cos\gamma = (1 - e^{-t}) + e^{-t}\cos\theta \tag{8}$$

is valid. We get therefore

$$\begin{aligned} h_\gamma(z) &= e^t \cdot h_\theta(w(z,t)) = \frac{e^t w}{1-w^2}\left(\frac{1-w^2}{1-2\cos\theta\cdot w + w^2}\right) \\ &= \frac{e^t w}{1-w^2}\left(1 + 2\sum_{k=1}^{\infty} w^k \cos k\theta\right) = W_0(z,t) + 2\sum_{k=1}^{\infty} W_k(z,t)\cos k\theta \quad (9) \\ &= W_0(z,t) + 2\sum_{k=1}^{\infty}\left(\sum_{n=k}^{\infty}\Lambda_k^n(t) z^{n+1}\right)\cos k\theta\ . \end{aligned}$$

It is easily seen that (9) remains valid for the pair $(\theta, \gamma) = (0, 0)$, corresponding to the representation

$$K(z) = W_0(z,t) + 2\sum_{k=1}^{\infty} W_k(z,t)\ .$$

Since on the other hand $h_\gamma(z)$ has the Taylor expansion

$$h_\gamma(z) = \frac{z}{1-2\cos\gamma\cdot z + z^2} = \sum_{n=0}^{\infty}\frac{\sin(n+1)\gamma}{\sin\gamma} z^{n+1}\ ,$$

equating the coefficients of z^{n+1} in (9) we get the identity

$$\frac{\sin(n+1)\gamma}{\sin\gamma} = \Lambda_0^n(t) + 2\sum_{k=1}^{n}\Lambda_k^n(t)\cos k\theta\ .$$

Hence we have discovered (see also [19], (2))

Theorem 1. (Fourier Expansion) *The Weinstein functions* $\Lambda_k^n(t)$ *satisfy the functional equation*

$$\begin{aligned} U_n\Big((1-e^{-t})+e^{-t}\cos\theta\Big) &= C_n^1\Big((1-e^{-t})+e^{-t}\cos\theta\Big) \\ &= \Lambda_0^n(t)+2\sum_{k=1}^{n}\Lambda_k^n(t)\cos k\theta\ , \end{aligned} \tag{10}$$

where $U_n(x)$ *denote the Chebyshev polynomials of the second kind.*

Proof. This is an immediate consequence of the identity

$$C_n^1(\cos\gamma)=U_n(\cos\gamma)=\frac{\sin(n+1)\gamma}{\sin\gamma}$$

(see e.g. [1], (22.3.16), (22.5.34)).

3. THE WEINSTEIN FUNCTIONS AS JACOBI POLYNOMIAL SUMS

In this section, we show that the Weinstein functions $\Lambda_k^n(t)$ can be represented as Jacobi polynomial sums.

Theorem 2. (Jacobi Sum) *The Weinstein functions have the representation*

$$\Lambda_k^n(t)=e^{-kt}\sum_{j=0}^{n-k}P_j^{(2k,0)}(1-2e^{-t})\ , \qquad (0\le k\le n)\ . \tag{11}$$

Proof. A calculation shows that $w(z,t)$ has the explicit representation

$$w(z,t)=\frac{4e^{-t}z}{\left(1-z+\sqrt{1-2xz+z^2}\right)^2}\ . \tag{12}$$

Here we use the abbreviation $x=1-2e^{-t}$. Furthermore, from

$$W_0(z,t)=\frac{e^t w}{1-w^2}=K(z)\,\frac{1-w}{1+w}\ ,$$

we get the explicit representation

$$W_0(z,t)=\frac{z}{1-z}\,\frac{1}{\sqrt{1-2xz+z^2}} \tag{13}$$

for $W_0(z,t)$. By the definition of $W_k(z,t)$, we have moreover

$$W_k(z,t) = \frac{e^t w^{k+1}}{1-w^2} = w^k\, W_0(z,t)\ .$$

Hence, by (12)–(13) we deduce the explicit representation

$$W_k(z,t) = e^{-kt}\,\frac{z^{k+1}}{1-z}\,\frac{4^k}{\sqrt{1-2xz+z^2}}\,\frac{1}{\left(1-z+\sqrt{1-2xz+z^2}\right)^{2k}} \tag{14}$$

for $W_k(z,t)$.

Since the Jacobi polynomials $P_j^{(\alpha,\beta)}(x)$ have the generating function

$$\begin{aligned}\sum_{j=0}^{\infty} P_j^{(\alpha,\beta)}(x)\, z^j \\ = \frac{2^{\alpha+\beta}}{\sqrt{1-2xz+z^2}}\,\frac{1}{\left(1-z+\sqrt{1-2xz+z^2}\right)^{\alpha}}\,\frac{1}{\left(1+z+\sqrt{1-2xz+z^2}\right)^{\beta}}\end{aligned} \tag{15}$$

(see e.g. [1], (22.9.1)), comparison with (14) yields

$$W_k(z,t) = e^{-kt}\,\frac{z^{k+1}}{1-z}\sum_{j=0}^{\infty} P_j^{(2k,0)}(x)\, z^j\ .$$

Using the Cauchy product

$$\frac{1}{1-z}\sum_{j=0}^{\infty} P_j^{(2k,0)}(x)\, z^j = \sum_{n=0}^{\infty}\sum_{j=0}^{n} P_j^{(2k,0)}(x)\, z^n\ ,$$

we finally have

$$W_k(z,t) = e^{-kt}\, z^{k+1}\sum_{n=0}^{\infty}\sum_{j=0}^{n} P_j^{(2k,0)}(x)\, z^n = \sum_{n=k}^{\infty}\Lambda_k^n(t)\, z^{n+1} = \sum_{n=0}^{\infty}\Lambda_k^{n+k}(t)\, z^{n+k+1}\ .$$

Equating coefficients gives the result.

4. ASKEY-GASPER INEQUALITY FOR THE WEINSTEIN FUNCTIONS

We would like to utilize the Fourier expansion (10) of Theorem 1 to find new representations for the Weinstein functions, hence by Theorem 2 for the Jacobi polynomial sum on the left hand side of (4). Hence, we have the need to find a representation for $C_n^1\Big((1-e^{-t})+e^{-t}\cos\theta\Big)$.

We do a little more, and give a representation for

$$C_n^1\Big(xy+\sqrt{1-x^2}\sqrt{1-y^2}\,\zeta\Big)\ , \tag{16}$$

from which the above expression is the special case $x=y=\sqrt{1-e^{-t}},\zeta=\cos\theta$. Actually, in the next section, an even more general expression is considered, see Theorem 5. Here we outline the deduction for our particular case.

The function given by (16) as a function of the variable ζ is a polynomial of degree n. Hence it can be expanded by Gegenbauer polynomials $C_j^\lambda(\zeta)$ $(j=0,\ldots,n)$. We choose $\lambda=1/2$, i.e. we develop in terms of Legendre polynomials $P_j(\zeta)=C_j^{1/2}(\zeta)$ (see e.g. [1], (22.5.36)),

$$C_n^1\Big(xy+\sqrt{1-x^2}\sqrt{1-y^2}\,\zeta\Big)=\sum_{m=0}^{n}A_m^n(x,y)\,C_m^{1/2}(\zeta) \tag{17}$$

with A_j^n depending on x and y. By the orthogonality of the Gegenbauer polynomials,

$$\int_{-1}^{1}C_j^{1/2}(\zeta)\,C_m^{1/2}(\zeta)\,d\zeta=\begin{cases}\frac{2}{2j+1} & \text{if } j=m\\ 0 & \text{otherwise}\end{cases}$$

multiplying (17) by $C_j^{1/2}(\zeta)$, and integrating from $\zeta=-1$ to $\zeta=1$, we get therefore

$$A_j^n(x,y)=\frac{2j+1}{2}\int_{-1}^{1}C_n^1\Big(xy+\sqrt{1-x^2}\sqrt{1-y^2}\,\zeta\Big)C_j^{1/2}(\zeta)\,d\zeta\ . \tag{18}$$

To eliminate the second (oscillating) factor $C_j^{1/2}(\zeta)$, we utilize the identity

$$\begin{aligned}&\int_{-1}^{1}f(\zeta)\,C_j^\lambda(\zeta)(1-\zeta^2)^{\lambda-1/2}\,d\zeta\\ &\qquad=\frac{2^j}{j!}\frac{\Gamma(j+\lambda)\Gamma(j+2\lambda)}{\Gamma(\lambda)\Gamma(2j+2\lambda)}\int_{-1}^{1}f^{(j)}(\zeta)\,(1-\zeta^2)^{\lambda+j-1/2}\,d\zeta\ ,\end{aligned} \tag{19}$$

which is valid for any j times continuously differentiable function f, and which can easily be proved by iterative partial integration (see e.g. [9], Chapter VII, p. 140). Choosing $\lambda = 1/2$ and

$$f(\zeta) := C_n^1\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\,\zeta\right)$$

we get (with the Gamma duplication formula (29))

$$\begin{aligned}
&\int_{-1}^{1} C_n^1\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\zeta\right) C_j^{1/2}(\zeta)\,d\zeta \\
&\qquad = \frac{1}{2^j\, j!}\int_{-1}^{1}(1-\zeta^2)^j\,\frac{d^j}{d\zeta^j}C_n^1\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\zeta\right)d\zeta.
\end{aligned} \tag{20}$$

Since furthermore

$$\frac{d^j}{d\zeta^j}C_n^\nu(\zeta) = 2^j\,(\nu)_j\,C_{n-j}^{\nu+j}(\zeta) \tag{21}$$

(see e.g. [17], p. 179), we get moreover

$$\begin{aligned}
&\frac{1}{2^j\, j!}\int_{-1}^{1}(1-\zeta^2)^j\,\frac{d^j}{d\zeta^j}C_n^1\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\,\zeta\right)d\zeta \\
&\qquad = (1-x^2)^{j/2}\,(1-y^2)^{j/2}\,Q_j^n(x,y)
\end{aligned} \tag{22}$$

with

$$Q_j^n(x,y) := \int_{-1}^{1}(1-\zeta^2)^j\,C_{n-j}^{j+1}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\,\zeta\right)d\zeta\,.$$

Now observe that $Q_j^n(x,y)$ is a polynomial in the variables x and y, of degree $n-j$ each. In the next section we will show that the integral $Q_j^n(x,y)$ has zeros at both the zeros of $C_{n-j}^{j+1}(x)$ and $C_{n-j}^{j+1}(y)$, hence, as a polynomial of degree $n-j$ in x and y respectively, must be a multiple of the product $C_{n-j}^{j+1}(x)\,C_{n-j}^{j+1}(y)$. An initial value gives

$$Q_j^n(x,y) = \frac{2^{2(j+1)}\,j!^2\,(n-j)!}{2(n+j+1)!}\,C_{n-j}^{j+1}(x)\,C_{n-j}^{j+1}(y)\,. \tag{23}$$

Note that the complete proof of a generalization of statement (17)/(23) will be given in the next section.

Therefore finally, combining (18)–(23), we have discovered the identity

$$A_j^n(x,y) = (2j+1)\frac{2^{2j}\,j!^2\,(n-j)!}{(n+j+1)!}(1-x^2)^{j/2}(1-y^2)^{j/2}\,C_{n-j}^{j+1}(x)\,C_{n-j}^{j+1}(y)\,. \quad (24)$$

As a first step this leads to the following Askey-Gasper type representation for the Fourier series (10).

Theorem 3. *The Fourier series (10) has the representation*

$$\begin{aligned} &C_n^1\Big((1-e^{-t})+e^{-t}\cos\theta\Big) \\ &\quad = \sum_{j=0}^{n} A_j^n\left(\sqrt{1-e^{-t}},\sqrt{1-e^{-t}}\right)\,C_j^{1/2}(\cos\theta) \qquad (25) \\ &\quad = \sum_{j=0}^{n}(2j+1)\frac{4^j\,j!^2\,(n-j)!}{(n+j+1)!}\,e^{-jt}\,\left(C_{n-j}^{j+1}\left(\sqrt{1-e^{-t}}\right)\right)^2\,P_j(\cos\theta)\,. \end{aligned}$$

Proof. Set $x = y = \sqrt{1-e^{-t}}$ and $\zeta = \cos\theta$ in (24).

Since by a simple function theoretic argument the Legendre polynomials $P_j(\cos\theta)$ on the right hand side of (25) can be written as

$$P_j(\cos\theta) = \sum_{l=0}^{j} g_l g_{j-l}\cos(j-2l)\theta\,, \quad (26)$$

with positive coefficients

$$g_l = \frac{(2l)!}{4^l\,l!^2} \quad (27)$$

(see e.g. [15], (4.9.3)), we have at this stage the

Corollary 7. *The Weinstein functions satisfy the inequalities (7),*

$$\Lambda_k^n(t) \geq 0 \qquad (t \in \mathbb{R}^+, \quad 0 \leq k \leq n)\,.$$

Proof. Combining Theorems 1 and 2 with (26)–(27) gives the result.

Theorem 3 together with (26) immediately yields sum representations for the Weinstein functions in terms of the Gegenbauer polynomials,

$$\Lambda_{2m}^n(t) = \sum_{j=m}^{[n/2]} 4^{2j}\, \frac{\Gamma(n+1-2j)(2j)!^2}{\Gamma(n+2+2j)}\,(4j+1)\, g_{j-m}\, g_{j+m}\, e^{-2jt}\left(C_{n-2j}^{2j+1}\left(\sqrt{1-e^{-t}}\right)\right)^2$$

for $m = 0, 1, \ldots, [n/2]$, and

$$\begin{aligned}\Lambda_{2m+1}^n(t) \quad &= \quad \sum_{j=m}^{[(n-1)/2]} 4^{2j+1}\, \frac{\Gamma(n-2j)(2j+1)!^2}{\Gamma(n+3+2j)}\,(4j+3)\, g_{j-m}\, g_{j+1+m}\, e^{-(2j+1)t} \\ &\qquad \times \left(C_{n-2j-1}^{2j+2}\left(\sqrt{1-e^{-t}}\right)\right)^2\end{aligned}$$

for $m = 0, 1, \ldots, [(n-1)/2]$. Another form of this statement will be given in Section 6.

5. ADDITION THEOREM FOR THE GEGENBAUER POLYNOMIALS

In this section, we fill the gap that remained in the last section by proving a generalization of (17)/(23), the *addition theorem* for the Gegenbauer polynomials (see e.g. [7]).

Theorem 5. (Addition Theorem for the Gegenbauer Polynomials) *For $\nu > 1/2$, $x, y \in [-1, 1]$, and $\zeta \in \mathbb{C}$, the Gegenbauer polynomials satisfy the identity*

$$\begin{aligned}C_n^\nu\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\,\zeta\right) \quad &= \quad \Gamma(2\nu-1)\sum_{j=0}^{n} \frac{4^j\,(n-j)!}{\Gamma(n+2\nu+j)}\left((\nu)_j\right)^2 \\ &\qquad \times (2\nu+2j-1)(1-x^2)^{j/2}\,(1-y^2)^{j/2} \\ &\qquad \times C_{n-j}^{\nu+j}(x)\, C_{n-j}^{\nu+j}(y)\, C_j^{\nu-1/2}(\zeta)\,.\end{aligned}$$

Proof. The function

$$C_n^\nu\big(xy + \sqrt{1-x^2}\sqrt{1-y^2}\,\zeta\big)$$

as a function of ζ is a polynomial of degree n. Therefore, for any $\lambda > 0$, we can expand it in terms of Gegenbauer polynomials $C_j^\lambda(\zeta)$,

$$C_n^\nu\big(xy + \sqrt{1-x^2}\sqrt{1-y^2}\,\zeta\big) = \sum_{m=0}^{n} A_m^n(x,y)\, C_m^\lambda(\zeta)\,, \tag{28}$$

the coefficients A_j^n being functions of the parameters x and y.

The orthogonality relation of the system $C_j^\lambda(\zeta)$ is given by

$$\int_{-1}^{1} (1-\zeta^2)^{\lambda-1/2}\, C_j^\lambda(\zeta)\, C_m^\lambda(\zeta)\, d\zeta = \begin{cases} \frac{\pi\, 2^{1-2\lambda}\,\Gamma(j+2\lambda)}{j!\,(j+\lambda)\,\Gamma(\lambda)^2} & \text{if } j = m \\ 0 & \text{otherwise} \end{cases}$$

(see e.g. [1], (22.2.3)). Multiplying (28) by $(1-\zeta^2)^{\lambda-1/2}\, C_j^\lambda(\zeta)$, and integrating from $\zeta = -1$ to $\zeta = 1$, we get therefore

$$\int_{-1}^{1} (1-\zeta^2)^{\lambda-1/2}\, C_n^\nu\big(xy+\sqrt{1-x^2}\sqrt{1-y^2}\,\zeta\big) C_j^\lambda(\zeta)\, d\zeta = A_j^n(x,y)\, \frac{\pi\, 2^{1-2\lambda}\,\Gamma(j+2\lambda)}{j!\,(j+\lambda)\,\Gamma(\lambda)^2}\,.$$

Utilizing identity (19) with

$$f(\zeta) := C_n^\nu\Big(xy+\sqrt{1-x^2}\sqrt{1-y^2}\,\zeta\Big)\,,$$

we get

$$\begin{aligned} A_j^n(x,y) &= \frac{2^{j+2\lambda-1}\,\Gamma(\lambda)\,\Gamma(j+\lambda+1)}{\pi\,\Gamma(2j+2\lambda)} \\ &\quad \times \int_{-1}^{1} (1-\zeta^2)^{j+\lambda-1/2}\, \frac{d^j}{d\zeta^j} C_n^\nu\Big(xy+\sqrt{1-x^2}\sqrt{1-y^2}\zeta\Big)\, d\zeta\,. \end{aligned}$$

The derivative identity (21) then yields

$$\begin{aligned} A_j^n(x,y) &= \frac{2^{2j+2\lambda-1}\,(\nu)_j\,\Gamma(\lambda)\,\Gamma(j+\lambda+1)}{\pi\,\Gamma(2j+2\lambda)} (1-x^2)^{j/2}\,(1-y^2)^{j/2} \\ &\quad \times \int_{-1}^{1} (1-\zeta^2)^{j+\lambda-1/2}\, C_{n-j}^{\nu+j}\Big(xy+\sqrt{1-x^2}\sqrt{1-y^2}\zeta\Big)\, d\zeta\,. \end{aligned}$$

Now we choose $\lambda := \nu - 1/2$ (hence our assumption $\nu > 1/2$). This choice is motivated by the calculation involving the differential equation that follows later, for which the desired simplification occurs exactly when $\lambda = \nu - 1/2$. Using the duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\,\Gamma(z)\,\Gamma(z+1/2) \tag{29}$$

of the Gamma function to simplify the factor in front of the integral, we finally arrive at the representation

$$\begin{aligned} A_j^n(x,y) &= 2^{1-2\nu}(2j+2\nu-1)\frac{\Gamma(2\nu-1)}{\Gamma(\nu)^2}(1-x^2)^{j/2}(1-y^2)^{j/2} \\ &\quad \times \int_{-1}^{1}(1-\zeta^2)^{j+\nu-1}\,C_{n-j}^{\nu+j}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\zeta\right)d\zeta \end{aligned}$$

for the coefficients $A_j^n(x,y)$. Hence, we consider the function

$$Q_j^n(x,y) := \int_{-1}^{1}(1-\zeta^2)^{j+\nu-1}\,C_{n-j}^{\nu+j}\left(xy+\sqrt{1-x^2}\sqrt{1-y^2}\zeta\right)d\zeta$$

in detail. Observe that $Q_j^n(x,y)$ is a polynomial in the variables x and y, of degree $n-j$ each. Note furthermore that $Q_j^n(x,y)$ is symmetric, i.e. $Q_j^n(x,y) = Q_j^n(y,x)$.

In the following we will show that the integral $Q_j^n(x,y)$ has zeros at both the zeros of $C_{n-j}^{\nu+j}(x)$ and $C_{n-j}^{\nu+j}(y)$, hence, as a polynomial of degree $n-j$ in x and y respectively, must be a constant multiple of the product $C_{n-j}^{\nu+j}(x)\,C_{n-j}^{\nu+j}(y)$.

By the symmetry of $Q_j^n(x,y)$ it is enough to show that $Q_j^n(x,y)$ has zeros at the zeros of $C_{n-j}^{\nu+j}(x)$. Since $C_{n-j}^{\nu+j}(x)$ is a solution of the differential equation

$$(1-x^2)\,p''(x) - (2\nu+2j+1)\,x\,p'(x) + (n-j)(n+j+2\nu)\,p(x) = 0\ , \tag{30}$$

and since any *polynomial* solution $p(x)$ of (30) must be a multiple of $C_{n-j}^{\nu+j}(x)$ (see e.g. [15], Theorem 4.2.2 in combination with [1], (22.5.27)), we have only to check that $p(x) := Q_j^n(x,y)$ satisfies (30).

We write $\eta(x) := xy + \sqrt{1-x^2}\sqrt{1-y^2}\zeta$, and note that

$$\eta'(x) = y - \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}\,x\,\zeta$$

so that

$$x\eta'(x) = xy - \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}\,x^2\,\zeta = \eta(x) - \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}\,\zeta\ .$$

Hence we deduce

$$-(2\nu+2j+1)\,x\,\frac{\partial}{\partial x}Q_j^n(x,y)$$

$$=\int_{-1}^{1}-(2\nu+2j+1)\,\eta(x)\left(C_{n-j}^{\nu+j}\right)'(\eta(x))(1-\zeta^2)^{j+\nu-1}\,d\zeta$$

$$+(2\nu+2j+1)\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}\int_{-1}^{1}\zeta\,(1-\zeta^2)^{j+\nu-1}\left(C_{n-j}^{\nu+j}\right)'(\eta(x))\,d\zeta.$$

Similarly, using the identity

$$\left(y\sqrt{1-x^2}-x\sqrt{1-y^2}\,\zeta\right)^2=(1-\eta(x)^2)-(1-y^2)\,(1-\zeta^2)\,,$$

we get

$$(1-x^2)\,\frac{\partial^2}{\partial x^2}Q_j^n(x,y)=\int_{-1}^{1}(1-\eta(x)^2)\left(C_{n-j}^{\nu+j}\right)''(\eta(x))(1-\zeta^2)^{j+\nu-1}\,d\zeta$$

$$-\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}\int_{-1}^{1}\sqrt{1-x^2}\sqrt{1-y^2}\,(1-\zeta^2)^{j+\nu}\left(C_{n-j}^{\nu+j}\right)''(\eta(x))\,d\zeta$$

$$-\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}\int_{-1}^{1}\zeta\,(1-\zeta^2)^{j+\nu-1}\left(C_{n-j}^{\nu+j}\right)'(\eta(x))\,d\zeta.$$

Combining these results, we arrive at the representation

$$(1-x^2)\,\frac{\partial^2}{\partial x^2}Q_j^n(x,y)-(2\nu+2j+1)\,x\,\frac{\partial}{\partial x}Q_j^n(x,y)+(n-j)(n+j+2\nu)\,Q_j^n(x,y)$$

$$=\int_{-1}^{1}(1-\zeta^2)^{j+\nu-1}\Big((1-\eta^2)\left(C_{n-j}^{\nu+j}\right)''(\eta)-(2\nu+2j+1)\,\eta\left(C_{n-j}^{\nu+j}\right)'(\eta)$$

$$+(n-j)(n+j+2\nu)\,C_{n-j}^{\nu+j}(\eta)\Big)\,d\zeta$$

$$+\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}\left(\int_{-1}^{1}2(j+\nu)\zeta(1-\zeta^2)^{j+\nu-1}\left(C_{n-j}^{\nu+j}\right)'(\eta)d\zeta\right.$$

$$- \int_{-1}^{1} \sqrt{1-x^2}\sqrt{1-y^2}(1-\zeta^2)^{j+\nu} \left(C_{n-j}^{\nu+j} \right)''(\eta) d\zeta \Bigg) .$$

The first integral obviously vanishes since $C_{n-j}^{\nu+j}(x)$ satisfies the differential equation (30). The vanishing of the final parenthesized expression follows easily by partial integration. Therefore, we have proved that $Q_j^n(x,y)$ is a solution of (30), as announced.

Hence,

$$Q_j^n(x,y) = a\, C_{n-j}^{\nu+j}(x)\, C_{n-j}^{\nu+j}(y) \tag{31}$$

with a constant a (not depending on x and y). For $y = 1$, we deduce

$$Q_j^n(x,1) = \int_{-1}^{1} (1-\zeta^2)^{j+\nu-1}\, C_{n-j}^{\nu+j}(x)\, d\zeta = 2^{2j+2\nu-1} \frac{\Gamma(j+\nu)^2}{\Gamma(2j+2\nu)}\, C_{n-j}^{\nu+j}(x) \tag{32}$$

by an evaluation of the Beta type integral. On the other hand, by (31),

$$Q_j^n(x,1) = a\, C_{n-j}^{\nu+j}(x)\, C_{n-j}^{\nu+j}(1) = a\, C_{n-j}^{\nu+j}(x) \binom{n+j+2\nu-1}{n-j}$$

(see e.g. [1], (22.4.2)), so that we get

$$a = 2^{2j+2\nu-1} \frac{\Gamma(j+\nu)^2}{\Gamma(2j+2\nu)} \Big/ \binom{n+j+2\nu-1}{n-j} = 2^{2j+2\nu-1} \frac{(n-j)!\,\Gamma(j+\nu)^2}{\Gamma(n+j+2\nu)} .$$

Hence

$$Q_j^n(x,y) = 2^{2j+2\nu-1} \frac{(n-j)!\,\Gamma(j+\nu)^2}{\Gamma(n+j+2\nu)}\, C_{n-j}^{\nu+j}(x)\, C_{n-j}^{\nu+j}(y) ,$$

implying

$$\begin{aligned} A_j^n(x,y) &= \Gamma(2\nu-1) \frac{2^{2j}(n-j)!}{\Gamma(n+j+2\nu)}\, \frac{\Gamma(j+\nu)^2}{\Gamma(\nu)^2} \\ &\quad \times (2j+2\nu-1)\, (1-x^2)^{j/2}\, (1-y^2)^{j/2}\, C_{n-j}^{\nu+j}(x)\, C_{n-j}^{\nu+j}(y), \end{aligned}$$

and we are done.

As a consequence, taking the limit $\nu \to 1/2$, we get the following

Corollary 6. (Addition Theorem for the Legendre Polynomials) *For $x, y \in [-1,1]$, $\zeta \in \mathbb{C}$, the Legendre polynomials satisfy the identities*

$$P_n\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\,\zeta\right)$$

$$= P_n(x)\,P_n(y) + 2\sum_{j=1}^{n} 4^j\,\frac{(n-j)!}{(n+j)!}\Big((1/2)_j\Big)^2\,(1-x^2)^{j/2}\,(1-y^2)^{j/2}$$

$$\times C_{n-j}^{1/2+j}(x)\,C_{n-j}^{1/2+j}(y)\,T_j(\zeta) \tag{33}$$

$$= P_n(x)\,P_n(y) + 2\sum_{j=1}^{n} \frac{(n-j)!}{(n+j)!}\,P_n^j(x)\,P_n^j(y)\,T_j(\zeta)\,, \tag{34}$$

where $T_j(\zeta)$ denote the Chebyshev polynomials of the first kind, and

$$P_n^j(x) = (-1)^j\,(1-x^2)^{j/2}\,\frac{\partial^j}{\partial x^j}P_n(x) \tag{35}$$

denote the associated Legendre functions (see e.g. [1], (8.6.6)).

In particular, for $y = x$, one has

$$P_n(x^2 + (1-x^2)\cos\theta) = P_n(x)^2 + 2\sum_{j=1}^{n}\frac{(n-j)!}{(n+j)!}\,P_n^j(x)^2\,\cos j\theta\,. \tag{36}$$

Proof. Since

$$C_n^0(x) = \lim_{\lambda\to 0}\frac{C_n^\lambda(x)}{\lambda} \qquad \text{and} \qquad C_n^\alpha(x) = \lim_{\lambda\to\alpha} C_n^\lambda(x) \quad \text{for all} \quad \alpha > 0$$

(see e.g. [1], (22.5.4)), for $\nu \to 1/2$ Theorem 5 implies

$$C_n^{1/2}\Big(xy + \sqrt{1-x^2}\sqrt{1-y^2}\,\zeta\Big)$$

$$= C_n^{1/2}(x)\,C_n^{1/2}(y) + \sum_{j=1}^{n} 4^j\,\frac{(n-j)!}{(n+j)!}\,\Big((1/2)_j\Big)^2$$

$$\times (1-x^2)^{j/2}\,(1-y^2)^{j/2}\,C_{n-j}^{1/2+j}(x)\,C_{n-j}^{1/2+j}(y)\,j\,C_j^0(\zeta)\,.$$

With $C_n^{1/2}(x) = P_n(x)$, and $j\,C_j^0(\zeta) = 2\,T_j(\zeta)$ (see e.g. [1], (22.5.35), (22.5.33)), we get (33). An application of (21) and (35) yields (34).

Using

$$T_n(\cos\theta) = \cos n\theta$$

(see e.g. [1], (22.3.15)) finally yields (36).

Note that Weinstein used (36) in his proof of Milin's conjecture.

6. ASKEY–GASPER IDENTITY FOR THE WEINSTEIN FUNCTIONS

Here, we combine the above results to deduce a sum representation with nonnegative summands for the Weinstein functions, and therefore by Theorem 2 for the Jacobi polynomial sum.

By Theorem 3 we have

$$\begin{aligned} C_n^1\Big((1-e^{-t})+e^{-t}\cos\theta\Big) &= \sum_{j=0}^{n}(2j+1)\frac{4^j\,j!^2\,(n-j)!}{(n+j+1)!}\,e^{-jt} \\ &\qquad \times\left(C_{n-j}^{j+1}\left(\sqrt{1-e^{-t}}\right)\right)^2 P_j(\cos\theta)\ , \end{aligned}$$

and, expanding $P_j(\cos\theta)$ using (33) with $x=y=0$, $\zeta=\cos\theta$, this gives

$$\begin{aligned} &= \sum_{j=0}^{n}(2j+1)\frac{4^j j!^2(n-j)!}{(n+j+1)!}e^{-jt}\left(C_{n-j}^{j+1}\left(\sqrt{1-e^{-t}}\right)\right)^2 \\ &\qquad \times 2{\sum_{k=0}^{j}}{}'\,4^k\frac{(j-k)!}{(j+k)!}\Big((1/2)_k\Big)^2 C_{j-k}^{1/2+k}(0)^2 T_k(\cos\theta), \end{aligned}$$

where Σ' indicates that the summand for $k=0$ is to be taken with a factor $1/2$. Interchanging the order of summation, and using $T_k(\cos\theta)=\cos k\theta$, gives

$$\begin{aligned} &= 2{\sum_{k=0}^{n}}{}'\sum_{j=k}^{n}(2j+1)4^k\frac{4^j j!^2(n-j)!}{(n+j+1)!}\frac{(j-k)!}{(j+k)!}\Big((1/2)_k\Big)^2 e^{-jt}C_{j-k}^{1/2+k}(0)^2 \\ &\qquad \times\left(C_{n-j}^{j+1}\left(\sqrt{1-e^{-t}}\right)\right)^2\cos k\theta. \end{aligned}$$

Comparing with Theorem 1,

$$C_n^1\Big((1-e^{-t})+e^{-t}\cos\theta\Big) = 2\sum_{k=0}^{n}{}'\Lambda_k^n(t)\cos k\theta\ ,$$

and equating coefficients yields for the Weinstein functions

$$\begin{aligned}\Lambda_k^n(t) &= \sum_{j=k}^{n}(2j+1)4^k\frac{4^j j!^2(n-j)!}{(n+j+1)!}\frac{(j-k)!}{(j+k)!}\\ &\quad\times\Big((1/2)_k\Big)^2 e^{-jt}C_{j-k}^{1/2+k}(0)^2\left(C_{n-j}^{j+1}\left(\sqrt{1-e^{-t}}\right)\right)^2\ .\end{aligned}$$

Replacing n by $k+n$, and then making the index shift $j_{\text{new}} := j_{\text{old}} - k$ finally leads to

$$\begin{aligned}\Lambda_k^{k+n}(t) &= \sum_{j=0}^{n}(2j+2k+1)\frac{4^{j+2k}(j+k)!^2(n-j)!j!((1/2)_k)^2}{(2k+n+j+1)!(j+2k)!}\\ &\quad\times e^{-(j+k)t}C_j^{1/2+k}(0)^2\Big(C_{n-j}^{j+k+1}\Big(\sqrt{1-e^{-t}}\Big)\Big)^2.\end{aligned}$$

Setting $y := \sqrt{1-e^{-t}}$, by Theorem 2

$$\begin{aligned}\sum_{j=0}^{n}P_j^{(2k,0)}(2y^2-1) &= \sum_{j=0}^{n}(2j+2k+1)\frac{4^{j+2k}(j+k)!^2(n-j)!j!((1/2)_k)^2}{(2k+n+j+1)!(j+2k)!}\\ &\quad\times(1-y^2)^j C_j^{1/2+k}(0)^2\Big(C_{n-j}^{j+k+1}(y)\Big)^2.\end{aligned}$$

This is an Askey-Gasper type representation different from (4) that was given by Gasper ([6], (8.17), and (8.18) with $x=0$). Note that Gasper's formula ([6], (8.18)) interpolates between these two representations. Whereas Askey's and Gasper's deductions of the given formulas prove the results for all $\alpha > -2$, our deduction has the disadvantage that it is only valid for $\alpha = 2k, k \in \mathbb{N}_0$. On the other hand, the advantage of our presentation is that it embeds this result in a natural way in Weinstein's proof of Milin's conjecture using only elementary properties of classical orthogonal polynomials.

7. CLOSED FORM REPRESENTATION OF WEINSTEIN FUNCTIONS

Note that nowhere in our deduction we needed the explicit representation of the de Branges functions = Weinstein functions, compare Henrici's comment [8], p. 602: "At the time of this writing, the only way to verify $\dot{\tau}_k^n(t) \le 0$ appears to be to solve the system explicitly, and to manipulate the solution".

In this connection we would like to mention that in [11] we proved the identity (3), which connects de Branges' with Weinstein's functions, by a pure application of the de Branges differential equations system (see also [14]), and without the use of an explicit representation of the de Branges functions.

In this section we give a simple method to generate this explicit representation which was used by de Branges, see also [19].

Since $(1-e^{-t}) + e^{-t}\cos\theta = 1 - 2e^{-t}\sin^2\frac{\theta}{2}$, Taylor expansion gives using (21) and ([1], (22.4.2))

$$
\begin{aligned}
C_n^1\Big((1-e^{-t}) + e^{-t}\cos\theta\Big) &= C_n^1\Big(1 - 2e^{-t}\sin^2\frac{\theta}{2}\Big) \\
&= \sum_{j=0}^{n} \frac{C_n^{1\,(j)}(1)}{j!}\,(-1)^j\,2^j\,e^{-jt}\left(\sin^2\frac{\theta}{2}\right)^j \\
&= \sum_{j=0}^{n} C_{n-j}^{j+1}(1)\,2^{2j}\,(-1)^j\,e^{-jt}\left(\sin^2\frac{\theta}{2}\right)^j \\
&= \sum_{j=0}^{n} \binom{n+j+1}{n-j}\,2^{2j}\,(-1)^j\,e^{-jt}\left(\sin^2\frac{\theta}{2}\right)^j .
\end{aligned}
$$

An elementary argument shows that

$$
\left(\sin^2\frac{\theta}{2}\right)^j = 2\sum_{k=0}^{j}{}' \frac{(-1)^k}{2^{2j}}\binom{2j}{j-k} T_{2k}\left(\cos\frac{\theta}{2}\right) = 2\sum_{k=0}^{j}{}' \frac{(-1)^k}{2^{2j}}\binom{2j}{j-k}\cos k\theta
$$

(see e.g. [17], p. 189). Changing the order of summation, we get therefore

$$
\begin{aligned}
C_n^1\Big((1-e^{-t})+e^{-t}\cos\theta\Big) &= 2\sum_{k=0}^{n}{}'\sum_{j=k}^{n}(-1)^{j+k}\binom{n+j+1}{n-j}\binom{2j}{j-k}e^{-jt}\cos k\theta \\
&= 2\sum_{k=0}^{n}{}'\Lambda_k^n(t)\cos k\theta
\end{aligned}
$$

by (10). Hence

$$
\begin{aligned}
\Lambda_k^n(t) &= \sum_{j=k}^{n}(-1)^{j+k}\binom{n+j+1}{n-j}\binom{2j}{j-k}e^{-jt} \\
&= e^{-kt}\binom{n+k+1}{n-k}\,{}_3F_2\left(\begin{matrix}n+k+2,k+1/2,-n+k\\ k+3/2,2k+1\end{matrix}\middle| e^{-t}\right).
\end{aligned}
$$

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