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On Nonvanishing Univalent Functions with Real Coefficients*

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Using the well-known Brickman representation for univalent functions, it is shown that the extreme points of the set $S_0(R)$ of nonvanishing univalent functions with real coefficients omit only real values. Furthermore a support point of $S_0(R)$ is shown to have the same property.

1. Introduction

Let A be the set of analytic functions of the unit disk $\mathbb{D}$. A is a locally convex linear space, so that the Krein-Mil'man theorem applies; i.e. the extreme points of a compact family F span the closed convex hull: $\overline{\text{co}}\, EF = \overline{\text{co}}\, F$. For an introduction look for example in [5].

Recently Duren and Schober [4] examined the set S_0 of univalent functions which are normalized by the conditions

$$f(0)=1, \quad 0 \notin f(\mathbb{D}).$$

$S_0 \cup \{1\}$ is a compact subset of A. Duren and Schober had been interested in extreme points and support points of S_0. Recall that a support point of a family F is a function which maximizes the real part of some continuous linear functional, that is not constant over F.

We shall give a characterization of the extreme points and support points of the subfamily $S_0(R)$ of nonvanishing univalent functions whose Taylor expansions at the origin have real coefficients.

2. Extreme Points of $S_0(R)$

Using the usual Brickman representation we get:

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Lemma. *Let be $f \in S_0(R)$. If f omits some nonreal value a, then f has a proper convex representation in $S_0(R)$.*

Proof. Because f is univalent and has real coefficients, the range of f is symmetric with respect to the real axis, so that f omits $\bar{a}$, too.

Because $a \notin \mathbb{R}$, we have now two omitted values which lie on an ellipse with foci in 0 and 1, so that f has the Brickman representation ([1], see [4], Theorem 1)

$$f = tf_1 + (1-t)f_2,$$

with $t \in]0, 1[$ and $(k=1,2)$

$$f_k := \psi_k \circ f \in S_0, \quad \psi_{1,2}(w) := \frac{w \pm \psi(w) \mp \psi(0)}{1 + \psi(1) \mp \psi(0)}$$

where

$$\psi(w) := \sqrt{(w-a)(w-\bar{a})}.$$

Now it remains to show that f_k $(k=1,2)$ or equivalently ψ (expanded at the point 1) has real coefficients.

For $w \in \mathbb{R}$ one has $\psi(w) \in \mathbb{R}$, implying the result. □

Now we are able to state the following result about the extreme points.

Theorem 1. *Every extreme point of $S_0(R) \cup \{1\}$ has the form*

$$f(z) = \frac{(1+z)^2}{(1-yz)(1-\bar{y}z)}, \quad y \in \partial\mathbb{D} \setminus \{-1\} \quad \textit{or}$$

$$f(z) = \frac{(1-z)^2}{(1-yz)(1-\bar{y}z)}, \quad y \in \partial\mathbb{D} \setminus \{1\}.$$

Proof. Because of the Lemma an extreme point of $S_0(R)$ omits only real values.

Thus with the origin all negative real numbers are omitted (because the range is simply connected). We next show that an extreme point of $S_0(R) \cup \{1\}$ omits no interval of the form $]-\infty, \varepsilon]$ for $\varepsilon > 0$. In this case there would be a representation

$$g = \frac{1}{1+\varepsilon} \cdot f + \frac{\varepsilon}{1+\varepsilon} \cdot 1$$

with a certain function $f \in S_0$ with similar range. This is a representation within $S_0 \cup \{1\}$, thus g is not extreme.

So an extreme point omits $]-\infty, 0]$ and possibly a second real interval $[1+\varepsilon, \infty[$, $\varepsilon > 0$. But the functions having this geometric property are exactly of the desired form. □

3. Support Points of $S_0(R)$

Using the result about the extreme points we are able to give the following result about the support points of $S_0(R)$.

Theorem 2. *Every support point of* $S_0(R)$ *has the form*

$$f(z)=1+\kappa\frac{z}{(1-yz)(1-\bar{y}z)}$$

for some $y\in\partial\mathbb{D}$ *and* $\kappa\in[-2(1-\operatorname{Re}y),\, 2(1+\operatorname{Re}y)]$, $\kappa\neq 0$.

Proof. Let L be a continuous linear functional over A which is not constant within $S_0(R)$.

If g is a support point of $S_0(R)$ with respect to L, we have

$$M:=\operatorname{Re}Lg=\max_{h\in S_0(R)}\operatorname{Re}Lh.$$

Because of Theorem 1 the function g has the Choquet representation (see e.g. [5])

$$g(z)=\int_{\partial\mathbb{D}}\frac{(1+z)^2}{(1-yz)(1-\bar{y}z)}\,d\mu_+(y)+\int_{\partial\mathbb{D}}\frac{(1-z)^2}{(1-yz)(1-\bar{y}z)}\,d\mu_-(y) \tag{1}$$

with positive measures μ_+ and μ_-, $\mu_+(\partial\mathbb{D})+\mu_-(\partial\mathbb{D})=1$, which are supported by the sets

$$\left\{y\in\partial\mathbb{D}\,\middle|\,\frac{(1\pm z)^2}{(1-yz)(1-\bar{y}z)}\in E(S_0(R)\cup\{1\})\right\}$$

respectively.

Therefore it follows that μ_+-a.e. and μ_--a.e. respectively

$$\operatorname{Re}L\left\{\frac{(1+z)^2}{(1-yz)(1-\bar{y}z)}\right\}=M \quad\text{and}$$
$$\operatorname{Re}L\left\{\frac{(1-z)^2}{(1-yz)(1-\bar{y}z)}\right\}=M. \tag{2}$$

Let $H_\pm$ be the subsets of $\partial\mathbb{D}$ in which (2) hold. The functions $l_\pm$ defined by

$$l_\pm(y):=L\left\{\frac{(1\pm z)^2}{(1-yz)(1-z/y)}\right\}$$

are analytic in a neighborhood of $\partial\mathbb{D}$.

Furthermore, let $g_\pm(y):=\frac{1}{2}(l_\pm(y)+\overline{l_\pm(1/\bar{y})})$. Then $g_\pm$ is analytic in a neighborhood of $\partial\mathbb{D}$ and $g_\pm(y)=\operatorname{Re}l_\pm(y)$ whenever $|y|=1$.

Assume now, for example H_+ were infinite. Then g_+ takes the value M infinitely often in its domain of analycity and is thus constant, in particular $\operatorname{Re}l_+(y)=M$ whenever $|y|=1$.

But then we get substituting $y=-1$, that the constant function 1 is a support point with respect to L. Because of the representations

$$1=\frac{1+z^k}{2}+\frac{1-z^k}{2},\qquad k\geqq 1$$

it follows that $1 \pm z^k$ are support points with respect to L for all $k \in \mathbb{N}$. Therefore the Toeplitz coefficients b_k of L (see e.g. [5], p. 36) vanish for all $k \in \mathbb{N}$.

Thus L is constant in $S_0(R)$, which contradicts the assumption.

So H_+ and – as a similar construction shows – H_- are finite, and (1) becomes a finite convex representation.

If it is a proper convex representation with a most two points, then the represented function g is either multi-valued, because g has poles on $\partial \mathbb{D}$ of order at least 4 (see [2], p. 103), or g is of the form ($t \in]0, 1[$)

$$g(z) = t \frac{(1+z)^2}{(1-yz)(1-\bar{y}z)} + (1-t) \frac{(1-z)^2}{(1-yz)(1-\bar{y}z)}$$

which gives the desired result. □

4. Application to Other Normalizations

Originally Brickman obtained a representation for the family S of univalent functions, normalized by

$$f(0) = 0, \quad f'(0) = 1.$$

Because the construction is similar, our method works also in the family $S(R) := \{f \in S \mid f \text{ has real MacLaurin coefficients}\}$.

Corollary. *Every extreme point of $S(R)$ is of the form*

$$f(z) = \frac{z}{(1-yz)(1-\bar{y}z)}, \quad |y| = 1. \tag{3}$$

Proof. Using the Brickman representation in S (see e.g. [3], Theorem 9.5) one gets similarly as in our Lemma, that an extreme point of $S(R)$ only omits real values, which is equivalent to representation (3). □

We remark that this is a refinement of a result due to Brickman, MacGregor and Wilken [2], Theorem 4, who showed representation (3) for an extreme point of the closed convex hull of $S(R)$. Because of a general result due to Mil'man (see e.g. [5]), one knows *a priori* that

$$E\,\overline{\text{co}}\,S(R) \subset ES(R).$$

[2], Theorem 4, gives also a proof of the statement

$$\{f \in S(R) \mid f \text{ has representation (3)}\} \subset E\,\overline{\text{co}}\,S(R),$$

so that all families are equal:

$$E\,\overline{\text{co}}\,S(R) = ES(R) = \{f \in S(R) \mid f \text{ has representation (3)}\}.$$

Our method also applies to other normalizations, for example to the Montel classes with

$$f(z_1)=w_1,\ f(z_2)=w_2, \qquad z_1, z_2, w_1, w_2 \in \mathbb{R},$$

because in this case there is a Brickman representation, too (see [5], Theorem 8.5). Thus here also the extreme points omit only real values.

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