Integral Transforms and Special Functions
Vol. 17, No. 12, December 2006, 863–875

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A generalization of Student's *t*-distribution from the viewpoint of special functions

WOLFRAM KOEPF*† and MOHAMMAD MASJED-JAMEI‡

†Department of Mathematics, University of Kassel, Heinrich-Plett-Str. 40, D-34132 Kassel, Germany
‡Department of Mathematics, K.N.Toosi University of Technology, Sayed Khandan, Jolfa Av., Tehran, Iran

(Received 1 July 2005)

Student's t-distribution has found various applications in mathematical statistics. One of the main properties of the t-distribution is to converge to the normal distribution as the number of samples tends to infinity. In this paper, by using a Cauchy integral we introduce a generalization of the t-distribution function with four free parameters and show that it converges to the normal distribution again. We provide a comprehensive treatment of mathematical properties of this new distribution. Moreover, since the Fisher F-distribution has a close relationship with the t-distribution, we also introduce a generalization of the F-distribution and prove that it converges to the chi-square distribution as the number of samples tends to infinity. Finally, some particular sub-cases of these distributions are considered.

Keywords: Probability distributions; Cauchy integral; Dominated convergence theorem; Pearson distribution family; Student's t-distribution; Fisher F-distribution; Normal distribution; Gamma distribution; Chi-square distribution

MSC 2000: 60E05, 62E20, 33C45

1. Introduction

The Pearson differential equation is given as

$$\frac{\mathrm{d}W}{\mathrm{d}x} = \frac{dx+e}{ax^2+bx+c} W(x), \tag{1}$$

which is intimately connected with classical orthogonal polynomials and defines their weight functions $W(x)$, see *e.g.* [1], and its solution

$$W(x) = W\left(\begin{matrix} d & & e \\ a & b & c \end{matrix} \,\middle|\, x\right) = \exp \int \frac{dx+e}{ax^2+bx+c} \mathrm{d}x, \tag{2}$$

*Corresponding author. Email: koepf@mathematik.uni-kassel.de

Integral Transforms and Special Functions
ISSN 1065-2469 print/ISSN 1476-8291 online © 2006 Taylor & Francis
http://www.tandf.co.uk/journals
DOI: 10.1080/10652460600856419

where a, b, c, d, e are all real parameters. There are several special sub-cases of equation (2). One of them is the Beta distribution, which is usually represented by the integral [2]

$$\int_C (L_1(t))^a (L_2(t))^b \mathrm{d}t, \tag{3}$$

where $L_1(t)$ and $L_2(t)$ are linear functions, a, b are complex numbers and C is an appropriate contour. The Euler and Cauchy integrals [3] are two important sub-classes of Beta-type integrals which are often used in applied mathematics. The Euler integral is given by

$$\int_a^b (t-a)^{c-1}(t-b)^{d-1}\mathrm{d}t = \frac{\Gamma(c)\Gamma(d)}{\Gamma(c+d)}(a+b)^{c+d-1} \qquad (\mathrm{Re}\, c > 0, \quad \mathrm{Re}\, d > 0, \quad a > 0, b > 0), \tag{4}$$

while the Cauchy integral is represented by the formula

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(a+it)^c(b-it)^d} = \frac{\Gamma(c+d-1)}{\Gamma(c)\Gamma(d)}(a+b)^{1-(c+d)}, \tag{5}$$

in which $i = \sqrt{-1}$, $\mathrm{Re}\,(c+d) > 1$, $\mathrm{Re}\, a > 0$ and $\mathrm{Re}\, b > 0$. Note that in both relations (4) and (5), $\Gamma(a) = \int_0^\infty x^{a-1}\mathrm{e}^{-x}\mathrm{d}x$ denotes the Gamma function.

The relation (5) is a suitable tool to compute some different-looking definite integrals. For this purpose, we use the relation

$$\left(\frac{a-ib}{a+ib}\right)^{iq} = \exp\left(2q \arctan\frac{b}{a}\right) \qquad (a, b, q \in \mathbf{R}) \tag{6}$$

which rewrites the complex left-hand side in terms of the real right-hand side. Consequently,

$$(b-it)^{p+iq}(b+it)^{p-iq} = (b^2+t^2)^p \exp\left(2q \arctan\frac{t}{b}\right). \tag{7}$$

Now if equation (7) is substituted, then the integral (5) changes towards

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} (b^2+t^2)^p \exp\left(2q \arctan\frac{t}{b}\right)\mathrm{d}t = \frac{\Gamma(-2p-1)}{\Gamma(-p+iq)\Gamma(-p-iq)}(2b)^{2p+1}. \tag{8}$$

The above integral plays a key role to introduce a generalization of the t-distribution.

2. A generalization of the t-distribution

The Student t-distribution [4, 5] having the probability density function (pdf)

$$T(t,m) = \frac{\Gamma((m+1)/2)}{\sqrt{m\pi}\,\Gamma(m/2)}\left(1+\frac{t^2}{m}\right)^{-((m+1)/2)} \qquad (-\infty < t < \infty, \quad m \in \mathbf{N}) \tag{9}$$

is perhaps one of the most important distributions in the sampling problems of normal populations. According to a theorem in mathematical statistics, if $\bar{X}$ and S^2 are, respectively, the mean value and variance of a stochastic sample with the size m of a normal population having the expected value μ and variance σ^2, then the random variable $T = (\bar{X}-\mu)/(S/\sqrt{m})$ has the probability density function (9) with $(m-1)$ degrees of freedom [5]. This theorem is used in the test of hypotheses and interval estimation theory when the size of the sample is small, for instance less than 30.

Now, by using equation (8), one can extend the pdf of the t-distribution. To meet this goal, we substitute $t \to t/\sqrt{m}$, $b = 1$, $p = -((m+1)/2)$ and $q \to q/2$ in equation (8). This yields

$$\int_{-\infty}^{\infty} \left(1+\frac{t^2}{m}\right)^{-((m+1)/2)} \exp\left(q \arctan\frac{t}{\sqrt{m}}\right) \mathrm{d}t = \frac{\sqrt{m}2^{1-m}\Gamma(m)\pi}{\Gamma((1+m+iq)/2)\Gamma((1+m-iq)/2)}. \tag{10}$$

Since the right-hand side of equation (10) is an even function with respect to the variable q, we take a linear combination and get accordingly

$$\int_{-\infty}^{\infty} \left(1+\frac{t^2}{m}\right)^{-((m+1)/2)} \left(\lambda_1 \exp\left(q \arctan\frac{t}{\sqrt{m}}\right) + \lambda_2 \exp\left(-q \arctan\frac{t}{\sqrt{m}}\right)\right) \mathrm{d}t$$
$$= \frac{(\lambda_1+\lambda_2)\sqrt{m}\, 2^{1-m}\Gamma(m)\pi}{\Gamma((1+m+iq)/2)\Gamma((1+m-iq)/2)}. \tag{11}$$

Therefore, the above integral can be used to generalize distribution (9) by

$$T(t,m,q,\lambda_1,\lambda_2) = \frac{\Gamma((1+m+iq)/2)\Gamma((1+m-iq)/2)}{(\lambda_1+\lambda_2)\sqrt{m}\, 2^{1-m}\Gamma(m)\pi} \left(1+\frac{t^2}{m}\right)^{-((m+1)/2)}$$
$$\times \left(\lambda_1 \exp\left(q \arctan\frac{t}{\sqrt{m}}\right) + \lambda_2 \exp\left(-q \arctan\frac{t}{\sqrt{m}}\right)\right) \tag{12}$$

where $-\infty < t < \infty$, $m \in \mathbf{N}$, q is a complex number and $\lambda_1, \lambda_2 \geq 0$.

Note that $\lambda_1, \lambda_2 \geq 0$ is a necessary condition, because the probability density function must always be positive. Also note that the normalization constant

$$\frac{\Gamma((1+m+iq)/2)\Gamma((1+m-iq)/2)}{(\lambda_1+\lambda_2)\sqrt{m}\, 2^{1-m}\Gamma(m)\pi}$$

of equation (12) is real, because the corresponding integrand is a real function on $(-\infty, \infty)$. It is clear that for $q = 0$ in equation (12), the usual t-distribution is derived. Moreover, for $q = 0$ the normalization constant of distribution (12) is equal to the normalization constant of the t-distribution. This fact can be proved by applying Legendre's duplication formula [3], *i.e.*

$$\Gamma\left(\frac{z}{2}\right)\Gamma\left(\frac{z+1}{2}\right) = \frac{\sqrt{\pi}}{2^{z-1}}\Gamma(z). \tag{13}$$

But, according to one of the basic theorems in sampling theory, $T(t, m)$ converges to the pdf of the standard normal distribution $N(t, 0, 1)$ as $m \to \infty$ [2, 5], that is

$$\lim_{m\to\infty} T(t,m) = N(t,0,1). \tag{14}$$

Here, we intend to show that this matter is also valid for the generalized distribution $T(t, m, q, \lambda_1, \lambda_2)$. To prove this claim, we use the dominated convergence theorem (DCT) [6] to the real sequence of functions

$$S_m^{(1)}(t,q,\lambda_1,\lambda_2)=\left(1+\frac{t^2}{m}\right)^{-((m+1)/2)}\left(\lambda_1\exp\left(q\arctan\frac{t}{\sqrt{m}}\right)\right.$$
$$\left.+\lambda_2\exp\left(-q\arctan\frac{t}{\sqrt{m}}\right)\right). \tag{15}$$

For every $m\in\mathbf{N}$, it is not difficult to see that

$$|S_m^{(1)}(t,q,\lambda_1,\lambda_2)|\le(\lambda_1+\lambda_2)\exp\left(|q|\frac{\pi}{2}\right)\quad(t\in\mathbf{R}). \tag{16}$$

Therefore, we have

$$\lim_{m\to\infty}\left(1+\frac{t^2}{m}\right)^{-((m+1)/2)}\left(\lambda_1\exp\left(q\arctan\frac{t}{\sqrt{m}}\right)+\lambda_2\exp\left(-q\arctan\frac{t}{\sqrt{m}}\right)\right)$$
$$=(\lambda_1+\lambda_2)\exp\left(-\frac{t^2}{2}\right). \tag{17}$$

The DCT states that if for a continuous and integrable function $g(x)$, we have $|f_m(x)|\le g(x)$, then

$$\lim_{m\to\infty}\int_a^b f_m(x)\mathrm{d}x=\int_a^b\lim_{m\to\infty}f_m(x)\mathrm{d}x. \tag{18}$$

Considering the limit relation (17), we therefore obtain

$$\begin{aligned}
&\lim_{m\to\infty}T(t,m,q,\lambda_1,\lambda_2)\\
&=\frac{\lim_{m\to\infty}(1+t^2/m)^{-((m+1)/2)}(\lambda_1\exp(q\arctan(t/\sqrt{m}))+\lambda_2\exp(-q\arctan(t/\sqrt{m})))}{\int_{-\infty}^{\infty}\lim_{m\to\infty}(1+t^2/m)^{-((m+1)/2)}(\lambda_1\exp(q\arctan(t/\sqrt{m}))+\lambda_2\exp(-q\arctan(t/\sqrt{m})))\mathrm{d}t}\\
&=\frac{(\lambda_1+\lambda_2)\exp(-t^2/2)}{\int_{-\infty}^{\infty}(\lambda_1+\lambda_2)\exp(-t^2/2)\mathrm{d}t}\\
&=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}t^2\right)=N(t,0,1).
\end{aligned} \tag{19}$$

Remark 1 Taking the limit on both sides of (11) as $m\to\infty$, the following asymptotic relation is obtained for the Gamma function

$$\lim_{x\to\infty}\frac{\Gamma(x+iy)\Gamma(x-iy)}{2^{-(2x-1)}\sqrt{2x-1}\,\Gamma(2x-1)}=\frac{2}{\sqrt{2\pi}}. \tag{20}$$

To compute the expected value of the distribution given by the pdf (12), it is sufficient to consider the definite integral

$$\int_{-\infty}^{\infty}t\left(1+\frac{t^2}{m}\right)^{-((m+1)/2)}\exp\left(q\arctan\frac{t}{\sqrt{m}}\right)\mathrm{d}t$$
$$=\frac{\sqrt{m}\,2^{1-m}\Gamma(m)\pi}{\Gamma((1+m+iq)/2)\Gamma((1+m-iq)/2)}\left(\frac{q\sqrt{m}}{m-1}\right), \tag{21}$$

which gives the expected value of (12) as

$$E[T] = \left(\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\right)\frac{q\sqrt{m}}{m-1}. \tag{22}$$

In contrast, as $E[1 + T^2/m]$ can be easily computed, after some calculations, we get for the variance measure of (12)

$$\mathrm{Var}[T] = E[T^2] - E^2[T] = \frac{m(q^2 + m - 1)}{(m-2)(m-1)} - \left(\frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2}\right)^2\left(\frac{mq^2}{(m-1)^2}\right). \tag{23}$$

It is valuable to point out that as expected, $q = 0$ in equations (22) and (23) gives the expected value and variance of the usual t-distribution, respectively.

It is known that the t-distribution has a close relationship with the Fisher F-distribution [1], defined by its pdf

$$F(x, m, k) = \frac{\Gamma((m+k)/2)(k/m)^{k/2}}{\Gamma(k/2)\Gamma(m/2)}x^{k/2-1}\left(1 + \frac{k}{m}x\right)^{-((m+k)/2)} \quad (m, k \in \mathbf{N},\ \ 0 < x < \infty), \tag{24}$$

where $x = t^2$ and $k = 1$ in equation (24). In other words, we have

$$T(t, m) = F(t^2, m, 1) \tag{25}$$

By referring to the above relation and the fact that the t-distribution was generalized by relation (12), it is now natural to generalize the pdf of the F-distribution (24) as follows

$$\begin{aligned} F(x, m, k, q, \lambda_1, \lambda_2) = Bx^{k/2-1}\left(1 + \frac{k}{m}x\right)^{-((m+k)/2)}\Bigg(&\lambda_1 \exp\left(q \arctan\sqrt{\frac{k}{m}x}\right) \\ &+ \lambda_2 \exp\left(-q \arctan\sqrt{\frac{k}{m}x}\right)\Bigg), \end{aligned} \tag{26a}$$

where

$$\begin{aligned} \frac{1}{B} = \int_0^\infty x^{k/2-1}\left(1 + \frac{k}{m}x\right)^{-((m+k)/2)}\Bigg(&\lambda_1 \exp\left(q \arctan\sqrt{\frac{k}{m}x}\right) \\ &+ \lambda_2 \exp\left(-q \arctan\sqrt{\frac{k}{m}x}\right)\Bigg)\,\mathrm{d}x. \end{aligned} \tag{26b}$$

For $q = 0$, equation (26a) is the usual F-distribution defined in equation (24).

According to the following theorem, the generalized function (26) converges to a special case of the Gamma distribution [5], defined by

$$G(x, \alpha, \beta) = \frac{\beta^{-\alpha}}{\Gamma(\alpha)}x^{\alpha-1}\exp\left(\frac{-x}{\beta}\right) \quad (\alpha, \beta > 0,\ \ 0 < x < \infty). \tag{27}$$

THEOREM 1 *If the Gamma distribution is given by definition* (27), *then we have*

$$\lim_{m\to\infty} F(x, m, k, q, \lambda_1, \lambda_2) = G\left(x, \alpha = \frac{k}{2}, \beta = 2\right) = \chi_k^2,$$

where χ_k^2 *denotes the pdf of the chi-square distribution.*

Proof Let us define the sequence

$$S_m^{(2)}(x,k,q,\lambda_1,\lambda_2) = x^{k/2-1}\left(1+\frac{k}{m}x\right)^{-((m+k)/2)}\left(\lambda_1 \exp\left(q \arctan\sqrt{\frac{k}{m}x}\right)\right.$$
$$\left. + \lambda_2 \exp\left(-q \arctan\sqrt{\frac{k}{m}x}\right)\right).$$

It is easy to show that

$$|S_m^{(2)}(x,k,q,\lambda_1,\lambda_2)| \le (\lambda_1+\lambda_2)x^{(k/2)-1}\exp\left(|q|\frac{\pi}{2}\right) \quad (x\in[0,\infty),\ \ k\in\mathbf{N}), \tag{28}$$

and

$$\lim_{m\to\infty} S_m^{(2)}(x,k,q,\lambda_1,\lambda_2) = (\lambda_1+\lambda_2)x^{(k/2)-1}\exp(-x/2). \tag{29}$$

Therefore, according to the DCT, we have

$$\lim_{m\to\infty} F(x,m,k,q,\lambda_1,\lambda_2) = \frac{\lim_{m\to\infty} S_m^{(2)}(x,k,q,\lambda_1,\lambda_2)}{\int_0^\infty \lim_{m\to\infty} S_m^{(2)}(x,k,q,\lambda_1,\lambda_2)\mathrm{d}x}$$
$$= \frac{x^{(k/2)-1}\exp(-x/2)}{\int_0^\infty x^{(k/2)-1}\exp(-x/2)\mathrm{d}x} = G\left(x,\frac{k}{2},2\right). \tag{30}$$

Moreover, it is not difficult to show that

$$F(t^2,m,1,q,\lambda_1,\lambda_2) = T(t,m,q,\lambda_1,\lambda_2). \tag{31}$$

■

3. Some particular sub-cases of the generalized *t* (and *F*) distribution

In this section, we will study some symmetric and asymmetric sub-cases of the generalized distributions (12) and (26).

3.1 *A symmetric generalization of the* t*-distribution, the case $q = ib$ and $\lambda_1 = \lambda_2 = 1/2$*

If the special case $q = ib$ and $\lambda_1 = \lambda_2 = 1/2$ is considered in (12), then

$$T\left(t,m,ib,\frac{1}{2},\frac{1}{2}\right) = T_S(t,m,b) = \frac{\Gamma((1+m+b)/2)\Gamma((1+m-b)/2)}{\sqrt{m}\,2^{1-m}\Gamma(m)\pi}$$
$$\times\left(1+\frac{t^2}{m}\right)^{-((m+1)/2)}\cos\left(b\arctan\frac{t}{\sqrt{m}}\right) \tag{32}$$

is a symmetric generalization of the ordinary t-distribution in which $-1 \le b \le 1$.

The usual pdf of the t-distribution is obviously derived by $b = 0$ in (32). Note that according to the Legendre duplication formula we reach the normalization constant of the t-distribution if $b = 0$ is considered in equation (32). In other words, we have

$$b = 0 \Longrightarrow \frac{\Gamma^2((1+m)/2)}{\sqrt{m}\,2^{1-m}\Gamma(m)\pi} = \frac{\Gamma((1+m)/2)}{\sqrt{m\pi}\,\Gamma(m/2)}. \tag{33}$$

Also note that the parameter b in the generalized distribution (32) must belong to $[-1,1]$, because the pdf must always be positive and, therefore, we ought to have

$\cos(b\arctan(t/\sqrt{m})) \geq 0$. In contrast, as for $-(\pi/2) \leq \theta \leq \pi/2$ we have $\cos\theta \geq 0$, therefore, to prove $\cos(b\arctan(t/\sqrt{m})) \geq 0$, it is sufficient to prove that

$$-1 \leq b \leq 1 \Longleftrightarrow b\arctan\frac{t}{\sqrt{m}} \subseteq \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \quad (t \in \mathbf{R}, \quad m \in \mathbf{N}). \tag{34}$$

For this purpose, let us consider the sequence $U_m(t) = \arctan t/\sqrt{m}$. We have

$$\begin{aligned} U_m'(t) &= \left(\frac{1}{\sqrt{m}}\right) \Big/ \left(1 + \frac{t^2}{m}\right) > 0 \Longrightarrow [\min U_m(t), \max U_m(t)] \\ &= [U_m(-\infty), U_m(\infty)] = \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \end{aligned} \tag{35}$$

Now, if we demand the sequence $bU_m(t) = b\arctan t/\sqrt{m}$ to belong to $[-\pi/2, \pi/2]$, it is clear that we must have $|b| \leq 1$, which proves relation (34). Figures 1 and 2 clarify this matter for $b \in [-1, 1]$ and $b \notin [-1, 1]$ in the interval $(-10, 10)$.

Figure 1 shows the pdf $T_S(t, 4, 1/2)$ with normalization constant $35\sqrt{2}/128$ and figure 2 shows the non-positive function $T_S(t, 4, 3) = (4/\pi)(1 + t^2/4)^{-5/2}\cos(3\arctan(t/2))$ in the interval $(-10, 10)$. As these figures show, the generalized distribution (32) is symmetric, *i.e.*

$$T_S(-t, m, b) = T_S(t, m, b) \quad (t \in \mathbf{R}). \tag{36}$$

Moreover, according to equations (22) and (23), the expected value and variance of distribution (32) take the forms

$$E[t] = 0, \quad \mathrm{Var}[t] = \frac{m(m - 1 - b^2)}{(m-1)(m-2)}. \tag{37}$$

Clearly, $b = 0$ in these relations gives the expected value and variance of the t-distribution.

THEOREM 2 *$T_S(t, m, q)$ converges to $N(t, 0, 1)$ as $m \to \infty$.*

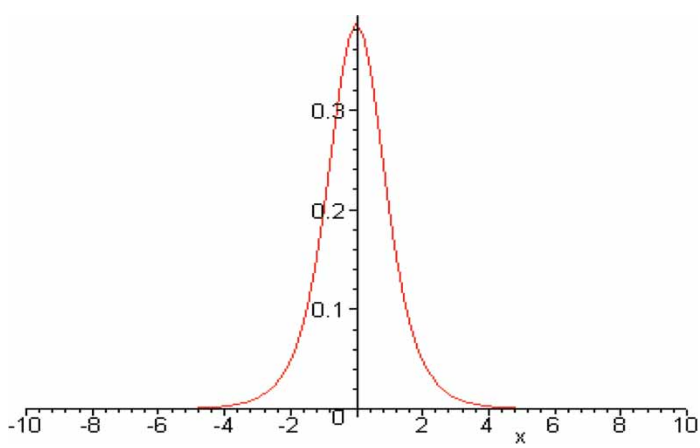

Figure 1. $b = 1/2, m = 4$.

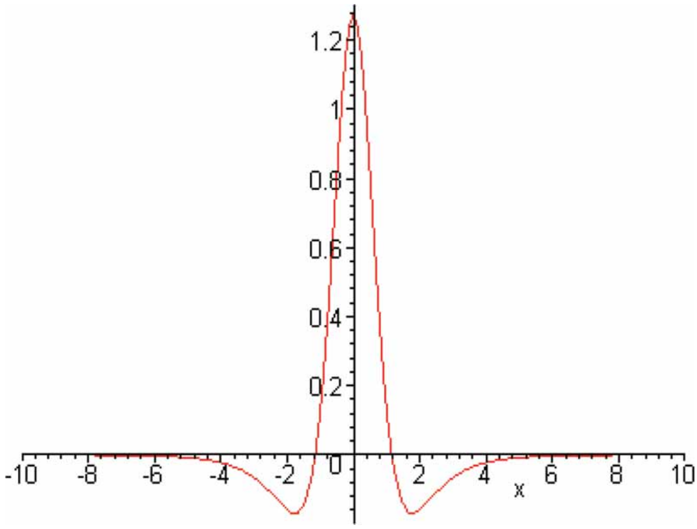

Figure 2. $b = 3, m = 4$.

Proof If the sequence $S_m^{(3)}(t, b) = \cos(b \arctan t/\sqrt{m})(1 + t^2/m)^{-((m+1)/2)}$ is considered, then one can show that

$$|S_m^{(3)}(t, b)| = \left|\cos\left(b \arctan \frac{t}{\sqrt{m}}\right)\right| \cdot \left|\left(1 + \frac{t^2}{m}\right)^{-((m+1)/2)}\right| \leq 1 \quad (t \in \mathbf{R}). \tag{38}$$

Consequently, we have

$$\begin{aligned} \lim_{m\to\infty} T_S(t, m, b) &= \lim_{m\to\infty} \frac{\cos(b \arctan(t/\sqrt{m}))(1 + t^2/m)^{-((m+1)/2)}}{\int_{-\infty}^{\infty} \cos(b \arctan(t/\sqrt{m}))(1 + t^2/m)^{-((m+1)/2)} \mathrm{d}t} \\ &= \frac{\lim_{m\to\infty} \cos(b \arctan(t/\sqrt{m}))(1 + t^2/m)^{-((m+1)/2)}}{\int_{-\infty}^{\infty} \lim_{m\to\infty} \cos(b \arctan(t/\sqrt{m}))(1 + t^2/m)^{-((m+1)/2)} \mathrm{d}t} \\ &= \frac{\exp(-t^2/2)}{\int_{-\infty}^{\infty} \exp(-t^2/2) \mathrm{d}t} = N(t, 0, 1). \end{aligned} \tag{39}$$

By referring to equation (26a), we can now define the generalized F-distribution corresponding to the first given sub-case as follows

$$\begin{aligned} F\left(x, m, k, ib, \frac{1}{2}, \frac{1}{2}\right) = F_1(x, m, k, b) = Bx^{(k/2)-1}\left(1 + \frac{k}{m}x\right)^{-((m+k)/2)} \\ \times \cos\left(b \arctan \sqrt{\frac{k}{m}x}\right) \quad (-1 \leq b \leq 1), \end{aligned} \tag{40a}$$

where

$$\begin{aligned} \frac{1}{B} &= \int_0^{\infty} x^{(k/2)-1}\left(1 + \frac{k}{m}x\right)^{-((m+k)/2)} \cos\left(b \arctan \sqrt{\frac{k}{m}x}\right) \mathrm{d}x \\ &= 2\left(\frac{m}{k}\right)^{k/2} \int_0^{\pi/2} \sin^{(k-1)} \theta \cos^{(m-1)} \theta \cos(b\theta) \mathrm{d}\theta. \end{aligned} \tag{40b}$$

■

THEOREM 3 *$F_1(x, m, k, b)$ converges to the chi-square distribution as $m \to \infty$.*

Proof We define the sequence $S_m^{(4)}(x, k, q) = x^{k/2-1}(1 + (k/m)\ x)^{-((m+k)/2)} \cos(b \arctan \sqrt{k/m\, x})$ to get

$$|S_m^{(4)}(x, k, b)| \leq x^{(k/2)-1} \quad (x \in [0, \infty),\ \ k \in \mathbf{N},\ \ |b| < 1). \tag{41}$$

Hence, according to DCT, we find out that

$$\begin{aligned}\lim_{m\to\infty} F_1(x, m, k, b) &= \lim_{m\to\infty} \frac{S_m^{(4)}(x, k, b)}{\int_0^\infty S_m^{(4)}(x, k, b)\mathrm{d}x} = \frac{x^{(k/2)-1}\exp(-x/2)}{\int_0^\infty x^{(k/2)-1}\exp(-x/2)\mathrm{d}x} \\ &= G\left(x, \frac{k}{2}, 2\right).\end{aligned} \tag{42}$$

It is not difficult to verify that the generalized distributions $T_S(t, m, b)$ and $F_1(x, m, k, b)$ are related with each other as follows

$$F_1(t^2, m, 1, b) = T_S(t, m, b). \tag{43}$$

■

Remark 2 Here is a good position to mention that in ref. [7], a class of orthogonal polynomials is studied, whose weight function [8] corresponds to the ordinary t-distribution. The related polynomials are defined as

$$I_n^{(p)}(x) = n! \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{p-1}{n-k} \binom{n-k}{k} (2x)^{n-2k}. \tag{44a}$$

and satisfy the orthogonality relation

$$\begin{aligned}&\int_{-\infty}^{\infty} (1 + x^2)^{-(p-1/2)} I_n^{(p)}(x) I_m^{(p)}(x)\mathrm{d}x \\ &\quad = \left(\frac{n! 2^{2n-1}\sqrt{\pi}\,\Gamma^2(p)\Gamma(2p-2n)}{(p-n-1)\Gamma(p-n)\Gamma(p-n+1/2)\Gamma(2p-n-1)}\right)\delta_{n,m}\end{aligned} \tag{44b}$$

where $m, n = 0, 1, 2, \ldots, N < p - 1$ and $\delta_{n,m} = \begin{cases} 0 & \text{if} \quad n \neq m \\ 1 & \text{if} \quad n = m \end{cases}$.

For $n = m = 0$ in equation (44b), an integral is derived that corresponds to the t-distribution. Furthermore, the mentioned comment holds for the F-distribution. In [7], a sequence of orthogonal polynomials is studied which is defined by

$$M_n^{(p,q)}(x) = (-1)^n n! \sum_{k=0}^{n} \binom{p-(n+1)}{k} \binom{q+n}{n-k} (-x)^k, \tag{45}$$

and satisfies the orthogonality relation

$$\int_0^\infty \frac{x^q}{(1+x)^{p+q}} M_n^{(p,q)}(x) M_m^{(p,q)}(x)\mathrm{d}x = \left(\frac{n!(p-(n+1))!(q+n)!}{(p-(2n+1))(p+q-(n+1))!}\right)\delta_{n,m} \tag{46}$$

where $m, n = 0, 1, 2, \ldots, N < p - (1/2), q > -1$.

Clearly, the weight function of integral (46) corresponds to the usual F-distribution in the case $n = m = 0$.

3.2 *An asymmetric generalization of the* t*-distribution, the case* $\lambda_2 = 0$

From the orthogonality relations (44b) and (46), it can be concluded that the category of Pearson distributions must have a related class of orthogonal polynomials. In ref. [9], a class of orthogonal polynomials is studied, whose weight function is a specific case of (2) and is represented by

$$W_n^{(p,q)}(x; a, b, c, d) = ((ax+b)^2 + (cx+d)^2)^{-p} \exp\left(q \arctan\frac{ax+b}{cx+d}\right) \quad (-\infty < x < \infty), \tag{47}$$

where a, b, c, d, p, q are all real parameters. This function is a sub-case of the Pearson distribution (2), because the logarithmic derivative of function (47) is a rational function. Hence, equation (47) is a special case of the Pearson distribution family. For convenience, we chose a particular sub-case of (47) in ref. [9] to generalize the usual t-distribution.

If $a = 1/\sqrt{m}$, $b = 0$, $c = 0$, $d = 1$ and $p = -((m+1)/2)$ $(m \in \mathbf{N})$ is selected in equation (47), one gets

$$W^{(-(m+1)/2,q)}\left(t; \frac{1}{\sqrt{m}}, 0, 0, 1\right) = \left(1 + \frac{t^2}{m}\right)^{-((m+1)/2)} \exp\left(q \arctan\frac{t}{\sqrt{m}}\right) \quad (m \in \mathbf{N},\ q \in \mathbf{R}). \tag{48}$$

As

$$\int_{-\infty}^{\infty} \left(1 + \frac{t^2}{m}\right)^{-((m+1)/2)} \exp\left(q \arctan\frac{t}{\sqrt{m}}\right) \mathrm{d}t = \sqrt{m} \int_{-\pi/2}^{\pi/2} \mathrm{e}^{q\theta} \cos^{(m-1)}\theta \, \mathrm{d}\theta, \tag{49}$$

we have

$$\int_{-\pi/2}^{\pi/2} \mathrm{e}^{q\theta} \cos^{(m-1)}\theta \, \mathrm{d}\theta = \frac{(m-1)!(q - (1 + (-1)^m/2)(q-1))(\mathrm{e}^{q\pi/2} + (-1)^m \mathrm{e}^{-q\pi/2})}{\prod_{k=0}^{\lfloor (m-1)/2 \rfloor} (q^2 + (m-2k-1)^2)}. \tag{50}$$

Hence, an asymmetric generalization of the t-distribution may be defined as

$$T_A(t, m, q) = K \left(1 + \frac{t^2}{m}\right)^{-((m+1))/2} \exp\left(q \arctan\frac{t}{\sqrt{m}}\right) \quad (-\infty < t < \infty,\ \ m \in \mathbf{N}, q \in \mathbf{R}) \tag{51a}$$

such that

$$K = \frac{\prod_{k=0}^{\lfloor (m-1)/2 \rfloor} (q^2 + (m-2k-1)^2)}{\sqrt{m}(m-1)!(q - (1 + (-1)^m/2)(q-1))(\mathrm{e}^{q\pi/2} + (-1)^m \mathrm{e}^{-q\pi/2})}. \tag{51b}$$

The distribution (51a) with normalization constant given by (51b) was defined in ref. [9] based on this particular approach. But here we can modify and simplify it. For this task, we set $\lambda_2 = 0$ in (12), and get

$$T_A(t, m, q) = \frac{\Gamma((1+m+iq)/2)\Gamma((1+m-iq)/2)}{\sqrt{m}\, 2^{1-m}\Gamma(m)\pi} \left(1 + \frac{t^2}{m}\right)^{-((m+1)/2)} \times \exp\left(q \arctan\frac{t}{\sqrt{m}}\right). \tag{52a}$$

This is in fact an explicit representation of the asymmetric generalization of the t-distribution. For this distribution, we clearly have

$$T_A(-t, m, q) = T_A(t, m, -q). \tag{52b}$$

The asymmetry is also shown by figures 3 and 4.

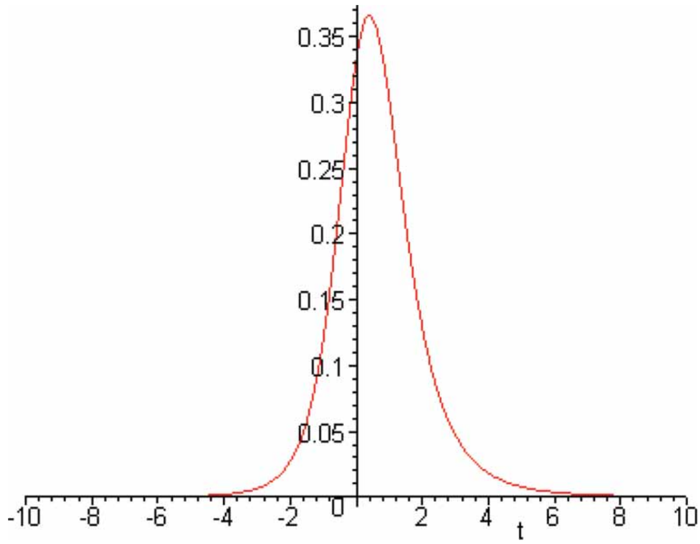

Figure 3. $q = 1, m = 4$.

According to equations (51a) and (51b), the explicit definitions of the two mentioned figures have, respectively, the forms

From figure 3,

$$T_A(t, 4, 1) = \frac{5}{6\cosh(\pi/2)}\left(1 + \frac{t^2}{4}\right)^{-5/2} \mathrm{e}^{\arctan(t/2)}$$

From figure 4,

$$T_A(t, 3, 1) = \frac{5\sqrt{3}}{12\sinh(\pi/2)}\left(1 + \frac{t^2}{3}\right)^{-2} \mathrm{e}^{\arctan(t/\sqrt{3})}$$

The following statements (A1–A5) collect the properties of the asymmetric distribution (52).

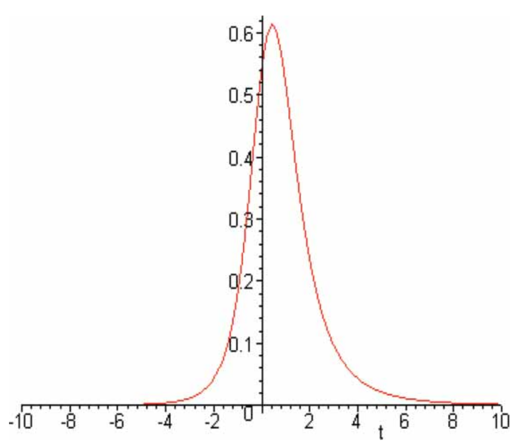

Figure 4. $q = 1, m = 3$.

(A1) The expected value and variance of (52) are, respectively, represented by

$$E[t] = \frac{q\sqrt{m}}{m-1}, \quad \mathrm{Var}[t] = \frac{m(q^2+(m-1)^2)}{(m-2)(m-1)^2}, \tag{53}$$

$q = 0$ in these relations gives the expected value and variance of the t-distribution.

(A2) $T_A(t, m, q)$ converges to $N(t, 0, 1)$ as $m \to \infty$.
The proof is similar to the first case if one chooses $\lambda_2 = 0$ and $\lambda_1 = 1$ in the defined sequence $S_m^{(1)}(t, q, \lambda_1, \lambda_2)$.

(A3) By the definition (26) and considering the case $\lambda_2 = 0$, we can define

$$F(x, m, k, q, \lambda_1, 0) = F_2(x, m, k, q) = D\, x^{k/2-1}\left(1 + \frac{k}{m}x\right)^{-((m+k)/2)} \times \exp\left(q \arctan\sqrt{\frac{k}{m}x}\right) \quad (q \in \mathbf{R},\ m, k \in \mathbf{N},\quad 0 < x < \infty), \tag{54a}$$

where

$$\begin{aligned}\frac{1}{D} &= \int_0^\infty x^{k/2-1}\left(1+\frac{k}{m}x\right)^{-((m+k)/2)} \exp\left(q \arctan\sqrt{\frac{k}{m}x}\right)\mathrm{d}x \\ &= 2\left(\frac{m}{k}\right)^{k/2}\int_0^{\pi/2} \sin^{(k-1)}\theta \cos^{(m-1)}\theta\, \mathrm{e}^{q\theta}\,\mathrm{d}\theta.\end{aligned} \tag{54b}$$

(A4) $F_2(x, m, k, q)$ converges to the chi-square distribution as $m \to \infty$.
The proof is similar to the proof of Theorem 1 when $\lambda_2 = 0$ and $\lambda_1 = 1$.

(A5) The distributions $F_2(x, m, k, q)$ and $T_A(t, m, q)$ are related to each other by

$$F_2(t^2, m, 1, q) = T_A(t, m, q). \tag{55}$$

Remark 3 There is another symmetric generalization of the t-distribution when we set $\lambda_1 = \lambda_2$ in definition (12). Its pdf is given as

$$T(-t, m, q, \lambda_1, \lambda_1) = T(t, m, q, \lambda_1, \lambda_1) = \frac{\Gamma((1+m+iq)/2)\Gamma((1+m-iq)/2)}{\sqrt{m}\,2^{1-m}\Gamma(m)\pi} \times \left(1+\frac{t^2}{m}\right)^{-(m+1)/2} \cosh\left(q \arctan\frac{t}{\sqrt{m}}\right). \tag{56}$$

Therefore, to summarize the last section, we in fact considered the three following particular sub-cases of the general distribution (12),

(a) $q = ib$ and $\lambda_1 = \lambda_2 = 1/2$; symmetric case
(b) $\lambda_2 = 0$; asymmetric case
(c) $\lambda_1 = \lambda_2$; symmetric case

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