Functions satisfying holonomic q-differential equations

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Abstract. In a similar manner as in the papers [7] and [8], where explicit algorithms for finding the differential equations satisfied by holonomic functions were given, in this paper we deal with the space of the q-*holonomic* functions which are the solutions of linear q-differential equations with polynomial coefficients. The sum, product and the composition with power functions of q-holonomic functions are also q-holonomic and the resulting q-differential equations can be computed algorithmically.

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1 Preliminaries

The purpose of this paper is to continue the research exposed in [7] and [8]. There, the authors discussed *holonomic* functions which are the solutions of homogeneous linear differential equations with polynomial coefficients.

In the present investigation, we consider a similar problem from the point of view of q-calculus. As general references for q-calculus see [2] and [4]. We begin with a few definitions.

Let $q \neq 1$. The q-complex number $[a]_q$ is given by

$$[a]_q := \frac{1-q^a}{1-q}, \quad a \in \mathbb{C}.$$

Of course

$$\lim_{q\to 1} [a]_q = a \ .$$

The q-factorial of a positive integer $[n]_q$ and the q-binomial coefficient are defined by

$$[0]_q! := 1, \quad [n]_q! := [n]_q[n-1]_q \cdots [1]_q, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

The q-Pochammer symbol is given as

$$(a;q)_0 = 1\ ,$$
$$(a;q)_k = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{k-1}), \quad k=1,2,\dots\ ,$$
$$(a;q)_\infty = \lim_{k\to\infty}(1-a)(1-aq)(1-aq^2)\cdots(1-aq^{k-1})$$

and

$$(a;q)_\lambda = \frac{(a;q)_\infty}{(aq^\lambda;q)_\infty} \qquad (|q|<1,\ \lambda\in\mathbb{C}).$$

The q-*derivative* of a function $f(x)$ is defined by

$$D_q f(x) := \frac{f(x)-f(qx)}{x-qx}\ \ (x\neq 0), \quad D_q f(0) := \lim_{x\to 0} D_q f(x), \tag{1}$$

and higher order q-derivatives are defined recursively

$$D_q^0 f := f, \quad D_q^n f := D_q D_q^{n-1} f, \quad n=1,2,3,\dots. \tag{2}$$

Of course, if f is differentiable at x, then

$$\lim_{q\to 1} D_q f(x) = f'(x)\ .$$

The next four lemmas are well-known in q-calculus and their proofs can be seen, for example, in [3] or [4].

Lemma 1.1. *For an arbitrary pair of functions* $u(x)$ *and* $v(x)$ *and constants* $\alpha,\beta\in\mathbb{C}$ *and* $q\neq 1$*, we have linearity and product rules*

$$\begin{aligned} D_q\big(\alpha u(x)+\beta v(x)\big) &= \alpha D_q u(x) + \beta D_q v(x),\\ D_q\big(u(x)\cdot v(x)\big) &= u(qx)D_q v(x) + v(x) D_q u(x)\\ &= u(x)D_q v(x) + v(qx) D_q u(x)\ . \end{aligned}$$

Lemma 1.2. *The Leibniz rule for the higher order* q*-derivatives of a product of functions is given as*

$$D_q^n\big(u(x)\cdot v(x)\big) = \sum_{k=0}^{n} \begin{bmatrix} n\\ k\end{bmatrix}_q D_q^{n-k} u(q^k x)\ D_q^k v(x).$$

Lemma 1.3. *For an arbitrary function* $u(x)$ *and for* $t(x) = cx^k$ $(c,k\in\mathbb{C},\ q^k\neq 1)$ *we have for the composition with* $t(x)$

$$D_q(u\circ t)(x) = D_{q^k} u(t)\cdot D_q t(x).$$

Lemma 1.4. *The values of the function for the shifted argument and for higher* q*-derivatives are connected by the two relations:*

$$f(q^n x) \ = \ \sum_{k=0}^{n} (-1)^k (1-q)^k \begin{bmatrix} n\\ k\end{bmatrix}_q q^{\binom{k}{2}} x^k\ D_q^k f(x), \tag{3}$$

$$D_q^n f(x) \ = \ \frac{1}{(1-q)^n x^n} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n\\ k\end{bmatrix}_q q^{\binom{k}{2}-(n-1)k} f(q^k x). \tag{4}$$

For our further work, it is useful to write the product rule in slightly different form.

Lemma 1.5. *The product rule for the q-derivative can be written in the form*

$$D_q\big(u(x)\cdot v(x)\big) = u(x)D_qv(x) + v(x)D_qu(x) - (1-q)xD_qu(x)D_qv(x)\ . \tag{5}$$

In the same manner, higher q-derivatives can be expressed by

$$D_q^n\big(u(x)\cdot v(x)\big) = \sum_{\nu=0}^{n}\sum_{\mu=0}^{n}\alpha_{\nu\mu}(x)D_q^{\nu}u(x)D_q^{\mu}v(x)\ ,$$

where $\alpha_{\nu\mu}(x)$ are appropriate polynomials.

Let us finally recall that the q-hypergeometric series is given by ([2], [6])

$$_r\phi_s\left(\begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{array}\middle|\, q, x\right) := \sum_{k=0}^{\infty}\frac{\prod_{j=1}^{r}(a_j;\, q)_k}{\prod_{j=1}^{s}(b_j;\, q)_k}\,\frac{x^k}{(q;\, q)_k}\,\left((-1)^k\, q^{\binom{k}{2}}\right)^{1+s-r}.$$

2 On q-holonomic functions

For every function $f(x)$ which is a solution of a *polynomial homogeneous linear q-differential equation*

$$\sum_{k=0}^{n}\tilde{p}_k(x)\ D_q^k f(x) = 0 \quad (\tilde{p}_n \not\equiv 0) \qquad (\tilde{p}_k \in \mathbb{K}(q)[x],\ n\in\mathbb{N}) \tag{6}$$

we say that $f(x)$ is a *q-holonomic function.* The smallest such n is called the *holonomic order* of $f(x)$. Here $\mathbb{K}$ is a field, typically $\mathbb{K} = \mathbb{Q}(a_1, a_2, \ldots)$ or $\mathbb{K} = \mathbb{C}(a_1, a_2, \ldots)$ where $a_1, a_2, \ldots$ denote some parameters. An equation of type (6) is called *a q-holonomic equation*.

Example 2.1. Since

$$D_q x^s = [s]_q\, x^{s-1} \quad (x, \alpha, s\in\mathbb{R}),$$

we have

$$f(x) = x^s\ \Rightarrow\ xD_qf(x) - [s]_qf(x) = 0\ ,$$

or

$$(q-1)\,x\,D_qf(x) - (q^s-1)\,f(x) = 0\ ,$$

i.e. the power function is (for integer s) a q-holonomic function of first order.

Example 2.2. For $0<|q|<1$, $\lambda\in\mathbb{R}$, $x\neq 0,1$, we have

$$D_q\big((x;q)_\lambda\big) = -[\lambda]_q\ (qx;q)_{\lambda-1} = \frac{-[\lambda]_q}{1-x}(x;q)_\lambda\ .$$

Hence

$$f(x) = (x;q)_\lambda\ \Rightarrow\ (x-1)D_qf(x) - [\lambda]_qf(x) = 0$$

or

$$(q-1)\,(x-1)\,D_q f(x) - (q^\lambda - 1)\,f(x) = 0\ .$$

Therefore the q-Pochhammer symbol is (for integer λ) also q-holonomic of first order.

Similarly, from

$$D_q\big((x;q)_\infty\big) = -(1-q)^{-1}(qx;q)_\infty = -\frac{1}{1-q}\frac{1}{1-x}(x;q)_\infty,$$

we get

$$f(x) = (x;q)_\infty \quad \Rightarrow \quad (1-x)D_q f(x) + \frac{1}{1-q} f(x) = 0.$$

Example 2.3. The small q-exponential function

$$e_q(x) = {}_1\phi_0\left(\begin{array}{c} 0 \\ - \end{array} \middle|\, q, x\right) = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n}\, x^n\ , \quad |x|<1,\ 0<|q|<1\ ,$$

has q-derivative

$$\begin{aligned}
D_q e_q(x) &= \frac{e_q(x) - e_q(qx)}{x - qx} \\
&= \frac{1}{x-qx}\left(\sum_{n=0}^{\infty}\frac{1}{(q;q)_n}\, x^n - \sum_{n=0}^{\infty}\frac{1}{(q;q)_n}\,(qx)^n\right) \\
&= \frac{1}{x-qx}\ \sum_{n=0}^{\infty}\frac{x^n-(qx)^n}{(q;q)_n} \\
&= \frac{1}{x-qx}\ \left\{x + \sum_{n=2}^{\infty}\frac{1-q^n}{(1-q)(1-q^2)\cdots(1-q^{n-1})(1-q^n)}x^n\right\} \\
&= \frac{x}{x-qx}\left\{1 + \sum_{k=1}^{\infty}\frac{1}{(1-q)(1-q^2)\cdots(1-q^k)}x^k\right\} \\
&= \frac{1}{1-q}\ e_q(x),
\end{aligned}$$

i.e. the small q-exponential function is q-holonomic of first order:

$$f(x) = e_q(x)\ \Rightarrow\ (1-q)D_q f(x) - f(x) = 0.$$

Note that this q-differential equation as well the resulting q-differential equations of the next four examples and similar ones can be obtained completely automatically by the `qsumdiffeq` command of the Maple package `qsum` by Böing and Koepf [1]. The above equation, e.g., is obtained using the command

```
qsumdiffeq(1/qpochhammer(q,q,n)*xˆn,q,n,f(x))}
```

Example 2.4. The big q-exponential function

$$E_q(x) = {}_0\phi_0 \left(\begin{array}{c} - \\ - \end{array} \middle| q, -x \right) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} \, x^n \; , \quad 0 < |q| < 1$$

has q-derivative

$$D_q E_q(x) = \frac{1}{x - qx} \left(\sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} \, x^n - \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} \, (qx)^n \right) = \frac{1}{1-q} \, E_q(qx).$$

which can obtained be in a similar way as in Example 2.3. Since

$$f(qx) = f(x) - (1-q)x(D_q f)(x),$$

we conclude that the big q-exponential function is also q-holonomic of first order:

$$f(x) = E_q(x) \;\Rightarrow\; (1-q)(x+1) D_q f(x) - f(x) = 0.$$

Example 2.5. For $0 < |q| < 1$, q-sine and q-cosine functions

$$\sin_q(x) = \frac{e_q(\mathrm{i}x) - e_q(-\mathrm{i}x)}{2\mathrm{i}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(q;q)_{2n+1}} \, x^{2n+1} \; ,$$
$$\cos_q(x) = \frac{e_q(\mathrm{i}x) + e_q(-\mathrm{i}x)}{2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(q;q)_{2n}} \, x^{2n} \; ,$$

satisfy

$$(1-q)^2 D_q^2 f(x) + f(x) = 0$$

and are therefore q-holonomic of second order.

Example 2.6. The q-hypergeometric series ${}_r\phi_s$ is q-holonomic. The `qsumdiffeq` command computes in particular for

$$f(x) = {}_2\phi_1 \left(\begin{array}{c} a, b \\ c \end{array} \middle| q, x \right)$$

the q-holonomic equation

$$\begin{aligned} 0 \;\; = \;\; & (xabq - c)x(q-1)^2 \, D_q^2 f(x) \\ & + (-xb - xa + 1 + xabq - c + xab)(q-1) D_q f(x) \\ & + (-1+a)(-1+b) f(x) \; . \end{aligned}$$

Example 2.7 Many q-orthogonal polynomials are q-holonomic. The Big q-Jacobi polynomials (see e.g. [5], 3.5) are given by

$$f(x) = P_n(x; a, b, c; q) = {}_3\phi_2 \left(\begin{array}{c} q^{-n}, abq^{n+1}, x \\ aq, cq \end{array} \middle| q, q \right) \; .$$

They satisfy the q-holonomic equation

$$
\begin{aligned}
0 \;=\; & q^n a(bqx-c)(q-1)^2(1-qx)\, D_q^2 f(x) \\
& +(q-1)(abq^{n+1}+abq^{2n+1}x+x-q^n a-q^n c-abq^{n+1}x-abq^{n+2}x+q^{n+1}ac)D_q f(x) \\
& +(q^n-1)(abq^{n+1}-1)f(x)
\end{aligned}
$$

which is again easily determined by the `qsumdiffeq` command.

The following lemma will be the crucial tool for the investigations of the next section.

Lemma 2.1. *If $f(x)$ is a function satisfying a holonomic equation (6) of order n, then the functions $D_q^l f(x)\,(l=n,n+1,\ldots)$ can be expressed as*

$$
D_q^l f(x) = \sum_{k=0}^{n-1} p_k^{(l)}(x) D_q^k f(x), \tag{7}
$$

where $p_k^{(l)}(x)$ are rational functions defined by

$$
p_k^{(l)}(x) \;=\; \begin{cases} \delta_{kl}\ , & 0 \le l < n-1\ , \\ -\dfrac{\tilde{p}_k(x)}{\tilde{p}_n(x)}\ , & l = n \\ p_{k-1}^{(l-1)}(qx) + D_q p_k^{(l-1)}(x) + p_{n-1}^{(l-1)}(qx) p_k^{(n)}(x)\ , & l > n\ , \end{cases}
$$

for $0 \le k \le n-1$ and 0 for other k's.

Proof. The representations (7) and the corresponding coefficients are evident by Equation (6) for $l = 0,1,\ldots,n$. By q-deriving and using Lemma 1.1, from

$$
D_q^n f(x) = \sum_{k=0}^{n-1} p_k^{(n)}(x) D_q^k f(x)
$$

we get

$$
\begin{aligned}
D_q^{n+1} f(x) &= \sum_{k=0}^{n-1} D_q\big(p_k^{(n)}(x) D_q^k f(x)\big) \\
&= \sum_{k=0}^{n-1} p_k^{(n)}(qx) D_q^{k+1} f(x) + \sum_{k=0}^{n-1} D_q\big(p_k^{(n)}(x)\big) D_q^k f(x) \\
&= \sum_{k=0}^{n-1} \big(p_{k-1}^{(n)}(qx) + D_q\big(p_k^{(n)}(x)\big) D_q^k f(x)\big) + p_{n-1}^{(n)}(x) D_q^n f(x) \\
&= \sum_{k=0}^{n-1} p_k^{(n+1)}(x) D_q^k f(x),
\end{aligned}
$$

with

$$p_k^{(n+1)}(x) = p_{k-1}^{(n)}(qx) + D_q p_k^{(n)}(x) + p_{n-1}^{(n)}(qx) p_k^{(n)}(x) \qquad (0 \leq k \leq n-1).$$

Repeating the procedure, we get the representation and coefficients for arbitrary $l > n$. $\diamondsuit$

We finish this section by noting that there are functions which are not q-holonomic.

Lemma 2.2. *The exponential function $f(x) = a^x$ $(a > 0, a \neq 1)$ is not q-holonomic.*

Proof. Taking successive q-derivatives of $f(x) := a^x$ generates iteratively the functions of the list $L := \{a^x, a^{qx}, a^{q^2 x}, \ldots\}$. Since the members of L are linearly independent over $\mathbb{K}(q)[x]$, the linear space over $\mathbb{K}(q)[x]$ generated by L has infinite dimension. This is equivalent to the fact that there is no q-holonomic equation for $f(x)$. $\diamondsuit$

3 Operations with q-holonomic functions

In this section, we will formulate and prove a few theorems about q-holonomic functions provided by derivation, addition or multiplication of the given q-holonomic functions.

Theorem 3.1. *If $f(x)$ is a q-holonomic function of order n, then the function $h_m(x) = D_q^m f(x)$ is a q-holonomic function of order at most n for every $m \in \mathbb{N}$. Furthermore, there is an algorithm to compute the corresponding q-differential equation.*

Proof. If we prove the statement for $m = 1$, the final conclusion follows by mathematical induction.

Let $h(x) = D_q f(x)$, where the function $f(x)$ satisfies (6). If $\tilde{p}_0(x) \equiv 0$, then immediately $h(x)$ is a q-holonomic function of order $n-1$.

Hence, let $\tilde{p}_0(x) \not\equiv 0$. Then, by Lemma 2.1, we have

$$D_q^n f(x) = \sum_{k=0}^{n-1} p_k^{(n)}(x) D_q^k f(x) \; ,$$

wherefrom

$$\begin{aligned} f(x) &= \frac{1}{p_0^{(n)}(x)} \Big(D_q^n f(x) - \sum_{k=1}^{n-1} p_k^{(n)}(x) D_q^k f(x) \Big) \\ &= \frac{1}{p_0^{(n)}(x)} \Big(D_q^{n-1} h(x) - \sum_{k=0}^{n-2} p_{k+1}^{(n)}(x) D_q^k h(x) \Big). \end{aligned}$$

Also, by q-deriving, we get

$$
\begin{aligned}
D_q^n h(x) & \\
&= D_q^{n+1} f(x) = \sum_{k=0}^{n-1} p_k^{(n+1)}(x) D_q^k f(x) \\
&= p_0^{(n+1)}(x) f(x) + \sum_{k=1}^{n-1} p_k^{(n+1)}(x) D_q^{k-1} h(x) \\
&= \frac{p_0^{(n+1)}(x)}{p_0^{(n)}(x)} \Big(D_q^{n-1} h(x) - \sum_{k=0}^{n-2} p_{k+1}^{(n)}(x) D_q^k h(x) \Big) + \sum_{k=0}^{n-2} p_{k+1}^{(n+1)}(x) D_q^k h(x).
\end{aligned}
$$

Hence,

$$D_q^n h(x) = \sum_{k=0}^{n-1} P_k(x;h)\ D_q^k h(x)\ ,$$

where

$$
\begin{aligned}
P_k(x;h) &= p_{k+1}^{(n+1)}(x) - \frac{p_0^{(n+1)}(x)}{p_0^{(n)}(x)} p_{k+1}^{(n)}(x)\ , \quad k = 0, 1, \ldots n-2, \\
P_{n-1}(x;h) &= \frac{p_0^{(n+1)}(x)}{p_0^{(n)}(x)}\ .
\end{aligned}
$$

By multiplying with the common denominator of the rational functions $\{P_k(x;h),\ k = 0, 1, \ldots, n-1\}$, we can conclude that $h(x)$ satisfies the equation

$$\sum_{k=0}^{n} \tilde{P}_k(x;h)\ D_q^k h(x) \ = \ 0\ ,$$

i.e it is a q-holonomic function of order $\leq n$. $\diamondsuit$

Example 3.1. In Example 2.2, for the q-Pochhammer symbol we proved that it satisfies

$$f(x) = (x;q)_\infty \quad \Rightarrow \quad (1-x) D_q f(x) + \frac{1}{1-q} f(x) = 0.$$

Now, we have

$$h_m(x) = D_q^m \big((x;q)_\infty \big) \quad \Rightarrow \quad (1 - q^m x) D_q h_m(x) + \frac{q^m}{1-q} h_m(x) = 0 \quad (m \in \mathbb{N}_0).$$

Theorem 3.2. *If $u(x)$ and $v(x)$ are q-holonomic functions of order n and m respectively, then the functions $u(x) + v(x)$ are q-holonomic functions of order at most $m+n$ and there is an algorithm to compute the corresponding q-differential equations.*

Proof. If $u(x)$ and $v(x)$ are q-holonomic functions of order n and m respectively, they satisfy holonomic equations

$$\sum_{k=0}^{n} \tilde{p}_k(x)\ D_q^k u(x) = 0\ , \qquad \sum_{j=0}^{m} \tilde{r}_j(x)\ D_q^j v(x) = 0\ , \tag{8}$$

where $\tilde{p}_k(x)$ and $\tilde{r}_j(x)$ are polynomials and $\tilde{p}_n \not\equiv 0$, $\tilde{r}_m \not\equiv 0$. According to Lemma 2.1, $D_q^l u(x)$ and $D_q^l v(x)$ can be represented as

$$D_q^l u(x) = \sum_{k=0}^{n-1} p_k^{(l)}(x)\, D_q^k u(x)\ , \qquad D_q^l v(x) = \sum_{j=0}^{m-1} r_j^{(l)}(x)\, D_q^j v(x)\ , \tag{9}$$

where $p_k^{(l)}(x)$ and $r_j^{(l)}(x)$ are rational functions as mentioned lemma.

Let $h(x) = u(x) + v(x)$. Then, according to (9), we have

$$D_q^l h(x) = \sum_{k=0}^{n-1} p_k^{(l)}(x)\, D_q^k u(x) + \sum_{j=0}^{m-1} r_j^{(l)}(x)\, D_q^j v(x)\ , \quad l = 0, 1, \ldots, m+n. \tag{10}$$

Taking the values for $l = 0, 1, \ldots, m+n-1$ in the above identities and expressing q-derivatives of $u(x)$ and $v(x)$ by q-derivatives of $h(x)$, we get

$$\begin{aligned} D_q^k u(x) &= \sum_{l=0}^{m+n-1} a_k^{(l)}(x)\, D_q^l h(x)\ , \quad k = 0, 1, \ldots, n-1\ , \\ D_q^j v(x) &= \sum_{l=0}^{m+n-1} b_j^{(l)}(x)\, D_q^l h(x)\ , \quad j = 0, 1, \ldots, m-1\ . \end{aligned}$$

By eliminating $D_q^k u(x)$ $(k = 0, 1, \ldots, n-1)$ and $D_q^j v(x)$ $(j = 0, 1, \ldots, m-1)$ from the last identity $(l = m+n)$ of (10), we get

$$D_q^{m+n} h(x) = \sum_{l=0}^{m+n-1} c_l(x) D_q^l h(x)\ ,$$

where

$$c_l(x) = \sum_{k=0}^{n-1} p_k^{(l)}(x) a_k^{(l)}(x) + \sum_{j=0}^{m-1} r_j^{(l)}(x) b_j^{(l)}(x)\ .$$

By multiplying with the common denominator of $\{c_l(x),\ l = 0, 1, \ldots m+n-1\}$, we get the holonomic equation for $h(x)$

$$\sum_{l=0}^{m+n} \tilde{c}_l(x) D_q^l h(x) \ = \ 0\ .$$

This proves that the q-holonomic order of $u(x)+v(x)$ is at most $m+n$, but can be less. An iterative version of the given algorithm will determine the q-holonomic equation of lowest order for $u(x)+v(x)$. $\diamondsuit$

Note that the algorithm given in Theorem 3.2 finds a q-differential equation which is not only valid for $u(x)+v(x)$, but also for every linear combination $\lambda_1 u(x) + \lambda_2 v(x)$, in particular for $u(x) - v(x)$.

Example 3.2. The small q-exponential function from Example 2.3 is q-holonomic of first order and satisfies

$$u(x) = e_q(x) \ \Rightarrow \ D_q^k u(x) = \frac{1}{(1-q)^k} u(x) \quad (k = 0, 1, \ldots).$$

Also, the q-sine from Example 2.5 is q-holonomic of second order and satisfies

$$v(x) = \sin_q(x) \ \Rightarrow \ D_q^{k+2} v(x) = \frac{-1}{(1-q)^2} D_q^k v(x) \quad (k = 0, 1, \ldots).$$

Now, by the algorithm given in the proof of Theorem 3.2, the function $h(x) = u(x) + v(x)$ satisfies

$$D_q^3 h(x) = \frac{1}{1-q} D_q^2 h(x) - \frac{1}{(1-q)^2} D_q h(x) + \frac{1}{(1-q)^3} h(x).$$

i.e., it is q-holonomic of third order.

Theorem 3.3. *If $u(x)$ and $v(x)$ are q-holonomic functions of order n and m respectively, then the function $u(x) \cdot v(x)$ is q-holonomic of order at most $m \cdot n$ and there is an algorithm to compute the corresponding q-differential equation.*

Proof. If $u(x)$ and $v(x)$ are q-holonomic functions of order n and m respectively, they satisfy holonomic equations (8), and their q-derivatives (9).

Let $h(x) = u(x) \cdot v(x)$. Then, according to (1.5), we have

$$\begin{aligned} D_q^l h(x) &= \sum_{\nu=0}^{l} \sum_{\mu=0}^{l} \alpha_{\nu\mu}(x) D_q^\nu u(x) D_q^\mu v(x) \\ &= \sum_{\nu=0}^{l} \sum_{\mu=0}^{l} \alpha_{\nu\mu}(x) \Big(\sum_{k=0}^{n-1} p_k^{(\nu)}(x) \ D_q^k u(x) \Big) \Big(\sum_{j=0}^{m-1} r_j^{(\mu)}(x) \ D_q^j v(x) \Big), \end{aligned}$$

i.e.

$$D_q^l h(x) = \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} \beta_{kj}^{(l)}(x) D_q^k u(x) D_q^j v(x) \qquad (l = 0, 1, \ldots, mn), \tag{11}$$

where

$$\beta_{kj}^{(l)}(x) = \sum_{\nu=0}^{l} \sum_{\mu=0}^{l} \alpha_{\nu\mu}(x) p_k^{(\nu)}(x) r_j^{(\mu)}(x) \ .$$

Taking the relations for (11) for $l = 0, 1, \ldots, mn-1$ and expressing the q-derivatives $D_q^k u(x) D_q^j v(x)$ by q-derivatives of $h(x)$, we get

$$D_q^k u(x) D_q^j v(x) = \sum_{l=0}^{mn-1} \gamma_{kj}^{(l)}(x) \ D_q^l h(x) \qquad (0 \le k \le n-1; \ 0 \le j \le m-1).$$

Eliminating all those $D_q^k u(x) D_q^j v(x)$ from the last identity $(l = mn)$ of (11), it becomes

$$D_q^{mn} h(x) = \sum_{l=0}^{mn-1} \sigma_l(x) D_q^l h(x) ,$$

where

$$\sigma_l(x) = \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} \beta_{kj}^{(l)}(x) \gamma_{kj}^{(l)}(x) .$$

By multiplying with the common denominator of $\{\sigma_l(x), l = 0, 1, \dots mn-1\}$, we get the q-holonomic equation for $h(x)$

$$\sum_{l=0}^{mn} \tilde{\sigma}_l(x) D_q^l h(x) \ = \ 0 .$$

This proves that the q-holonomic order of $u(x) \cdot v(x)$ is at most mn, but can be less. An iterative version of the given algorithm will determine the q-holonomic equation of lowest order for $u(x) \cdot v(x)$. $\diamondsuit$

Note furthermore that by Lemma 2.2 there is no similar algorithm for the quotient $u(x)/v(x)$.

Example 3.3. We use again $u(x) = e_q(x)$ and $v(x) = \sin_q(x)$. Now, by the given algorithm the function $h(x) = u(x) \cdot v(x)$ satisfies

$$(1-q)^2 D_q^2 h(x) - (1-q^2) D_q h(x) + \big(qx^2 - (1+q)(x-1)\big) h(x) = 0,$$

i.e., it is q-holonomic of second order.

Theorem 3.4. *If $u(x)$ is a q-holonomic function of order n, then the function $w(x) = u(x^\nu)$ $(\nu \in \mathbb{N})$ is a q-holonomic function of order at most n and there is an algorithm to compute the corresponding q-differential equation.*

Proof. By assumption $u(t)$ satisfies a q-holonomic equation

$$\sum_{k=0}^{n} \tilde{p}_k(t) \ D_q^k u(t) = 0 , \tag{12}$$

where $\tilde{p}_k(t)$ are polynomials and $\tilde{p}_n \not\equiv 0$. Then, by Lemma 2.1, $D_q^l u(t)$ can be represented as

$$D_q^l u(t) = \sum_{k=0}^{n-1} p_k^{(l)}(t) \ D_q^k u(t) , \tag{13}$$

where $p_k^{(l)}(t)$ are rational functions determined by that lemma.

Let $t = x^\nu$. Using Lemma 1.3, we have

$$D_q w(x) = D_{q^\nu} \ u(t) \ D_q(x^\nu) = \frac{u(t) - u(q^\nu t)}{(1-q^\nu)t} \ [\nu]_q \, x^{\nu - 1} .$$

According to (4), we get

$$D_q w(x) = \sum_{j=1}^{\nu} e_{j,\nu}(x) D_q^j u(t) ,$$

where

$$e_{j,\nu}(x) = (-1)^{j-1}(1-q)^{j-1} \begin{bmatrix} \nu \\ j \end{bmatrix}_q q^{\binom{j}{2}} x^{\nu j-1}, \qquad j = 1, 2, \ldots, \nu. \tag{14}$$

By (13), we can write

$$D_q w(x) = \sum_{j=1}^{\nu} e_{j,\nu}(x) \sum_{k=0}^{n-1} p_k^{(j)}(t)\ D_q^k u(t) = \sum_{k=0}^{n-1} f_{k,\nu}^{(1)}(x)\ D_q^k u(t) ,$$

where

$$f_{k,\nu}^{(1)}(x) = \sum_{j=1}^{\nu} p_k^{(j)}(x^\nu) e_{j,\nu}(x) , \quad k = 0, 1, \ldots, n-1 . \tag{15}$$

Furthermore,

$$\begin{aligned} D_q^2 w(x) &= \sum_{k=0}^{n-1} D_q\big(f_{k,\nu}^{(1)}(x) D_q^k u(t)\big) \\ &= \sum_{k=0}^{n-1} D_q f_{k,\nu}^{(1)}(x)\ D_q^k u(t) + \sum_{k=0}^{n-1} f_{k,\nu}^{(1)}(qx) D_q\big(D_q^k u(t)\big) .. \end{aligned}$$

As before, the second sum in the above term can be transformed to

$$\begin{aligned} \sum_{i=0}^{n-1} f_{i,\nu}^{(1)}(qx) D_q\big(D_q^i u(t)\big) &= \sum_{i=0}^{n-1} f_{i,\nu}^{(1)}(qx) \sum_{j=1}^{\nu} e_{j,\nu}(x) D_q^j\big(D_q^i u(t)\big) \\ &= \sum_{i=1}^{n-1} \sum_{j=1}^{\nu} f_{i,\nu}^{(1)}(qx) e_{j,\nu}(x) D_q^{i+j} u(t) \\ &= \sum_{i=1}^{n-1} \sum_{j=1}^{\nu} f_{i,\nu}^{(1)}(qx) e_{j,\nu}(x) \sum_{k=0}^{n-1} p_k^{(i+j)}(t)\ D_q^k u(t) . \end{aligned}$$

Hence,

$$D_q^2 w(x) = \sum_{k=0}^{n-1} f_{k,\nu}^{(2)}(x)\ D_q^k u(t) ,$$

where

$$f_{k,\nu}^{(2)}(x) = D_q f_{k,\nu}^{(1)}(x) + \sum_{i=0}^{n-1} \sum_{j=1}^{\nu} f_{i,\nu}^{(1)}(qx) e_{j,\nu}(x) p_k^{(i+j)}(x^\nu), \quad k = 0, 1, \ldots, n-1.$$

By induction, we obtain the representations

$$D_q^l w(x) = \sum_{k=0}^{n-1} f_{k,\nu}^{(l)}(x)\; D_q^k u(t)\;, \quad l = 0,1,2,\ldots,n \tag{16}$$

where $f_{k,\nu}^{(0)}(x) = \delta_{k0}$, $f_{k,\nu}^{(1)}(x)$ is given in (15) and

$$f_{k,\nu}^{(l)}(x) = D_q f_{k,\nu}^{(l-1)}(x) + \sum_{i=0}^{n-1}\sum_{j=1}^{\nu} f_{i,\nu}^{(l-1)}(qx) e_{j,\nu}(x) p_k^{(i+j)}(x^\nu)\;. \tag{17}$$

Taking the first n of the identities (16), we can determine

$$D_q^k u(t) = \sum_{l=0}^{n-1} b_{l,\nu}^{(k)}(x) D_q^l w(x)\;, \quad k = 0,1,\ldots,n-1,$$

where $b_{l,\nu}^{(k)}(x)$ are rational functions. Substituting this in identity (16), we get

$$D_q^n w(x) = \sum_{k=0}^{n-1} f_{k,\nu}^{(l)}(x) \sum_{l=0}^{n-1} b_{l,\nu}^{(k)}(x) D_q^l w(x) = \sum_{l=0}^{n-1} c_{l,\nu}(x) D_q^l w(x)\;,$$

where

$$c_{l,\nu}(x) = \sum_{k=0}^{n-1} f_{k,\nu}^{(l)}(x) b_{l,\nu}^{(k)}(x).$$

By multiplying with the common denominator of $\{c_{l,\nu}(x),\ l = 0,1,\ldots,n-1\}$, we obtain

$$\sum_{l=0}^{n} \tilde{c}_{l,\nu}(x) D_q^l w(x) \;=\; 0\,. \quad \diamondsuit$$

Example 3.4. In Example 2.1, it was proved that

$$u(x) = x^s \;\Rightarrow\; (q-1)\, x D_q u(x) - (q^s - 1)\, u(x) = 0.$$

Hence $n = 1$, $\tilde{p}_1(x) = (q-1)\,x$ and $\tilde{p}_0(x) = -(q^s - 1)$. By the procedure of Theorem 3.4, we get for $w(x) = u(x)^\nu$

$$D_q w(x) = f_{0,\nu}^{(1)}(x) w(x) \qquad \text{where} \qquad f_{0,\nu}^{(1)}(x) = \frac{q^{s\nu} - 1}{q-1} \cdot \frac{1}{x}.$$

Finally,

$$w(x) = u(x^\nu) \;\Rightarrow\; (q-1)\, x D_q w(x) - (q^{s\nu} - 1)\, w(x) = 0.$$

Example 3.5. In Example 2.2, it was proved that

$$u(x) = (x;q)_\lambda \;\Rightarrow\; (q-1)\,(x-1) D_q u(x) - (q^\lambda - 1)\, u(x) = 0.$$

Using our algorithm we get for $w(x) = u(x^2) = (x^2; q)_\lambda$ the q-holonomic equation

$$(q-1)(x-1)(x+1)(x^2q-1)\, D_q w(x) - x(q^\lambda - 1)(x^2q^{\lambda+1} - q - 1 + x^2q)\, f(x) = 0$$

and similar, but more complicated, equations for $(x^\nu; q)_\lambda$ for higher $\nu \in \mathbb{N}$.

Example 3.6. In Example 2.5, for the q-sine function, we got

$$u(x) = \sin_q(x) \;\Rightarrow\; (1-q)^2 D_q^2 u(x) + u(x) = 0.$$

Now, for $w(x) = u(x^2)$, we have

$$D_q w(x) = f_{0,2}^{(1)}(x)u(t) + f_{1,2}^{(1)}(x)D_q u(t),$$

with

$$f_{0,2}^{(1)}(x) = \frac{qx^3}{1-q}, \qquad f_{1,2}^{(1)}(x) = (1+q)x$$

and

$$D_q^2 w(x) = f_{0,2}^{(2)}(x)u(t) + f_{1,2}^{(2)}(x)D_q u(t),$$

with

$$f_{0,2}^{(2)}(x) = \frac{(qx)^2(-2 - q - q^2 + q^3x^4)}{(1-q)^2}$$
$$f_{1,2}^{(2)}(x) = \frac{(1+q)(1 - q + q^2(1+q^2)x^4)}{1-q}.$$

By eliminating $D_q u(t)$, we get

$$D_q^2 w(x) = c_{0,2}(x)w(x) + c_{1,2}(x)D_q w(x),$$

wherefrom we get for the function $w(x) = u(x^2)$ the following equation

$$xD_q^2 w(x) - \Big(1 + q^2\frac{1+q^2}{1-q}x^4\Big)D_q w(x) + qx^3\Big(\frac{1-q^4}{(1-q)^3} + \frac{q^2}{(1-q)^2}x^4\Big)w(x) = 0.$$

4 Sharpness of the algorithms

In the previous section we proved that the sum, product and composition with powers of q-holonomic functions are q-holonomic too. In this section we show that the given bounds for the orders are sharp in all algorithms considered.

Example 4.1. The functions $u(x) = x^2$ and $v(x) = x^3$ are q-holonomic of first order. According to Theorem 3.2, the function $h(x) = u(x) + v(x)$ is q-holonomic of order at most two. However, all polynomials are q-holonomic functions of first order, and we find that $h(x)$ satisfies the equation

$$x(1+x)D_q h(x) - ([2]_q + [3]_q x)h(x) = 0.$$

This example shows that the order of the sum of some q-holonomic functions can be strictly less than the sum of their orders. This applies if the two functions $u(x)$ and $v(x)$ are linearly dependent over $\mathbb{K}(q)(x)$.

However, we will prove that for every algorithm given in the previous section there are functions for which the maximal order is attained.

Lemma 4.1. *The functions* $E_q(x^\mu)$ $(\mu = 1, 2, \ldots, n)$ *are linearly independent over* $\mathbb{K}(q)(x)$.

Proof. Let us consider a linear combination

$$r_1 E_q(x) + r_2 E_q(x^2) + \cdots + r_n E_q(x^n) = 0 \,,$$

where $r_\mu = r_\mu(x)$ $(\mu = 1, 2, \ldots, n)$ are rational functions and suppose that $r_\nu \not\equiv 0$. Then,

$$r_\nu E_q(x^\mu) = -\sum_{\substack{\mu=0\\ \mu\neq\nu}}^{n} r_\mu E_q(x^\mu) \,,$$

i.e.,

$$\sum_{\substack{\mu=0\\ \mu\neq\nu}}^{n} \frac{r_\mu}{r_\nu}\, \frac{E_q(x^\mu)}{E_q(x^\nu)} = -1 \,. \tag{18}$$

Since

$$A(m) = \lim_{x\to\infty} \frac{\displaystyle\sum_{n=0}^{m} \frac{q^{\binom{n}{2}}}{(q;q)_n}\, (x^\mu)^n}{\displaystyle\sum_{n=0}^{m} \frac{q^{\binom{n}{2}}}{(q;q)_n}\, (x^\nu)^n} = \lim_{x\to\infty} x^{m(\mu-\nu)} = \begin{cases} +\infty\,, & \mu > \nu\,,\\ 0\,, & \mu < \nu\,,\end{cases}$$

we have

$$\lim_{x\to\infty} \frac{E_q(x^\mu)}{E_q(x^\nu)} = \lim_{m\to\infty} A(m) = \begin{cases} +\infty\,, & \mu > \nu\,,\\ 0\,, & \mu < \nu\,.\end{cases}$$

This is a contradiction with (18). Hence, it follows that $r_\mu \equiv 0$ for all $\mu = 1, 2, \ldots, n$, i.e. $E_q(x^\mu)$ $(\mu = 1, 2, \ldots, n)$ are linearly independent over $\mathbb{K}(q)[x]$. $\Diamond$

Lemma 4.2. *The function*

$$F_n(x) = \sum_{\mu=1}^{n} E_q(x^\mu) \tag{19}$$

is q*-holonomic of order* n.

Proof. The function $E_q(x)$ satisfies the q-holonomic equation of first order (see Example 2.4)

$$(1-q)(t+1) D_q f(t) - f(t) = 0 \,.$$

With respect to Theorem 3.4, for each $\mu \in \mathbb{N}$, the function $E_q(x^\mu)$ is q-holonomic of first order and one has

$$D_q^l\big(E_q(x^\mu)\big) = f_{0,\mu}^{(l)}(x) E_q(x^\mu) \,, \quad l = 0, 1, \ldots \,, \tag{20}$$

where $f_{0,\mu}^{(l)}(x)$ are rational functions given as in (17).

According to Theorem 3.2, the function $F_n(x)$ is q-holonomic of order at most n. Therefore

$$D_q^l F_n(x) = \sum_{\mu=1}^{n} D_q^l\big(E_q(x^\mu)\big) = \sum_{\mu=1}^{n} f_{0,\mu}^{(l)}(x) E_q(x^\mu) \; .$$

Let us suppose that the function $F_n(x)$ satisfies a q-holonomic equation of order m, i.e.

$$D_q^m F_n(x) + \sum_{i=0}^{m-1} A_i D_q^i F_n(x) = 0 \; . \tag{21}$$

This equation can be represented in the form

$$\sum_{\mu=1}^{n} \Big(f_{0,\mu}^{(m)}(x) + \sum_{i=0}^{m-1} A_i f_{0,\mu}^{(i)}(x) \Big) E_q(x^\mu) = 0.$$

Since $E_q(x^\mu)$ $(\mu = 1, 2, \ldots, n)$ are linearly independent over $\mathbb{K}(q)[x]$, it follows that

$$f_{0,\mu}^{(m)}(x) + \sum_{i=0}^{m-1} A_i f_{0,\mu}^{(i)}(x) = 0 \; , \quad \mu = 1, 2, \ldots, n \; .$$

This can be written in the form of the system of equations

$$\sum_{i=0}^{m-1} A_i f_{0,\mu}^{(i)}(x) = -f_{0,\mu}^{(m)}(x) \; , \quad \mu = 1, 2, \ldots, n$$

with unknown rational functions $A_i = A_i(x)$.

If $m < n$, then the system is overdetermined and has no solution. Hence it follows that $m = n$. $\Diamond$

Theorem 4.3. *For each $n \in \mathbb{N}$ there is a function F which is q-holonomic of order n, such that $H = D_q F$ is q-holonomic of order n.*

Proof. The function defined by (19) satisfies the statement. $\Diamond$

Theorem 4.4. *For each $n, m \in \mathbb{N}$ there are functions U and V that are q-holonomic of order n and m respectively, such that $H = U + V$ is q-holonomic of order $n + m$.*

Proof. Consider the functions

$$U(x) = \sum_{\mu=1}^{n} E_q(x^\mu) \qquad \text{and} \qquad V(x) = \sum_{\mu=n+1}^{n+m} E_q(x^\mu) \; . \tag{22}$$

According to Lemma 4.2, they are q-holonomic of order n and m respectively, and the function

$$H(x) = U(x) + V(x) = \sum_{\mu=1}^{n+m} E_q(x^\mu)$$

is q-holonomic of order $n + m$. $\Diamond$

Theorem 4.5. *For each $n, m \in \mathbb{N}$ there are functions U and V that are q-holonomic of order n and m respectively, such that $H = U \cdot V$ is q-holonomic of order $n \cdot m$.*

Proof. The statement is valid for the functions defined by (22), because in the function

$$H(x) = U(x) \cdot V(x) = \sum_{\mu=1}^{n} \sum_{\nu=n+1}^{n+m} E_q(x^\mu) E_q(x^\nu)$$

there are nm linearly independent summands $E_q(x^\mu)E_q(x^\nu)$ $(\mu = 1, 2, \ldots, n;\ \nu = n+1, n+2, \ldots, n+m)$ over $\mathbb{K}(q)[x]$. The proof of their independence is again based on Lemma 4.1. $\diamondsuit$

Theorem 4.6. *For each $n \in \mathbb{N}$ there is a function F which is q-holonomic of order n, such that $W(x) = F(x^\nu)$ is q-holonomic of order n.*

Proof. Starting from the function $F_n(x)$ defined by (19), we can form

$$W(x) = F_n(x^\nu) = \sum_{\mu=1}^{n} E_q(x^{\mu\nu})$$

which is of the same type as $F_n(x)$. $\diamondsuit$

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