On Generating Symmetric Orthogonal Polynomials

Mohammad Masjed-Jamei [a] **Wolfram Koepf** [b,*]

[a] *Department of Mathematics, K.N.Toosi University of Technology, P.O.Box 16315-1618, Tehran, Iran, E-mail: mmjamei@kntu.ac.ir , mmjamei@yahoo.com*

[b,*] *Institute of Mathematics, University of Kassel, Heinrich-Plett-Str. 40, D-34132 Kassel, Germany, E-mail: koepf@mathematik.uni-kassel.de, Corresponding author* [*]

Abstract. In this paper, we show how to generate symmetric sequences of orthogonal polynomials whose moments are given. The advantage of this method is that only one Hankel determinant must be calculated. Three illustrative examples are presented in this sense.

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1. Introduction. Let $\bar{P}_n(x) = x^n + \sum_{k=0}^{n-1} \lambda_k x^k$ be a sequence of monic polynomials orthogonal with respect to the weight function $w^*(x)$ on $[a,b]$, i.e.

$$\left\langle \bar{P}_n, \bar{P}_m \right\rangle_{w^*} = \int_a^b w^*(x)\,\bar{P}_n(x)\,\bar{P}_m(x)\,dx = \left\langle \bar{P}_n, \bar{P}_n \right\rangle_{w^*} \delta_{n,m} = \left\| \bar{P}_n \right\|_2^2 \begin{cases} 0 & (n \neq m), \\ 1 & (n = m). \end{cases} \tag{1}$$

Since the norm square value of a sequence of orthogonal polynomials is in general minimal [2,9], the coefficients $\lambda_k = \lambda_k(\{w_j\}_{j=0}^{2n-1})$ in $\bar{P}_n(x)$ can be computed by solving the following linear system

$$\begin{bmatrix} w_0 & w_1 & \cdots & w_{n-1} \\ w_1 & w_2 & \cdots & w_n \\ \vdots & \vdots & & \vdots \\ w_{n-1} & w_n & \cdots & w_{2n-2} \end{bmatrix} \begin{bmatrix} \lambda_0(\{w_j\}_{j=0}^{2n-1}) \\ \lambda_1(\{w_j\}_{j=0}^{2n-1}) \\ \vdots \\ \lambda_{n-1}(\{w_j\}_{j=0}^{2n-1}) \end{bmatrix} = -\begin{bmatrix} w_n \\ w_{n+1} \\ \vdots \\ w_{2n-1} \end{bmatrix}, \tag{2}$$

in which $w_j = \int_a^b x^j w^*(x)\,dx$ denote the moments of order j and $w^*(x)$ is the weight function over $[a,b]$. To solve the linear system (2), clearly $n+1$ separate determinants must be calculated.

On the other hand, according to Favard's theorem [1,9], if $\{P_n(x)\}_{n=0}^{\infty}$ is defined by a three-term recurrence relation of the form

$$x\,P_n(x) = A_n\,P_{n+1}(x) + B_n P_n(x) + C_n P_{n-1}(x) \qquad (n = 0,1,2,...)\ , \tag{3}$$

where $P_{-1}(x) = 0$, $P_0(x) = 1$, A_n, B_n, C_n real and $A_n C_{n+1} > 0$ for $n \in \mathbb{N}$, then there exists a weight function, say $\rho(x)$, such that

$$\int_{-\infty}^{\infty} \rho(x) P_n(x)\, P_m(x)\, dx = \left(\prod_{i=0}^{n-1} \frac{C_{i+1}}{A_i} \int_{-\infty}^{\infty} \rho(x)\, dx \right) \delta_{n,m}\, . \tag{4}$$

It is clear that relations (3) and (4) are also valid for the symmetric case $A_n = 1$, $B_n = 0$, $C_n = -C_n^*$ and $(-\infty,\infty) \to [-\alpha,\alpha]$ provided that $-C_{n+1}^* > 0$. In this case, we encounter with monic symmetric orthogonal polynomials satisfying the equation

$$\overline{P}_{n+1}(x) = x\,\overline{P}_n(x) + C_n^*\,\overline{P}_{n-1}(x) \quad \text{with} \quad \begin{cases} \overline{P}_0(x) = 1,\ \overline{P}_1(x) = x, \\ C_{n+1}^* < 0,\ n \in \mathbb{N}, \end{cases} \tag{5}$$

and also

$$\overline{P}_n(-x) = (-1)^n\,\overline{P}_n(x)\, .$$

Hence, there is an even weight function, say $w(x)$, such that the solution of the symmetric recurrence equation (5) satisfies the following orthogonality relation

$$\int_{-\alpha}^{\alpha} w(x) \overline{P}_n(x)\, \overline{P}_m(x)\, dx = \left((-1)^n \prod_{i=1}^{n} C_i^* \int_{-\alpha}^{\alpha} w(x)\, dx \right) \delta_{n,m}\, . \tag{6}$$

Relations (5) and (6) form the basis of the algorithm proposed in this paper. In other words, instead of explicitly computing $\lambda_k(\{w_j\}_{j=0}^{2n-1})$ in the linear system (2), we establish a symmetric three-term recurrence relation of type (5) for $\overline{P}_n(x)$ and then compute its norm square value by relation (6).

First of all, we should note since the weight function corresponding to recurrence relation (5) and orthogonality relation (6) is even, we automatically have

$$w_{2j+1} = \int_{-\alpha}^{\alpha} x^{2j+1} w(x)\, dx = 0 \quad \text{for every } \ j = 0,1,2,.... \, . \tag{7}$$

This result helps to reduce the volume of calculations in (2) as much as possible, because substituting (7) in (2) yields

$$\begin{bmatrix} w_0 & 0 & w_2 & \dots \\ 0 & w_2 & 0 & \dots \\ w_2 & 0 & w_4 & \dots \\ 0 & w_4 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \lambda_0(\{w_j\}_{j=0}^{2n-1}) \\ \lambda_1(\{w_j\}_{j=0}^{2n-1}) \\ \vdots \\ \lambda_{n-1}(\{w_j\}_{j=0}^{2n-1}) \end{bmatrix} = -\begin{bmatrix} w_n \\ w_{n+1} \\ \vdots \\ 0 \end{bmatrix}. \tag{8}$$

The key point here is that solving the above linear system will eventually lead to either $\lambda_{2k}(\{w_j\}_{j=0}^{2n-1}) = 0$ or $\lambda_{2k+1}(\{w_j\}_{j=0}^{2n-1}) = 0$, because if we expand the recurrence relation (5), then the general shape of $\overline{P}_n(x)$ will finally be $\overline{P}_n(x) = \sum_{k=0}^{[n/2]} \Phi_k(n)\, x^{n-2k}$ where $\Phi_0(n) = 1$ and $\Phi_k(n) = 0$ for $k > [n/2]$. Hence, we deal with two different cases. In the first case, if $n = 2m$, then (8) is transformed to

$$\begin{bmatrix} w_0 & 0 & \dots & 0 \\ 0 & w_2 & \dots & w_{2m} \\ w_2 & 0 & \dots & 0 \\ 0 & w_4 & \dots & w_{2m+2} \\ \vdots & \vdots & \vdots & \vdots \\ w_{2m-2} & 0 & \dots & 0 \\ 0 & w_{2m} & \dots & w_{4m-2} \end{bmatrix} \begin{bmatrix} \lambda_0(\{w_{2j}\}_{j=0}^{2m-1}) \\ \lambda_1(\{w_{2j}\}_{j=0}^{2m-1}) \\ \vdots \\ \lambda_{2m-1}(\{w_{2j}\}_{j=0}^{2m-1}) \end{bmatrix} = -\begin{bmatrix} w_{2m} \\ 0 \\ w_{2m+2} \\ 0 \\ \vdots \\ w_{4m-2} \\ 0 \end{bmatrix}. \tag{9}$$

Now, by collapsing the matrix form of the linear system (9), it can be reduced to the following matrix representation

$$\begin{bmatrix} w_0 & w_2 & \dots & w_{2m-2} \\ w_2 & w_4 & \dots & w_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ w_{2m-4} & w_{2m-2} & \dots & w_{4m-6} \\ w_{2m-2} & w_{2m} & \dots & w_{4m-4} \end{bmatrix} \begin{bmatrix} \lambda_0(\{w_{2j}\}_{j=0}^{2m-1}) \\ \lambda_2(\{w_{2j}\}_{j=0}^{2m-1}) \\ \vdots \\ \lambda_{2m-2}(\{w_{2j}\}_{j=0}^{2m-1}) \end{bmatrix} = -\begin{bmatrix} w_{2m} \\ w_{2m+2} \\ \vdots \\ w_{4m-2} \end{bmatrix}. \tag{10}$$

For instance, set $m = 2$ in (9) to get

$$\begin{cases} w_0\lambda_0 + w_2\lambda_2 = -w_4, \\ w_2\lambda_1 + w_4\lambda_3 = 0, \\ w_2\lambda_0 + w_4\lambda_2 = -w_6, \\ w_4\lambda_1 + w_6\lambda_3 = 0. \end{cases} \tag{11}$$

Since $w_2 = \int_{-\alpha}^{\alpha} x^2 w(x)\, dx = 2\int_0^{\alpha} x^2 w(x)\, dx > 0$ and $w_4 > 0$, (11) implies that $\lambda_1 = \lambda_3 = 0$. Therefore we have $w_0\lambda_0 + w_2\lambda_2 = -w_4$ and $w_2\lambda_0 + w_4\lambda_2 = -w_6$, which is the same as (10) for $m = 2$.

Similar considerations can be done for $n=2m+1$. One can verify that the matrix representation of the case $n=2m+1$ does directly depend on the previous case $n=2m$, because when $n=2m+1$, (8) takes the form

$$\begin{bmatrix} w_0 & 0 & \dots & w_{2m} \\ 0 & w_2 & \dots & 0 \\ w_2 & 0 & \dots & w_{2m+2} \\ 0 & w_4 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & w_{2m} & \dots & 0 \\ w_{2m} & 0 & \dots & w_{4m} \end{bmatrix} \begin{bmatrix} \lambda_0(\{w_{2j}\}_{j=0}^{2m}) \\ \lambda_1(\{w_{2j}\}_{j=0}^{2m}) \\ \vdots \\ \lambda_{2m}(\{w_{2j}\}_{j=0}^{2m}) \end{bmatrix} = -\begin{bmatrix} 0 \\ w_{2m+2} \\ 0 \\ w_{2m+4} \\ \vdots \\ w_{4m} \\ 0 \end{bmatrix},$$

which is simplified as

$$\begin{bmatrix} w_2 & w_4 & \dots & w_{2m} \\ w_4 & w_6 & \dots & w_{2m+2} \\ \vdots & \vdots & \vdots & \vdots \\ w_{2m-2} & w_{2m} & \dots & w_{4m-4} \\ w_{2m} & w_{2m+2} & \dots & w_{4m-2} \end{bmatrix} \begin{bmatrix} \lambda_1(\{w_{2j}\}_{j=0}^{2m}) \\ \lambda_3(\{w_{2j}\}_{j=0}^{2m}) \\ \vdots \\ \lambda_{2m-1}(\{w_{2j}\}_{j=0}^{2m}) \end{bmatrix} = -\begin{bmatrix} w_{2m+2} \\ w_{2m+4} \\ \vdots \\ w_{4m} \end{bmatrix}. \qquad (12)$$

By comparing (10) and (12) we observe that just w_{2j} in (10) has been substituted by w_{2j+2} in (12). Hence, solving these two linear systems finally results in

$$\begin{cases} \overline{P}_{2m}(x) = x^{2m} + \sum_{k=1}^{m} \lambda_{2m-2k}\left(\{w_{2j}\}_{j=0}^{2m-1}\right) x^{2m-2k}, \\ \overline{P}_{2m+1}(x) = x^{2m+1} + \sum_{k=1}^{m} \lambda_{2m-2k}\left(\{w_{2j+2}\}_{j=0}^{2m-1}\right) x^{2m+1-2k}. \end{cases}$$

For instance, if $n=1,2,\ldots,5$ then the related polynomials respectively take the forms

$$\overline{P}_1(x) = x, \quad \overline{P}_2(x) = x^2 + \lambda_0(\{w_{2j}\}_{j=0}^{1}) = x^2 - \frac{w_2}{w_0},$$

$$\overline{P}_3(x) = x^3 + \lambda_0(\{w_{2j+2}\}_{j=0}^{1})\, x = x^3 - \frac{w_4}{w_2} x,$$

$$\overline{P}_4(x) = x^4 + \frac{\begin{vmatrix} w_0 & -w_4 \\ w_2 & -w_6 \end{vmatrix}}{\begin{vmatrix} w_0 & w_2 \\ w_2 & w_4 \end{vmatrix}} x^2 + \frac{\begin{vmatrix} -w_4 & w_2 \\ -w_6 & w_4 \end{vmatrix}}{\begin{vmatrix} w_0 & w_2 \\ w_2 & w_4 \end{vmatrix}} = x^4 - \frac{w_6 w_0 - w_4 w_2}{w_4 w_0 - w_2^2} x^2 + \frac{w_6 w_2 - w_4^2}{w_4 w_0 - w_2^2},$$

$$\bar{P}_5(x) = x^5 + \frac{\begin{vmatrix} w_2 & -w_6 \\ w_4 & -w_8 \end{vmatrix}}{\begin{vmatrix} w_2 & w_4 \\ w_4 & w_6 \end{vmatrix}} x^3 + \frac{\begin{vmatrix} -w_6 & w_4 \\ -w_8 & w_6 \end{vmatrix}}{\begin{vmatrix} w_2 & w_4 \\ w_4 & w_6 \end{vmatrix}} x = x^5 - \frac{w_8 w_2 - w_6 w_4}{w_6 w_2 - w_4^2} x^3 + \frac{w_8 w_4 - w_6^2}{w_6 w_2 - w_4^2} x.$$

Now, the question is: *How to find the sequence* C_n^* *in the symmetric recurrence relation (5) when the moments* $\{w_j\}_{j=0}$ *are known.*

To answer this question, we first refer to the coefficient matrix of system (2) and define the Hankel determinant

$$\Delta_n = \det(w_{i+j})_{i,j=0}^n = \begin{vmatrix} w_0 & w_1 & \cdots & w_n \\ w_1 & w_2 & \cdots & w_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ w_n & w_{n+1} & \cdots & w_{2n} \end{vmatrix}. \qquad (13)$$

Using the orthogonality relation (1) and taking a weighted inner product from both sides of the recurrence equation (5) we get

$$\int_{-\alpha}^{\alpha} w(x)\bar{P}_{n+1}(x)\,x^{n-1}dx = \int_{-\alpha}^{\alpha} w(x)\bar{P}_n(x)\,x^n dx + C_n^* \int_{-\alpha}^{\alpha} w(x)\bar{P}_{n-1}(x)\,x^{n-1}\,dx.$$

Therefore

$$C_n^* = -\frac{\int_{-\alpha}^{\alpha} w(x)\bar{P}_n(x)\,x^n dx}{\int_{-\alpha}^{\alpha} w(x)\bar{P}_{n-1}(x)\,x^{n-1}\,dx} = -\frac{\langle \bar{P}_n, \bar{P}_n \rangle_w}{\langle \bar{P}_{n-1}, \bar{P}_{n-1} \rangle_w} = -\frac{\Delta_n / \Delta_{n-1}}{\Delta_{n-1} / \Delta_{n-2}} = -\frac{\Delta_n \Delta_{n-2}}{\Delta_{n-1}^2}. \qquad (14)$$

This gives us the final form of the recurrence relation (5) as

$$\bar{P}_{n+1}(x) = x\,\bar{P}_n(x) - \frac{\Delta_n \Delta_{n-2}}{\Delta_{n-1}^2}\,\bar{P}_{n-1}(x) \quad \text{for} \quad \bar{P}_0(x) = 1,\ \bar{P}_1(x) = x,\ \Delta_{-1} = 1 \ \text{ and } \ n = 1, 2, ... \qquad (15)$$

Relation (15) shows that to compute C_n^* we only need to compute one Hankel determinant Δ_n in which $w_{2j+1} = 0$ for any $j = 0, 1, ..., n-1$. For instance, if $n = 1,2,3,4$ then

$$C_1^* = -\frac{\Delta_1}{\Delta_0^2} = -\frac{w_2}{w_0},$$
$$C_2^* = -\frac{\Delta_2 \Delta_0}{\Delta_1^2} = -\frac{w_4}{w_2} + \frac{w_2}{w_0},$$
$$C_3^* = -\frac{\Delta_3\,\Delta_1}{\Delta_2^2} = -\frac{w_6 w_0 - w_4 w_2}{w_4 w_0 - w_2^2} + \frac{w_4}{w_2},$$

$$C_4^* = -\frac{\Delta_4\,\Delta_2}{\Delta_3^2} = -\frac{w_8 w_2 - w_6 w_4}{w_6 w_2 - w_4^2} + \frac{w_6 w_0 - w_4 w_2}{w_4 w_0 - w_2^2}.$$

After computing the explicit values C_n^*, one can directly obtain the explicit form of the symmetric monic polynomials $\overline{P}_n(x)$ by solving the recurrence equation

$$\overline{P}_{n+1}(x) = x\,\overline{P}_n(x) + C_n^*\,\overline{P}_{n-1}(x) \quad \text{with} \quad (\overline{P}_0(x) = 1 \text{ s.t. } C_0^* = 0) \quad \text{and} \quad \overline{P}_1(x) = x\,. \tag{16}$$

In detail, by knowing that the solution of equation (16) is monic and symmetric, we can assume that

$$\overline{P}_n(x) = \sum_{k=0}^{[n/2]} \Phi_k(n)\, x^{n-2k}\,. \tag{17}$$

This can be substituted in equation (16) to eventually reach the functional equation

$$\Phi_k(n+1) - \Phi_k(n) = C_n^*\, \Phi_{k-1}(n-1)\,,$$

where $\Phi_0(n) = 1$ and $\Phi_k(n) = 0$ for $k > [n/2]$. The above equation can straightforwardly be solved by induction [8] so that we finally have

$$\Phi_k(n) = \sum_{j_1=0}^{n+1-2k} C_{j_1}^* \sum_{j_2=j_1+2}^{n+3-2k} C_{j_2}^* \sum_{j_3=j_2+2}^{n+5-2k} C_{j_3}^* \;\cdots \sum_{j_k=j_{k-1}+2}^{n-1} C_{j_k}^*\,.$$

The next question is: *How to convert a general three-term recurrence relation of type (3) to the symmetric type (5) (or conversely).*

This question was first answered by Chihara [1] as follows: Since the solution of recurrence equation (5) is a symmetric polynomial, we can first assume that

$$\overline{P}_{2n}(x) = \overline{Q}_n(x^2) \quad \text{and} \quad \overline{P}_{2n+1}(x) = x\,\overline{R}_n(x^2)\,, \tag{18}$$

in which $\overline{Q}_n(x)$ and $\overline{R}_n(x)$ are two monic polynomials of degree *n*.

On the other hand, in (15) we obtained the relation

$$\overline{P}_{n+1}(x) = x\,\overline{P}_n(x) + C_n^*\,\overline{P}_{n-1}(x) \;\text{ where }\; C_n^* = -\frac{\Delta_n \Delta_{n-2}}{\Delta_{n-1}^2} \;\text{ and }\; C_0^* = 0\,. \tag{19}$$

Therefore if *n* is respectively given as $2m$ or $2m+1$ in (19), respectively, by using (18) we get

$$\begin{cases} x\,\overline{R}_m(x^2) = x\,\overline{Q}_m(x^2) + C_{2m}^*\, x\,\overline{R}_{m-1}(x^2) \\ \overline{Q}_{m+1}(x^2) = x^2\,\overline{R}_m(x^2) + C_{2m+1}^*\,\overline{Q}_m(x^2) \end{cases} \Rightarrow \begin{cases} \overline{Q}_m(x) = \overline{R}_m(x) - C_{2m}^*\,\overline{R}_{m-1}(x), \\ x\,\overline{R}_{m-1}(x) = \overline{Q}_m(x) - C_{2m-1}^*\,\overline{Q}_{m-1}(x). \end{cases} \tag{20}$$

Combining these two latter relations in (20) one gets

$$x\,\bar{R}_{m-1}(x) = \left(\bar{R}_m(x) - C^*_{2m}\,\bar{R}_{m-1}(x)\right) - C^*_{2m-1}\left(\bar{R}_{m-1}(x) - C^*_{2m-2}\,\bar{R}_{m-2}(x)\right),$$

which is equivalent to

$$\bar{R}_{m+1}(x) = \left(x - \frac{\Delta_{2m+2}\Delta_{2m}}{\Delta^2_{2m+1}} - \frac{\Delta_{2m+1}\Delta_{2m-1}}{\Delta^2_{2m}}\right)\bar{R}_m(x) - \frac{\Delta_{2m+1}\Delta_{2m-2}}{\Delta_{2m}\Delta_{2m-1}}\bar{R}_{m-1}(x). \qquad (21)$$

Similarly for $\bar{Q}_m(x)$ we get

$$x\,\bar{Q}_m(x) = \left(\bar{Q}_{m+1}(x) - C^*_{2m+1}\,\bar{Q}_m(x)\right) - C^*_{2m}\left(\bar{Q}_m(x) - C^*_{2m-1}\,\bar{Q}_{m-1}(x)\right),$$

which is equivalent to

$$\bar{Q}_{m+1}(x) = \left(x - \frac{\Delta_{2m+1}\Delta_{2m-1}}{\Delta^2_{2m}} - \frac{\Delta_{2m}\Delta_{2m-2}}{\Delta^2_{2m-1}}\right)\bar{Q}_m(x) - \frac{\Delta_{2m}\Delta_{2m-3}}{\Delta_{2m-1}\Delta_{2m-2}}\bar{Q}_{m-1}(x), \qquad (22)$$

where $C^*_{-1} = C^*_0 = 0$.

Since now the recurrence relations of $\bar{Q}_n(x)$ and $\bar{R}_n(x)$ have been determined in (21) and (22), one can determine their orthogonality relations with related weight functions. For this purpose, we reconsider (6) in the form

$$\int_{-\alpha}^{\alpha} w(x)\bar{P}_n(x)\,\bar{P}_m(x)\,dx = \left((-1)^n \prod_{i=1}^{n} C^*_i \int_{-\alpha}^{\alpha} w(x)\,dx\right)\delta_{n,m}. \qquad (23)$$

By noting that $w(x)$ is an even function, substituting $n \to 2n$ and $m \to 2m$, (23) changes to

$$\begin{aligned}\int_{-\alpha}^{\alpha} w(x)\bar{P}_{2n}(x)\,\bar{P}_{2m}(x)\,dx &= \int_{-\alpha}^{\alpha} w(x)\bar{Q}_n(x^2)\,\bar{Q}_m(x^2)\,dx = 2\int_{0}^{\alpha} w(x)\bar{Q}_n(x^2)\,\bar{Q}_m(x^2)\,dx \\ &= \int_{0}^{\alpha^2} \frac{w(\sqrt{t})}{\sqrt{t}}\bar{Q}_n(t)\bar{Q}_m(t)\,dt = \left(\prod_{i=1}^{2n} C^*_i \int_{-\alpha}^{\alpha} w(x)\,dx\right)\delta_{n,m}.\end{aligned} \qquad (24)$$

This means that $\rho_1(t) = w(\sqrt{t})/\sqrt{t}$, which is equivalent to $w(x) = |x|\,\rho_1(x^2)$, is the weight function corresponding to the orthogonal polynomials $\bar{Q}_n(x)$ on $[0, \alpha^2]$.

Similarly, if $n \to 2n+1$ and $m \to 2m+1$ in (23) then

$$\begin{aligned}\int_{-\alpha}^{\alpha} w(x)\bar{P}_{2n+1}(x)\,\bar{P}_{2m+1}(x)\,dx &= \int_{-\alpha}^{\alpha} x^2 w(x)\,\bar{R}_n(x^2)\,\bar{R}_m(x^2)\,dx = 2\int_{0}^{\alpha} x^2 w(x)\,\bar{R}_n(x^2)\,\bar{R}_m(x^2)\,dx \\ &= \int_{0}^{\alpha^2} \sqrt{t}\,w(\sqrt{t})\,\bar{R}_n(t)\,\bar{R}_m(t)\,dt = \left(-\prod_{i=1}^{2n+1} C^*_i \int_{-\alpha}^{\alpha} w(x)\,dx\right)\delta_{n,m}.\end{aligned} \qquad (25)$$

Therefore, $\rho_2(t)=\sqrt{t}\,w(\sqrt{t})$, which is equivalent to $w(x)=\rho_2(x^2)/|x|$, is the weight function corresponding to the orthogonal polynomials $\overline{R}_n(x)$ on $[0,\alpha^2]$.
We can now summarize our approach for generating the sequence of orthogonal polynomials whose moments with respect to the given weight function are given as follows:

Step 1. Suppose that a weight function $\rho(x)$ (not necessarily even) is given on $[0,\alpha^2]$ and note that by a simple linear change of variable all arbitrary closed intervals can be transformed to $[0,\alpha^2]$.

Step 2. Instead of considering $\rho(x)$ on $[0,\alpha^2]$ consider one of the two following even weight functions on $[-\alpha,\alpha]$, which we call the dual weight functions corresponding to $\rho(x)$:

$$\begin{cases} w_1(x)=|x|\,\rho(x^2), \\ w_{-1}(x)=|x|^{-1}\rho(x^2)=\rho(x^2)/|x|. \end{cases} \tag{26}$$

Step 3. Obtain the explicit values of the even moments related to (for example) $w_1(x)$ as

$$w_{2j}=\int_{-\alpha}^{\alpha}x^{2j}w_1(x)\,dx=\int_{-\alpha}^{\alpha}x^{2j}|x|\,\rho(x^2)\,dx=2\int_0^{\alpha}x^{2j+1}\,\rho(x^2)\,dx=\int_0^{\alpha^2}t^j\rho(t)\,dt\,,$$

and substitute them in (13) to explicitly compute the Hankel determinant.

Step 4. Substitute the obtained Hankel determinant in (15) to reach the recurrence relation of type (4).

Step 5. Obtain the explicit form of the orthogonal polynomials by (17) using the obtained recurrence relation.

Step 6. Obtain the orthogonality relation corresponding to (6).

Step 7. Follow the explained symmetrization process for the orthogonality relation obtained in step 5 by using the two orthogonality relations (24) and (25) and obtain the explicit forms of the polynomials $\overline{Q}_n(x)$ and $\overline{R}_n(x)$ using (18) and (17) as follows

$$\overline{Q}_n(x)=\sum_{k=0}^{[n/2]}\Phi_k(2n)\,x^{n-k} \qquad \text{and} \qquad \overline{R}_n(x)=\sum_{k=0}^{[n/2]}\Phi_k(2n+1)\,x^{n-k}\,.$$

2. Illustrative Examples

In this section we focus on three step-by-step examples to clarify the above-mentioned method.

Example 1. Suppose that the non-symmetric weight function $\rho(x)=\exp(-\sqrt{x})$ is given on $[0,\infty)$. Here, one of its dual weight functions is $w_1(x)=|x|\exp(-|x|)$ on $(-\infty,\infty)$ and the moments corresponding to it can be explicitly computed as

$$w_{2j+1}=0 \quad \text{and} \quad w_{2j}=\int_{-\infty}^{\infty} x^{2j}|x|\exp(-|x|)\,dx=2\int_{0}^{\infty} x^{2j+1}\exp(-x)\,dx=2(2j+1)!\,.$$

By substituting the above values in (13), the Hankel determinant for $n=2m$ and $n=2m+1$ respectively take the forms

$$A(m)=\Delta_{2m}=2^{2m+1}\begin{vmatrix} 1! & 0 & \dots & 0 & (2m+1)! \\ 0 & 3! & \dots & (2m+1)! & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & (2m+1)! & \dots & (4m-1)! & 0 \\ (2m+1)! & 0 & \dots & 0 & (4m+3)! \end{vmatrix},$$

and

$$B(m)=\Delta_{2m+1}=2^{2m+2}\begin{vmatrix} 1! & 0 & \dots & (2m+1)! & 0 \\ 0 & 3! & \dots & 0 & (2m+3)! \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (2m+1)! & 0 & \dots & (4m+1)! & 0 \\ 0 & (2m+3)! & \dots & 0 & (4m+3)! \end{vmatrix}.$$

According to (14) we have

$$\begin{cases} C_{2m}^{*}=-\dfrac{A(m)\,A(m-1)}{B^{2}(m-1)}, \\ C_{2m+1}^{*}=-\dfrac{B(m)\,B(m-1)}{A^{2}(m)}. \end{cases}$$

After computing the coefficients $\left\{C_j^*\right\}_{j=1}^{n}=\left\{C_j^*\left\{\,|x|\exp(-|x|)\right\}\right\}_{j=1}^{n}$ for this example, we encounter with a symmetric recurrence relation as

$$\bar{P}_{n+1}(x)=x\,\bar{P}_n(x)+C_n^*\left\{\,|x|\exp(-|x|)\right\}\bar{P}_{n-1}(x)\,,$$

satisfying the orthogonality relation

$$\int_{-\infty}^{\infty}|x|\exp(-|x|)\,\bar{P}_n(x)\,\bar{P}_m(x)\,dx=\left((-1)^n\prod_{j=1}^{n}C_j^*\left\{\,|x|\exp(-|x|)\right\}\int_{-\infty}^{\infty}|x|\exp(-|x|)\,dx\right)\delta_{n,m}. \qquad (27)$$

At this stage, step 7 should be employed by changing $n\to 2n$ and $m\to 2m$ in (27) to get

$$\int_{-\infty}^{\infty} |x|\exp(-|x|)\,\bar{P}_{2n}(x)\,\bar{P}_{2m}(x)\,dx = 2\int_{0}^{\infty} x e^{-x}\,\bar{Q}_n(x^2)\,\bar{Q}_m(x^2)\,dx = \int_{0}^{\infty} e^{-\sqrt{t}}\,\bar{Q}_n(t)\,\bar{Q}_m(t)\,dt$$
$$= 2\left(\prod_{i=1}^{2n} C_i^*\left\{|x|\exp(-|x|)\right\}\right)\delta_{n,m}.$$

The sequence $\bar{Q}_n(x)$ is the monic polynomial solution, which is orthogonal with respect to the initial weight function $\rho(x) = \exp(-\sqrt{x})$ on $[0,\infty)$ satisfying the non-symmetric recurrence equation

$$\bar{Q}_{m+1}(x) = \left(x + C_{2m+1}^*\{|x|\exp(-|x|)\} + C_{2m}^*\{|x|\exp(-|x|)\}\right)\bar{Q}_m(x)$$
$$- C_{2m}^*\{|x|\exp(-|x|)\}C_{2m-1}^*\{|x|\exp(-|x|)\}\bar{Q}_{m-1}(x).$$

Example 2. Suppose that the non-symmetric weight function $\rho(x;a,b,c) = x^a(1-x^b)^c$ is given for $a,c \in \mathbb{R}$ and $b>0$ on [0,1]. For $b=1$ it reduces to the well-known shifted Jacobi weight function [1,3]. According to (26), one of the dual weight functions corresponding to $\rho(x;a,b,c)$ is the symmetric measure $w_1(x;a,b,c) = |x|^{2a+1}(1-x^{2b})^c$ on $[-1,1]$. Therefore, the moments of $w_1(x;a,b,c)$ must be respectively computed as $w_{2j+1}=0$ and

$$\begin{aligned} w_{2j} &= \int_{-1}^{1} x^{2j}\,|x|^{2a+1}(1-x^{2b})^c\,dx = 2\int_0^1 x^{2j+2a+1}(1-x^{2b})^c\,dx \\ &= \frac{1}{b}\int_0^1 t^{\frac{j+a+1}{b}-1}(1-t)^c\,dt = \frac{1}{b}\mathbf{B}\left(\frac{j+a+1}{b};c+1\right), \end{aligned} \tag{28}$$

in which $\mathbf{B}(\lambda_1,\lambda_2)$ denotes the Beta integral [7] having various definitions as

$$\mathbf{B}(\lambda_1;\lambda_2) = \int_0^1 x^{\lambda_1-1}(1-x)^{\lambda_2-1}dx = \int_{-1}^{1} x^{2\lambda_1-1}(1-x^2)^{\lambda_2-1}dx = \int_0^\infty \frac{x^{\lambda_1-1}}{(1+x)^{\lambda_1+\lambda_2}}dx$$
$$= 2\int_0^{\pi/2} \sin^{(2\lambda_1-1)}x\,\cos^{(2\lambda_2-1)}x\,dx = \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda_1+\lambda_2)} = \mathbf{B}(\lambda_2;\lambda_1),$$

and

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx \qquad \mathrm{Re}(z)>0,$$

is the well-known Gamma function satisfying the fundamental recurrence relation $\Gamma(z+1) = z\Gamma(z)$.

By substituting (28) in the Hankel determinant, a symmetric recurrence relation of type (15) will be derived for the desired polynomials that satisfy the orthogonality relation

$$\int_{-1}^{1} |x|^{2a+1}(1-x^{2b})^c\,\bar{P}_n(x;a,b,c)\,\bar{P}_m(x;a,b,c)\,dx = \left((-1)^n \prod_{j=1}^{n} C_j^*\left\{|x|^{2a+1}(1-x^{2b})^c\right\}\int_{-1}^{1}|x|^{2a+1}(1-x^{2b})^c\,dx\right)\delta_{n,m}$$

$$= \left((-1)^n \frac{1}{b}\mathbf{B}(\frac{a+1}{b};c+1)\prod_{j=1}^{n} C_j^*\left\{|x|^{2a+1}(1-x^{2b})^c\right\}\right)\delta_{n,m}. \qquad (29)$$

At this stage, step 7 should be again employed by changing $n \to 2n$ and $m \to 2m$ in (29) to get

$$\int_{-1}^{1} |x|^{2a+1}(1-x^{2b})^c\,\bar{P}_{2n}(x;a,b,c)\,\bar{P}_{2m}(x;a,b,c)\,dx = 2\int_{0}^{1} x^{2a+1}(1-x^{2b})^c\,\bar{Q}_n(x^2;a,b,c)\,\bar{Q}_m(x^2;a,b,c)\,dx$$

$$= \int_{0}^{1} t^a(1-t^b)^c\,\bar{Q}_n(t;a,b,c)\,\bar{Q}_m(t;a,b,c)\,dt = \left(\frac{1}{b}\mathbf{B}(\frac{a+1}{b};c+1)\prod_{j=1}^{2n} C_j^*\left\{|x|^{2a+1}(1-x^{2b})^c\right\}\right)\delta_{n,m}.$$

Remark 1. It may be interesting to remark that for $b=1$ in (29), the subject is explicitly known, because according to [5] the generalized ultraspherical polynomials orthogonal with respect to the weight function $w_1(x;a-1/2,1,b) = x^{2a}(1-x^2)^b$ on $[-1,1]$ are in fact a particular case of a main class of symmetric orthogonal polynomials [5] defined by

$$S_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle|\, x\right) = \sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\left(\prod_{i=0}^{[n/2]-(k+1)}\frac{\left(2i+(-1)^{n+1}+2[n/2]\right)p+r}{\left(2i+(-1)^{n+1}+2\right)q+s}\right)x^{n-2k}, \qquad (30)$$

for $(p,q,r,s) = (-1,1,-2a-2b-2,2a)$, that satisfy a recurrence relation of symmetric type (5) as

$$\bar{S}_{n+1}(x) = x\,\bar{S}_n(x) + C_n\begin{pmatrix} r & s \\ p & q \end{pmatrix}\bar{S}_{n-1}(x) \quad ; \quad \bar{S}_0(x)=1,\ \bar{S}_1(x)=x,\ n\in\mathbb{N},$$

where

$$C_n\begin{pmatrix} r & s \\ p & q \end{pmatrix} = \frac{pqn^2 + \left((r-2p)q-(-1)^n ps\right)n + (r-2p)\,s(1-(-1)^n)/2}{(2pn+r-p)(2pn+r-3p)},$$

and

$$\bar{S}_n(x) = \bar{S}_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle|\, x\right) = \prod_{i=0}^{[n/2]-1}\frac{\left(2i+(-1)^{n+1}+2\right)q+s}{\left(2i+(-1)^{n+1}+2[n/2]\right)p+r}\,S_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle|\, x\right).$$

Moreover, the weight function corresponding to symmetric polynomials (30) is defined by [5,6]:

$$W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle|\, x\right) = \exp\left(\int \frac{(r-2p)x^2+s}{x(px^2+q)}dx\right).$$

Consequently we have

$$\int_{-1}^{1} x^{2a}(1-x^2)^b\, \bar{S}_n\begin{pmatrix} -2a-2b-2, & 2a \\ -1, & 1 \end{pmatrix}\Bigg| x\Bigg)\, \bar{S}_m\begin{pmatrix} -2a-2b-2, & 2a \\ -1, & 1 \end{pmatrix}\Bigg| x\Bigg)\, dx$$

$$= \left((-1)^n \prod_{i=1}^{n} C_i \begin{pmatrix} -2a-2b-2, & 2a \\ -1, & 1 \end{pmatrix} \int_{-1}^{1} x^{2a}(1-x^2)^b\, dx \right) \delta_{n,m},$$

in which

$$C_n^* = C_n\begin{pmatrix} -2a-2b-2, & 2a \\ -1, & 1 \end{pmatrix} = \frac{-n^2-(2b+2(1-(-1)^n)a)n-2a(a+b)(1-(-1)^n)}{(2n+2a+2b-1)(2n+2a+2b+1)}$$

$$= \frac{-(n+(1-(-1)^n)a)(n+(1-(-1)^n)a+2b)}{(2n+2a+2b-1)(2n+2a+2b+1)},$$

and

$$\int_{-1}^{1} x^{2a}(1-x^2)^b\, dx = B(a+\frac{1}{2}, b+1) = \frac{\Gamma(a+1/2)\Gamma(b+1)}{\Gamma(a+b+3/2)} \; .$$

In general, there are four classes of symmetric orthogonal polynomials that can be derived from the main sequence (30). Two of them are infinitely orthogonal (namely the generalized ultraspherical polynomials and generalized Hermite polynomials [4,5]) and two other ones, which are less known [5,6], are finitely orthogonal as the following table shows.

Table 1: Four particular classes of $S_n(p,q,r,s;x)$

Definition	Weight function	Interval and Kind
$S_n\begin{pmatrix} -2a-2b-2, & 2a \\ -1, & 1 \end{pmatrix}\Big\| x\Big)$	$W\begin{pmatrix} -2a-2b-2, & 2a \\ -1, & 1 \end{pmatrix}\Big\| x\Big) = x^{2a}(1-x^2)^b$	$[-1,1]$, Infinite
$S_n\begin{pmatrix} -2, & 2a \\ 0, & 1 \end{pmatrix}\Big\| x\Big)$	$W\begin{pmatrix} -2, & 2a \\ 0, & 1 \end{pmatrix}\Big\| x\Big) = x^{2a}\exp(-x^2)$	$(-\infty,\infty)$, Infinite
$S_n\begin{pmatrix} -2a-2b+2, & -2a \\ 1, & 1 \end{pmatrix}\Big\| x\Big)$	$W\begin{pmatrix} -2a-2b+2, & -2a \\ 1, & 1 \end{pmatrix}\Big\| x\Big) = \frac{x^{-2a}}{(1+x^2)^b}$	$(-\infty,\infty)$, Finite
$S_n\begin{pmatrix} -2a+2, & 2 \\ 1, & 0 \end{pmatrix}\Big\| x\Big)$	$W\begin{pmatrix} -2a+2, & 2 \\ 1, & 0 \end{pmatrix}\Big\| x\Big) = x^{-2a}\exp(-1/x^2)$	$(-\infty,\infty)$, Finite

Example 3. Let the non-symmetric weight function $\rho(x;a,b) = x^a \exp(-x^b)$ be given for $b \neq 0$ and $a > 0$ on $[0,\infty)$. For $b = 1$, it reduces to the well-known Laguerre weight function [3]. However, by considering $w_1(x;a,b) = |x|^{2a+1}\exp(-x^{2b})$ on $(-\infty,\infty)$ as one of the dual weight functions corresponding to $\rho(x;a,b)$, the moments of $w_1(x;a,b)$ are computed respectively as $w_{2j+1} = 0$ and

$$w_{2j} = \int_{-\infty}^{\infty} x^{2j} |x|^{2a+1} \exp(-x^{2b})\, dx = 2\int_0^{\infty} x^{2j+2a+1} \exp(-x^{2b})\, dx$$
$$= \frac{1}{b}\int_0^{\infty} t^{\frac{j+a+1}{b}-1} \exp(-t)\, dt = \frac{1}{b}\Gamma(\frac{j+a+1}{b}).$$

By substituting the above values in the Hankel determinant, eventually a symmetric relation of type (15) will be derived for the desired polynomials satisfying

$$\int_{-\infty}^{\infty} |x|^{2a+1} \exp(-x^{2b})\, \bar{P}_n(x;a,b)\, \bar{P}_m(x;a,b)\, dx$$
$$= \left((-1)^n \prod_{j=1}^{n} C_j^* \left\{ |x|^{2a+1} \exp(-x^{2b}) \right\} \int_{-\infty}^{\infty} |x|^{2a+1} \exp(-x^{2b})\, dx \right) \delta_{n,m}$$
$$= \left((-1)^n \frac{1}{b}\Gamma(\frac{a+1}{b}) \prod_{j=1}^{n} C_j^* \left\{ |x|^{2a+1} \exp(-x^{2b}) \right\} \right) \delta_{n,m}. \tag{31}$$

Again, step 7 should be here employed by changing $n \to 2n$ and $m \to 2m$ in (31) to get

$$\int_{-\infty}^{\infty} |x|^{2a+1} \exp(-x^{2b})\, \bar{P}_{2n}(x;a,b)\, \bar{P}_{2m}(x;a,b)\, dx = 2\int_0^{\infty} x^{2a+1} \exp(-x^{2b})\, \bar{Q}_n(x^2;a,b)\, \bar{Q}_m(x^2;a,b)\, dx$$
$$= \int_0^{\infty} t^a \exp(-t^b)\, \bar{Q}_n(t;a,b)\, \bar{Q}_m(t;a,b)\, dt = \left(\frac{1}{b}\Gamma(\frac{a+1}{b}) \prod_{j=1}^{2n} C_j^* \left\{ |x|^{2a+1} \exp(-x^{2b}) \right\} \right) \delta_{n,m}.$$

Remark 2. Similarly to the previous remark, note that for the case $b=1$, the subject is explicitly known, because it generates the generalized Hermite polynomials which are orthogonal with respect to the weight function $w_1(x; a-1/2, 1) = x^{2a} \exp(-x^2)$ on $(-\infty,\infty)$, and are a particular case of the main orthogonal polynomials (30) for $(p,q,r,s) = (0,1,-2,2a)$. In other words, we have

$$\int_{-\infty}^{\infty} x^{2a} e^{-x^2}\, \bar{S}_n\left(\begin{matrix} -2 & 2a \\ 0 & 1 \end{matrix} \middle| x\right) \bar{S}_m\left(\begin{matrix} -2 & 2a \\ 0 & 1 \end{matrix} \middle| x\right) dx = \left((-1)^n \prod_{i=1}^{n} C_i \begin{pmatrix} -2 & 2a \\ 0 & 1 \end{pmatrix} \right) \Gamma(a+\frac{1}{2})\ \delta_{n,m},$$

in which

$$C_n^* = C_n \begin{pmatrix} -2 & 2a \\ 0 & 1 \end{pmatrix} = -\frac{1}{2}n - \frac{1-(-1)^n}{2}a.$$

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