Symbolic Computation of Some Power-Trigonometric Series

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Abstract

Let $f^*(z) = \sum_{j=0}^{\infty} a_j^* z^j$ be a convergent series in which $\{a_j^*\}_{j=0}^{\infty}$ are known real numbers. In this paper, by referring to Osler's lemma [8], we obtain explicit forms of the two bivariate series

$$\sum_{j=0}^{\infty} a_{n\,j+m}^* \, r^j \cos(\alpha + j)\theta \quad \textit{and} \quad \sum_{j=0}^{\infty} a_{n\,j+m}^* \, r^j \sin(\alpha + j)\theta,$$

where r, θ are real variables, $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$ and $m \in \{0, 1, \ldots, n-1\}$. With some illustrative examples, we also show how to obtain the explicit form of a trigonometric series when $f^*(z)$ is explicitly given. Three new integral formulae are derived in this direction.

Key words: Bivariate series of power-trigonometric type, trigonometric series, power series, convergence radius.

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1. Introduction

Let

$$f^*(z) = \sum_{j=0}^{\infty} a_j^* \, z^j, \tag{1}$$

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be a series convergent in a specific region, say $|z| < A^*$ and $\{a_j^*\}_{j=0}^{\infty}$ are known real numbers. According to Abel's theorem [2, p. 7], the convergence of two bivariate series [6]

$$C_\alpha(f^*; r, \theta, m, n) = \sum_{j=0}^{\infty} a^*_{n\,j+m}\, r^j \cos(\alpha + j)\theta, \tag{2}$$

and

$$S_\alpha(f^*; r, \theta, m, n) = \sum_{j=0}^{\infty} a^*_{n\,j+m}\, r^j \sin(\alpha + j)\theta, \tag{3}$$

in which r, θ are real variables, $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$ and $m \in \{0, 1, \ldots, n-1\}$, directly depends on the convergence of the reduced series $\sum\limits_{j=0}^{\infty} a^*_{n\,j+m}\, r^j$, because after applying the ratio test for the series (2) and (3), the values

$$\lim_{j\to\infty} \frac{\cos(\alpha + j + 1)\theta}{\cos(\alpha + j)\theta} \quad \textit{and} \quad \lim_{j\to\infty} \frac{\sin(\alpha + j + 1)\theta}{\sin(\alpha + j)\theta},$$

are not generally known. Moreover, if $r \in [-1, 1]$ and $\sum\limits_{j=0}^{\infty} \left|a^*_{n\,j+m}\right|$ is bounded, then

$$|C_\alpha(f^*; r, \theta, m, n)| \le \sum_{j=0}^{\infty} \left|a^*_{n\,j+m}\right| \quad \textit{and} \quad |S_\alpha(f^*; r, \theta, m, n)| \le \sum_{j=0}^{\infty} \left|a^*_{n\,j+m}\right|.$$

For the series (1), Osler proved the following lemma [8].

Lemma. *Let* $n \in \mathbb{N}$, $m \in \{0, 1, n-1\}$ *and* $f^*(z) = \sum\limits_{j=0}^{\infty} a_j^*\, z^j$ *be a convergent series in* $|z| < A^*$. *Then we have*

$$\sum_{j=0}^{\infty} a^*_{n\,j+m}\, z^{n\,j+m} = \frac{1}{n} \sum_{k=0}^{n-1} e^{-\frac{2m\pi i}{n}\,k} f^*(e^{\frac{2\pi i}{n}\,k} z), \tag{4}$$

which can be rewritten as

$$\sum_{j=0}^{\infty} a^*_{n\,j+m}\, z^j = \frac{1}{n} z^{-\frac{m}{n}} \sum_{k=0}^{n-1} e^{-\frac{2m\pi i}{n}\,k} f^*(e^{\frac{2\pi i}{n}\,k} z^{\frac{1}{n}}).$$

In order to compute the two series (2) and (3) via Osler's lemma, suppose in (1) that

$$z = x + iy = r\, e^{i\theta} \qquad (i = \sqrt{-1}),$$

is a complex variable so that

$$r=\sqrt{x^2+y^2},\quad \theta=\arctan\frac{y}{x}\quad and\quad x,y\in\mathbb{R},y>0.$$

It can be directly concluded from (1) that

$$\operatorname{Im}\left(z^\lambda f^*(z)+\bar{z}^\lambda f^*(\bar{z})\right)=\operatorname{Im}(\frac{z^\lambda f^*(z)-\bar{z}^\lambda f^*(\bar{z})}{i})=0\,,\quad \forall\lambda\in\mathbb{R}\,. \tag{5}$$

In other words, $z^\lambda f^*(z)+\bar{z}^\lambda f^*(\bar{z})$ and $(z^\lambda f^*(z)-\bar{z}^\lambda f^*(\bar{z}))/i$ are always two real functions if (1) holds. Now by applying this result to Osler's identity and doing some computations, we can obtain explicit forms of two real bivariate series of types (2) and (3) as

$$\begin{aligned}&C_0(f^*;r,\theta,m,n)=\sum_{j=0}^{\infty}a^*_{n\,j+m}\,r^j\cos j\theta\\&=\tfrac{1}{2n}r^{-\frac{m}{n}}\sum_{k=0}^{n-1}\Big(e^{-\frac{m}{n}(2k\pi+\theta)\,i}f^*(r^{\frac{1}{n}}e^{\frac{(2k\pi+\theta)\,i}{n}})+e^{-\frac{m}{n}(2k\pi-\theta)\,i}f^*(r^{\frac{1}{n}}e^{\frac{(2k\pi-\theta)\,i}{n}})\Big)\,,\end{aligned} \tag{6}$$

and

$$\begin{aligned}&S_0(f^*;r,\theta,m,n)=\sum_{j=0}^{\infty}a^*_{n\,j+m}\,r^j\sin j\theta\\&=\tfrac{1}{2ni}r^{-\frac{m}{n}}\sum_{k=0}^{n-1}\Big(e^{-\frac{m}{n}(2k\pi+\theta)\,i}f^*(r^{\frac{1}{n}}e^{\frac{(2k\pi+\theta)\,i}{n}})-e^{-\frac{m}{n}(2k\pi-\theta)\,i}f^*(r^{\frac{1}{n}}e^{\frac{(2k\pi-\theta)\,i}{n}})\Big)\,.\end{aligned} \tag{7}$$

For example, substituting $n=2,3$ in relations (6) and (7), respectively, yields

$$\begin{aligned}
&4C_0(f^*;r,\theta,0,2)=f^*(r^{\frac{1}{2}}e^{\frac{\theta}{2}i})+f^*(r^{\frac{1}{2}}e^{-\frac{\theta}{2}i})+f^*(-r^{\frac{1}{2}}e^{\frac{\theta}{2}i})+f^*(-r^{\frac{1}{2}}e^{-\frac{\theta}{2}i}),\\
&4i\,S_0(f^*;r,\theta,0,2)=f^*(r^{\frac{1}{2}}e^{\frac{\theta}{2}i})-f^*(r^{\frac{1}{2}}e^{-\frac{\theta}{2}i})+f^*(-r^{\frac{1}{2}}e^{\frac{\theta}{2}i})-f^*(-r^{\frac{1}{2}}e^{-\frac{\theta}{2}i})\,,\\
&4r^{\frac{1}{2}}C_0(f^*;r,\theta,1,2)=e^{-\frac{\theta}{2}i}f^*(r^{\frac{1}{2}}e^{\frac{\theta}{2}i})+e^{\frac{\theta}{2}i}f^*(r^{\frac{1}{2}}e^{-\frac{\theta}{2}i})-e^{-\frac{\theta}{2}i}f^*(-r^{\frac{1}{2}}e^{\frac{\theta}{2}i})-e^{\frac{\theta}{2}i}f^*(-r^{\frac{1}{2}}e^{-\frac{\theta}{2}i}),\\
&4ir^{\frac{1}{2}}S_0(f^*;r,\theta,1,2)=e^{-\frac{\theta}{2}i}f^*(r^{\frac{1}{2}}e^{\frac{\theta}{2}i})-e^{\frac{\theta}{2}i}f^*(r^{\frac{1}{2}}e^{-\frac{\theta}{2}i})-e^{-\frac{\theta}{2}i}f^*(-r^{\frac{1}{2}}e^{\frac{\theta}{2}i})+e^{\frac{\theta}{2}i}f^*(-r^{\frac{1}{2}}e^{-\frac{\theta}{2}i}),\\
&6\,C_0(f^*;r,\theta,0,3)=f^*(r^{\frac{1}{3}}e^{\frac{\theta}{3}i})+f^*(r^{\frac{1}{3}}e^{-\frac{\theta}{3}i})+f^*(r^{\frac{1}{3}}e^{\frac{\theta+2\pi}{3}i})+f^*(r^{\frac{1}{3}}e^{-\frac{\theta-2\pi}{3}i})\\
&\qquad+f^*(r^{\frac{1}{3}}e^{\frac{\theta+4\pi}{3}i})+f^*(r^{\frac{1}{3}}e^{-\frac{\theta-4\pi}{3}i})\,,\\
&6i\,S_0(f^*;r,\theta,0,3)=f^*(r^{\frac{1}{3}}e^{\frac{\theta}{3}i})-f^*(r^{\frac{1}{3}}e^{-\frac{\theta}{3}i})+f^*(r^{\frac{1}{3}}e^{\frac{\theta+2\pi}{3}i})-f^*(r^{\frac{1}{3}}e^{-\frac{\theta-2\pi}{3}i})\\
&\qquad+f^*(r^{\frac{1}{3}}e^{\frac{\theta+4\pi}{3}i})-f^*(r^{\frac{1}{3}}e^{-\frac{\theta-4\pi}{3}i})\,,\\
&6\,r^{\frac{1}{3}}\,C_0(f^*;r,\theta,1,3)=e^{-\frac{\theta}{3}i}f^*(r^{\frac{1}{3}}e^{\frac{\theta}{3}i})+e^{\frac{\theta}{3}i}f^*(r^{\frac{1}{3}}e^{-\frac{\theta}{3}i})+e^{-\frac{\theta+2\pi}{3}i}f^*(r^{\frac{1}{3}}e^{\frac{\theta+2\pi}{3}i})\\
&\qquad+e^{\frac{\theta-2\pi}{3}i}f^*(r^{\frac{1}{3}}e^{-\frac{\theta-2\pi}{3}i})+e^{-\frac{\theta+4\pi}{3}i}f^*(r^{\frac{1}{3}}e^{\frac{\theta+4\pi}{3}i})+e^{\frac{\theta-4\pi}{3}i}f^*(r^{\frac{1}{3}}e^{-\frac{\theta-4\pi}{3}i})\,,
\end{aligned} \tag{8}$$

$$
\begin{aligned}
6i\, r^{\frac{1}{3}}\, S_0(f^*; r, \theta, 1, 3) &= e^{-\frac{\theta}{3} i} f^*(r^{\frac{1}{3}} e^{\frac{\theta}{3} i}) - e^{\frac{\theta}{3} i} f^*(r^{\frac{1}{3}} e^{-\frac{\theta}{3} i}) + e^{-\frac{\theta+2\pi}{3} i} f^*(r^{\frac{1}{3}} e^{\frac{\theta+2\pi}{3} i}) \\
&\quad - e^{\frac{\theta-2\pi}{3} i} f^*(r^{\frac{1}{3}} e^{-\frac{\theta-2\pi}{3} i}) + e^{-\frac{\theta+4\pi}{3} i} f^*(r^{\frac{1}{3}} e^{\frac{\theta+4\pi}{3} i}) - e^{\frac{\theta-4\pi}{3} i} f^*(r^{\frac{1}{3}} e^{-\frac{\theta-4\pi}{3} i}) ,
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
6\, r^{\frac{2}{3}}\, C_0(f^*; r, \theta, 2, 3) &= e^{-\frac{2\theta}{3} i} f^*(r^{\frac{1}{3}} e^{\frac{\theta}{3} i}) + e^{\frac{2\theta}{3} i} f^*(r^{\frac{1}{3}} e^{-\frac{\theta}{3} i}) + e^{-\frac{2\theta+4\pi}{3} i} f^*(r^{\frac{1}{3}} e^{\frac{\theta+2\pi}{3} i}) \\
&\quad + e^{\frac{2\theta-4\pi}{3} i} f^*(r^{\frac{1}{3}} e^{-\frac{\theta-2\pi}{3} i}) + e^{-\frac{2\theta+8\pi}{3} i} f^*(r^{\frac{1}{3}} e^{\frac{\theta+4\pi}{3} i}) + e^{\frac{2\theta-8\pi}{3} i} f^*(r^{\frac{1}{3}} e^{-\frac{\theta-4\pi}{3} i}) ,
\end{aligned}
$$

$$
\begin{aligned}
6i\, r^{\frac{2}{3}}\, S_0(f^*; r, \theta, 2, 3) &= e^{-\frac{2\theta}{3} i} f^*(r^{\frac{1}{3}} e^{\frac{\theta}{3} i}) - e^{\frac{2\theta}{3} i} f^*(r^{\frac{1}{3}} e^{-\frac{\theta}{3} i}) + e^{-\frac{2\theta+4\pi}{3} i} f^*(r^{\frac{1}{3}} e^{\frac{\theta+2\pi}{3} i}) \\
&\quad - e^{\frac{2\theta-4\pi}{3} i} f^*(r^{\frac{1}{3}} e^{-\frac{\theta-2\pi}{3} i}) + e^{-\frac{2\theta+8\pi}{3} i} f^*(r^{\frac{1}{3}} e^{\frac{\theta+4\pi}{3} i}) - e^{\frac{2\theta-8\pi}{3} i} f^*(r^{\frac{1}{3}} e^{-\frac{\theta-4\pi}{3} i}) .
\end{aligned}
$$

On the other hand, since

$$
C_\alpha(f^*; r, \theta, m, n) = \cos\alpha\theta\, C_0(f^*; r, \theta, m, n) - \sin\alpha\theta\, S_0(f^*; r, \theta, m, n) , \tag{10}
$$

and

$$
S_\alpha(f^*; r, \theta, m, n) = \sin\alpha\theta\, C_0(f^*; r, \theta, m, n) + \cos\alpha\theta\, S_0(f^*; r, \theta, m, n) , \tag{11}
$$

one can extend the results (6) and (7) as follows

$$
\begin{aligned}
C_\alpha(f^*; r, \theta, m, n) &= \sum_{j=0}^{\infty} a^*_{n\,j+m}\, r^j \cos(\alpha + j)\theta \\
&= \frac{1}{2n} r^{-\frac{m}{n}} \sum_{k=0}^{n-1} \Big(e^{(\alpha\theta - \frac{m}{n}(2k\pi+\theta))\, i} f^*(r^{\frac{1}{n}} e^{\frac{(2k\pi+\theta)\, i}{n}}) + e^{-(\alpha\theta + \frac{m}{n}(2k\pi-\theta))\, i} f^*(r^{\frac{1}{n}} e^{\frac{(2k\pi-\theta)\, i}{n}}) \Big) ,
\end{aligned} \tag{12}
$$

and

$$
\begin{aligned}
S_\alpha(f^*; r, \theta, m, n) &= \sum_{j=0}^{\infty} a^*_{n\,j+m}\, r^j \sin(\alpha + j)\theta \\
&= \frac{1}{2ni} r^{-\frac{m}{n}} \sum_{k=0}^{n-1} \Big(e^{(\alpha\theta - \frac{m}{n}(2k\pi+\theta))\, i} f^*(r^{\frac{1}{n}} e^{\frac{(2k\pi+\theta)\, i}{n}}) - e^{-(\alpha\theta + \frac{m}{n}(2k\pi-\theta))\, i} f^*(r^{\frac{1}{n}} e^{\frac{(2k\pi-\theta)\, i}{n}}) \Big) .
\end{aligned} \tag{13}
$$

It is clear that the extensions (12) and (13) are valid only in the convergence region of r for any $\alpha \in \mathbb{R}$ and $\theta \in [-\pi, \pi]$. Moreover, the initial condition $\lim_{j\to\infty} a^*_{n\,j+m} r^j = 0$ must be always satisfied in order to automatically have

$$
\lim_{j\to\infty} a^*_{n\,j+m} r^j \cos(\alpha + j)\theta = 0 \qquad \textit{and} \qquad \lim_{j\to\infty} a^*_{n\,j+m} r^j \sin(\alpha + j)\theta = 0.
$$

Relations (10) and (11) show that for computing

$$
\sum_{j=0}^{\infty} a^*_{n\,j+m}\, r^j \cos(\alpha + j)\theta \qquad \textit{and} \qquad \sum_{j=0}^{\infty} a^*_{n\,j+m}\, r^j \sin(\alpha + j)\theta,
$$

we should just compute $C_0(f; r, \theta, m, n)$ and $S_0(f; r, \theta, m, n)$.

Example 1. Assume that $f^*(z) = -\frac{\ln(1-z)}{z} = \sum_{j=0}^{\infty} \frac{z^j}{j+1}$. Hence $a^*_{nj+m} = \frac{1}{nj+m+1}$ and the two bivariate series

$$C_\alpha(-\frac{\ln(1-z)}{z}; r, \theta, m, n) = \sum_{j=0}^{\infty} \frac{r^j}{nj+m+1} \cos(\alpha+j)\theta,$$

and

$$S_\alpha(-\frac{\ln(1-z)}{z}; r, \theta, m, n) = \sum_{j=0}^{\infty} \frac{r^j}{nj+m+1} \sin(\alpha+j)\theta,$$

can be explicitly computed via the expansion of the given function. For instance, to compute

$$\sum_{j=0}^{\infty} \frac{r^{3j+2}}{3j+2} \cos(3j+2)\theta = r^2 C_{2/3}(-\frac{\ln(1-z)}{z}; r^3, 3\theta, 1, 3),$$

we first have from (10), (8) and (9) that

$$C_{2/3}(f^*; r^3, 3\theta, 1, 3) = \cos 2\theta\, C_0(f^*; r^3, 3\theta, 1, 3) - \sin 2\theta\, S_0(f^*; r^3, 3\theta, 1, 3),$$

in which

$$\begin{aligned} 6\,r\,C_0(f^*; r^3, 3\theta, 1, 3) = {} & e^{-\theta i} f^*(re^{\theta i}) + e^{\theta i} f^*(re^{-\theta i}) + e^{-(\theta+\frac{2\pi}{3})i} f^*(re^{(\theta+\frac{2\pi}{3})i}) \\ & + e^{(\theta-\frac{2\pi}{3})i} f^*(re^{-(\theta-\frac{2\pi}{3})i}) + e^{-(\theta+\frac{4\pi}{3})i} f^*(re^{(\theta+\frac{4\pi}{3})i}) \\ & + e^{(\theta-\frac{4\pi}{3})i} f^*(re^{-(\theta-\frac{4\pi}{3})i}), \end{aligned} \tag{14}$$

and

$$\begin{aligned} 6i\,r\,S_0(f^*; r^3, 3\theta, 1, 3) = {} & e^{-\theta i} f^*(re^{\theta i}) - e^{\theta i} f^*(re^{-\theta i}) + e^{-(\theta+\frac{2\pi}{3})i} f^*(re^{(\theta+\frac{2\pi}{3})i}) \\ & - e^{(\theta-\frac{2\pi}{3})i} f^*(re^{-(\theta-\frac{2\pi}{3})i}) + e^{-(\theta+\frac{4\pi}{3})i} f^*(re^{(\theta+\frac{4\pi}{3})i}) \\ & - e^{(\theta-\frac{4\pi}{3})i} f^*(re^{-(\theta-\frac{4\pi}{3})i}). \end{aligned} \tag{15}$$

By noting the identity

$$\begin{aligned} -\frac{\ln(1-x-iy)}{x+iy} &= f^*(x+iy) \\ = \frac{-\frac{x}{2}\ln\left((1-x)^2+y^2\right) + y \arctan\frac{y}{1-x}}{x^2+y^2} &+ i\,\frac{x\arctan\frac{y}{1-x} + \frac{y}{2}\ln\left((1-x)^2+y^2\right)}{x^2+y^2}, \end{aligned} \tag{16}$$

the aforesaid series can be finally computed as

$$\sum_{j=0}^{\infty} \frac{r^{3j+2}}{3j+2} \cos(3j+2)\theta = \\ -\frac{1}{12} \ln \frac{(1+r^2-2r\cos\theta)^2}{4r^2\cos^2\theta + 2r(r^2+1)\cos\theta + r^4 - r^2 + 1} - \frac{\sqrt{3}}{6} \arctan \frac{\sqrt{3}\,(r^2+2\,r\cos\theta)}{-r^2+2r\cos\theta+2}, \tag{17}$$

which is valid for any $|r| < 1$ and $\theta \in [-\pi, \pi]$.

Here the important question is whether one can directly obtain the result (17) without using the relations (14) and (15)? In other words, how to compute the explicit form of a trigonometric series when $f^*(z)$ is given? We explain the solution of this problem by some particular examples.

Example 2. To directly compute the two series

$$\sum_{j=0}^{\infty} \frac{r^j}{j+1} \cos(\alpha+j)\theta \qquad \textit{and} \qquad \sum_{j=0}^{\infty} \frac{r^j}{j+1} \sin(\alpha+j)\theta,$$

by using identity (16) and the fact that $\sum\limits_{j=0}^{\infty} \frac{z^j}{j+1} = -\frac{\ln(1-z)}{z}$ we have

$$\begin{aligned} &C_\alpha(-\tfrac{\ln(1-z)}{z}; r, \theta, 0, 1) = \sum_{j=0}^{\infty} \tfrac{r^j}{j+1} \cos(\alpha+j)\theta \\ &= \tfrac{1}{2}\cos\alpha\theta \sum_{j=0}^{\infty} \tfrac{(re^{i\theta})^j + (re^{-i\theta})^j}{j+1} - \tfrac{1}{2i}\sin\alpha\theta \sum_{j=0}^{\infty} \tfrac{(re^{i\theta})^j - (re^{-i\theta})^j}{j+1} \\ &= -\frac{1}{2}\cos\alpha\theta \left(\frac{\ln(1-re^{i\theta})}{re^{i\theta}} + \frac{\ln(1-re^{-i\theta})}{re^{-i\theta}} \right) - \frac{1}{2i}\sin\alpha\theta \left(-\frac{\ln(1-re^{i\theta})}{re^{i\theta}} + \frac{\ln(1-re^{-i\theta})}{re^{-i\theta}} \right) \\ &= -\frac{\sin(\alpha-1)\theta}{r} \arctan(\frac{r\sin\theta}{1-r\cos\theta}) - \frac{\cos(\alpha-1)\theta}{2r} \ln\left(1+r^2-2r\cos\theta\right), \end{aligned} \tag{18}$$

as well as

$$\begin{aligned} &S_\alpha(-\tfrac{\ln(1-z)}{z}; r, \theta, 0, 1) = \sum_{j=0}^{\infty} \tfrac{r^j}{j+1} \sin(\alpha+j)\theta \\ &= \tfrac{1}{2}\sin\alpha\theta \sum_{j=0}^{\infty} \tfrac{(re^{i\theta})^j + (re^{-i\theta})^j}{j+1} + \tfrac{1}{2i}\cos\alpha\theta \sum_{j=0}^{\infty} \tfrac{(re^{i\theta})^j - (re^{-i\theta})^j}{j+1} \\ &= -\frac{1}{2}\sin\alpha\theta \left(\frac{\ln(1-re^{i\theta})}{re^{i\theta}} + \frac{\ln(1-re^{-i\theta})}{re^{-i\theta}} \right) + \frac{1}{2i}\cos\alpha\theta \left(-\frac{\ln(1-re^{i\theta})}{re^{i\theta}} + \frac{\ln(1-re^{-i\theta})}{re^{-i\theta}} \right) \\ &= \frac{\cos(\alpha-1)\theta}{r} \arctan(\frac{r\sin\theta}{1-r\cos\theta}) - \frac{\sin(\alpha-1)\theta}{2r} \ln\left(1+r^2-2r\cos\theta\right). \end{aligned} \tag{19}$$

Example 3. To directly compute the two series

$$\sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} r^{2j} \cos j\theta \qquad \textit{and} \qquad \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} r^{2j} \sin j\theta,$$

by using equality $\sum\limits_{j=0}^{\infty} \frac{(-z^2)^j}{2j+1} = \frac{\arctan z}{z}$ and the fact that

$$\frac{\arctan(x+iy)}{x+iy} = \frac{1}{2(x^2+y^2)} \left(\frac{y}{2}\, \ln \frac{x^2+(1+y)^2}{x^2+(1-y)^2} + x \arctan \frac{2x}{1-x^2-y^2} \right) + i\, \frac{1}{2(x^2+y^2)} \left(\frac{x}{2}\, \ln \frac{x^2+(1+y)^2}{x^2+(1-y)^2} - y \arctan \frac{2x}{1-x^2-y^2} \right),$$

we have

$$\begin{aligned} &C_0(\tfrac{\arctan z}{z}; r, \theta, 0, 1) = \sum_{j=0}^{\infty} \tfrac{(-1)^j}{2j+1} r^{2j} \cos j\theta \\ &= \frac{1}{2} \sum_{j=0}^{\infty} \frac{\left(-(re^{i\theta/2})^2\right)^j + \left(-(re^{-i\theta/2})^2\right)^j}{2j+1} = \frac{1}{2} \left(\frac{\arctan(re^{i\theta/2})}{re^{i\theta/2}} + \frac{\arctan(re^{-i\theta/2})}{re^{-i\theta/2}} \right) \\ &= \frac{1}{2r} \left(\frac{\sin(\theta/2)}{2}\, \ln \frac{1+r^2+2r\sin(\theta/2)}{1+r^2-2r\sin(\theta/2)} + \cos(\theta/2) \arctan \frac{2r\cos(\theta/2)}{1-r^2} \right), \end{aligned}$$

as well as

$$\begin{aligned} &S_0(\tfrac{\arctan z}{z}; r, \theta, 0, 1) = \sum_{j=0}^{\infty} \tfrac{(-1)^j}{2j+1} r^{2j} \sin j\theta \\ &= \frac{1}{2i} \sum_{j=0}^{\infty} \frac{\left(-(re^{i\theta/2})^2\right)^j - \left(-(re^{-i\theta/2})^2\right)^j}{2j+1} = \frac{1}{2i} \left(\frac{\arctan(re^{i\theta/2})}{re^{i\theta/2}} - \frac{\arctan(re^{-i\theta/2})}{re^{-i\theta/2}} \right) \\ &= \frac{1}{2r} \left(\frac{\cos(\theta/2)}{2}\, \ln \frac{1+r^2+2r\sin(\theta/2)}{1+r^2-2r\sin(\theta/2)} - \sin(\theta/2) \arctan \frac{2r\cos(\theta/2)}{1-r^2} \right). \end{aligned}$$

Example 4. Obtain the series (17) directly. For this purpose, one should first apply Osler's lemma (4) to obtain the explicit form of $\sum\limits_{j=0}^{\infty} \frac{z^{3j+2}}{3j+2}$ via the equality $-\ln(1-z) = \sum\limits_{j=0}^{\infty} \frac{z^{j+1}}{j+1}$ and then apply the technique used in examples 2 and 3. In other words, first via Osler's lemma we obtain

$$\sum_{j=0}^{\infty} \frac{z^{3j+2}}{3j+2} = -\frac{1}{6} \ln \frac{(1-z)^2}{z^2+z+1} - \frac{\sqrt{3}}{3} \arctan \frac{\sqrt{3}\, z}{z+2}, \quad |z|<1. \tag{20}$$

Then, by applying the explained computational technique to (20), we have

$$\begin{aligned} &\sum_{j=0}^{\infty} \tfrac{r^{3j+2}}{3j+2} \cos(3j+2)\theta = \tfrac{1}{2} \sum_{j=0}^{\infty} \tfrac{(re^{i\theta})^{3j+2} + (re^{-i\theta})^{3j+2}}{3j+2} \\ &= -\frac{1}{12} \ln \frac{(1-re^{i\theta})^2}{r^2e^{2i\theta}+re^{i\theta}+1} - \frac{\sqrt{3}}{6} \arctan \frac{\sqrt{3}\, re^{i\theta}}{re^{i\theta}+2} - \frac{1}{12} \ln \frac{(1-re^{-i\theta})^2}{r^2e^{-2i\theta}+re^{-i\theta}+1} - \frac{\sqrt{3}}{6} \arctan \frac{\sqrt{3}\, re^{-i\theta}}{re^{-i\theta}+2} \quad (21) \\ &= -\frac{1}{12} \ln \frac{(1+r^2-2r\cos\theta)^2}{4r^2\cos^2\theta + 2r(r^2+1)\cos\theta + r^4 - r^2 + 1} - \frac{\sqrt{3}}{6} \arctan \frac{\sqrt{3}\,(r^2+2\,r\cos\theta)}{-r^2+2r\cos\theta+2}, \end{aligned}$$

which is an extension of the series (20) for $\theta = 0$ and $r = z$, because

$$\frac{1}{2}\arctan\frac{\sqrt{3}\,(z^2+2\,z)}{-z^2+2z+2} = \arctan\frac{\sqrt{3}\,z}{z+2}.$$

Note that the extended series (21) converges for any $|r| < 1$ and $\theta \in [-\pi, \pi]$.

Remark. All series given in examples 1 to 4 can be also represented by the generalized hypergeometric series [1,7]

$${}_pF_q\left(\begin{array}{c} a_1,\ a_2,\ \ldots,\ a_p \\ b_1,\ b_2,\ \ldots,\ b_q \end{array}\middle|\, z\right) = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k}{(b_1)_k(b_2)_k\cdots(b_q)_k}\,\frac{z^k}{k!},$$

in which $(r)_k = \prod\limits_{j=0}^{k-1}(r+j)$ and z may be a complex variable. For instance, series (20) can be represented as

$$\sum_{j=0}^{\infty}\frac{z^{3j+2}}{3j+2} = \frac{1}{2}z^2\,{}_2F_1\left(\begin{array}{c} 2/3,\ 1 \\ 5/3 \end{array}\middle|\, z^3\right).$$

2. Three new integral formulae derived by relations (12) and (13)

It is well known that $\{\cos(\alpha+j)\theta\}_{j=0}^{\infty}$ and $\{\sin(\alpha+j)\theta\}_{j=0}^{\infty}$ constitute two biorthogonal sequences [5] for $\alpha = 0$ and $\alpha = 1/2$ on $[-\pi, \pi]$ so that we have

$$\int_{-\pi}^{\pi}\cos j\theta\cos k\theta\,d\theta = 2\int_{0}^{\pi}\cos j\theta\cos k\theta\,d\theta = \pi\,\delta_{j,k} = \begin{cases} 0 & (j \neq k), \\ \pi & (j = k), \end{cases} \tag{22}$$

$$\int_{-\pi}^{\pi}\sin j\theta\sin k\theta\,d\theta = 2\int_{0}^{\pi}\sin j\theta\sin k\theta\,d\theta = \pi\,\delta_{j,k},$$

$$\int_{-\pi}^{\pi}\sin j\theta\cos k\theta\,d\theta = 0, \quad \forall\, j,k \in \mathbb{Z}\,,$$

and

$$\int_{-\pi}^{\pi}\cos(j+\tfrac{1}{2})\theta\cos(k+\tfrac{1}{2})\theta\,d\theta = 2\int_{0}^{\pi}\cos(j+\tfrac{1}{2})\theta\cos(k+\tfrac{1}{2})\theta\,d\theta = \pi\,\delta_{j,k},$$

$$\int_{-\pi}^{\pi}\sin(j+\tfrac{1}{2})\theta\sin(k+\tfrac{1}{2})\theta\,d\theta = 2\int_{0}^{\pi}\sin(j+\tfrac{1}{2})\theta\sin(k+\tfrac{1}{2})\theta\,d\theta = \pi\,\delta_{j,k}, \tag{23}$$

$$\int_{-\pi}^{\pi}\sin(j+\tfrac{1}{2})\theta\cos(k+\tfrac{1}{2})\theta\,d\theta = 0, \quad \forall\, j,k \in \mathbb{Z}\,,$$

where $[-\pi, \pi]$ can be changed to any other arbitrary interval of length 2π by a simple linear transform. By referring to the relations (12) and (13), now define the finite sequence

$$T_l(\theta) = \sum_{j=0}^{l} a^*_{n\,j+m}\, r^j \begin{Bmatrix} \cos(\alpha+j)\theta \\ \sin(\alpha+j)\theta \end{Bmatrix},$$

and assume that there exists a real function $t(\theta)$ (independent of l) such that $|T_l(\theta)| \leq t(\theta)$. Since $C_\alpha(f^*; r, \theta, m, n)$ and $S_\alpha(f^*; r, \theta, m, n)$ are respectively even and odd functions in θ, by using the dominated convergence theorem [1,7] and applying biorthogonality relations (22) and (23) we can respectively find the following formulae

$$\int_{-\pi}^{\pi} C_\alpha(f^*; r, \theta, m, n)\, \cos k\theta\, d\theta = \int_{-\pi}^{\pi} \left(\sum_{j=0}^{\infty} a^*_{n\,j+m} r^j \cos(\alpha+j)\theta \right) \cos k\theta\, d\theta$$
$$= \sum_{j=0}^{\infty} a^*_{n\,j+m} r^j \left(\int_{-\pi}^{\pi} \cos(\alpha+j)\theta \cos k\theta\, d\theta \right) = \pi\, a^*_{n\,(k-\alpha)+m}\, r^{k-\alpha} \quad (k-\alpha \in \mathbb{N} \cup \{0\}),$$

$$\int_{0}^{\pi} C_0(f^*; r, \theta, m, n)\, d\theta = \int_{0}^{\pi} \left(\sum_{j=0}^{\infty} a^*_{n\,j+m} r^j \cos j\theta \right) d\theta = \sum_{j=0}^{\infty} a^*_{n\,j+m} r^j \left(\int_{0}^{\pi} \cos j\theta d\theta \right) = \pi\, a^*_m,$$

and

$$\int_{-\pi}^{\pi} S_\alpha(f^*; r, \theta, m, n)\, \sin k\theta\, d\theta = \int_{-\pi}^{\pi} \left(\sum_{j=0}^{\infty} a^*_{n\,j+m} r^j \sin(\alpha+j)\theta \right) \sin k\theta\, d\theta$$
$$= \sum_{j=0}^{\infty} a^*_{n\,j+m} r^j \left(\int_{-\pi}^{\pi} \sin(\alpha+j)\theta \sin k\theta\, d\theta \right) = \pi\, a^*_{n\,(k-\alpha)+m}\, r^{k-\alpha} \quad (k-\alpha \in \mathbb{N} \cup \{0\}).$$

Similarly, corresponding to relations (23) we have

$$\int_{-\pi}^{\pi} C_{1/2}(f^*; r, \theta, m, n)\, \cos(k+\frac{1}{2})\theta\, d\theta = \int_{-\pi}^{\pi} \left(\sum_{j=0}^{\infty} a^*_{n\,j+m} r^j \cos(j+\frac{1}{2})\theta \right) \cos(k+\frac{1}{2})\theta\, d\theta$$
$$= \sum_{j=0}^{\infty} a^*_{n\,j+m} r^j \left(\int_{-\pi}^{\pi} \cos(j+\frac{1}{2})\theta \cos(k+\frac{1}{2})\theta\, d\theta \right) = \pi\, a^*_{n\,k+m}\, r^k \quad (k \in \mathbb{N} \cup \{0\}),$$

and

$$\int_{-\pi}^{\pi} S_{1/2}(f^*;r,\theta,m,n)\,\sin(k+\frac{1}{2})\theta\,d\theta = \int_{-\pi}^{\pi}\left(\sum_{j=0}^{\infty} a^*_{n\,j+m} r^j \sin(j+\frac{1}{2})\theta\right)\sin(k+\frac{1}{2})\theta\,d\theta$$
$$= \sum_{j=0}^{\infty} a^*_{n\,j+m} r^j \left(\int_{-\pi}^{\pi} \sin(j+\frac{1}{2})\theta \sin(k+\frac{1}{2})\theta\,d\theta\right) = \pi\, a^*_{n\,k+m}\, r^k \quad (k \in \mathbb{N}\cup\{0\}).$$

Corollary. *If $k \in \mathbb{N}\cup\{0\}$ and $j \in \mathbb{Z}\cup\{1/2\}$ except the case $k = j = 0$, then three following formulae are valid in the convergence region of r:*

$$\int_0^{\pi} C_j(f^*;r,\theta,m,n)\,\cos(k+j)\theta\,d\theta = \frac{\pi}{2}\, a^*_{n\,k+m}\, r^k,$$
$$\int_0^{\pi} C_0(f^*;r,\theta,m,n)\,d\theta = \pi\, a^*_m,$$
$$\int_0^{\pi} S_j(f^*;r,\theta,m,n)\,\sin(k+j)\theta\,d\theta = \frac{\pi}{2} a^*_{n\,k+m}\, r^k\,.$$

Example 5. By noting the results (18) and (19), replacing them in the above corollary yields

$$\int_0^{\pi}\left(\frac{\sin(j-1)\theta}{r}\arctan\left(\frac{r\sin\theta}{1-r\cos\theta}\right)+\frac{\cos(j-1)\theta}{2r}\ln\left(1+r^2-2r\cos\theta\right)\right)\cos(k+j)\theta\,d\theta = -\frac{\pi}{2}\,\frac{r^k}{k+1}\,,$$
$$\int_0^{\pi}\left(\frac{\sin\theta}{r}\arctan\left(\frac{r\sin\theta}{1-r\cos\theta}\right)-\frac{\cos\theta}{2r}\ln\left(1+r^2-2r\cos\theta\right)\right)d\theta = \pi,$$

and

$$\int_0^{\pi}\left(\frac{\cos(j-1)\theta}{r}\arctan\left(\frac{r\sin\theta}{1-r\cos\theta}\right)-\frac{\sin(j-1)\theta}{2r}\ln\left(1+r^2-2r\cos\theta\right)\right)\sin(k+j)\theta\,d\theta = \frac{\pi}{2}\,\frac{r^k}{k+1}\,,$$

which are valid for any $|r| < 1$, $\theta \in [0,\pi]$, $k \in \mathbb{N}\cup\{0\}$ and $j \in \mathbb{Z}\cup\{1/2\}$ except the case $k = j = 0$.

Remark. Many other series in mathematics can be extended via the approach presented in this paper. For instance, we can extend to trigonometric expressions

i) generalized hypergeometric series as in [6];
ii) all classical orthogonal polynomial expansions [in preparation];
iii) Laurent series and classical generating functions of polynomial families [in preparation].

3. Conclusion

We showed in this paper that if f^* in the series $f^*(z) = \sum_{j=0}^{\infty} a_j^* z^j$ is known, then two power trigonometric series

$$\sum_{j=0}^{\infty} a^*_{n\,j+m}\, r^j \cos(\alpha + j)\theta \qquad \textit{and} \qquad \sum_{j=0}^{\infty} a^*_{n\,j+m}\, r^j \sin(\alpha + j)\theta,$$

can be explicitly computed in which r, θ are real variables, $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$ and $m \in \{0, 1, \ldots, n-1\}$.

4. Appendix

The algorithms covered in this paper can be directly implemented in Maple. As an example, in this section we show how Example 4 can be derived.
We use the following two short programs implementing Osler's lemma (4) and the real part of it.

```
> nfold:=(f,z,n,shift)->1/n*add(exp(-2*Pi*I*shift/n*k)*
> subs(z=exp(k*I*2*Pi/n)*z,f),k=0..n-1):
> computeu:=proc(f,z,r,theta)
> local fpolar;
> fpolar:=subs(z=r*exp(I*theta),f);
> simplify(Re(evalc(fpolar))) assuming r>0,theta::real;
> end proc:
```

We start with the function

```
> f:=-log(1-z);
```

$$f := -\ln(1-z)$$

The FPS algorithm (see [3]) gives the representation

```
> convert(f,FormalPowerSeries);
```

$$\sum_{k=0}^{\infty} \frac{z^{(k+1)}}{k+1}$$

We compute the m-fold part for $m = 3$ and shift $= 2$, and get first the complex form

```
> res:=nfold(f,z,3,2);
```

$$\mathit{res} := -\frac{1}{3}\ln(1-z) - \frac{1}{3}\left(-\frac{1}{2} + \frac{1}{2} I \sqrt{3}\right) \ln\left(1 - \left(-\frac{1}{2} + \frac{1}{2} I \sqrt{3}\right) z\right) - \frac{1}{3}\left(-\frac{1}{2} - \frac{1}{2} I \sqrt{3}\right) \ln\left(1 - \left(-\frac{1}{2} - \frac{1}{2} I \sqrt{3}\right) z\right)$$

which we bring in real form

```
> res:=Re(evalc(res)) assuming z>0,z<1;
```

$$\mathit{res} := -\frac{1}{3}\ln(1-z) + \frac{1}{6}\ln\left(\left(1 + \frac{z}{2}\right)^2 + \frac{3z^2}{4}\right) - \frac{1}{3}\sqrt{3}\arctan\left(\frac{z\sqrt{3}}{2\left(1 + \frac{z}{2}\right)}\right)$$

The following computation using the FPS algorithm shows that this result is correct.

```
> simplify(convert(res,FormalPowerSeries));
```

$$\sum_{k=0}^{\infty} \frac{z^{(3\,k+2)}}{3\,k+2}$$

To obtain (17), we must just compute the real part:

```
> res1:=computeu(res,z,r,theta);
```

$$res1 := -\frac{1}{6}$$
$$\sqrt{3}\arctan\left(\frac{r\sqrt{3}\,(r+2\cos(\theta))}{\mathrm{r}^2+4\,\mathrm{r}\cos(\theta)+4}, -\frac{2\,r\sin(\theta)\sqrt{3}-4\,r\cos(\theta)-r^2-4}{\mathrm{r}^2+4\,\mathrm{r}\cos(\theta)+4}\right)$$
$$+\frac{1}{6}$$
$$\sqrt{3}\arctan\left(-\frac{r\sqrt{3}\,(r+2\cos(\theta))}{\mathrm{r}^2+4\,\mathrm{r}\cos(\theta)+4}, \frac{2\,r\sin(\theta)\sqrt{3}+4\,r\cos(\theta)+r^2+4}{\%1}\right)$$
$$-\frac{1}{6}\ln(1+r^2-2\,r\cos(\theta))$$
$$+\frac{1}{12}\ln(4\,r^2\cos(\theta)^2+2\,r^3\cos(\theta)+r^4+2\,r\cos(\theta)-r^2+1)$$

Here is our term (17):

```
> term17:=-1/12*log((1+r^2-2*r*cos(theta))^2/(4*r^2*cos(theta)^2+
> 2*r*(r^2+1)*cos(theta)+r^4-r^2+1))-
> sqrt(3)/6*arctan(sqrt(3)*(r^2+2*r*cos(theta))/(-r^2+2*r*cos(theta)+2));
```

$$term17 := -\frac{1}{12}\ln\left(\frac{(1+r^2-2\,r\cos(\theta))^2}{4\,r^2\cos(\theta)^2+2\,r\,(r^2+1)\cos(\theta)+r^4-r^2+1}\right)$$
$$-\frac{1}{6}\sqrt{3}\arctan\left(\frac{\sqrt{3}\,(r^2+2\,r\cos(\theta))}{-r^2+2\,r\cos(\theta)+2}\right)$$

Unfortunately it is not easy to prove that the two terms agree.

```
> simplify(res1-term17) assuming r>0,r<1,theta::real;
```

$$-\frac{1}{6}\sqrt{3}\arctan(r\sqrt{3}\,(r+2\cos(\theta)),\ -2\,r\sin(\theta)\sqrt{3}+4\,r\cos(\theta)+r^2+4)+\frac{1}{6}$$
$$\sqrt{3}\arctan(-r\sqrt{3}\,(r+2\cos(\theta)),\ 2\,r\sin(\theta)\sqrt{3}+4\,r\cos(\theta)+r^2+4)$$
$$-\frac{1}{6}\ln(1+r^2-2\,r\cos(\theta))$$
$$+\frac{1}{12}\ln(4\,r^2\cos(\theta)^2+2\,r^3\cos(\theta)+r^4+2\,r\cos(\theta)-r^2+1)$$
$$+\frac{1}{12}\ln((2\,r\cos(\theta)-r^2-1)^2)$$
$$+\frac{1}{12}\ln\left(\frac{1}{4\,r^2\cos(\theta)^2+2\,r^3\cos(\theta)+r^4+2\,r\cos(\theta)-r^2+1}\right)$$
$$+\frac{1}{6}\sqrt{3}\arctan\left(\frac{\sqrt{3}\,r\,(r+2\cos(\theta))}{-r^2+2\,r\cos(\theta)+2}\right)$$

However, computing the derivative w. r. t. r of the difference of the two terms, we easily get a proof:

```
> simplify(diff(res1-term17,r));
```

$$0$$

Computing an initial value finished the proof:

```
> simplify(eval(res1-term17,r=0));
```

$$0$$

This is also recognized by the FPS algorithm.

```
> convert(res1-term17,FormalPowerSeries,r);
```

$$0$$

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