On structure formulas for Wilson polynomials

P. Njionou Sadjang[*], W. Koepf[†] [‡], and M. Foupouagnigni[§] [¶]

Abstract

By studying various properties of some divided difference operators, we prove that Wilson polynomials are solutions of a second order difference equation of hypergeometric type. Next, some new structure relations are deduced, the inversion and the connection problems are solved using an algorithmic method.

Keywords: Wilson polynomials, Quadratic lattices, Difference equations, Hypergeometric representation, Structure relation, Inversion formula, Connection coefficients.

AMS Subject Classification (2010): 33C20; 33C25; 42C05.

We define the difference operator $\mathbf{D}$ (see [1, 2, 3, 4]) and its companion operator $\mathbf{S}$ as follows:

$$\mathbf{D}f(x) = \frac{f(x+\frac{i}{2}) - f(x-\frac{i}{2})}{2ix} \qquad \mathbf{S}f(x) = \frac{f(x+\frac{i}{2}) + f(x-\frac{i}{2})}{2}.$$

The operator $\mathbf{D}$ transforms a polynomial of degree n ($n \geq 1$) in x^2 into a polynomial of degree $n-1$ in x^2 and a polynomial of degree 0 into the zero polynomial. The operator $\mathbf{S}$ transforms a polynomial of degree n in x^2 into a polynomial of degree n in x^2. We introduce the basis $\vartheta_n(a,x) = (a-ix, a+ix)_n$.

Here, the symbol $(a)_n$ denotes the so-called Pochhammer symbol which is defined by

$$(a)_m = \begin{cases} 1 \quad \text{if} \quad m = 0 \\ a(a+1)\dots(a+m-1) \quad \text{if} \quad m = 1, 2, \dots \end{cases}$$

and $(a_1, a_2, \dots, a_p)_n = (a_1)_n (a_2)_n \cdots (a_p)_n$.

By studying various properties of $\mathbf{D}$ and $\mathbf{S}$, we prove, following previous works ([5, 6, 7]), that the Wilson polynomials defined in this paper by (2) (and the Continuous Dual Hahn polynomials as special case of Wilson polynomials defined by (3)) are solutions of the divided difference equation

$$\phi(x^2)\mathbf{D}^2 y(x^2) + \psi(x^2)\mathbf{SD}y(x^2) + \lambda y(x^2) = 0, \tag{1}$$

where ϕ and ψ are polynomials of degree 2 and 1, respectively, and λ is a constant depending on the degree of the polynomial solution and the four parameters a, b, c and d

[*]Faculty of Industrial Engineering, University of Douala, Douala, Cameroon; [b]Institute of Mathematics

[†]Institute of Mathematics, University of Kassel, Heinrich-Plett Str. 40, 34132 Kassel, Germany

[‡]Corresponding author. Email: koepf@mathematik.uni-kassel.de

[§]African Institute for Mathematical Sciences, Limbé, Cameroon

[¶]Department of Mathematics, Higher Teachers' Training College, University of Yaounde I, Cameroon

and are given in (19)-(21). Here, it should be mentioned that the Wilson and Continuous Dual Hahn polynomials are defined respectively as (see [8]):

$$\frac{W_n(x^2;a,b,c,d)}{(a+b,a+c,a+d)_n} = {}_4F_3\left(\begin{array}{c} -n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{array}\middle| 1\right), \tag{2}$$

$$\frac{S_n(x^2;a,b,c)}{(a+b,a+c)_n} = {}_3F_2\left(\begin{array}{c} -n, a-ix, a+ix \\ a+b, a+c \end{array}\middle| 1\right). \tag{3}$$

As consequences of (1), we derive the three-term recurrence relation and the first structure relation. Next, inversion and connection problems are solved for those families. It should be noted that the *connection problem* is the problem of finding the coefficients $C_m(n)$ in the expansion

$$P_n(x) = \sum_{m=0}^{n} C_m(n) Q_m(x).$$

Note that, in this setting, the polynomials P_n and Q_m may belong to two different polynomial families. If $P_n(x) = \mathcal{V}_n(x)$ (where $\mathcal{V}_n(x)$ appears in the expansion $Q_n(x) = \sum\limits_{m=0}^{n} B_m(n)\mathcal{V}_m(x)$), we are faced with the so-called *inversion problem* for the family $Q_m(x)$.

The paper is organized as follows:

1. In Section 2, we present some identities for the operators **D** and **S** and the basis $\vartheta_n(a,x)$. Next, using these identities, we state the second order divided difference equation satisfied by the Wilson polynomials. Also we recover the coefficients of the three-term recurrence and give some structural relations for the Wilson polynomials. Similar results are deduced for the Continuous Dual Hahn polynomials by a limiting process.

2. In Section 3, we prove by solving the divided difference equation stated in Section 2, using the algorithm described in [9, 10], that we can obtain the hypergeometric representation of the Wilson polynomials. This clarifies that the Wilson polynomials can also be defined by this divided difference equation. Next, by an algorithmic method, we solve the inversion and the connection problems for the Wilson and the Continuous Dual Hahn polynomials. Note that some results of this section can also be obtained by limit considerations. However, we present here a selfcontained method to obtain these formulas.

To the best of our knowledge, the identities (7)-(11), (18), (26), (29), (31), (39), (42) appear here for the first time. The other formulas are already given in the literature and are proved here using an algorithmic method.

1 Difference equation and structure relations

In order to establish the divided difference equation satisfied by the Wilson polynomials, we state the following results.

1.1 Some miscellaneous results

Proposition 1. *The basis $\vartheta_n(a,x)$ fulfills the following relations*

$$x^2\vartheta_n(a,x) = \vartheta_{n+1}(a,x) + \nu(a,n)\vartheta_n(a,x), \tag{4}$$

$$\vartheta_1(a,x)\vartheta_n(a+1,x) = \vartheta_{n+1}(a,x), \tag{5}$$

$$\vartheta_1(a,x)\vartheta_n(a,x) = \vartheta_{n+1}(a,x) + \mu(a,n)\vartheta_n(a,x), \tag{6}$$

$$\mathbf{D}\vartheta_n(a,x) = n\vartheta_{n-1}\left(a+\frac{1}{2},x\right), \tag{7}$$

$$\mathbf{D}^{\ell}\vartheta_n(a,x) = \frac{n!}{(n-l)!}\vartheta_{n-\ell}\left(a+\frac{\ell}{2},x\right), \, 0 \le \ell \le n; \tag{8}$$

$$\vartheta_1(a,x)\ \mathbf{D}^2\vartheta_n(a,x) = \delta(n)\vartheta_{n-1}(a,x), \tag{9}$$

$$\mathbf{S}\vartheta_n(a,x) = \vartheta_n(a+\frac{1}{2},x) + \varepsilon_0(a,n)\vartheta_{n-1}(a+\frac{1}{2},x), \tag{10}$$

$$\vartheta_1(a,x)\mathbf{S}\mathbf{D}\vartheta_n(a,x) = n\vartheta_n(a,x) + \varepsilon_1(a,n)\vartheta_{n-1}(a,x) \tag{11}$$

where $\delta(n) = n(n-1)$, $\varepsilon_0(a,n) = -n\left(n+a-\frac{1}{2}\right)$, $\nu(a,n) = -(a+n)^2$, $\mu(a,n) = -(n^2+2an)$, $\varepsilon_1(a,n) = -n(n-1)(n+a-1)$.

Proof. The proof is obtained by direct computation. □

Proposition 2. *The operators* **D** *and* **S** *satisfy the following product rules*

$$\mathbf{D}(fg) = \mathbf{D}f\mathbf{S}g + \mathbf{S}f\mathbf{D}g, \tag{12}$$

$$\mathbf{S}(fg) = -x^2\mathbf{D}f\mathbf{D}g + \mathbf{S}f\mathbf{S}g, \tag{13}$$

$$\mathbf{DS} = \mathbf{SD} - \frac{1}{2}\mathbf{D}^2, \tag{14}$$

$$\mathbf{S}^2 = -x^2\mathbf{D}^2 - \frac{1}{2}\mathbf{SD} + \boldsymbol{I}, \tag{15}$$

where $\boldsymbol{I}f = f$.

Proof. By using the definition of the operators **D** and **S**, the proof follows. □

Proposition 3. *The following relation is valid*

$$\mathbf{D}W_n(x^2;a,b,c,d) = -n(n+a+b+c+d-1)W_{n-1}\left(x^2;a+\frac{1}{2},b+\frac{1}{2},c+\frac{1}{2},d+\frac{1}{2}\right). \tag{16}$$

Proof. Using the relation (7) of Proposition 1 and the fact that $(-n)_{m+1} = -n(-n+1)_m$, the proof follows. We would like to mention that an equivalent form of (16) is given in [11] without using the operator **D**. □

Proposition 4 (see [8], P. 187)**.** *The following relation is valid.*

$$\begin{aligned} &\mathbf{D}[\omega(x;a,b,c,d)W_n(x^2;a,b,c,d)] \\ = \;&\omega\left(x;a-\frac{1}{2},b-\frac{1}{2},c-\frac{1}{2},d-\frac{1}{2}\right)W_{n+1}\left(x^2;a-\frac{1}{2},b-\frac{1}{2},c-\frac{1}{2},d-\frac{1}{2}\right). \end{aligned} \tag{17}$$

where

$$\omega(x;a,b,c,d) = \frac{f(a,x)f(b,x)f(c,x)f(d,x)}{2ix\Gamma(2ix)\Gamma(-2ix)}$$

with $f(r,x) = \Gamma(r+ix)\Gamma(r-ix)$, *for* $r \in \{a,b,c,d\}$.

1.2 The second order difference equation

Theorem 5. *The Wilson polynomials are solutions of the second order difference equation*

$$\phi(x^2)\mathbf{D}^2 y(x) + \psi(x^2)\mathbf{S}\mathbf{D}y(x) + \lambda_n y(x) = 0, \tag{18}$$

where

$$\phi(x^2) = x^4 - (ab + ac + bc + cd + ad + bd)\, x^2 + abcd, \tag{19}$$

$$\psi(x^2) = (a + b + c + d)x^2 - (acd + bad + bca + bcd) \tag{20}$$

and

$$\lambda_n = -n(n + a + b + c + d - 1). \tag{21}$$

Proof. First combine (16) and (17) to get the relation

$$\begin{aligned}
&\mathbf{D}\left[\omega\left(x; a+\frac{1}{2}, b+\frac{1}{2}, c+\frac{1}{2}, d+\frac{1}{2}\right)\mathbf{D}W_n(x^2; a, b, c, d)\right] \\
&= -n(n+a+b+c+d-1)\mathbf{D}\left[\omega\left(x; a+\frac{1}{2}, b+\frac{1}{2}, c+\frac{1}{2}, d+\frac{1}{2}\right)\right. \\
&\qquad\qquad \left.\times W_{n-1}\left(x^2; a+\frac{1}{2}, b+\frac{1}{2}, c+\frac{1}{2}, d+\frac{1}{2}\right)\right] \\
&= -n(n+a+b+c+d-1)\omega(x; a, b, c, d)W_n(x^2; a, b, c, d).
\end{aligned}$$

Next, use the property (12) to write the left-hand side as

$$\begin{aligned}
&\mathbf{D}\left[\omega\left(x; a+\frac{1}{2}, b+\frac{1}{2}, c+\frac{1}{2}, d+\frac{1}{2}\right)\mathbf{D}W_n(x^2; a, b, c, d)\right] \\
&= \mathbf{S}\omega\left(x; a+\frac{1}{2}, b+\frac{1}{2}, c+\frac{1}{2}, d+\frac{1}{2}\right)\mathbf{D}^2W_n(x^2; a, b, c, d) \\
&\quad + \mathbf{D}\omega\left(x; a+\frac{1}{2}, b+\frac{1}{2}, c+\frac{1}{2}, d+\frac{1}{2}\right)\mathbf{S}\mathbf{D}W_n(x^2; a, b, c, d) \\
&= -n(n+a+b+c+d-1)\omega(x; a, b, c, d)W_n(x^2; a, b, c, d).
\end{aligned}$$

Since

$$\frac{\mathbf{S}\omega\left(x; a+\frac{1}{2}, b+\frac{1}{2}, c+\frac{1}{2}, d+\frac{1}{2}\right)}{\omega(x; a, b, c, d)} = -\phi(x^2), \quad \text{and} \quad \frac{\mathbf{D}\omega\left(x; a+\frac{1}{2}, b+\frac{1}{2}, c+\frac{1}{2}, d+\frac{1}{2}\right)}{\omega(x; a, b, c, d)} = -\psi(x^2),$$

we have proved (18). □

Remark 6. *It should be noted that (18) is equivalent to the well-known difference equation (see [8], page 187)*

$$n(n + a + b + c + d - 1)y(x) = B(x)y(x + i) - [B(x) + D(x)]y(x) + D(x)y(x - i),$$

where

$$y(x) = W_n(x^2; a, b, c, d)$$

and

$$\begin{cases} B(x) = \dfrac{(a - ix)(b - ix)(c - ix)(d - ix)}{2ix(2ix - 1)} \\[2ex] D(x) = \dfrac{(a + ix)(b + ix)(c + ix)(d + ix)}{2ix(2ix + 1)} \end{cases}$$

The form (18) is closer to the second order differential equation satisfied by the classical continuous orthogonal polynomials. Therefore, following similar methods to the classical case many interesting structure relations for the Wilson polynomials are recovered.

1.3 Three-term recurrence relation

In order to recover the three-term recurrence relation satisfied by the Wilson polynomials, we adapt the algorithm given in [10]. For this purpose, we expand $P_n(x)$ (where $P_n(x) = W_n(x^2; a, b, c, d)$) in the basis $\vartheta_n(a, x)$ as:

$$P_n(x) = k_n \vartheta_n(a, x) + k'_n \vartheta_{n-1}(a, x) + k''_n \vartheta_{n-2}(a, x) + \dots . \tag{22}$$

and then we substitute it in the difference equation (1). Next we multiply the obtained equation by $\vartheta_1(a, x)$ and use Proposition 1.
Equating the coefficients of $\vartheta_{n+1}(a, x)$ gives

$$\lambda_n = -\psi_1 n - \phi_2 n\,(n-1) = -n(n + a + b + c + d - 1).$$

Equating the coefficients of $\vartheta_n(a, x)$ gives

$$k'_n = -\frac{k_n n\,(n-1+a+d)\,(n-1+a+c)\,(n-1+a+b)}{2\,n-2+d+c+b+a} \tag{23}$$

and equating the coefficients of $\vartheta_{n-1}(a, x)$ gives

$$\begin{aligned} k''_n = {} & k_n n\,(n-1)\,(n-1+a+d)\,(n-2+a+d) \\ & \times \frac{(n-1+a+c)\,(n+a+c-2)\,(n-1+a+b)\,(n-2+a+b)}{2\,(2\,n-2+d+c+b+a)\,(2\,n-3+d+c+b+a)} \end{aligned} \tag{24}$$

Proposition 7. *The Wilson polynomials satisfy the three-term recurrence relation*

$$\begin{aligned} & -\vartheta_1(a, x) W_n(x^2; a, b, c, d) \\ & \quad = a_n W_{n+1}(x^2; a, b, c, d) + b_n W_n(x^2; a, b, c, d) + c_n W_{n-1}(x^2; a, b, c, d), \end{aligned} \tag{25}$$

with

$$\begin{aligned} a_n = {} & \frac{k_n}{k_{n+1}} \\ b_n = {} & \Big(a^4 + 6\,n^2a^2 + 4\,an^3 - 5\,n^2d - 5\,n^2c - 5\,n^2b + 4\,n^3d + 4\,n^3c + 4\,n^3b \\ & -2\,cab - 2\,a^3 - 2\,da^2 + 2\,da^3 + 2\,ca^3 + 2\,ba^3 + an + 2\,n^2d^2 + 2\,n^2c^2 \\ & +2\,n^2b^2 + 6\,ndab + 6\,ncab - 4\,dbn + 6\,n^2da + 2\,n^2 + d^2a^2 + c^2a^2 + a^2b^2 \\ & +6\,ndcb + 6\,ndca + 6\,dcn^2 + 6\,n^2ca - 4\,cbn + 6\,n^2ab - 4\,dcn + 6\,dbn^2 \\ & +6\,cbn^2 + 6\,nda^2 + 6\,nca^2 + 6\,nba^2 - 4\,n^3 + 2\,n^4 + 4\,dcab - 2\,dcb - 2\,dca \\ & -2\,dab - 2\,ca^2 - 2\,ba^2 - 5\,an^2 - 5\,na^2 + nb + nc + nd + d^2cb + d^2ca + dc^2b \\ & +dc^2a + 2\,dc^2n + 2\,d^2cn + db^2a + d^2ba + db^2c + 2\,db^2n + 2\,d^2bn + 2\,d^2an \\ & +cb^2a + c^2ba + 2\,cb^2n + 2\,c^2bn + 2\,c^2an + 2\,ab^2n - 4\,nab - 4\,nca - 4\,nda \\ & +3\,dca^2 + 3\,da^2b + 3\,ca^2b + 4\,na^3 - nb^2 - nd^2 - nc^2\Big)\Big/ \\ & (2\,n-2+d+c+b+a)\,(2\,n+d+c+b+a) \\ c_n = {} & \frac{k_n}{k_{n-1}}\,\frac{(n-1+c+d)\,(n-1+b+d)\,(n-1+b+c)}{(2\,n-1+d+c+b+a)} \\ & \times \frac{(b+d-2+c+a+n)\,n\,(n-1+a+d)\,(n-1+a+c)\,(n-1+a+b)}{(2\,n-3+d+c+b+a)\,(2\,n-2+d+c+b+a)^2} \end{aligned}$$

Proof. In order to get those coefficients, we substitute the expression of $W_n(x^2; a, b, c, d)$ given by (22) in (25). Next we multiply both sides of the resulting equation by $\vartheta_1(a, x)$ and use the relation (6) to get simplifications. Next equating the coefficients of $\vartheta_{n+1}(a, x)$, we get a_n, equating the coefficients of $\vartheta_n(a, x)$ we get b_n and equating the coefficients of $\vartheta_{n-1}(a, x)$, we get c_n. $\square$

1.4 Further structure relations

The approaches used to prove Proposition 8, Theorem 9, and Proposition 10 are similar to those used in [6].

Proposition 8. *Let f be a function satisfying (1). Then, the function $\mathbf{D}f$ is solution of the equation*

$$\phi^{(1)}(x)\mathbf{D}^2 y(x) + \psi^{(1)}(x)\mathbf{S}\mathbf{D}y(x) + \lambda^{(1)} y(x) = 0, \tag{26}$$

with

$$\phi^{(1)} = \mathbf{S}\phi - x^2\mathbf{D}\psi - \frac{1}{2}\mathbf{S}\psi, \quad \psi^{(1)} = \mathbf{D}\phi + \mathbf{S}\phi + \frac{1}{2}\mathbf{D}\psi, \quad \lambda^{(1)} = \mathbf{D}\psi + \lambda.$$

Proof. We apply the operator $\mathbf{D}$ to the difference equation (1) and use the relations (12), (13), (14) and (15) to obtain the result. $\square$

A computation shows that

$$\phi^{(1)}(x) = \phi_2^{(1)} x^4 + \phi_1^{(1)} x^2 + \phi_0^{(1)}, \quad \psi^{(1)}(x) = \psi_1^{(1)} x^2 + \psi_0^{(1)}, \quad \lambda^{(1)} = \lambda + \psi_1 \tag{27}$$

with

$$\phi_2^{(1)} = \phi_2, \quad \phi_1^{(1)} = -\frac{3}{2}\phi_2 - \frac{3}{2}\psi_1 + \phi_1, \quad \phi_0^{(1)} = -\frac{1}{4}\phi_1 + \phi_0 + \frac{1}{16}\phi_2 + \frac{1}{8}\psi_1 - \frac{1}{2}\psi_0$$

and

$$\psi^{(1)} = \psi_1 + 2\phi_2, \qquad \psi_0^{(1)} = \frac{1}{4}\psi_1 - \frac{1}{2}\phi_2 + \phi_1 + \psi_0.$$

From Proposition 8, we deduce that the equation

$$\begin{aligned} &-\vartheta_1(a, x)\mathbf{D}\bar{W}_n(x; a, b, c, d) \\ &\quad = \alpha_n^\star \mathbf{D}\bar{W}_{n+1}(x; a, b, c, d) + \beta_n^\star \mathbf{D}\bar{W}_n(x; a, b, c, d) + \gamma_n^\star \mathbf{D}\bar{W}_{n-1}(x; a, b, c, d), \quad n \geq 1 \end{aligned} \tag{28}$$

namely the recurrence relation for $\mathbf{D}\bar{W}_n(x; a, b, c, d)$ is valid and from (27), it follows that

$$\alpha_n^\star = a_n\left(\phi_2^{(1)}, \phi_1^{(1)}, \phi_0^{(1)}, \psi_1^{(1)}, \psi_0^{(1)}\right), \quad \beta_n^\star = b_n\left(\phi_2^{(1)}, \phi_1^{(1)}, \phi_0^{(1)}, \psi_1^{(1)}, \psi_0^{(1)}\right),$$
$$\text{and} \qquad \gamma_n^\star = c_n\left(\phi_2^{(1)}, \phi_1^{(1)}, \phi_0^{(1)}, \psi_1^{(1)}, \psi_0^{(1)}\right).$$

where $\alpha_n(\phi_2, \phi_1, \phi_0, \psi_1, \psi_0)$, $\beta_n(\phi_2, \phi_1, \phi_0, \psi_1, \psi_0)$ and $\gamma_n(\phi_2, \phi_1, \phi_0, \psi_1, \psi_0)$ are given by (25).

Theorem 9. *Assume that $P_n(x)$ is a solution of (1). Then a structure formula of the type*

$$\mathbf{S}P_n(x) = \hat{\alpha}_n \mathbf{D}P_{n+1}(x) + \hat{\beta}_n \mathbf{D}P_n(x) + \hat{\gamma}_n \mathbf{D}P_{n-1}(x) \quad (\hat{\alpha}_n, \hat{\beta}_n, \hat{\gamma}_n \in \mathbb{R} \text{ for } n \in \mathbb{N}) \tag{29}$$

is valid for $P_n(x)$. The coefficients $\hat{\alpha}_n$, $\hat{\beta}_n$ and $\hat{\gamma}_n$ are related to the coefficients a_n, b_n and c_n of (32) and the coefficients $\alpha_n^\star$, $\beta_n^\star$ and $\gamma_n^\star$ by

$$\hat{\alpha}_n = \alpha_n^\star - a_n, \quad \hat{\beta}_n = \beta_n^\star - b_n + \frac{1}{4}, \quad \hat{\gamma}_n = \gamma_n^\star - c_n. \tag{30}$$

Proof. First we remark that

$$\mathbf{D}(\vartheta_1(a,x)P_n(x)) = \mathbf{D}\vartheta_1(a,x)\mathbf{S}P_n + \mathbf{S}\vartheta_1(a,x)\mathbf{D}P_n(x) = \mathbf{S}P_n(x) + \left(\vartheta_1(a,x) - \frac{1}{4}\right)\mathbf{D}P_n(x).$$

Applying $\mathbf{D}$ to the recurrence equation (25), we get

$$-\left(\mathbf{S}P_n(x) + \left(\vartheta_1(a,x) - \frac{1}{4}\right)\mathbf{D}P_n(x)\right) = a_n\mathbf{D}P_{n+1}(x) + b_n\mathbf{D}P_n + c_n\mathbf{D}P_{n-1}.$$

Next we use (28) to obtain the result. □

Proposition 10. *Let f be a function satisfying (1), and $m \geq 1$ an integer, then the function $\mathbf{D}^m f$ is solution of the equation*

$$\phi^{(m)}(x)\mathbf{D}^2 y(x) + \psi^{(m)}(x)\mathbf{S}\mathbf{D}y(x) + \lambda^{(m)}y(x) = 0, \tag{31}$$

where $\phi^{(m+1)} = \mathbf{S}\phi^{(m)} - x^2\mathbf{D}\psi^{(m)} - \frac{1}{2}\mathbf{S}\psi^{(m)}$, $\psi^{(m+1)} = \mathbf{D}\phi^{(m)} + \mathbf{S}\phi^{(m)} + \frac{1}{2}\mathbf{D}\psi^{(m)}$, $\lambda^{(m+1)} = \mathbf{D}\psi^{(m)} + \lambda^{(m)}$, with $\phi^{(0)} = \phi$, $\psi^{(0)} = \psi$ and $\lambda^{(0)} = \lambda$.

Proof. We apply the operator $\mathbf{D}$ to the difference equation (1) and use the relations (12), (13), (14) and (15) to obtain the result. □

1.5 Three-term recurrence relation for $\mathbf{D}^2 W_n(x^2;a,b,c,d)$

In order to solve the inversion problem, we will need the following.

Proposition 11. *The second-order divided differences of the Wilson polynomials satisfy the following three-term recurrence relation*

$$\begin{aligned}\vartheta_1(a,x)&\mathbf{D}^2W_n(x^2;a,b,c,d)\\ &= a_n^\star\mathbf{D}^2W_{n+1}(x^2;a,b,c,d) + b_n^\star\mathbf{D}^2W_n(x^2;a,b,c,d) + c_n^\star\mathbf{D}^2W_{n-1}(x^2;a,b,c,d),\end{aligned} \tag{32}$$

with

$$\begin{aligned}
a_n^\star &= \frac{n-1}{n+1}\frac{k_n}{k_{n+1}},\\
b_n^\star &= \Big(an + 2n^2 - 2cab + a^4 - nb^2 - nc^2 - nd^2 - 5n^2d - 5n^2c - 5n^2b + 4n^3d\\
&\quad +4n^3c + 4n^3b + 4na^3 + 6n^2a^2 + 4an^3 + 2ab^2n + d^2ac + 2d^2an + 2dc^2n\\
&\quad +2d^2cn + 2c^2an + c^2bd + c^2ba + 2cb^2n + 2c^2bn + c^2ad + db^2a + d^2bc\\
&\quad +d^2ba + db^2c + 2db^2n + 2d^2bn + cb^2a + 6cbn^2 - 4dbn - 4dcn + 6nba^2\\
&\quad +6n^2da + 6n^2ca + 6dbn^2 + 6dcn^2 + 6nca^2 + 6n^2ab - 4cbn + 6nda^2\\
&\quad +2n^2d^2 + 2n^2c^2 + 2n^2b^2 + c^2a^2 + a^2b^2 + d^2a^2 - 2a^3 + 2a^3c + 2a^3d\\
&\quad -2ba^2 - 2dcb - 2dca - 2dab + 4dcab - 4nab - 4nda - 4nca + 3dca^2\\
&\quad +3da^2b + 3ca^2b - 4n^3 + 2n^4 + 2a^3b + nb + nc + nd - 5an^2 - 5na^2\\
&\quad -2da^2 - 2ca^2 + 6ndcb + 6ndca + 6ndab + 6ncab\Big)\Big/\\
&\quad (d+c+b+a+2n-2)(d+c+b+a+2n)\\
c_n^\star &= \frac{k_n}{k_{n-1}}\frac{(d+n-1+c)(d+b+n-1)(b+c-1+n)}{(2n-1+d+c+b+a)}\\
&\quad\times\frac{(-2+n+d+c+b+a)\,n\,(a-1+n+d)(a+c-1+n)(a+n+b-1)}{(2n-3+d+c+b+a)(2n-2+d+c+b+a)^2}.
\end{aligned}$$

Proof. Since the Wilson polynomials satisfy the difference equation (26), the three-term recurrence relation (32) is valid. The coefficients $a_n^\star$, $b_n^\star$ and $c_n^\star$ can be computed as in Proposition 7. □

1.6 The Continuous Dual Hahn Polynomials

It is well known that the Continuous Dual Hahn polynomials are special cases of the Wilson polynomials. More precisely, we have (see [8, P. 199])

$$S_n(x^2;a,b,c) = \lim_{d\to\infty} \frac{W_n(x^2;a,b,c,d)}{(a+d)_n}. \tag{33}$$

The divided difference equation and the three-term recurrence relation for the Continuous Dual Hahn polynomials are obtained from the ones of the Wilson polynomials by a limiting process.

Theorem 12. *The Continuous Dual Hahn polynomials are solutions of the second order difference equation*

$$\left(-(a+b+c)x^2 + abc\right)\mathbf{D}^2 y(x) + \left(x^2 - ab - ac - bc\right)\mathbf{SD}y(x) - ny(x) = 0. \tag{34}$$

Proposition 13. *The Continuous Dual Hahn polynomials satisfy the three-term recurrence relation*

$$\vartheta_1(a,x)S_n(x^2;a,b,c) = a_n S_{n+1}(x^2;a,b,c) + b_n S_n(x^2;a,b,c) + c_n S_{n-1}(x^2;a,b,c), \tag{35}$$

with

$$\begin{aligned}
a_n &= \frac{k_n}{k_{n+1}},\\
b_n &= -n + 2\,an + a^2 + 2\,n^2 + cb + ca + ab + 2\,nc + 2\,nb\\
c_n &= \frac{k_n}{k_{n-1}} n\left(-1+c+a+n\right)\left(b+a-1+n\right)\left(-1+b+n+c\right).
\end{aligned}$$

Proposition 14. *The second-order divided differences of the Continuous Dual Hahn polynomials satisfy the following three-term recurrence relation*

$$\vartheta_1(a,x)\mathbf{D}^2 S_n(x^2;a,b,c) = a_n^\star \mathbf{D}^2 S_{n+1}(x^2;a,b,c) + b_n^\star \mathbf{D}^2 S_n(x^2;a,b,c) + c_n^\star \mathbf{D}^2 S_{n-1}(x^2;a,b,c),$$

with

$$\begin{aligned}
a_n^\star &= \frac{(n-1)}{n+1}\frac{k_n}{k_{n+1}}\\
b_n^\star &= 1 - 2\,a - 2\,b - 3\,n - 2\,c + 2\,an + a^2 + 2\,n^2 + cb + ca + ab + 2\,nc + 2\,nb,\\
c_n^\star &= \frac{k_n}{k_{n-1}} n\left(-1+c+a+n\right)\left(b+a-1+n\right)\left(-1+b+n+c\right).
\end{aligned}$$

2 Hypergeometric representation, Inversion formula and Connection coefficients

2.1 Hypergeometric Representation

Of course, the hypergeometric representation for the Wilson polynomials is well known and is given as definition in this paper. Here, we recover this hypergeometric representation from the divided difference equation algorithmically.
We assume that the solution of (1) is of the form

$$y_n(x) = \sum_{m=0}^{n} C_m(n)\vartheta_m(a,x). \tag{36}$$

Theorem 15. *Let $y_n(s)$ be a polynomial system given by the q-differential equation (1) with $\phi(x) = \phi_2 x^4 + \phi_1 x^2 + \phi_0$ and $\psi(x) = \psi_1 x^2 + \psi_0$ given by (19) and (20). Then, the power series coefficients $C_m(n)$ given by (36) satisfy the second-order recurrence equation:*

$$\begin{aligned}
&(m+2)\,(m+1)\,(m+2\,a+1)\,(a+m+d+1)\,(a+m+1+c)\,(1+m+b+a)\,C_{m+2}(n)\\
&-(m+1)\Big(-mnd+a+b+n+m+d+c-mnb+2\,a^2+cab+6\,am^2+2\,bm^2\\
&+2\,dm^2+2\,cm^2+5\,a^2m+2\,m^3+a^3+da^2+2\,bm+2\,cm+2\,dm+an-n^2+4\,am\\
&+2\,ab+dcb+dca+dab+2\,da+2\,ca+4\,abm+dcm+dbm+4\,dam+cbm+4\,cam\\
&-mna+nm-mnc-mn^2+ca^2+ba^2-2\,an^2-2\,na^2-nb-nc-nd-2\,nab\\
&-2\,nca-2\,nda+2\,m^2\Big)C_{m+1}(n)+(m-n)\,(m-1+b+d+c+a+n)\,C_m(n)=0.
\end{aligned}$$

Hence, the Wilson polynomials have the following hypergeometric representation

$$\begin{aligned}
W_n(x^2;a,b,c,d) &= K_n \sum_{m=0}^{n} \frac{(-n,n+a+b+c+d-1)_m}{m!(a+b,a+c,a+d)_m}\vartheta_m(a,x) \qquad (37)\\
&= K_n\, {}_4F_3\left(\left.\begin{array}{c} -n,n+a+b+c+d-1,a-ix,a+ix \\ a+b,a+c,a+d \end{array}\right| 1\right).
\end{aligned}$$

Taking $K_n = (a+b,a+c,a+d)_n$ gives the usual form of the Wilson polynomials ([see [8], P. 185]).

Proof. Substitute $y_n(x)$ by the expression (36) in the equation (1), then multiply the resulting equation by $\vartheta_1(a,x)$. Next use the relations (4)-(11) for simplification. The desired recurrence relation follows by equating the coefficients of $\vartheta_n(a,x)$. Next, by Petkovšek-van-Hoeij's algorithm ([9], Chapter 9) via the Maple command `LREtools[hypergeomsols]`, we obtain the coefficient $C_m(n)$ up to a constant factor. □

Theorem 16. *Let $y_n(s)$ be a polynomial system given by the q-differential equation (34), then the power series coefficients $C_m(n)$ given by (36) satisfy the recurrence equation*

$$\begin{aligned}
&2(m+2)(m+b+c+1)C_{m+2}(n)+\Big(-4\,m^2+(-4-4\,a+2\,n-4\,c-4\,b)m\\
&+(2\,b+2\,c+2)\,n-2\,b-2-2\,ca-2\,c-2\,a^2-2\,ab-2\,cb\Big)C_{m+1}(n)\\
&+2\,\frac{(a+m+b)\,(a+m+c)\,(m-n)}{m+1}C_m(n)=0.
\end{aligned}$$

Hence, the Continuous Dual Hahn polynomials have the following hypergeometric representation

$$\begin{aligned}
S_n(x^2;a,b,c) &= K_n \sum_{m=0}^{n} \frac{(-n)_m}{m!(a+b,a+c)_m}\vartheta_m(a,x)\\
&= K_n\, {}_3F_2\left(\left.\begin{array}{c} -n,a-ix,a+ix \\ a+b,a+c \end{array}\right| 1\right). \qquad (38)
\end{aligned}$$

Taking $K_n = (a+b,a+c)_n$ gives the usual form of the Continuous Dual Hahn polynomials ([see [8], P. 196]).

Remark 17. *Note that (38) can be deduced from (37) by a limiting process.*

2.2 Connection and Linearization formulas for $(\vartheta_n(a,x))_n$.

Theorem 18. *The basis $(\vartheta_n(a,x))_n$ satisfies the following linearization formula*

$$\vartheta_n(a,x)\vartheta_m(a_1,x) = \sum_{k=0}^{m} J_{n+k}(m,n,a,a_1)\vartheta_{n+k}(a,x), \quad m,n \in \mathbb{N}, \tag{39}$$

with

$$J_{n+k}(m,n,a,a_1) = \binom{m}{k}(a_1 - a - n, a_1 + a + n + k)_{m-k}, \quad k = 0,\ldots,m.$$

Proof. We first remark that

$$\vartheta_n(a,x) = \prod_{j=0}^{n-1}((a+j)^2 + x^2).$$

Hence, for $x = \xi_j(a) = i(a+j)$, it happens that

$$\vartheta_n(a,\xi_j(a)) = 0, \quad j = 0,1,\ldots,n-1, \quad \text{and} \quad \vartheta_n(a,\xi_n(a)) \neq 0.$$

We now expand the product $\vartheta_n(a,x)\vartheta_m(a_1,x)$ in the basis $\vartheta_k(a,x)$

$$\vartheta_n(a,x)\vartheta_m(a_1,x) = \sum_{k=0}^{n+m} J_k(m,n,a,a_1)\vartheta_k(a,x). \tag{40}$$

We get

$$\begin{aligned} 0 &= \vartheta_n(a,\xi_0(a))\vartheta_m(a_1,\xi_0(a)) \\ &= J_0(m,n,a,a_1) + \sum_{k=1}^{n+m} J_k(m,n,a,a_1)\vartheta_k(a,\xi_0(a)) = J_0(m,n,a,a_1). \end{aligned}$$

Hence, we can write

$$\vartheta_n(a,x)\vartheta_m(a_1,x) = \sum_{k=1}^{n+m} J_k(m,n,a,a_1)\vartheta_k(a,x).$$

By the same procedure, we get

$$J_1(m,n,a,a_1)\vartheta_1(a,x) = \vartheta_n(a,\xi_1(a))\vartheta_m(a_1,\xi_1(a)) = 0,$$

and hence we get $J_1(m,n,a,a_1) = 0$. Progressively, we prove that

$$J_0(m,n,a,a_1) = J_1(m,n,a,a_1) = \cdots = J_j(m,n,a,a_1) = 0, \quad j \leq n-1.$$

We can actually write

$$\vartheta_n(a,x)\vartheta_m(a_1,x) = \sum_{k=n}^{n+m} J_k(m,n,a,a_1)\vartheta_{n+k}(a,x) = \sum_{k=0}^{m} J_k(m,n,a,a_1)\vartheta_{n+k}(a,x). \tag{41}$$

First of all, we have

$$\vartheta_n(a,\xi_n(a))\vartheta_m(a_1,\xi_n(a)) = J_n(m,n,a,a_1)\vartheta_n(a,\xi_n(a)),$$

and hence

$$J_n(m,n,a,a_1) = \vartheta_m(a_1,\xi_n(a)) = (a_1 + a + n, a_1 - a - n)_m.$$

Using (41), we can write

$$\vartheta_m(a_1,x) = \sum_{k=n}^{m} J_{n+k}(m,n,a,a_1)\frac{\vartheta_{n+k}(a,x)}{\vartheta_n(a,x)} = \sum_{k=0}^{m} J_{n+k}(m,n,a,a_1)\vartheta_k(a+n,x).$$

The use of the relation (8) yields

$$\frac{m!}{(m-l)!}\vartheta_{m-l}\left(a_1+\frac{l}{2},x\right) = \sum_{k=l}^{m} J_{n+k}(m,n,a,a_1)\frac{k!}{(k-l)!}\vartheta_{k-l}\left(a+n+\frac{l}{2},x\right).$$

Taking $k=l$ and $x=\xi_0\left(a+n+\frac{l}{2}\right)$, it follows that

$$J_{n+l}(m,n,a,a_1) = \binom{m}{l}\vartheta_{m-l}\left(a_1+\frac{l}{2},\xi_0\left(a+n+\frac{l}{2}\right)\right).$$

The required result follows by an easy simplification. □

Corollary 19. *The following connection formula between $(\vartheta_n(a,x))_n$ and $(\vartheta_m(a_1,x))_m$ is valid*

$$\vartheta_m(a,x) = \sum_{k=0}^{m}\binom{m}{k}(a-a_1,a+a_1+k)_{m-k}\vartheta_k(a_1,x), \quad m\in\mathbb{N}. \tag{42}$$

Remark 20. *If we take $a_1=a$, then the formula (39) becomes*

$$\vartheta_m(a,x)\vartheta_n(a,x) = \sum_{k=0}^{m}\binom{m}{k}(-n,2a+n+k)_{m-k}\vartheta_{n+k}(a,x), \quad m,n\in\mathbb{N}. \tag{43}$$

The case $m=1$ gives relation (6).

2.3 Inversion Formula

2.3.1 Structure relation for the basis $\vartheta_n(a,x)$.

Proposition 21. *The basis $\vartheta_n(a,x)$ fulfils the following structure relation.*

$$\vartheta_1(a,x)\mathbf{D}^2\vartheta_n(a,x) = \frac{n-1}{n+1}\mathbf{D}^2\vartheta_{n+1}(a,x) - (n-1)(n-1+2a)\mathbf{D}^2\vartheta_n(a,x). \tag{44}$$

Proof. From relation (9), we have $\vartheta_1(a,x)\mathbf{D}^2\vartheta_{n+1}(a,x) = \delta(n+1)\vartheta_n(a,x)$. Hence we deduce

$$\vartheta_{n-1}(a,x) = \frac{1}{\delta(n)}\vartheta_1(a,x)\mathbf{D}^2\vartheta_n(a,x), \quad \vartheta_n(a,x) = \frac{1}{\delta(n+1)}\vartheta_1(a,x)\mathbf{D}^2\vartheta_{n+1}(a,x).$$

Finally, the use of the three-term recurrence relation (6) yields the result. □

2.3.2 The inversion formula.

Next we find the coefficients $I_m(n)$ in the expansion

$$\vartheta_n(a,x) = \sum_{m=0}^{n} I_m(n)y_m(x), \tag{45}$$

where $y_n(x)$ is $W_n(x^2;a,b,c,d)$ or $S_n(x^2;a,b,c,d)$.

Theorem 22. *The following inversion formulas are valid:*

$$\vartheta_n(a,x) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m (a+b+m, a+c+m, a+d+m)_{n-m}}{(a+b+c+d+m-1)_m (a+b+c+d+2m)_{n-m}} W_m(x^2; a,b,c,d) \qquad (46)$$

$$\vartheta_n(a,x) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} (a+b+m, a+c+m)_{n-m} S_m(x^2; a,b,c). \qquad (47)$$

Proof. Substituting the expression of $\vartheta_n(a,x)$ given by (45) in (6) and in (44), and using the three-term recurrence relations (32) and (11) we get by an appropriate shift of indices the following two cross-rules

$$I_m(n+1) + \mu(a,n) I_m(n) = a_{m-1} I_{m-1}(n) + b_m I_m(n) + c_{m+1} I_{m+1}(n)$$
$$\frac{\delta(n)}{\delta(n+1)} I_m(n+1) + \mu(a,n-1) I_m(n) = a^\star_{m-1} I_{m-1}(n) + b^\star_m I_m(n) - c^\star_{m+1} I_{m+1}(n).$$

By linear algebra, we eliminate the term $I_m(n+1)$ and get a pure recurrence equation with respect to m in $I_m(n)$. Next, by Petkovšek-van-Hoeij's algorithm ([9], Chapter 9) via the Maple command `LREtools[hypergeomsols]`. Identification of the coefficient of $\vartheta_n(a,x)$ on both sides gives the desired constant. □

2.4 Connection formulas

In this subsection, we give an explicit formula for the coefficients $D_k(n)$ in the expansion

$$P_n(x) = \sum_{k=0}^{n} D_k(n) P_k(x),$$

where $P_n(x) = W_n(x^2; a,b,c,d)$ or $P_n(x) = S_n(x^2; a,b,c)$, the parameters a, b, c and d may be different.
First note the following

$$P_n(x) = \sum_{m=0}^{n} C_m(n) \vartheta_m(a,x) \quad \text{and} \quad \vartheta_m(a,x) = \sum_{k=0}^{m} I_k(m) P_k(x).$$

Combining those two relations we get

$$P_n(x) = \sum_{k=0}^{n} D_k(n) P_k(x), \quad \text{with} \quad D_k(n) = \sum_{m=0}^{n-k} C_{m+k}(n) I_k(m+k).$$

Proposition 23. *The following connections are valid*

$$\begin{aligned} W_n(x^2; a,b,c,d) &= \sum_{k=0}^{n} \binom{n}{k} \frac{(a+b+k, a+c+k, a+d+k)_{n-k}}{(a+\beta+\gamma+\delta+k-1)_k} \\ &\quad \times (n+a+b+c+d-1)_k A_k W_k(x^2; a,\beta,\gamma,\delta) \end{aligned} \qquad (48)$$

$$\begin{aligned} S_n(x^2; a,b,c) &= \sum_{k=0}^{n} \binom{n}{k} (a+b+k, a+c+k)_{n-k} \\ &\quad \times {}_3F_2 \left(\left. \begin{matrix} k-n, a+\beta+k, a+\gamma+k \\ a+b+k, a+c+k \end{matrix} \right| 1 \right) S_k(x^2; a,\beta,\gamma). \end{aligned} \qquad (49)$$

with

$$A_k = {}_5F_4 \left(\left. \begin{matrix} k-n, n+a+b+c+d+k-1, a+k+\beta, a+k+\gamma, a+k+\delta \\ 2k+\delta+\gamma+\beta+a, a+b+k, a+c+k, a+d+k \end{matrix} \right| 1 \right).$$

Proof. We start by writing the inversion formula (46) for $W_n(x^2;a,\beta,\gamma,\delta)$. Combining this inversion formula with the expansion of $W_n(x^2;a,b,c,d)$ in the basis $\vartheta_n(a,x)$, we get (48). The result for the Continuous Dual Hahn polynomials follows in the same manner or can be obtained from (48) by a limiting process. $\square$

Proposition 24. *The Wilson polynomials $W_n(x^2,a,b,c,d)$ and the Continuous Dual Hahn polynomials $S_n(x^2,a,b,c)$ have the following representation in the basis $(\vartheta_n(\alpha,x))$*

$$W_n(x^2;a,b,c,d)=\sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^m(a+b,a+c,a+d)_n(n+a+b+c+d-1)_m}{(a+b,a+c,a+d)_m}$$
$$\times{}_4F_3\left(\left.\begin{matrix}m-n,n+a+b+c+d+m-1,a+\alpha+m,a-\alpha\\ a+b+m,a+c+m,a+d+m\end{matrix}\right|1\right)\vartheta_m(\alpha,x),\quad(50)$$

$$S_n(x^2;a,b,c)=\sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^m(a+b,a+c)_n}{(a+b,a+c)_m}{}_3F_2\left(\left.\begin{matrix}m-n,a+\alpha+m,a-\alpha\\ a+b+m,a+c+m\end{matrix}\right|1\right)\vartheta_m(\alpha,x).\quad(51)$$

Proof. Combining (42) with (38), (50) follows. (51) is obtained from (50) by a limiting process. $\square$

Proposition 25. *The following connection formulas are valid.*

$$\begin{aligned}&W_n(x^2;a,b,c,d) \qquad (52)\\ &=\sum_{m=0}^{n}\left\{\binom{n}{m}\frac{(n+a+b+c+d-1)_m(m+a+b,m+a+c,m+a+d)_{n-m}}{(m+\alpha+\beta+\gamma+\delta-1)_m}\right.\\ &\times\sum_{k=0}^{n-m}\frac{(m-n,m+n+a+b+c+d-1,m+\alpha+\beta,m+\alpha+\gamma,m+\alpha+\delta)_k}{k!(m+a+b,m+a+c,m+a+d,2m+\alpha+\beta+\gamma+\delta)_k}\\ &\left.\times\,{}_4F_3\left(\left.\begin{matrix}k+m-n,a+\alpha+k+m,a-\alpha,n+a+b+c+d+m+k-1\\ k+m+a+b,k+m+a+c,k+m+a+d\end{matrix}\right|1\right)\right\}\\ &\times W_m(x^2;\alpha,\beta,\gamma,\delta),\end{aligned}$$

$$S_n(x^2,a,b,c)=\sum_{m=0}^{n}\binom{n}{m}\sum_{k=0}^{n-m}\frac{(m-n,m+\alpha+\beta,m+\alpha+\gamma)_k(a+b+m,a+c+m)_{n-m}}{k!(a+b+m,a+c+m)_k}$$
$$\times{}_3F_2\left(\left.\begin{matrix}m+k-n,a+\alpha+k+m,a-\alpha\\ a+b+m+k,a+c+m+k\end{matrix}\right|1\right)S_m(x^2,\alpha,\beta,\gamma),$$

Proof. We start by writing the inversion formula (46) for $W_n(x^2;\alpha,\beta,\gamma,\delta)$. Combining this inversion formula with the expansion of $W_n(x^2;a,b,c,d)$ in the basis $\vartheta_n(\alpha,x)$ as in (50), we get (52). The result for the Continuous Dual Hahn polynomials follows in the same manner or can be obtained from (52) by a limiting process. $\square$

Remark 26. *1. Connection formula (48) has been given by Jorge Sánchez-Ruiz and Jesús S. Dehesa in [14] using the following formula derived by Fields and Wimp [12] (see also [13], p. 7)*

$$
\begin{aligned}
{}_{p+r+1}F_{q+s}\left(\begin{array}{c} -n, [a_p], [c_r] \\ \\ [b_q], [d_s] \end{array}\middle| zw\right) &= \sum_{k=0}^{n}\binom{n}{k}\frac{([a_p])_k([\alpha_t])_k z^k}{([b_p])_k([\beta_u])_k(k+\lambda)_k} \\
&\times {}_{p+t+1}F_{q+u+1}\left(\begin{array}{c} k-n, [k+a_p], [k+\alpha_t] \\ \\ 2k+\lambda+1, [k+b_q], [k+\beta_t] \end{array}\middle| z\right) \\
&\times {}_{r+u+2}F_{s+t}\left(\begin{array}{c} -k, k+\lambda, [c_r], [\beta_u] \\ \\ [d_s], [\alpha_t] \end{array}\middle| w\right).
\end{aligned}
$$

2. *Connection formula (52) generalizes (48).*

3. *Connection formulas* (48) *and* (52) *were proved in [7] by a limiting process using the connection formulas for the Askey-Wilson polynomials.*

Acknowledgements

This work has been partially supported by the STIBET fellowship of DAAD for P. Njionou Sadjang, the Institute of Mathematics of the University of Kassel (Germany) and a Research-Group Linkage Programme 2009-2012 between the University of Kassel (Germany) and the University of Yaounde I (Cameroon) sponsored by the Alexander von Humboldt Foundation. All these institutions receive our sincere thanks.
We would like to thank the anonymous reviewer of this paper for very carefully reading the manuscript, and also for his valuable comments and suggestions which improved the paper significantly.

References

[1] Askey R, Wilson J. *Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials,* Mem. Amer. Math. Soc., **319**, 1985.

[2] Atakishiyev NM, Rahman M, Suslov SK. *On classical orthogonal polynomials*, Constr. Approx. **11**, (1995) 181-226.

[3] Cooper S. *The Askey-Wilson operator and the ${}_6\phi_5$ summation formula*, preprint, December 2012.

[4] Suslov SK. *The theory of difference analogues of special functions of hypergeometric type*, Russian Math. Surveys **44**, (1989) 227-278.

[5] Foupouagnigni M. *On difference equations for orthogonal polynomials on non-uniform lattices*, J. Diff. Eqn. Appl. **14**, (2008) 127-174.

[6] Foupouagnigni M, Kenfack-Nangho M, Mboutngam S. *Characterization theorem of classical orthogonal polynomials on non-uniform lattices: The functional approach*, Integral Transforms Spec. Funct. **22**, (2011) 739-758.

[7] Foupouagnigni M, Koepf W, Tcheutia DD. *Connection and Linearization Coefficients of the Askey-Wilson Polynomials*, J. Symbolic Comput. **53**, (2013) 96-118.

[8] Koekoek R, Lesky PA, Swarttouw RF, *Hypergeometric Orthogonal Polynomials and Their q-Analogues*, Springer, 2010.

[9] Koepf W. *Hypergeometric Summation – An algorithmic approach to summation and special function identities*, Second Edition, Springer, 2014.

[10] Koepf W, Schmersau D. *Representations of orthogonal polynomials*, J. Comput. Appl. Math. **90**, (1998) 57-94.

[11] Miller W. *A note on Wilson polynomials*, SIAM J. Math. Anal. **18**, (1987) 1221-1226.

[12] Fields JL, Wimp J. *Expansions of hypergeometric functions in hypergeometric functions*, Math. Comp. **15**, (1961) 390-395.

[13] Luke. YL, *The special functions and their approximations*, Academic Press, New York, 1969.

[14] Sánchez-Ruiz J, Dehesa JS. *Some connection and linearization problems for the polynomials in and beyond the Askey scheme*, J. Comput. Appl. Math. **133**, (2001) 579-591.