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Recurrence equations and their classical orthogonal polynomial solutions

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Abstract

The classical orthogonal polynomials are given as the polynomial solutions $p_n(x)$ of the differential equation

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda_n y(x) = 0,$$

where $\sigma(x)$ is a polynomial of at most second degree and $\tau(x)$ is a polynomial of first degree.

In this paper a general method to express the coefficients A_n, B_n and C_n of the recurrence equation

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x)$$

in terms of the given polynomials $\sigma(x)$ and $\tau(x)$ is used to present an algorithm to determine the classical orthogonal polynomial solutions of any given holonomic three-term recurrence equation, i.e., a homogeneous linear three-term recurrence equation with polynomial coefficients.

In a similar way, classical discrete orthogonal polynomial solutions of holonomic three-term recurrence equations can be determined by considering their corresponding difference equation

$$\sigma(x)\Delta\nabla y(x) + \tau(x)\Delta y(x) + \lambda_n y(x) = 0,$$

where $\Delta y(x) = y(x+1) - y(x)$ and $\nabla y(x) = y(x) - y(x-1)$ denote the forward and backward difference operators, respectively, and a similar approach applies to classical q-orthogonal polynomials, being solutions of the q-difference equation

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$$\sigma(x)D_qD_{1/q}y(x) + \tau(x)D_qy(x) + \lambda_{q,n}y(x) = 0,$$

where

$$D_qf(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad q \neq 1,$$

denotes the q-difference operator.

Keywords: Computer algebra; Maple; Differential equation; Q-difference equation; Structure formula

1. Introduction

Families of orthogonal polynomials $p_n(x)$ (corresponding to a positive-definite measure) satisfy a three-term recurrence equation of the form

$$p_{n+1}(x) = (A_nx + B_n)p_n(x) - C_np_{n-1}(x) \quad (n \in \mathbb{N}_0, p_{-1} \equiv 0) \tag{1}$$

with $C_nA_nA_{n-1} > 0$, see e.g. [5, p. 20]. Moreover, Favard's theorem states that the converse is also true.

On the other hand, in practice one is often interested in an *explicit* solution of a given recurrence equation. Therefore it is an interesting question to ask whether a given recurrence equation has *classical* orthogonal polynomial solutions.

In this paper an algorithm is developed which answers this question for a large class of classical orthogonal polynomial systems. Furthermore, we present results of our corresponding Maple implementation `retode` and compare these with the Maple implementation `rec2ortho` of Koornwinder and Swarttouw [12]. These programs overlap, but `rec2ortho` does not cover Bessel, Hahn and q-polynomials, whereas `retode` does not include the Meixner–Pollaczek case.

2. Classical orthogonal polynomials

A family

$$y(x) = p_n(x) = k_nx^n + k'_nx^{n-1} + \cdots \quad (n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\},\ k_n \neq 0) \tag{2}$$

of polynomials of degree exactly n is a family of *classical continuous orthogonal polynomials* if it is the solution of a differential equation of the type

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda_ny(x) = 0, \tag{3}$$

where $\sigma(x) = ax^2 + bx + c$ is a polynomial of at most second-order and $\tau(x) = dx + e$ $(d \neq 0)$ is a polynomial of first-order ([3,13]). Since one demands that $p_n(x)$ has exact degree n, by equating the coefficients of x^n in (3) one gets

$$\lambda_n = -(an(n-1) + dn). \tag{4}$$

Similarly, a family $p_n(x)$ of polynomials of degree exactly n, given by (2), is a family of *classical discrete orthogonal polynomials* if it is the solution of a difference equation of the type

$$\sigma(x)\Delta\nabla y(x) + \tau(x)\Delta y(x) + \lambda_n y(x) = 0, \tag{5}$$

where

$$\Delta y(x) = y(x+1) - y(x) \quad \text{and} \quad \nabla y(x) = y(x) - y(x-1)$$

denote the forward and backward difference operators, respectively, and $\sigma(x) = ax^2 + bx + c$ and $\tau(x) = dx + e$ are again polynomials of at most second- and of first-order, respectively, see e.g. [18]. Again, (4) follows.

Finally, a family $p_n(x)$ of polynomials of degree exactly n, given by (2), is a family of *classical q-orthogonal polynomials* if it is the solution of a q-difference equation of the type

$$\sigma(x)D_q D_{1/q} y(x) + \tau(x)D_q y(x) + \lambda_{q,n} y(x) = 0, \tag{6}$$

where

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad q \neq 1,$$

denotes the q-difference operator [6], and $\sigma(x) = ax^2 + bx + c$ and $\tau(x) = dx + e$ are again polynomials of at most second- and of first-order, respectively. By equating the coefficients of x^n in (6) one gets

$$\lambda_{q,n} = -a[n]_{1/q}[n-1]q - d[n]q, \tag{7}$$

where the abbreviation

$$[n]_q = \frac{1-q^n}{1-q}$$

denotes the so-called q-brackets. Note that $\lim_{q\to 1}[n]_q = n$.

It can be shown (see e.g. [14]) that any solution $p_n(x)$ of either (3), (5) or (6) satisfies a recurrence equation (1).

The following is a general procedure to find the coefficients of the recurrence equation (as well as of similar structural formulas for classical orthogonal polynomials, see [10]) in terms of the coefficients a, b, c, d and e of $\sigma(x)$ and $\tau(x)$:

1. Substitute $p_n(x) = k_n x^n + k_n' x^{n-1} + k_n'' x^{n-2} + \cdots$ in the differential equation (3), in the difference equation (5) or in the q-difference equation (6), respectively.

2. Equating the coefficients of x^n yields λ_n, given by (4) and (7), respectively.
3. Equating the coefficients of x^{n-1} and x^{n-2} gives k_n', and k_n'', respectively, as rational multiples of k_n.
4. Substitute $p_n(x)$ in the proposed equation, and equate again the three highest coefficients. In the case of the recurrence equation (1), this yields

$$\frac{k_n}{k_{n+1}} A_n = 1, \quad \widetilde{B}_n = \frac{k_n}{k_{n+1}} B_n = \frac{k'_{n+1}}{k_{n+1}} - \frac{k'_n}{k_n}$$

and

$$\widetilde{C}_n = \frac{k_{n-1}}{k_{n+1}} C_n = \frac{k''_n}{k_n} - \frac{k''_{n+1}}{k_{n+1}} - \left(\frac{k'_n}{k_n}\right)^2 + \frac{k'_n}{k_n} \frac{k'_{n+1}}{k_{n+1}},$$

by linear algebra.
5. Substituting the values of k_n' and k_n'' given in step 3 in these equations yields the three unknowns in terms of $a, b, c, d, e, n, k_{n-1}, k_n$, and k_{n+1}.

With regard to the recurrence equation coefficients, we collect these results in the following theorem.

Theorem 1. *Let* $p_n(x) = k_n x^n + \cdots \ (n \in \mathbb{N}_0)$ *be a family of polynomial solutions of the differential equation* (3). *Then the recurrence equation* (1) *is valid with*

$$\frac{k_n}{k_{n+1}} A_n = 1, \tag{8}$$

$$\frac{k_n}{k_{n+1}} B_n = \frac{2bn(an + d - a) - e(-d + 2a)}{(d + 2an)(d - 2a + 2an)} \tag{9}$$

and

$$\begin{aligned} \frac{k_{n-1}}{k_{n+1}} C_n = & -\frac{(an + d - 2a)n}{(d - 2a + 2an)^2 (2an - 3a + d)(2an - a + d)} \cdot \big((an + d - 2a) \\ & \times n(4ca - b^2) + 4a^2c - ab^2 + ae^2 - 4acd + db^2 - bed + d^2c\big) \end{aligned} \tag{10}$$

in terms of the coefficients a, b, c, d and e of the given differential equation.

Let $p_n(x) = k_n x^n + \cdots (n \in \mathbb{N}_0)$ *be a family of polynomial solutions of the difference equation* (5). *Then the recurrence equation* (1) *is valid with*

$$\frac{k_n}{k_{n+1}} A_n = 1,$$

$$\frac{k_n}{k_{n+1}} B_n = \frac{n(d + 2b)(d + an - a) + e(d - 2a)}{(2an - 2a + d)(d + 2an)} \tag{11}$$

and

$$\frac{k_{n-1}}{k_{n+1}}C_n = -\frac{(an+d-2a)n}{(d-a+2an)(d+2an-3a)(2an-2a+d)^2}\cdot\big((n-1) \times(d+an-a)(and-db-ad+a^2n^2-2a^2n + 4ca+a^2+2ea-b^2)-dbe+d^2c+ae^2\big) \tag{12}$$

in terms of the coefficients a, b, c, d *and* e *of the given difference equation.*

Let $p_n(x) = k_n x^n + \cdots (n \in \mathbb{N}_0)$ *be a family of polynomial solutions of the* q*-difference equation* (6). *Then the recurrence equation* (1) *is valid with*

$$\frac{k_n}{k_{n+1}}A_n = 1,$$

$$\frac{k_n}{k_{n+1}}B_n = [n+1]_q\frac{b[n]_q+eq^n}{a[2n]_q+dq^{2n}} - [n]_q\frac{b[n-1]_q+eq^{n-1}}{a[2(n-1)]_q+dq^{2(n-1)}} \tag{13}$$

and $(N = q^n)$

$$\begin{aligned}\frac{k_{n-1}}{k_{n+1}}C_n = {} & ((N-1)(-Nd+Na+Nqd-aq^2)\\ & \times(-a^2cN^4-N^4d^2c+N^3beqd-abeNq^3-2adcN^2q^2\\ & +2aN^2q^2be-aN^3beq+beN^3q^3d-2beN^3q^2d\\ & +2dcN^2q^3a-2beN^2q^3a-2dcN^4qa+beNq^4a\\ & +beN^3aq^2-b^2q^3dN^2-a^2cq^4+N^2b^2q^2d-N^3b^2qd\\ & +aNq^3b^2-2ab^2N^2q^2-ae^2N^2q^2+aN^3b^2q+2aN^4dc\\ & +2a^2cq^2N^2+2d^2cN^4q-d^2cN^4q^2+b^2N^3q^2d\\ & +2e^2N^2q^3a-e^2N^2q^4a)Nq)/((dN^2q-aq+aN^2-dN^2)\\ & \times(dN^2q-dN^2+aN^2-aq^2)^2(dN^2q-dN^2+aN^2-aq^3))\end{aligned} \tag{14}$$

in terms of the coefficients a, b, c, d *and* e *of the given* q*-difference equation.*

3. The inverse characterization problem

It is well-known ([3], see also [4,13]) that polynomial solutions of (3) can be classified according to the zeros of $\sigma(x)$, leading to the normal forms of Table 1 besides linear transformations $x \mapsto Ax + B$. The type of differential equation that we consider is invariant under such a transformation.

This shows that the only orthogonal polynomial solutions are linear transforms of the Hermite, Laguerre, Bessel and Jacobi polynomials (for details see e.g. [10]), hence using a mathematical dictionary one can always deduce the recurrence equation. Note, however, that this approach except than

Table 1
Normal forms of polynomial solutions

	$\sigma(x)$	$\tau(x)$	$p_n(x)$	Family
1	0	x	x^n	
2	1	$-2x$	$H_n(x)$	Hermite polynomials
3	x	$-x+\alpha+1$	$L_n^{(\alpha)}(x)$	Laguerre polynomials
4a	x^2	0	x^n	
4b	x^2	$(\alpha+2)x+2$	$B_n^{(\alpha)}(x)$	Bessel polynomials
5	$(x+1)(x-1)$	$(\alpha+\beta+2)x+\alpha-\beta$	$P_n^{(\alpha,\beta)}(x)$	Jacobi polynomials

being tedious may require the work with radicals, namely the zeros of the quadratic polynomial $\sigma(x)$, whereas our approach is completely rational: Given $k_{n+1}/k_n \in \mathbb{Q}(n)$, the recurrence equation is given rationally by Theorem 1.

Moreover, Theorem 1 represents the recurrence equation by a unique formula. It is valid also in the cases of Table 1(1) and (4a), with the trivial solution $p_n(x) = x^n$. In both cases we have the recurrence equation $p_{n+1}(x) = xp_n(x)$.

Now, we will use the fact that these equations are given explicitly to solve an inverse problem.

Assume one knows that a polynomial system satisfies a differential equation (3). Then by the classification of Table 1 it is easy to identify the system. On the other hand, given an arbitrary holonomic three-term recurrence equation

$$q_n(x)P_{n+2}(x) + r_n(x)P_{n+1}(x) + s_n(x)P_n(x) = 0, \tag{15}$$

$(q_n(x), r_n(x), s_n(x) \in \mathbb{Q}[n,x])$, it is less obvious to find out whether there is a polynomial system

$$P_n(x) = k_nx^n + \cdots \quad (n \in \mathbb{N}_0, k_n \neq 0)$$

satisfying (15), being a linear transform of one of the classical systems (Hermite, Laguerre, Jacobi, Bessel), and to identify the system in the affirmative case. In this section we present an algorithm for this purpose. Note that Koornwinder and Swarttouw [12] have also considered this question and in their Maple implementation `rec2ortho` propose a solution based on the careful ad hoc analysis of the input polynomials (actually, they start with Eq. (19)). Their Maple implementation deals with the following families: Hermite, Charlier, Laguerre, Meixner–Pollaczek, Meixner, Krawtchouk, and Jacobi.

Let us start with a recurrence equation of type (15). Without loss of generality we assume that neither $q_{n-1}(x)$ nor $s_n(x)$ has a nonnegative integer zero w.r.t. n. Otherwise, a suitable shift can be applied, see Algorithm 1 and Example 1.

Therefore, in the sequel we assume that the recurrence equation

$$q_n(x)p_{n+2}(x) + r_n(x)p_{n+1}(x) + s_n(x)p_n(x) = 0, \tag{16}$$

$(q_n(x), r_n(x), s_n(x) \in \mathbb{Q}[n, x])$, is valid, but neither $q_{n-1}(x)$ nor $s_n(x)$ have non-negative integer zeros. We search for solutions

$$p_n(x) = k_n x^n + k_n' x^{n-1} + \cdots \quad (n \in \mathbb{N}_0, k_n \neq 0). \tag{17}$$

Next, we divide (16) by $q_n(x)$, and replace n by $n-1$. This brings (16) into the form

$$p_{n+1}(x) = t_n(x) p_n(x) + u_n(x) p_{n-1}(x) \quad (t_n(x), u_n(x) \in \mathbb{Q}(n, x)). \tag{18}$$

For $p_n(x)$ being a linear transform of a classical orthogonal system, there is a recurrence equation (1)

$$p_{n+1}(x) = (A_n x + B_n) p_n(x) - C_n p_{n-1}(x) \quad (A_n, B_n, C_n \in \mathbb{Q}(n), A_n \neq 0), \tag{19}$$

therefore (18) and (19) must agree. We would like to conclude that $t_n(x) = A_n x + B_n$, and $u_n(x) = -C_n$. This follows if we can show that $p_n(x)/p_{n-1}(x) \notin \mathbb{Q}(n, x)$. For a proof of this assertion, see [9].

Therefore we can conclude that $t_n(x) = A_n x + B_n$, and $u_n(x) = -C_n$. Hence if (18) does not have this form, i.e., if either $t_n(x)$ is not linear in x or $u_n(x)$ is not a constant with respect to x, we see that $p_n(x)$ cannot be a linear transform of a classical orthogonal polynomial system. In the positive case, we can assume the form (19).

Since we propose solutions (17), equating the coefficients of x^{n+1} in (19) we get

$$\frac{k_{n+1}}{k_n} = A_n = \frac{v_n}{w_n} \quad (v_n, w_n \in \mathbb{Q}[n]). \tag{20}$$

Hence the given $A_n = v_n/w_n \in \mathbb{Q}(n)$ generates the term ratio k_{n+1}/k_n. In particular k_n turns out to be a hypergeometric term, (i.e., k_{n+1}/k_n is rational) and is uniquely determined by (20) up to a normalization constant $k_0 = p_0(x)$. Since the zeros of w_n are a subset of the zeros of $q_{n-1}(x)$, k_n is defined by (20) for all $n \in \mathbb{N}$ from k_0.

In the next step we can eliminate the dependency of k_n by generating a recurrence equation for the corresponding monic polynomials $\widetilde{p}_n(x) = p_n(x)/k_n$. For $\widetilde{p}_n(x)$, we get by (20)

$$\widetilde{p}_{n+1}(x) = \left(x + \frac{B_n}{A_n}\right)\widetilde{p}_n(x) - \frac{C_n}{A_n A_{n-1}}\widetilde{p}_{n-1}(x) = \left(x + \widetilde{B}_n\right)\widetilde{p}_n(x) - \widetilde{C}_n \widetilde{p}_{n-1}(x)$$

with

$$\widetilde{B}_n = \frac{B_n}{A_n} \in \mathbb{Q}(n) \quad \text{and} \quad \widetilde{C}_n = \frac{C_n}{A_n A_{n-1}} \in \mathbb{Q}(n).$$

Then our formulas (9) and (10) read in terms of $\widetilde{B}_n$ and $\widetilde{C}_n$

$$\widetilde{B}_n = \frac{2bn(a(n-1)+d)+e(d-2a)}{(2a(n-1)+d)(2an+d)} \tag{21}$$

and

$$\widetilde{C}_n = \frac{-n(a(n-2)+d)}{(a(2n-1)+d)(a(2n-3)+d)} \times \left(c + \frac{b(n-1)+e}{(2a(n-1)+d)^2}((ae-bd)-ab(n-1)) \right), \tag{22}$$

and these are independent of k_n by construction.

Now we would like to deduce a, b, c, d and e from (21) and (22). Note that as soon as we have found these five values, we can apply a linear transform (according to the zeros of $\sigma(x)$) to bring the differential equation in one of the normal forms of Table 1 which finally gives us the desired information.

We can assume that $\widetilde{B}_n$ and $\widetilde{C}_n$ are in lowest terms. If the degree of either the numerator or the denominator of $\widetilde{B}_n$ is larger than 2, then by (21) $p_n(x)$ is not a classical system. Similarly, if the degree of either the numerator or the denominator of $\widetilde{C}_n$ is larger than 4, by (22) the same conclusion follows.

Otherwise we can multiply (21) and (22) by their common denominators, and bring them therefore in polynomial form. Both resulting equations must be polynomial identities in the variable n, hence all of their coefficients must vanish. This gives a nonlinear system of equations for the unknowns a, b, c, d and e. Any solution of this system with not both a and d being zero yields a differential equation (3), and hence given such a solution one can characterize it via Table 1. Therefore our question can be resolved in this case. In particular, if one of the cases Table 1(1) or (1.4a) applies, then there are no orthogonal polynomial solutions.

If the nonlinear system does *not* have such a solution, we deduce that no such values a, b, c, d and e exist, hence no such differential equation is satisfied by $p_n(x)$, implying that the system is not a linear transformation of a classical orthogonal polynomial system.

Hence the whole question boils down to decide whether the given nonlinear system has nontrivial solutions, and to find these solutions in the affirmative case. As a matter of fact, with Gröbner bases methods, this question can be decided algorithmically [15–17]. Such an algorithm is implemented, e.g., in the computer algebra system REDUCE [16], and Maple's `solve` command can also solve such a system.

Note that the solution of the nonlinear system is not necessarily unique. For example, the Chebyshev polynomials of the first and second kind $T_n(x)$ and $U_n(x)$ satisfy the same recurrence equation, but a different differential equation. We will consider this example in more detail later.

If we apply this algorithm to the recurrence equation $p_{n+2}(x) - xp_{n+1}(x)$ of the power $p_n(x) = x^n$, it generates the complete solution set, given by Table 1(1) and (4a).

The following statement summarizes the above considerations:

Algorithm 1. This algorithm decides whether a given holonomic three-term recurrence equation has shifted, linear transforms of classical orthogonal polynomial solutions, and returns their data if applicable.

1. *Input*: A holonomic three-term recurrence equation
$$q_n(x)p_{n+2}(x) + r_n(x)p_{n+1}(x) + s_n(x)p_n(x) = 0 \quad (q_n(x), r_n(x), s_n(x) \in \mathbb{Q}[n, x]).$$
2. *Shift*: Shift by
$$N := \begin{cases} 0 \quad \text{if } q_{n-1}(x) \text{ and } s_n(x) \text{ have no nonnegative integer zero,} \\ \max\{n \in \mathbb{N}_0 \mid n \text{ is a zero of } q_{n-1}(x) \text{ or } s_n(x)\} + 1 \quad \text{otherwise.} \end{cases} \tag{23}$$
3. *Rewriting*: Rewrite the recurrence equation in the form
$$p_{n+1}(x) = t_n(x)p_n(x) + u_n(x)p_{n-1}(x) \quad (t_n(x), u_n(x) \in \mathbb{Q}(n, x)).$$
If either $t_n(x)$ is not a polynomial of degree one in x or $u_n(x)$ is not constant with respect to x, then return "`no orthogonal polynomial solution exists`"; exit.
4. *Standardization*: Given now A_n, B_n and C_n by
$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x) \quad (A_n, B_n, C_n \in \mathbb{Q}(n),\ A_n \neq 0),$$
define
$$\frac{k_{n+1}}{k_n} := A_n = \frac{v_n}{w_n} \quad (v_n, w_n \in \mathbb{Q}[n])$$
according to (20).
5. *Make monic*: Set
$$\widetilde{B}_n := \frac{B_n}{A_n} \in \mathbb{Q}(n) \quad \text{and} \quad \widetilde{C}_n := \frac{C_n}{A_n A_{n-1}} \in \mathbb{Q}(n)$$
and bring these rational functions in lowest terms. If the degree of either the numerator or the denominator of $\widetilde{B}_n$ is larger than 2, or if the degree of either the numerator or the denominator of $\widetilde{C}_n$ is larger than 4, return "`no classical orthogonal polynomial solution exists`"; exit.
6. *Polynomial identities*: Set
$$\widetilde{B}_n = \frac{2bn(a(n-1)+d) + e(d-2a)}{(2a(n-1)+d)(2an+d)}$$

and

$$\widetilde{C}_n = \frac{-n(a(n-2)+d)}{(a(2n-1)+d)(a(2n-3)+d)} \times \left(c + \frac{b(n-1)+e}{(2a(n-1)+d)^2}((ae-bd)-ab(n-1))\right),$$

using the as yet unknowns a, b, c, d and e. Multiply these identities by their common denominators, and bring them therefore in polynomial form.

7. *Equating coefficients*: Equate the coefficients of the powers of n in the two resulting equations. This results in a nonlinear system in the unknowns a, b, c, d and e. Solve this system by Gröbner bases methods. If the system has no solution or only one with $a = d = 0$, then return "`no classical orthogonal polynomial solution exists`"; exit.
8. *Output*: Return the classical orthogonal polynomial solutions of the differential equations (3) given by the solution vectors (a, b, c, d, e) of the last step, according to the classification of Table 1, together with the information about the standardization given by (20). This information includes the density

$$\frac{\rho(x)}{C} = \frac{1}{\sigma(x)} \exp \int \frac{\tau(x)}{\sigma(x)}\, \mathrm{d}x$$

(see e.g. [13]), and the supporting interval through the zeros of $\sigma(x)$. [1]

Remark. Assume that a given recurrence equation contains parameters. Then our implementation determines for which values of the parameters there are orthogonal polynomial solutions, by solving not only for a, b, c, d and e, but moreover for those parameters.

Example 1. As a first example, we consider the recurrence equation

$$(n+2)P_{n+2}(x) - x(n+1)P_{n+1}(x) + nP_n(x) = 0.$$

Since $s_0(x) \equiv 0$, we see that the shift $p_n(x) := P_{n+1}(x)$ is necessary, i.e., $N = 1$ by (23). For $p_n(x)$, we have the recurrence equation

$$(n+3)p_{n+2}(x) - x(n+2)p_{n+1}(x) + (n+1)p_n(x) = 0. \tag{24}$$

In the first steps this recurrence equation is brought into the form

$$p_{n+1}(x) = \frac{n+1}{n+2} x p_n(x) - \frac{n}{n+2} p_{n-1}(x),$$

[1] If the zeros of $\sigma(x)$ are not real, then these orthogonal polynomials are not positive-definite. The Bessel system is never positive-definite [2].

hence

$$A_n = \frac{k_{n+1}}{k_n} = \frac{n+1}{n+2} = \frac{v_n}{w_n},$$

and therefore

$$k_n = \frac{1}{n+1} k_0.$$

Moreover, for monic $\widetilde{p}_n(x) = p_n(x)/k_n$ we get

$$\widetilde{p}_{n+1}(x) = x\widetilde{p}_n(x) + \widetilde{p}_{n-1}(x),$$

hence $\widetilde{B}_n = 0$ and $\widetilde{C}_n = 1$. The polynomial identities concerning $\widetilde{B}_n$ and $\widetilde{C}_n$ of step 5 of the algorithm yield $b = 0$, $c = -4a$, and either

$$d = a, \text{ or } d = 2a, \text{ or } d = 3a.$$

At this point we have already determined

$$\sigma(x) = ax^2 + bx + c = a(x^2 - 4).$$

Hence possible classical orthogonal polynomial solutions of (24) are defined in the interval $[-2, 2]$.

In the first of the above cases, i.e., for $d = a$, one gets $e = 0$ and the differential equation

$$(x^2 - 4)y''(x) + xy'(x) - n(n-2)y(x) = 0 \tag{25}$$

corresponding to the density

$$\rho(x) = -\frac{1}{\sigma(x)} \exp \int \frac{\tau(x)}{\sigma(x)}\, \mathrm{d}x = \frac{1}{\sqrt{4 - x^2}}.$$

The corresponding orthogonal polynomials are multiples of translated Chebyshev polynomials of the first kind

$$p_n(x) = k_n C_n(x) = \frac{p_0}{n+1} C_n(x) = \frac{2p_0}{n+1} T_n(x/2) \quad (n \geqslant 0) \tag{26}$$

(see e.g. [1], Table 22.2, and (22.5.11); $C_n(x)$ are monic, but $C_0 = 2$, see also Table 22.7), hence finally

$$P_n(x) = p_{n-1}(x) = \frac{2P_1}{n} T_{n-1}(x/2) \quad (n \geqslant 1).$$

In the second of the above cases, i.e., for $d = 2a$, one gets the equation

$$a^2(e - 2a)(e + 2a) = 0$$

with two possible solutions $e = \pm 2a$ that give the differential equations

$$(x^2-4)y''(x) + 2(x+1)y'(x) - n(n-3)y(x) = 0, \tag{27}$$

and

$$(x^2-4)y''(x) + 2(x-1)y'(x) - n(n-3)y(x) = 0. \tag{28}$$

They correspond to the densities

$$\rho(x) = \sqrt{\frac{4+x}{4-x}} \quad \text{and} \quad \rho(x) = \sqrt{\frac{4-x}{4+x}},$$

respectively, hence the orthogonal polynomials are multiples of the Jacobi polynomials $P_n^{(1/2,-1/2)}(x/2)$ and $P_n^{(-1/2,1/2)}(x/2)$.

Finally, in the third of the above cases, i.e., for $d = 3a$, we get again $e = 0$ and

$$(x^2-4)y''(x) + 3xy'(x) - n(n-4)y(x) = 0 \tag{29}$$

corresponding to the density

$$\rho(x) = -\frac{1}{\sigma(x)} \exp \int \frac{\tau(x)}{\sigma(x)}\, \mathrm{d}x = \sqrt{4-x^2}.$$

The corresponding orthogonal polynomials are multiples of translated Chebyshev polynomials of the second kind

$$p_n(x) = k_n S_n(x) = \frac{p_0}{n+1} S_n(x) = \frac{p_0}{n+1} U_n(x/2) \quad (n \geqslant 0) \tag{30}$$

(see e.g. [1], Table 22.2, and (22.5.13); $S_n(x)$ are monic, see also Table 22.8), hence

$$P_n(x) = p_{n-1}(x) = \frac{P_1}{n} U_{n-1}(x/2) \quad (n \geqslant 1).$$

We see that the recurrence equation (24) has four different (shifted) linearly transformed classical orthogonal polynomial solutions!

Using our implementation, these results are obtained by

```
> strict: = true:
> RE := (n + 3) * p(n + 2) - x * (n + 2) * p(n + 1) + (n + 1) * p(n) = 0;
```

$$RE := (n+3)\mathrm{p}(n+2) - x(n+2)\mathrm{p}(n+1) + (n+1)\mathrm{p}(n) = 0$$

```
> REtoDE(RE, p(n), x);
```

Warning: *several solutions found*

$$\left[\left[\%1 + 2(x-1)\left(\frac{\partial}{\partial x}\mathrm{p}(n,x)\right) - n(n+1)\mathrm{p}(n,x) = 0,\right.\right.$$

$$\left.\left.\left[I = [-2,2], \rho(x) = \frac{\sqrt{x+2}}{\sqrt{x-2}}\right]\right]\right],$$

$$\left[\%1 + 2(x+1)\left(\frac{\partial}{\partial x}\mathrm{p}(n,x)\right) - n(n+1)\mathrm{p}(n,x) = 0,\right.$$

$$\left.\left[I = [-2,2], \rho(x) = \frac{\sqrt{x-2}}{\sqrt{x+2}}\right]\right], \left[\%1 + x\left(\frac{\partial}{\partial x}\mathrm{p}(n,x)\right) - n^2\mathrm{p}(n,x) = 0,\right.$$

$$\left.\left[I = [-2,2], \rho(x) = \frac{\sqrt{x^2-4}}{(x-2)(x+2)}\right]\right],$$

$$\left[\%1 + 3x\left(\frac{\partial}{\partial x}\mathrm{p}(n,x)\right) - n(n+2)\mathrm{p}(n,x) = 0,\right.$$

$$\left.\left.\left[I = [-2,2], \rho(x) = \sqrt{x^2-4}\right]\right], \frac{k_{n+1}}{k_n} = \frac{n+1}{n+2}\right]$$

$$\%1 := (x-2)(x+2)\left(\frac{\partial^2}{\partial x^2}\mathrm{p}(n,x)\right)$$

which gives the corresponding differential equations, the intervals and densities, as well as the term ratio $k_{n+1}/k_n = (n+1)/(n+2)$.

With Koornwinder–Swarttouw's `rec2ortho`, these results are obtained by the statements `rec2ortho((n+2)/(n+1),0,n/(n+1))`, `rec2ortho((n+2)/(n+1),0,n/(n+1),4,0)`, `rec2ortho((n+2)/(n+1),0,n/(n+1),2,-1)`, and `rec2ortho((n+2)/(n+1),0,n/(n+1),2,1)`, respectively. Note that here the user must know the initial values to determine possible orthogonal polynomial solutions, whereas our approach finds all possible solutions at once.

Example 2. As a second example, we consider the recurrence equation

$$p_{n+2}(x) - (x-n-1)p_{n+1}(x) + \alpha(n+1)^2 p_n(x) = 0 \tag{31}$$

depending on the parameter $\alpha \in \mathbb{R}$. Here obviously the question arises whether or not there are any instances of this parameter for which there are classical orthogonal polynomial solutions. In step 6 of Algorithm 1 we therefore solve also for this unknown parameter. This gives a slightly more complicated nonlinear system, with the unique solution

$$\left\{b = 2c, c = c, d = -4c, e = 0, a = 0, \alpha = \frac{1}{4}\right\}.$$

Hence the only possible value for α with classical orthogonal polynomial solutions is $\alpha = 1/4$, in which case one gets the differential equation

$$\left(x+\frac{1}{2}\right)p_n''(x) - 2xp_n'(x) - 2np_n(x) = 0$$

with density

$$\rho(x) = 2\mathrm{e}^{-2x}$$

in the interval $[-1/2, \infty]$, corresponding to linearly transformed Laguerre polynomials.

Using our implementation, these results are obtained by

```
> strict: =false:
> REtoDE(RE,p(n),x);
```

Warning: *parameters have the values*,

$$\left\{d = -4c, b = 2c, c = c, e = 0, a = 0, \alpha = \frac{1}{4}\right\}$$

$$\left[\frac{1}{2}(2x+1)\left(\frac{\partial^2}{\partial x^2}\mathrm{p}(n,x)\right) - 2x\left(\frac{\partial}{\partial x}\mathrm{p}(n,x)\right) + 2n\mathrm{p}(n,x) = 0,\right.$$

$$\left.\left[I = \left[\frac{-1}{2}, \infty\right], \rho(x) = 2e^{(-2x)}\right], \frac{k_{n+1}}{k_n} = 1\right].$$

With Koornwinder–Swarttouw's `rec2ortho`, this result can also be obtained. On the other hand, the Bessel polynomials are not accessible with Koornwinder–Swarttouw's `rec2ortho`.

4. Classical discrete orthogonal polynomials

In this section, we give similar results for classical orthogonal polynomials of a discrete variable (see [18, Chapter 2]). The classical discrete orthogonal polynomials are given by a difference equation (5).

These polynomials can be classified similarly as in the continuous case according to the functions $\sigma(x)$ and $\tau(x)$; up to linear transformations the classical discrete orthogonal polynomials are classified according to Table 2 (compare [18, Chapter 2]). In particular, case (2a) corresponds to the non-orthogonal solution x^n in Table 1. Similarly as for the powers

$$\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1},$$

Table 2
Normal forms of discrete polynomials

	$\sigma(x)$	$\sigma(x)+\tau(x)$	$p_n(x)$	Family
1	1	$\alpha x+1+\beta$	$(-1)^n c_n^{(-1/\alpha)}\left(\frac{\alpha-1-\beta x}{\beta}\right)$	Translated Charlier
2a	x	0	$x^{\underline{n}}$	Falling factorial
2b	x	$\mu(\mu\neq 0)$	$c_n^{(\mu)}(x)$	Charlier polynomials
3	x	$\mu(\gamma+x)$	$m_n^{(\gamma,\mu)}(x)$	Meixner polynomials
4	x	$\frac{p}{1-p}(N-x)$	$k_n^{(p)}(x,N)$	Krawtchouk polynomials
5	$x(N+\alpha-x)$	$(x+\beta+1)(N-1-x)$	$h_n^{(\alpha,\beta)}(x,N)$	Hahn polynomials
6	$x(x+\mu)$	$(v+N-1-x)(N-1-x)$	$\widetilde{h}_n^{(\mu,v)}(x,N)$	Hahn–Eberlein polynomials

the *falling factorials* $x^{\underline{n}} := x(x-1)\cdots(x-n+1)$ satisfy

$$\Delta x^{\underline{n}} = n x^{\underline{n-1}}.$$

It turns out that they are connected with the Charlier polynomials by the limiting process

$$\lim_{\mu\to 0}(-1)^n \mu^n c_n^{(\mu)}(x) = \lim_{\mu\to 0}(x-n+1)_n \cdot {}_1F_1\left(\begin{array}{c} -n \\ x-n+1 \end{array}\middle|\mu\right)$$
$$= (-1)^n (x-n+1)_n = x^{\underline{n}},$$

where we used the hypergeometric representation given in [18, (2.7.9)].

Note, however, that other than in the differential equation case the above type of difference equation is *not* invariant under general linear transformations, but only under integer shifts. We will have to take this under consideration.

The classical discrete orthogonal polynomials satisfy a recurrence equation (1)

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x)$$

with A_n, B_n and C_n given by Theorem 1.

Similarly as in the continuous case, this information can be used to generate an algorithm to test whether or not a given holonomic recurrence equation has classical discrete orthogonal polynomial solutions. Obviously the first three steps of this algorithm agree with those given in Algorithm 1.

Algorithm 2. This algorithm decides whether a given holonomic three-term recurrence equation has classical discrete orthogonal polynomial solutions, and returns their data if applicable.

1. *Input*: A holonomic three-term recurrence equation
$$q_n(x)p_{n+2}(x) + r_n(x)p_{n+1}(x) + s_n(x)p_n(x) = 0 \quad (q_n(x), r_n(x), s_n(x) \in \mathbb{Q}[n,x]).$$
2. *Shift*: Shift by $\max\{n \in \mathbb{N}_0 \mid n \text{ is zero of either } q_{n-1}(x) \text{ or } s_n(x)\} + 1$ if necessary.
3. *Rewriting*: Rewrite the recurrence equation in the form
$$p_{n+1}(x) = t_n(x)p_n(x) + u_n(x)p_{n-1}(x) \quad (t_n(x), u_n(x) \in \mathbb{Q}(n,x)).$$
If either $t_n(x)$ is not a polynomial of degree one in x or $u_n(x)$ is not constant with respect to x, return "`no orthogonal polynomial solution exists`"; exit.
4. *Linear transformation*: Rewrite the recurrence equation by the linear transformation $x \mapsto (x-g)/f$ with (as yet) unknowns f and g.
5. *Standardization*: Given now A_n, B_n and C_n by
$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x) \quad (A_n, B_n, C_n \in \mathbb{Q}(n), A_n \neq 0),$$
define

$$\frac{k_{n+1}}{k_n} := A_n = \frac{v_n}{w_n} \quad (v_n, w_n \in \mathbb{Q}[n])$$

according to (8).

6. *Make monic*: Set

$$\widetilde{B}_n := \frac{B_n}{A_n} \in \mathbb{Q}(n) \quad \text{and} \quad \widetilde{C}_n := \frac{C_n}{A_n A_{n-1}} \in \mathbb{Q}(n)$$

and bring these rational functions in lowest terms. If the degree of either the numerator or the denominator of $\widetilde{B}_n$ is larger than 2, if the degree of the numerator of $\widetilde{C}_n$ is larger than 6, or if the degree of the denominator of $\widetilde{C}_n$ is larger than 4, then return "`no classical discrete orthogonal polynomial solution exists`"; exit.

7. *Polynomial identities*: Set

$$\widetilde{B}_n = \frac{k_n}{k_{n+1}} B_n$$

according to (11), and

$$\widetilde{C}_n = \frac{k_{n-1}}{k_{n+1}} C_n$$

according to (12), in terms of the unknowns a, b, c, d, e, f and g. Multiply these identities by their common denominators, and bring them therefore in polynomial form.

8. *Equating coefficients*: Equate the coefficients of the powers of n in the two resulting equations. This results in a nonlinear system in the unknowns a, b, c, d, e, f and g. Solve this system by Gröbner bases methods. If the system has no solution, then return "`no classical discrete orthogonal polynomial solution exists`"; exit.

9. *Output*: Return the classical orthogonal polynomial solutions of the difference equations (5) given by the solution vectors (a, b, c, d, e, f, g) of the last step, according to the classification given in Table 2, together with the information about the standardization given by (8). This information includes the necessary linear transformation $y = fx + g$, as well as the discrete weight function $\rho(x)$ given by

$$\frac{\rho(x+1)}{\rho(x)} = \frac{\sigma(x) + \tau(x)}{\sigma(x+1)}$$

(see e.g. [18]).

Proof. The proof is an obvious modification of Algorithm 1. The only difference is that we have to take a possible linear transformation $fx + g$ into consideration since the difference equation (5) is not invariant under those transformations. This leads to step 4 of the algorithm. □

Note that an application of Algorithm 2 to the recurrence equation $p_{n+2}(x) - (x - n - 1)p_{n+1}(x) = 0$ which is valid for the falling factorial $p_n(x) = x^{\underline{n}}$, generates the difference equation $x\Delta\nabla p_n(x) - x\Delta p_n(x) + np_n(x) = 0$ of Table 2(2a).

Example 3. We consider again the recurrence equation (31)

$$p_{n+2}(x) - (x - n - 1)p_{n+1}(x) + \alpha(n + 1)^2 p_n(x) = 0$$

depending on the parameter $\alpha \in \mathbb{R}$. This time, we are interested in classical discrete orthogonal polynomial solutions.

According to step 4 of Algorithm 2, we rewrite (31) using the linear transformation $x \mapsto (x - g)/f$ with as yet unknowns f and g. Step 5 yields the standardization

$$\frac{k_{n+1}}{k_n} = \frac{1}{f}.$$

In step 8, we solve the resulting nonlinear system for the variables $\{a, b, c, d, e, f, g, \alpha\}$, resulting in

$$\Bigg\{a = 0, b = b, c = -\frac{b(-e + d + b)}{d}, d = d, e = e,$$

$$f = -\frac{d + 2b}{d}, g = -\frac{e}{d}, \alpha = \frac{b(d + b)}{(d + 2b)^2}\Bigg\}. \tag{32}$$

This is a rational representation of the solution. However, since we assume α to be arbitrary, we solve the last equation for b. This yields

$$b = -\frac{d}{2}\left(1 \pm \frac{1}{\sqrt{1 - 4\alpha}}\right),$$

which cannot be represented without radicals. Substituting this into (32) yields the solution

$$\Bigg\{a = 0, b = -\frac{d}{2}\left(1 \pm \frac{1}{\sqrt{1 - 4\alpha}}\right), c = \frac{4\alpha e - e - 2\alpha d}{2(1 - 4\alpha)} \pm \frac{e}{2}\frac{1}{\sqrt{1 - 4\alpha}},$$

$$f = \mp\frac{1}{\sqrt{1 - 4\alpha}}, g = -\frac{e}{d}\Bigg\},$$

d and e being arbitrary. It turns out that for $\alpha < 1/4$ this corresponds to Meixner or Krawtchouk polynomials.

With Koornwinder–Swarttouw's `rec2ortho`, this result can be also obtained. Moreover, `rec2ortho` determines that for $\alpha > 1/4$ one gets Meixner–Pollaczek polynomials. These polynomials are not accessible by our approach.

Example 4. Here we want to discuss the possibility that a given recurrence equation might have several classical discrete orthogonal solutions. Whereas the recurrence equation of the Hahn polynomials $h_n^{(\alpha,\beta)}(x,N)$ has (besides several linear transformations) only this single classical discrete orthogonal solution, the case $\beta = -\alpha$ results in two essentially different solutions.

Here one has the recurrence equation

$$\begin{aligned}&(n+2+\alpha)(2+n)(2n+2)(n-N+1)p_{n+2}(x)+(3+2n)\Big(-6n\alpha-2n^2\alpha\\&\quad-4n^2x-12nx+2n^2N+6nN+4N-4\alpha-8x\Big)p_{n+1}(x)\\&\quad-(1+n)(n+1-\alpha)(2n+4)(n+N+2)p_n(x)=0.\end{aligned}$$

An application of Algorithm 2 shows that this recurrence equation corresponds to the two different difference equations

$$x(-x+1-\alpha+N)\Delta\nabla p_n(x)+(-2x+N+\alpha N)\Delta p_n(x)+n(n-3)p_n(x)=0$$

and

$$\begin{aligned}&(x+\alpha)(-x+1+N)\Delta\nabla p_n(x)-(2x-N+2\alpha+\alpha N)\Delta p_n(x)\\&\quad+n(n-3)p_n(x)=0.\end{aligned}$$

Using our implementation, these results are obtained by

```
> strict: = true:
> RE := (n + 2 + alpha) * (2 + n) * (2 * n + 2) * (n − N + 1) * p(n + 2) +
(3 + 2 * n)*
> (−6 * n * alpha − 2 * n^2 * alpha − 4 * n^2 * x − 2 * n * x + 2 * n^2 *
N + 6 * n * N + 4 * N − 4 * alpha − 8 * x) * p(n + 1)
> −(1 + n) * (n + 1 − alpha) * (2 * n + 4) * (n + N + 2) * p(n) = 0;
```

$$\begin{aligned}RE &:= (n+2+\alpha)(n+2)(2n+2)(n-N+1)\mathrm{p}(n+2)+(3+2n)\\&\quad\times(-6n\alpha-2\alpha n^2-4n^2x-12nx+2n^2N+6nN+4N-4\alpha-8x)\mathrm{p}(n+1)\\&\quad-(n+1)(n+1-\alpha)(2n+4)(n+N+2)\mathrm{p}(n)=0\end{aligned}$$

```
> REtodiscreteDE(RE, p(n), x);
```

Warning: *parameters have the values*,

$$\{b=-a+\alpha a-Na,e=2\alpha a-Na+\alpha aN,c=-\alpha aN-\alpha a,a=a,d=2a\}$$

Warning: *parameters have the values*,

$$\{b = -a + \alpha a - Na, c = 0, e = -Na - \alpha aN, a = a, d = 2a\}$$

`Warning: several solutions found`

$$\Big[[(x+\alpha)(x-1-N)\Delta(\mathrm{Nabla}(\mathrm{p}(n,x),x),x)$$
$$+ (2x - N + 2\alpha + \alpha N)\Delta(\mathrm{p}(n,x),x) - n(n+1)\mathrm{p}(n,x) = 0,$$
$$[\sigma(x) = (x+\alpha)(x-1-N), \sigma(x)+\tau(x) = (x+1)(x+\alpha-N)],$$
$$\rho(x) = \mathrm{Hyperterm}([1, -N+\alpha, 1], [1+\alpha, -N], 1, x)],$$
$$[x(x-1-N+\alpha)\Delta(\mathrm{Nabla}(\mathrm{p}(n,x),x),x)$$
$$+ (2x - N - \alpha N)\Delta(\mathrm{p}(n,x),x) - n(n+1)\mathrm{p}(n,x) = 0,$$
$$[\sigma(x) = x(x-1-N+\alpha), \sigma(x)+\tau(x) = (x+1+\alpha)(x-N)],$$
$$\rho(x) = \mathrm{Hyperterm}([1+\alpha, -N], [-N+\alpha], 1, x)],$$
$$\frac{k_{n+1}}{k_n} = 2\frac{2n+1}{(n+1+\alpha)(n-N)}\Big]$$

Note that `Hyperterm(upper,lower,z,x)` denotes the hypergeometric term (= summand) of the hypergeometric function `hypergeom(upper,lower,z)` with summation variable x, see [8].

Hahn polynomials are not accessible with Koornwinder–Swarttouw's `rec2ortho`.

5. Classical q-orthogonal polynomials

In this section, we consider the same problem for classical q-orthogonal polynomials ([6,11], see e.g. [7]). The classical q-orthogonal polynomials are given by a q-difference equation (6).

These polynomials can be classified similarly as in the continuous and discrete cases according to the functions $\sigma(x)$ and $\tau(x)$; up to linear transformations the classical q-orthogonal polynomials are classified according to Table 3.

For the sake of completeness we have included all families from [7], Chapter 3, although they overlap in several instances. The non-orthogonal polynomial solutions are the powers x^n and the q-Pochhammer functions

$$(x;q)_n := (1-x)(1-xq)\cdots(1-xq^{n-1}).$$

The classical q-orthogonal polynomials satisfy a recurrence equation (1)

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x)$$

with A_n, B_n and C_n given by Theorem 1.

Similarly as in the continuous and discrete cases, this information can be used to generate an algorithm to test whether or not a given holonomic recurrence equation has classical q-orthogonal polynomial solutions.

Table 3
Normal forms of q-polynomials

	$\sigma(x)$	$\tau(x)$	$p_n(x)$	family
1	0	x	x^n	
2	0	$1-x$	$(x;q)_n$	
3	1	$1-x$	$\widetilde{h}_n(x;q)$	Discrete q-Hermite II polynomials
4	1	$\frac{a+1-x}{a(q-1)}$	$V_n^{(a)}(x;q)$	Al-Salam–Carlitz II polynomials
5	x	$\frac{xq+a+q}{a(q-1)}$	$C_n(x;a;q)$	q-Charlier polynomials
6	x	$-q^{\alpha+1}x+\frac{q^{\alpha+1}-1}{q-1}$	$L_n^{(\alpha)}(x;q)$	q-Laguerre polynomials
7	x	$\frac{xq-1}{q-1}$	$S_n(x;q)$	Stieltjes–Wigert polynomials
8	$x-bq$	$\frac{xq-q-c+qbc}{c(q-1)}$	$M_n(x;b,c;q)$	q-Meixner polynomials
9	$x(x-1)$	$-\frac{x-1+aq}{q-1}$	$p_n(x;a\,\vert\,q)$	Little q-Laguerre polynomials
10	$x(x-1)$	$\frac{x+aqx-1}{q-1}$	$K_n(x;a;q)$	alternative q-Charlier polynomials
11	$x(x-1)$	$\frac{1-aq-x+xabq^2}{q-1}$	$p_n(x;a,b\,\vert\,q)$	Little q-Jacobi polynomials
12	$(x-1)(x+1)$	$-\frac{x}{q-1}$	$h_n(x;q)$	discrete q-Hermite I polynomials
13	$(x-1)(x-a)$	$\frac{a+1-x}{q-1}$	$U_n^{(a)}(x;q)$	Al-Salam–Carlitz I polynomials
14	$(x-aq)(x-bq)$	$\frac{aq+bq-abq^2-x}{q-1}$	$P_n(x;a,b;q)$	Big q-Laguerre polynomials
15	$(q^Nx-1)(x-\alpha q)$	$\frac{q^{N+2}\alpha\beta(x-1)+q^{N+1}\alpha-\alpha q+1-xq^N}{q-1}$	$Q_n(x;\alpha,\beta,N\,\vert\,q)$	q-Hahn polynomials
16	$(x-aq)(x-bq)$	$\frac{q(a+c-abq-acq)-x+abq^2x}{q-1}$	$P_n(x;a,b,c;q)$	Big q-Jacobi polynomials

Algorithm 3. This algorithm decides whether a given holonomic three-term recurrence equation has classical q-orthogonal polynomial solutions, and returns their data if applicable.

1. *Input*: A holonomic three-term recurrence equation

$$q_n(x)p_{n+2}(x) + r_n(x)p_{n+1}(x) + s_n(x)p_n(x) = 0$$
$$(q_n(x), r_n(x), s_n(x) \in \mathbb{Q}[q^n, q, x]).$$

2. *Shift*: Shift by $\max\{n \in \mathbb{N}_0 \mid n \text{ is zero of either } q_{n-1}(x) \text{ or } s_n(x)\} + 1$ if necessary.
3. *Rewriting*: Rewrite the recurrence equation in the form

$$p_{n+1}(x) = t_n(x)p_n(x) + u_n(x)p_{n-1}(x) \quad (t_n(x), u_n(x) \in \mathbb{Q}(q^n, q, x)).$$

   If either $t_n(x)$ is not a polynomial of degree one in x or $u_n(x)$ is not constant with respect to x, return "`no q-orthogonal polynomial solution exists`"; exit.
4. *Linear transformation*: Rewrite the recurrence equation by the linear transformation $x \mapsto (x-g)/f$ with (as yet) unknowns f and g.
5. *Standardization*: Given now A_n, B_n and C_n by

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x) \quad (A_n, B_n, C_n \in \mathbb{Q}(q^n, q), A_n \neq 0),$$

   define

$$\frac{k_{n+1}}{k_n} := A_n = \frac{v_n}{w_n} \quad (v_n, w_n \in \mathbb{Q}[q^n, q]).$$

6. *Make monic*: Set

$$\widetilde{B}_n := \frac{B_n}{A_n} \in \mathbb{Q}(q^n, q) \quad \text{and} \quad \widetilde{C}_n := \frac{C_n}{A_n A_{n-1}} \in \mathbb{Q}(q^n, q)$$

   and bring these rational functions in lowest terms. If the degree (w.r.t $N := q^n$) of the numerator of $\widetilde{B}_n$ is larger than 3, the degree of the denominator of $\widetilde{B}_n$ is larger than 4, the degree of the numerator of $\widetilde{C}_n$ is larger than 7, or the degree of the denominator of $\widetilde{C}_n$ is larger than 8, then return "`no classical q-orthogonal polynomial solution exists`"; exit.
7. *Polynomial identities*: Set

$$\widetilde{B}_n = \frac{k_n}{k_{n+1}} B_n$$

   according to (13), and

$$\widetilde{C}_n = \frac{k_{n-1}}{k_{n+1}} C_n$$

according to (14), in terms of the unknowns a, b, c, d, e, f and g. Multiply these identities by their common denominators, and bring them therefore in polynomial form.

8. *Equating coefficients*: Equate the coefficients of the powers of $N = q^n$ in the two resulting equations. This results in a nonlinear system in the unknowns a, b, c, d, e, f and g. Solve this system by Gröbner bases methods. If the system has no solution, then return "`no classical q-orthogonal polynomial solution exists`"; exit.
9. *Output*: Return the q-classical orthogonal polynomial solutions of the q-difference equations (6) given by the solution vectors (a, b, c, d, e, f, g) of the last step, according to the classification given in Table 3, together with the information about the standardization given by (8). This information includes the necessary linear transformation $y = fx + g$, as well as the q-discrete weight function $\rho(x)$ given by

$$\frac{\rho(qx)}{\rho(x)} = \frac{\sigma(x) + (q-1)x\tau(x)}{\sigma(qx)}.$$

Proof. The proof is an obvious modification of Algorithms 1 and 2. □

Example 5. We consider the recurrence equation

$$p_{n+2}(x) - xp_{n+1}(x) + \alpha q^n(q^{n+1} - 1)p_n(x) = 0$$

depending on the parameter $\alpha \in \mathbb{R}$. This time, we are interested in classical q-orthogonal polynomial solutions.

According to step 4 of Algorithm 3, we rewrite (31) using the linear transformation $x \mapsto (x - g)/f$ with as yet unknowns f and g. Step 5 yields the standardization

$$\frac{k_{n+1}}{k_n} = \frac{1}{f}.$$

In step 8, we solve the resulting nonlinear system for the variables $\{a, b, c, d, e, f, g, \alpha\}$, resulting in the following nontrivial solution

$$\{a = -dq + d, b = 0, c = -\alpha f^2 d(q-1), d = d, e = 0, f = f, g = 0, \alpha = \alpha\}$$

that corresponds – for $f = 1$ – to the q-difference equation

$$(x^2 + \alpha)D_q D_{1/q} y(x) - \frac{x}{q-1} D_q y(x) + \lambda_{q,n} y(x) = 0.$$

Hence for every $\alpha \in \mathbb{R}$ and every scale factor f there is a q-classical solution that corresponds to q-Hermite I polynomials, see Table 3, which have real support for $\alpha < 0$.

Using our implementation, these results are obtained by

```
> RE:=p(n+2)-x*p(n+1)+alpha*q^n*(q^(n+1)-1)*p(n)=0;
```

$$RE := \mathrm{p}(n+2) - x\mathrm{p}(n+1) + \alpha q^n (q^{(n+1)} - 1)\mathrm{p}(n) = 0$$

```
> REtoqDE(RE,p(n),q,x);
```

Warning: *parameters have the values*

$$\{e = 0, a = -dq + d, c = -\alpha qd + \alpha d, d = d, b = 0\}$$

$$\left[(x^2+\alpha)\mathrm{Dq}\left(\mathrm{Dq}\left(\mathrm{p}(n,x),\frac{1}{q},x\right),q,x\right) - \frac{x\mathrm{Dq}(\mathrm{p}(n,x),q,x)}{q-1} + \frac{q(-1+q^n)\mathrm{p}(n,x)}{(q-1)^2 q^n} = 0, \frac{\rho(qx)}{\rho(x)} = \frac{\alpha}{q^2x^2+\alpha}, \frac{k_{n+1}}{k_n} = 1\right].$$

Note that q-polynomials are not accessible with Koornwinder–Swarttouw's `rec2ortho`.

Note: The Maple implementation `retode`, and a worksheet `retode.mws` with the examples of this paper can be obtained from http://www.mathematik.uni-kassel.de/~koepf/Publikationen.

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