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# On Non-linear Characterizations of Classical Orthogonal Polynomials

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**Abstract.** Classical orthogonal polynomials are known to satisfy seven equivalent properties, namely the Pearson equation for the linear functional, the second-order differential/difference/$q$-differential/ divided-difference equation, the orthogonality of the derivatives, the Rodrigues formula, two types of structure relations, and the Riccati equation for the formal Stieltjes function. In this work, following previous work by Kil et al. (J Differ Equ Appl 4:145–162, 1998a; Kyungpook Math J 38:259–281, 1998b), we state and prove a non-linear characterization result for classical orthogonal polynomials on non-uniform lattices. Next, we give explicit relations for some families of these classes.

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## 1. Introduction

Univariate orthogonal polynomials (or orthogonal polynomials for short) are systems of polynomials $(p_n)_n$ with $\deg(p_n) = n$, satisfying a certain orthogonality relation. They are very useful in practice in various domains of mathematics, physics, engineering, image processing and so on, because of the many properties and relations they satisfy. As examples of areas where orthogonal polynomials play important roles, we could cite approximation theory (see for example [6,31]) and also numerical analysis (see [14,15]).

It is known that any family of orthogonal polynomials $(p_n)_{n\geq 0}$ satisfies a three-term recurrence relation of the form

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x), \quad p_{-1}(x) = 0. \qquad (1.1)$$

If $h_n = \langle \mathcal{L}, p_n^2 \rangle$, where $\mathcal{L}$ is the corresponding linear functional with respect to the sequence $(p_n)_{n\geq 0}$ and $k_n$ is the leading coefficient of $p_n(x)$ (see [17]), then

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$$A_n = \frac{k_{n+1}}{k_n}, \quad C_n = \frac{A_n}{A_{n+1}} \frac{h_n}{h_{n+1}}, \quad n \geq 1$$

and we set $C_0 = 1$.

The systems of orthogonal polynomials associated with the names of Hermite, Laguerre, Jacobi and Bessel (including the special cases named after Tchebychev, Legendre, and Gegenbauer) are the most extensively and widely applied systems.

An orthogonal polynomial system $(p_n)_{n\geq 0}$ with respect to a weight function $\rho(x)$ is called classical if it satisfies one of the equivalent assertions (see [17]):

- $(p_n)_{n\geq 0}$ satisfies a second-order linear differential equation of the Sturm–Liouville type

$$\phi(x)y''(x) + \psi(x)y'(x) + \lambda_n y(x) = 0, \tag{1.2}$$

  where $\phi(x)$ is a polynomial of degree $\leq 2$ and $\psi(x)$ is a polynomial of exact degree 1, both independent on $n$ and $\lambda_n$ is independent on $x$.
- The derivatives $(p'_{n+1})_{n\geq 0}$ form an orthogonal polynomial system.
- The $p_n$s have the Rodrigues representation

$$p_n(x) = \frac{D_n}{\rho(x)} \left(\phi^n(x)\rho(x)\right)^{(n)}, \quad n \geq 0. \tag{1.3}$$

- The weight function $\rho(x)$ satisfies a Pearson-type equation

$$(\phi(x)\rho(x))' = \psi(x)\rho(x). \tag{1.4}$$

- The $p_n$s satisfy a difference-differential equation (or structure relation) of the form

$$\pi(x)p'_n(x) = (\alpha_n x + \beta_n)p_n(x) + \gamma_n p_{n-1}(x). \tag{1.5}$$

In his paper [2], Al-Salam has obtained an expression for the derivative of the product of two consecutive Bessel polynomials and has shown that this expression does, in fact, characterize the Bessel polynomials. Based on this paper, McCarthy in [25] proved that there is an analogous characterization for very classical orthogonal polynomials (Hermite, Laguerre and Jacobi polynomials). This characterization can be stated as

- $(p_n)_{n\geq 0}$ satisfies a non-linear equation of the form:

$$\phi(x)\frac{\mathrm{d}}{\mathrm{d}x}(p_n(x)p_{n-1}(x)) = (\alpha_n x + \beta_n)p_n(x)p_{n-1}(x) + \gamma_n p_n^2(x) + \delta_n p_{n-1}^2(x), \tag{1.6}$$

  where $\alpha_n$, $\beta_n$, $\gamma_n$ and $\delta_n$ are independent on $x$.

Note that several other characterizations of classical orthogonal polynomials with respect to the derivative operator can be found in [23].

Very close to the very classical orthogonal polynomials (classical orthogonal polynomials of a continuous variable) are the classical orthogonal polynomials of a discrete variable. An orthogonal polynomial system $(p_n)_{n\geq 0}$ of a discrete variable with respect to a weight function $\rho(x)$ is called classical if it satisfies one of the equivalent assertions (see [1,8,13]):

- $(p_n)_{n\geq 0}$ satisfies a second-order linear difference equation of the Sturm–Liouville type

$$\phi(x)\Delta\nabla y(x) + \psi(x)\Delta y(x) + \lambda_n y(x) = 0, \qquad (1.7)$$

  where $\phi(x)$ is a polynomial of degree $\leq 2$ and $\psi(x)$ is a polynomial of exact degree 1, both independent on $n$ and $\lambda_n$ is independent on $x$.
- The sequence of difference polynomials $(\Delta p_{n+1})_{n\geq 0}$ form an orthogonal polynomial system of discrete variable.
- The $p_n$s have the Rodrigues representation

$$p_n(x) = \frac{D_n}{\rho(x)}\Delta^n\left(\phi^n(x)\rho(x)\right), \quad n \geq 0. \qquad (1.8)$$

- The weight function $\rho(x)$ satisfies a Pearson-type equation

$$\Delta[\phi(x)\rho(x)] = \psi(x)\rho(x). \qquad (1.9)$$

- The $p_n$s satisfy a difference equation (or structure relation) of the form

$$\pi(x)\nabla p_n(x) = (\alpha_n x + \beta_n)p_n(x) + \gamma_n p_{n-1}(x), \qquad (1.10)$$

  or otherwise stated (see [19])

$$\phi(x)\nabla p_n(x) = \tilde{\alpha}_n p_{n+1}(x) + \tilde{\beta}_n p_n(x) + \tilde{\gamma}_n p_{n-1}(x). \qquad (1.11)$$

- For each $n \geq 1$, $p_n$ and $p_{n-1}$ satisfy a relation of the form (see [21, Theorem 5.2])

$$\pi(x)\left[p_n(x)\nabla p_{n-1}(x) + p_{n-1}(x)\nabla p_n(x)\right]$$
$$= U_n p_n^2(x) + V_n p_{n-1}^2(x) + (W_n x + Y_n)p_n(x)p_{n-1}(x),$$

  where the coefficients $U_n$, $V_n$, $W_n$ and $Y_n$ are independent on $x$ and $\pi$ is a polynomial of degree less or equal to 2.

It should be noted that the operators $\Delta$ and $\nabla$ are respectively defined by

$$\Delta f(x) = f(x+1) - f(x),$$
$$\nabla f(x) = f(x) - f(x-1).$$

Close to the classical discrete orthogonal polynomials are classical orthogonal polynomials of a $q$-discrete variable. An orthogonal polynomial system $(p_n)_{n\geq 0}$ of a $q$-discrete variable with respect to a weight function $\rho(x)$ is called classical if it satisfies one of the equivalent assertions (see [8,18,19]):

- $(p_n)_{n\geq 0}$ satisfies a second-order linear $q$-difference equation of the Sturm–Liouville type

$$\phi(x)\mathcal{D}_q\mathcal{D}_{\frac{1}{q}}y(x) + \psi(x)\mathcal{D}_q y(x) + \lambda_n y(x) = 0, \qquad (1.12)$$

  where $\phi(x)$ is a polynomial of degree less than or equal to 2 and $\psi(x)$ is a polynomial of exact degree 1, both independent on $n$ and $\lambda_n$ is independent on $x$.
- The sequence of $q$-difference polynomials $(\mathcal{D}_q p_{n+1})_{n\geq 0}$ form an orthogonal polynomial system of a $q$-discrete variable.

- The $p_n$s have the Rodrigues representation

$$p_n(x) = \frac{D_n}{\rho(x)} \mathcal{D}_q^n \left( \phi^n(x)\rho(x) \right), \quad n \geq 0. \tag{1.13}$$

- The weight function $\rho(x)$ satisfies a Pearson-type equation

$$\mathcal{D}_q[\phi(x)\rho(x)] = \psi(x)\rho(x). \tag{1.14}$$

- The $p_n$s satisfy a $q$-difference equation (or structure relation) of the form (see [19])

$$\phi(x)\mathcal{D}_{\frac{1}{q}} p_n(x) = \tilde{\alpha}_n p_{n+1}(x) + \tilde{\beta}_n p_n(x) + \tilde{\gamma}_n p_{n-1}(x). \tag{1.15}$$

- For each $n \geq 1$, $p_n$ and $p_{n-1}$ satisfy a relation of the form (see [22, Theorem 3.5])

$$\tilde{\pi}(x) \left[ p_n(x)\mathcal{D}_{\frac{1}{q}} p_{n-1}(x) + p_{n-1}(x)\mathcal{D}_{\frac{1}{q}} p_n(x) \right]$$
$$= \tilde{U}_n p_n^2(x) + \tilde{V}_n p_{n-1}^2(x) + (\tilde{W}_n x + \tilde{Y}_n)p_n(x)p_{n-1}(x),$$

where the coefficients $\tilde{U}_n$, $\tilde{V}_n$, $\tilde{W}_n$ and $\tilde{Y}_n$ are independent on $x$ and $\tilde{\pi}$ is a polynomial of degree less than or equal to 2.

It should be noted that the $q$-derivative $D_q$ is defined as

$$D_q f(x) = \begin{cases} \dfrac{f(x) - f(qx)}{(1-q)x} & \text{if } q \neq 1 \text{ and } x \neq 0 \\ f'(0) & \text{if } x = 0 \end{cases}.$$

The difference operator $\Delta$ and the $q$-derivative $D_q$ are both special cases of the Hahn's operator $D_{q,\omega}$ (see [7]) which is defined as

$$D_{q,\omega} f(x) = \frac{f(qx + \omega) - f(x)}{(qx + \omega) - x}.$$

More precisely, $D_q = D_{q,0}$ and $\Delta = D_{1,1}$.

In this paper, we prove equivalent non-linear characterization results similar to (1.6) for classical orthogonal polynomials on non-uniform lattices (including Wilson and Askey–Wilson polynomials). Also, we prove such a non-linear characterization for Meixner–Pollaczek and Continuous Hahn polynomials. Indeed, we give explicitly the coefficients of these relations for some families of classical orthogonal polynomials on non-uniform lattices.

## 2. Preliminaries

This section contains some preliminary definitions and results that are useful for a better reading of this article. The $q$-hypergeometric series, a fractional $q$-derivative and fractional $q$-integral are defined. The reader will consult the reference [18] for more informations about these concepts.

## 2.1. The Hypergeometric Series

In what follows, the symbol $(a)_n$ denotes the so-called Pochhammer symbol and is defined by

$$(a)_m = \begin{cases} 1 & \text{if } m = 0 \\ a(a+1)\cdots(a+m-1) & \text{if } m = 1, 2, \ldots \end{cases}$$

and the hypergeometric series is defined as

$$_pF_q \left( \begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} \middle| x \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!}.$$

## 2.2. The $q$-Hypergeometric Series

The basic hypergeometric or $q$-hypergeometric series $_r\phi_s$ is defined by the series

$$_r\phi_s \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \middle| q; z \right) := \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n} \left( (-1)^n q^{\binom{k}{2}} \right)^{1+s-r} \frac{z^n}{(q; q)_n},$$

where

$$(a_1, \ldots, a_r; q)_n := (a_1; q)_n \cdots (a_r; q)_n,$$

with

$$(a_i; q)_n = \begin{cases} \prod_{j=0}^{n-1} (1 - a_i q^j) & \text{if } n = 1, 2, 3, \ldots \\ 1 & \text{if } n = 0 \end{cases}.$$

For $n = \infty$, we set

$$(a; q)_\infty = \prod_{n=0}^{\infty} (1 - a q^n), \ |q| < 1.$$

The notation $(a; q)_n$ is the so-called $q$-Pochhammer symbol.

## 2.3. Difference and Divided-Difference Operators

**2.3.1. The Operators $\mathcal{D}$ and $\mathcal{S}$.** We define the difference operator $\mathcal{D}$ (see [26,28]) and its companion operator $\mathcal{S}$ as follows:

$$\mathcal{D}f(x) = f\left( x + \frac{i}{2} \right) - f\left( x - \frac{i}{2} \right), \quad \mathcal{S}f(x) = \frac{f\left( x + \frac{i}{2} \right) + f\left( x - \frac{i}{2} \right)}{2},$$

with $i^2 = -1$.

The operator $\mathcal{D}$ transforms a polynomial of degree $n$ $(n \geq 1)$ in $x$ into a polynomial of degree $n - 1$ in $x$ and a polynomial of degree 0 into the zero polynomial. The operator $\mathcal{S}$ transforms a polynomial of degree $n$ in $x$ into a polynomial of degree $n$ in $x$.

The operators $\mathcal{D}$ and $\mathcal{S}$ fulfill the following properties.

**Proposition 2.1.** (See [26,30]) *The operators $\mathcal{D}$ and $\mathcal{S}$ satisfy the following product rules*

$$\mathcal{D}(fg) = \mathcal{D}f\mathcal{S}g + \mathcal{S}f\mathcal{D}g, \tag{2.1}$$

$$\mathcal{S}(fg) = \frac{1}{4}\mathcal{D}f\mathcal{D}g + \mathcal{S}f\mathcal{S}g, \tag{2.2}$$

$$\mathcal{D}\mathcal{S} = \mathcal{S}\mathcal{D}, \tag{2.3}$$

$$\mathcal{S}^2 = \frac{1}{4}\mathcal{D}^2 + \mathbf{I}, \tag{2.4}$$

*where* $\mathbf{I}f = f$.

**2.3.2. The Operators D and S.** We define the difference operator $\mathbf{D}$ (see [27]) and its companion operator $\mathbf{S}$ as follows:

$$\mathbf{D}f(x^2) = \frac{f\left((x+\frac{i}{2})^2\right) - f\left((x-\frac{i}{2})^2\right)}{2ix}, \quad \mathbf{S}f(x^2) = \frac{f\left((x+\frac{i}{2})^2\right) + f\left((x-\frac{i}{2})^2\right)}{2},$$

with $i^2 = -1$. The operator $\mathbf{D}$ transforms a polynomial of degree $n$ $(n \geq 1)$ in $x^2$ into a polynomial of degree $n-1$ in $x^2$ and a polynomial of degree 0 into the zero polynomial. The operator $\mathbf{S}$ transforms a polynomial of degree $n$ in $x^2$ into a polynomial of degree $n$ in $x^2$.

The operators $\mathbf{D}$ and $\mathbf{S}$ fulfill the following properties.

**Proposition 2.2.** (See [27]) *The operators $\mathbf{D}$ and $\mathbf{S}$ satisfy the following product rules*

$$\mathbf{D}(fg) = \mathbf{D}f\mathbf{S}g + \mathbf{S}f\mathbf{D}g, \tag{2.5}$$

$$\mathbf{S}(fg) = -x^2\mathbf{D}f\mathbf{D}g + \mathbf{S}f\mathbf{S}g, \tag{2.6}$$

$$\mathbf{D}\mathbf{S} = \mathbf{S}\mathbf{D} - \frac{1}{2}\mathbf{D}^2, \tag{2.7}$$

$$\mathbf{S}^2 = -x^2\mathbf{D}^2 - \frac{1}{2}\mathbf{S}\mathbf{D} + \mathbf{I}, \tag{2.8}$$

*where* $\mathbf{I}f = f$.

**2.3.3. The Operators $\mathbb{D}_x$ and $\mathbb{S}_x$.** We define the operator $\mathbb{D}_x$ (called divided-difference operator) and its companion operator $\mathbb{S}_x$ (called mean operator) as [5,9,11,29]

$$\mathbb{D}_x f(x(s)) = \frac{f(x(s+\frac{1}{2})) - f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})},$$

$$\mathbb{S}_x f(x(s)) = \frac{f(x(s+\frac{1}{2})) + f(x(s-\frac{1}{2}))}{2},$$

where $x(s)$ is a non-uniform lattice (see [9]). The operator $\mathbb{D}_x$ transforms a polynomial of degree $n$ $(n \geq 1)$ in $x(s)$ into a polynomial of degree $n-1$ in $x(s)$ and a polynomial of degree 0 into the zero polynomial. The operator $\mathbb{S}_x$ transforms a polynomial of degree $n$ in $x(s)$ into a polynomial of degree $n$ in $x(s)$.

The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ satisfy the product rules

$$\mathbb{D}_x \left( f(x(s))g(x(s)) \right) = \mathbb{S}_x f(x(s)) \, \mathbb{D}_x g(x(s)) + \mathbb{D}_x f(x(s)) \, \mathbb{S}_x g(x(s)), \qquad (2.9)$$

$$\mathbb{S}_x \left( f(x(s))g(x(s)) \right) = U_2(x(s)) \, \mathbb{D}_x f(x(s)) \, \mathbb{D}_x g(x(s)) + \mathbb{S}_x f(x(s)) \, \mathbb{S}_x g(x(s)),$$
$$(2.10)$$

$$\mathbb{D}_x \mathbb{S}_x f = \alpha \, \mathbb{S}_x \mathbb{D}_x f + U_1 \, \mathbb{D}_x^2 f, \qquad (2.11)$$

$$\mathbb{S}_x^2 f = U_1 \mathbb{S}_x \, \mathbb{D}_x f + \alpha U_2 \, \mathbb{D}_x^2 f + f, \qquad (2.12)$$

where $U_2$ is a polynomial of degree 2

$$U_2(x(s)) = (\alpha^2 - 1) \, x^2(s) + 2 \, \beta \, (\alpha + 1) \, x(s) + \delta_x, \qquad (2.13)$$

and $\delta_x$ is a constant depending on $\alpha$, $\beta$ and the initial values $x(0)$ and $x(1)$ of $x(s)$:

$$\delta_x = \frac{x^2(0) + x^2(1)}{4\alpha^2} - \frac{(2\alpha^2 - 1)}{2\alpha^2} x(0) \, x(1) - \frac{\beta \, (\alpha + 1)}{\alpha^2} (x(0) + x(1)) + \frac{\beta^2 \, (\alpha + 1)^2}{\alpha^2},$$

and

$$U_1(s) := U_1(x(s)) = (\alpha^2 - 1) \, x(s) + \beta \, (\alpha + 1), \quad U_2(s) := U_2(x(s)). \quad (2.14)$$

Note that

$$\mathbb{D}_x F_n(x(s)) = \gamma_n F_{n-1}(x(s)),$$

$$\mathbb{S}_x F_n(x(s)) = \alpha_n F_n(x(s)) + \frac{\gamma_n}{2} \nabla x_{n+1}(\varepsilon) F_{n-1}(x(s)),$$

where $F_n(x(s))$ is a function defined in [24]. More properties of the non-uniform lattices $x(s)$, the properties of the divided-difference operator $\mathbb{D}_x$ and its companion $\mathbb{S}_x$ can be found in [10–12, 16, 24] : $x(s)$ satisfies the conditions

$$x(s + k) - x(s) = \gamma_k \nabla x_{k+1}(s), \qquad (2.15)$$

$$\frac{x(s + k) - x(s)}{2} = \alpha_k x_k(s) + \beta_k, \qquad (2.16)$$

for $k = 0, 1, \ldots$, with

$$\alpha_0 = 1, \alpha_1 = \alpha, \beta_0 = 0, \beta_1 = \beta, \gamma_0 = 0, \gamma_1 = 1,$$

and the sequences $(\alpha_k)$, $(\beta_k)$, $(\gamma_k)$ satisfy the following relations

$$\alpha_{k+1} - 2\alpha\alpha_k + \alpha_{k-1} = 0,$$
$$\beta_{k+1} - 2\beta_k + \beta_{k-1} = 2\beta\alpha_k,$$
$$\gamma_{k+1} - \gamma_{k-1} = 2\alpha_k,$$

for $k = 0, 1, \ldots$.

## 3. Non-linear Characterization for Meixner–Pollaczek and Continuous Hahn Polynomials

The Meixner–Pollaczek polynomials $P_n^{(\lambda)}(x; \varphi)$ and the Continuous Hahn polynomials $p_n(x; a, b, c, d)$, respectively, have the hypergeometric representation (see [18]):

$$P_n^{(\lambda)}(x; \varphi) = \frac{(2\lambda)_n}{n!} e^{in\varphi}{}_2F_1 \left( \begin{matrix} -n, \lambda + ix \\ 2\lambda \end{matrix} \, \middle| \, 1 - e^{-2i\varphi} \right), \tag{3.1}$$

$$\frac{p_n(x; a, b, c, d)}{(a+d)_n(a+c)_n} = \frac{i^n}{n!}{}_3F_2 \left( \begin{matrix} -n, n+a+b+c+d-1, a+ix \\ a+c, a+d \end{matrix} \, \middle| \, 1 \right). \tag{3.2}$$

They are known to satisfy the second-order difference equation (see [30])

$$\phi(x)\mathcal{D}^2 y(x) + \psi(x)\mathcal{S}\mathcal{D}y(x) + \lambda_n y(x) = 0, \tag{3.3}$$

where $\phi$ and $\psi$ are polynomials of degree 2 and 1, respectively, and $\lambda$ is a constant depending on the degree of the polynomial solution and the parameters involved in the polynomials.

Note that for the Meixner–Pollaczek polynomials, we have (see [30])

$$\phi(x) = i(\lambda \sin \varphi - x \cos \varphi), \tag{3.4}$$

$$\psi(x) = 2(\lambda \cos \varphi + x \sin \varphi), \tag{3.5}$$

and

$$\lambda_n = -2in \sin \varphi,$$

and for the Continuous Hahn polynomials we have (see [30])

$$\phi(x) = -x^2 + \frac{i}{2}(a+b-c-d)x + \frac{1}{2}(ab+cd),$$

$$\psi(x) = -i(a+b+c+d)x + cd - ab,$$

and

$$\lambda = \lambda_n = -n(n+a+b+c+d-1).$$

**Theorem 3.1.** (Non-linear characterization) *Let* $(P_n)_{n\geq 0}$ *be a sequence of classical orthogonal polynomials on non-uniform lattice. Then, for* $n \geq 1$, $P_n(x)$ *and* $P_{n-1}(x)$ *satisfy*

$$\phi(x)\Big[ P_n(x)\mathcal{S}\mathcal{D}P_{n-1}(x) + P_{n-1}(x)\mathcal{S}\mathcal{D}P_n(x) \Big]$$

$$+ \psi(x)\Big[ P_n(x)\mathcal{S}^2 P_{n-1}(x) + P_{n-1}(x)\mathcal{S}^2 P_n(x) \Big]$$

$$= \Bigg[ (\psi_1 + 2i\phi_2)x + \psi_0 + \psi_1 \left( \frac{B_n}{2A_n} - \frac{B_{n-1}}{2A_{n-1}} \right)$$

$$+ i\phi_2 \left( n\frac{B_n}{A_n} - (n-2)\frac{B_{n-1}}{A_{n-1}} \right) \Bigg] P_n(x)P_{n-1}(x)$$

$$+ \frac{1}{A_{n-1}}(\psi_1 + (2n-3)i\phi_2)P_n^2(x) - \frac{C_n}{A_n}(\psi_1 + (2n-1)i\phi_2)P_{n-1}^2(x). \tag{3.6}$$

*Furthermore, if $(Q_n)_{n \in \mathbb{N}}$ is a sequence of polynomials such that $Q_0(x) = P_0(x)$ and, for $n \geq 1$, $Q_n(x)$ and $Q_{n-1}(x)$ satisfy (3.6). Then $Q_n(x) = P_n(x)$, for all $n \geq 0$.*

*Proof.* Using the fact that the sequence $(P_n)_{n \geq 0}$ is a classical orthogonal polynomial sequence, for all non-negative integer $n$, $P_{n+1}(x)$ satisfies (3.3), namely:

$$\phi(x)\mathcal{D}^2 P_{n+1}(x) + \psi(x)\mathcal{S}\mathcal{D}P_{n+1}(x) + \lambda_{n+1}P_{n+1}(x) = 0, \qquad (3.7)$$

with

$$\phi(x) = \phi_2 x^2 + \phi_1 x + \phi_0; \quad \psi(x) = \psi_1 x + \psi_0; \quad \lambda_n = n(n-1)\phi_2 - in\psi_1.$$

In (1.1), using the relations (2.1), (2.2), (2.3) and (2.4), we obtain:

$$\mathcal{D}^2 P_{n+1}(x) = 2iA_n \mathcal{S}\mathcal{D}P_n(x) + (A_n x + B_n)\mathcal{D}^2 P_n(x) - C_n \mathcal{D}^2 P_{n-1}(x) \quad (3.8)$$

and

$$\mathcal{S}\mathcal{D}P_{n+1}(x) = 2iA_n \mathcal{S}^2 P_n(x) - iA_n P_n(x) + (A_n x + B_n)\mathcal{S}\mathcal{D}P_n(x) - C_n \mathcal{S}\mathcal{D}P_{n-1}(x). \qquad (3.9)$$

Using (1.1), (3.8) and (3.9) to replace $\mathcal{D}^2 P_{n+1}(x)$, $\mathcal{S}\mathcal{D}P_{n+1}(x)$ and $P_{n+1}(x)$ in (3.7), we obtain:

$$\phi(x)\mathcal{S}\mathcal{D}P_n(x) + \psi(x)\mathcal{S}^2 P_n(x) = -\frac{C_n}{2iA_n}(\lambda_{n-1} - \lambda_{n+1})P_{n-1}(x)$$
$$+ \left[\frac{1}{2}\psi(x) + \left(\frac{1}{2i}x + \frac{B_n}{2iA_n}\right)(\lambda_n - \lambda_{n+1})\right]P_n(x). \qquad (3.10)$$

For $n \geq 2$, we replace $n$ by $n-1$ in (3.10) and obtain:

$$\phi(x)\mathcal{S}\mathcal{D}P_{n-1}(x) + \psi(x)\mathcal{S}^2 P_{n-1}(x)$$
$$= -\frac{C_{n-1}}{2iA_{n-1}}(\lambda_{n-2} - \lambda_n)P_{n-2}(x)$$
$$+ \left[\frac{1}{2}\psi(x) + \left(\frac{1}{2i}x + \frac{B_{n-1}}{2iA_{n-1}}\right)(\lambda_{n-1} - \lambda_n)\right]P_{n-1}(x). \quad (3.11)$$

We replace again $n$ by $n-1$ in (1.1) and use the resulting relation to replace $P_{n-2}(x)$ in (3.11) to obtain:

$$\phi(x)\mathcal{S}\mathcal{D}P_{n-1}(x) + \psi(x)\mathcal{S}^2 P_{n-1}(x)$$
$$= -\frac{1}{2iA_{n-1}}(\lambda_{n-2} - \lambda_n)P_n(x)$$
$$+ \left[\frac{1}{2}\psi(x) + \left(\frac{1}{2i}x + \frac{B_{n-1}}{2iA_{n-1}}\right)(\lambda_{n-1} - \lambda_{n-2})\right]P_{n-1}(x). \quad (3.12)$$

If we multiply (3.10) by $P_{n-1}(x)$, (3.12) by $P_n(x)$ and add the resulting expression, we get:

$$
\begin{aligned}
&\phi(x)\left[P_n(x)\mathcal{SD}P_{n-1}(x) + P_{n-1}(x)\mathcal{SD}P_n(x)\right] \\
&\quad + \psi(x)\left[P_n(x)\mathcal{S}^2 P_{n-1}(x) + P_{n-1}(x)\mathcal{S}^2 P_n(x)\right] \\
&= \left[(\psi_1 + 2i\phi_2)x + \psi_0 + \psi_1\left(\frac{B_n}{2A_n} - \frac{B_{n-1}}{2A_{n-1}}\right)\right. \\
&\qquad \left. + i\phi_2\left(n\frac{B_n}{A_n} - (n-2)\frac{B_{n-1}}{A_{n-1}}\right)\right]P_n(x)P_{n-1}(x) \\
&\qquad + \frac{1}{A_{n-1}}(\psi_1 + (2n-3)i\phi_2)P_n^2(x) - \frac{C_n}{A_n}(\psi_1 + (2n-1)i\phi_2)P_{n-1}^2(x).
\end{aligned}
$$

This proves the first part of Theorem 3.1.

Now, we prove the second part.

Let $(Q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of a quadratic variable such that $Q_0(x) = P_0(x)$ and, for $n \geq 1$, $Q_n(x)$ and $Q_{n-1}(x)$ satisfy

$$
\begin{aligned}
&\phi(x)\left[Q_n(x)\mathcal{SD}Q_{n-1}(x) + Q_{n-1}(x)\mathcal{SD}Q_n(x)\right] \\
&\quad + \psi(x)\left[Q_n(x)\mathcal{S}^2 Q_{n-1}(x) + Q_{n-1}(x)\mathcal{S}^2 Q_n(x)\right] \\
&= \left[(\psi_1 + 2i\phi_2)x + \psi_0 + \psi_1\left(\frac{B_n}{2A_n} - \frac{B_{n-1}}{2A_{n-1}}\right) + i\phi_2\left(n\frac{B_n}{A_n} - (2-n)\frac{B_{n-1}}{A_{n-1}}\right)\right] \\
&\qquad Q_n(x)Q_{n-1}(x) \\
&\qquad + \frac{1}{A_{n-1}}(\psi_1 + (2n-3)i\phi_2)Q_n^2(x) - \frac{C_n}{A_n}(\psi_1 + (2n-1)i\phi_2)Q_{n-1}^2(x). \quad (3.13)
\end{aligned}
$$

Let $b_n$ be the leading coefficient of $Q_n(x)$. We shall firstly show by induction that $k_n = b_n$ for all $n \geq 0$. We have $b_0 = k_0$ and we assume that $n \geq 1$ and $b_{n-1} = k_{n-1}$. If we compare the coefficients of $x^{2n}$ in (3.13), we find that we must consider two cases according as the degree of $\phi$ is less than two or equal to two.

- If the degree of $\phi$ is less than two then we have

$$
2\psi_1 b_n k_{n-1} = \psi_1 b_n k_{n-1} + \frac{\psi_1}{A_{n-1}}(b_n)^2
$$

and $b_n \neq 0$ implies that for the quadratic or $q$-quadratic variable, we have $b_n = A_{n-1}k_{n-1} = k_n$.

- If the degree of $\phi$ is equal to two, then we have

$$
\begin{aligned}
&\phi_2\left((n-1)ib_n k_{n-1} + nib_n k_{n-1}\right) + 2\psi_1 b_n k_{n-1} \\
&= (\psi_1 + 2i\phi_2)b_n k_{n-1} + \frac{1}{A_{n-1}}(\psi_1 + (2n-3)i\phi_2)b_n^2,
\end{aligned}
$$

and the regularity of the corresponding linear functional with respect to the sequence $(Q_n)_{n\geq0}$ implies that $\psi_1 + (2n-3)i\phi_2 \neq 0$ and $b_n \neq 0$ we have $b_n = A_{n-1}k_{n-1} = k_n$.

We have by assumption $Q_0(x) = P_0(x)$. Assume further that $n \geq 1$ and $Q_{n-1}(x) = P_{n-1}(x)$ but $Q_n(x) \neq P_n(x)$.

Then $Q_n(x) = P_n(x) + g(x)$ where $g(x) = c(x^r + \cdots)$, $c \neq 0$. Since $Q_n(x)$ and $P_n(x)$ have the same degree and the same leading coefficient, we must have $r < n$. From (3.13), we get

$$\phi(x) \left[ (P_n(x) + g(x))\mathcal{S}\mathcal{D}P_{n-1}(x) + P_{n-1}(x)\mathcal{S}\mathcal{D}(P_n(x) + g(x)) \right]$$

$$+ \psi(x) \left[ (P_n(x) + g(x))\mathcal{S}^2 P_{n-1}(x) + P_{n-1}(x)\mathcal{S}^2(P_n(x) + g(x)) \right]$$

$$= \left[ (\psi_1 + 2i\phi_2)x + \psi_0 + \psi_1 \left( \frac{B_n}{2A_n} - \frac{B_{n-1}}{2A_{n-1}} \right) \right.$$

$$\left. + i\phi_2 \left( n\frac{B_n}{A_n} - (2-n)\frac{B_{n-1}}{A_{n-1}} \right) \right] (P_n(x) + g(x))P_{n-1}(x)$$

$$+ \frac{1}{A_{n-1}}(\psi_1 + (2n-3)i\phi_2)(P_n(x) + g(x))^2$$

$$- \frac{C_n}{A_n}(\psi_1 + (2n-1)i\phi_2)P_{n-1}^2(x).$$

Using the fact that $P_n(x)$ and $P_{n-1}(x)$ satisfy (3.6) we obtain

$$\phi(x) \left[ g(x)\mathcal{S}\mathcal{D}P_{n-1}(x) + P_{n-1}(x)\mathcal{S}\mathcal{D}g(x) \right]$$

$$+ \psi(x) \left[ g(x)\mathcal{S}^2 P_{n-1}(x) + P_{n-1}(x)\mathcal{S}^2 g(x) \right]$$

$$= \left[ (\psi_1 + 2i\phi_2)x + \psi_0 + \psi_1 \left( \frac{B_n}{2A_n} - \frac{B_{n-1}}{2A_{n-1}} \right) \right.$$

$$\left. + i\phi_2 \left( n\frac{B_n}{A_n} - (2-n)\frac{B_{n-1}}{A_{n-1}} \right) \right] g(x)P_{n-1}(x)$$

$$+ \frac{1}{A_{n-1}}(\psi_1 + (2n-3)i\phi_2)(2P_n(x)g(x) + g(x)^2). \qquad (3.14)$$

We compare the coefficients of $x^{n+r}$ in (3.14). Let us consider two cases:

1. If the degree of $\phi$ is less than two, then we get

$$2\psi_1 cb_{n-1} = \psi_1 cb_{n-1} + 2\frac{b_n}{A_{n-1}}c\psi_1$$

which is equivalent to

$$2\psi_1 cb_{n-1} = \psi_1 cb_{n-1} + 2c\psi_1 b_{n-1}.$$

Then, the fact that $\psi_1 b_{n-1} \neq 0$ implies that this is impossible if $c \neq 0$.

2. If the degree of $\phi$ is equal to two, then we get

$$\phi_2 \left( ci(n-1)b_{n-1} + ricb_{n-1} \right) + 2c\psi_1 b_{n-1}$$

$$= (\psi_1 + 2i\phi_2)cb_{n-1} + \frac{2}{A_{n-1}} \left( \psi_1 + (2n-3)i\phi_2 \right) cb_n,$$

which is equivalent to

$$\phi_2 \left( ci(n-1)b_{n-1} + ricb_{n-1} \right) + 2c\psi_1 b_{n-1}$$

$$= (\psi_1 + 2i\phi_2)cb_{n-1} + 2 \left( \psi_1 + (2n-3)i\phi_2 \right) cb_{n-1}.$$

The regularity of the corresponding linear functional with respect to the sequence $(P_n)$ implies that $\psi_1 + (3n-3-r)i\phi_2 \neq 0$ and the previous equation is impossible if $c \neq 0$.

The proof is therefore completed. $\qquad\qquad\qquad\square$

The following corollaries give explicit coefficients for the non-linear characterization of the Meixner–Pollaczek and the Continuous Dual Hahn polynomials.

**Corollary 3.2.** *The Meixner–Pollaczek polynomials are characterized by the following non-linear difference equation*

$$i(\lambda \sin\varphi - x\cos\varphi)\left[P_n(x)\mathcal{SD}P_{n-1}(x) + P_{n-1}(x)\mathcal{SD}P_n(x)\right]$$
$$+ 2(\lambda\cos\varphi + x\sin\varphi)\left[P_n(x)\mathcal{S}^2 P_{n-1}(x) + P_{n-1}(x)\mathcal{S}^2 P_n(x)\right]$$
$$= [2\sin\varphi x + 3\lambda\cos\varphi]\,P_n(x)P_{n-1}(x) + nP_n^2(x) - (n + 2\lambda - 1)P_{n-1}^2(x).$$

**Corollary 3.3.** *The Continuous Hahn polynomials are characterized by the following non-linear difference equation*

$$\left(-x^2 + \frac{i}{2}(a + b - c - d)x + \frac{1}{2}(ab + cd)\right)\left[P_n(x)\mathcal{SD}P_{n-1}(x) + P_{n-1}(x)\mathcal{SD}P_n(x)\right]$$
$$+ (-i(a + b + c + d + 2)x + cd - ab)\left[P_n(x)\mathcal{S}^2 P_{n-1}(x) + P_{n-1}(x)\mathcal{S}^2 P_n(x)\right]$$
$$= -\frac{n(b + c + n - 1)(b + d + n - 1)}{2n + a + b + c + d - 2}P_{n-1}^2(x)$$
$$+ (-i(a + b + c + d + 2)x + D_n)\,P_n(x)P_{n-1}(x)$$
$$+ \frac{(n - 2 + a + b + c + d)(n - 1 + a + c)(n - 1 + a + d)}{2n + a + b + c + d - 2}P_n^2(x),$$

*where $D_n$ depends on $n$, $a$, $b$, $c$ and $d$.*

## 4. Non-linear Characterization for Wilson and Continuous Dual Hahn Polynomials

The Wilson polynomials $W_n(x^2; a, b, c, d)$ and Continuous Dual Hahn polynomials $S_n(x^2; a, b, c)$, respectively, have the hypergeometric representation (see [18]):

$$\frac{W_n(x^2; a, b, c, d)}{(a + b, a + c, a + d)_n} = {}_4F_3\left(\begin{array}{c}-n, n + a + b + c + d - 1, a + ix, a - ix \\ a + b, a + c, a + d\end{array}\middle|\, 1\right),\tag{4.1}$$

$$\frac{S_n(x^2; a, b, c)}{(a + b, a + c)_n} = {}_3F_2\left(\begin{array}{c}-n, a - ix, a + ix \\ a + b, a + c\end{array}\middle|\, 1\right).\tag{4.2}$$

They are known to satisfy the second-order divided-difference equation (see [27])

$$\phi(x^2)\mathbf{D}^2 y(x^2) + \psi(x^2)\mathbf{SD}y(x^2) + \lambda_n y(x^2) = 0,\tag{4.3}$$

and these two families satisfy the three-term recurrence relation

$$P_{n+1}(x^2) = (A_n x^2 + B_n)P_n(x^2) - C_n P_{n-1}(x^2), \quad P_{-1}(x^2) = 0.\tag{4.4}$$

**Theorem 4.1.** (Non-linear characterization) *Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of classical orthogonal polynomials on a non-uniform lattice. Then, for $n \geq 1$, $P_n(x^2)$ and $P_{n-1}(x^2)$ satisfy*

$$\phi(x^2)\Big[P_n(x^2)\mathbf{DS}P_{n-1}(x^2) + P_{n-1}(x^2)\mathbf{DS}P_n(x^2)\Big]$$

$$+\psi(x^2)\Big[P_n(x^2)\mathbf{S}^2 P_{n-1}(x^2) + P_{n-1}(x^2)\mathbf{S}^2 P_n(x^2)\Big]$$

$$= \Big[(\psi_1 + 2\phi_2)x^2 + \psi_0 + \psi_1\left(\frac{B_n}{A_n} - \frac{B_{n-1}}{A_{n-1}}\right) + \phi_2\left(n\frac{B_n}{A_n} + (2-n)\frac{B_{n-1}}{A_{n-1}}\right)\Big]$$

$$\times P_n(x^2)P_{n-1}(x^2) + \frac{1}{A_{n-1}}(\psi_1 + (2n-3)\phi_2)P_n^2(x^2)$$

$$-\frac{C_n}{A_n}(\psi_1 + (2n-1)\phi_2)P_{n-1}^2(x^2). \tag{4.5}$$

*Furthermore, if $(Q_n(x^2))_{n\in\mathbb{N}}$ is a sequence of polynomials such that $Q_0(x^2) = P_0(x^2)$ and, for $n \geq 1$, $Q_n(x)$ and $Q_{n-1}(x)$ satisfy (4.5). Then $Q_n(x^2) = P_n(x^2)$, for all $n \geq 0$.*

*Proof.* For all integers $n$, $P_{n+1}(x^2)$ satisfies (4.3), namely:

$$\phi(x^2)\mathbf{D}^2 P_{n+1}(x^2) + \psi(x^2)\mathbf{SD}P_{n+1}(x^2) + \lambda_{n+1}P_{n+1}(x^2) = 0, \tag{4.6}$$

with

$$\phi(x^2) = \phi_2 x^4 + \phi_1 x^2 + \phi_0; \quad \psi(x^2) = \psi_1 x^2 + \psi_0; \quad \lambda_n = -n(n-1)\phi_2 - n\psi_1.$$

From (4.4), using the relations (2.5), (2.6), (2.7) and (2.8), we obtain:

$$\mathbf{D}^2 P_{n+1}(x^2) = 2A_n\mathbf{DS}P_n(x^2) + (A_n x^2 + B_n)\mathbf{D}^2 P_n(x^2) - C_n\mathbf{D}^2 P_{n-1}(x^2) \tag{4.7}$$

and

$$\mathbf{SD}P_{n+1}(x^2) = 2A_n\mathbf{S}^2 P_n(x^2) - A_n P_n(x^2) + (A_n x^2 + B_n)\mathbf{SD}P_n(x^2) - C_n\mathbf{SD}P_{n-1}(x^2). \tag{4.8}$$

We use (4.4), (4.7) and (4.8) to replace $\mathbf{D}^2 P_{n+1}(x^2)$, $\mathbf{SD}P_{n+1}(x^2)$ and $P_{n+1}(x^2)$ in (4.6); we obtain:

$$\phi(x^2)\mathbf{DS}P_n(x^2) + \psi(x^2)\mathbf{S}^2 P_n(x^2)$$

$$= -\frac{C_n}{2A_n}(\lambda_{n-1} - \lambda_{n+1})P_{n-1}(x^2)$$

$$+ \left[\frac{1}{2}\psi(x^2) + \left(\frac{1}{2}x^2 + \frac{B_n}{2A_n}\right)(\lambda_n - \lambda_{n+1})\right]P_n(x^2). \tag{4.9}$$

For $n \geq 2$, we replace $n$ by $n-1$ in (4.9) and we obtain:

$$\phi(x^2)\mathbf{DS}P_{n-1}(x^2) + \psi(x^2)\mathbf{S}^2 P_{n-1}(x^2)$$

$$= -\frac{C_{n-1}}{2A_{n-1}}(\lambda_{n-2} - \lambda_n)P_{n-2}(x^2)$$

$$+ \left[\frac{1}{2}\psi(x^2) + \left(\frac{1}{2}x^2 + \frac{B_{n-1}}{2A_{n-1}}\right)(\lambda_{n-1} - \lambda_n)\right]P_{n-1}(x^2). \tag{4.10}$$

We replace again $n$ by $n-1$ in (4.4) and we use the resulting relation to replace $P_{n-2}(x^2)$ in (4.10) to obtain:

$$\phi(x^2)\mathbf{DS}P_{n-1}(x^2) + \psi(x^2)\mathbf{S}^2 P_{n-1}(x^2)$$
$$= \frac{1}{2A_{n-1}}(\lambda_{n-2} - \lambda_n)P_n(x^2)$$
$$+ \left[\frac{1}{2}\psi(x^2) + \left(\frac{1}{2}x^2 + \frac{B_{n-1}}{2A_{n-1}}\right)(\lambda_{n-1} - \lambda_{n-2})\right]P_{n-1}(x^2). \quad (4.11)$$

If we multiply (4.9) by $P_{n-1}(x^2)$, (4.11) by $P_n(x^2)$ and add the resulting expressions, we obtain:

$$\phi(x^2)\left[P_n(x^2)\mathbf{DS}P_{n-1}(x^2) + P_{n-1}(x^2)\mathbf{DS}P_n(x^2)\right]$$
$$+ \psi(x^2)\left[P_n(x^2)\mathbf{S}^2 P_{n-1}(x^2) + P_{n-1}(x^2)\mathbf{S}^2 P_n(x^2)\right]$$
$$= \left[(\psi_1 + 2\phi_2)x^2 + \psi_0 + \psi_1\left(\frac{B_n}{A_n} - \frac{B_{n-1}}{A_{n-1}}\right) + \phi_2\left(n\frac{B_n}{A_n} + (2-n)\frac{B_{n-1}}{A_{n-1}}\right)\right]$$
$$\times P_n(x^2)P_{n-1}(x^2) + \frac{1}{A_{n-1}}(\psi_1 + (2n-3)\phi_2)P_n^2(x^2)$$
$$- \frac{C_n}{A_n}(\psi_1 + (2n-1)\phi_2)P_{n-1}^2(x^2).$$

This proves the first part of Theorem 4.1.

Now, we prove the second part.

Let $(Q_n)_{n\in\mathbb{N}}$ be a sequence of polynomials of a quadratic variable such that $Q_0(x^2) = P_0(x^2)$ and, for $n \geq 1$, $Q_n(x^2)$ and $Q_{n-1}(x^2)$ satisfy (4.5). Let $b_n$ be the leading coefficient of $Q_n(x^2)$. We shall first show by induction that $k_n = b_n$ for all $n \geq 0$. We have $b_0 = k_0$ and we assume that $n \geq 1$ and $b_{n-1} = k_{n-1}$. If we compare the coefficients of $x^{4n}$ in (4.5), we find that we must consider two cases whether the degree of $\phi$ is less than two or equal to two.

- If the degree of $\phi$ is less than two then, we have

$$2\psi_1 b_n k_{n-1} = \psi_1 b_n k_{n-1} + \frac{\psi_1}{A_{n-1}}(b_n)^2$$

  and $b_n \neq 0$ implies that for the quadratic or $q$-quadratic variable, we have $b_n = A_{n-1}k_{n-1} = k_n$.
- If the degree of $\phi$ is equal to two then, we have

$$\phi_2\left((n-1)b_n k_{n-1} + nb_n k_{n-1}\right) + 2\psi_1 b_n k_{n-1}$$
$$= (\psi_1 + 2\phi_2)b_n k_{n-1} + \frac{1}{A_{n-1}}\left(\psi_1 + (2n-3)\phi_2\right)b_n^2,$$

  and the regularity of the corresponding linear functional with respect to the sequence $(Q_n)_{n\geq 0}$ implies that $\psi_1 + (2n-3)\phi_2 \neq 0$ and $b_n \neq 0$ we have $b_n = A_{n-1}k_{n-1} = k_n$.

We have by assumption $Q_0(x^2) = P_0(x^2)$. Assume further that $n \geq 1$ and $Q_{n-1}(x^2) = P_{n-1}(x^2)$ but $Q_n(x^2) \neq P_n(x^2)$.

Then $Q_n(x^2) = P_n(x^2) + g(x^2)$ where $g(x^2) = c(x^{2r} + \cdots)$, $c \neq 0$. Since $Q_n(x^2)$ and $P_n(x^2)$ have the same degree and the same leading coefficient, we must have $r < n$. From (4.5), we get

$$\phi(x^2) \left[ (P_n(x^2) + g(x^2))\mathbf{DS}P_{n-1}(x^2) + P_{n-1}(x^2)\mathbf{DS}(P_n(x^2) + g(x^2)) \right]$$
$$+ \psi(x^2) \left[ (P_n(x^2) + g(x^2))\mathbf{S}^2 P_{n-1}(x^2) + P_{n-1}(x^2)\mathbf{S}^2(P_n(x^2) + g(x^2)) \right]$$
$$= \left[ (\psi_1 + 2\phi_2)x^2 + \psi_0 + \psi_1 \left( \frac{B_n}{A_n} - \frac{B_{n-1}}{A_{n-1}} \right) \right.$$
$$+ \phi_2 \left. \left( n\frac{B_n}{A_n} + (2-n)\frac{B_{n-1}}{A_{n-1}} \right) \right] (P_n(x^2) + g(x^2))P_{n-1}(x^2)$$
$$+ \frac{1}{A_{n-1}}(\psi_1 + (2n-3)\phi_2)(P_n(x^2) + g(x^2))^2$$
$$- \frac{C_n}{A_n}(\psi_1 + (2n-1)\phi_2)P_{n-1}^2(x^2).$$

Using the fact that $P_n(x)$ and $P_{n-1}(x)$ satisfy (4.5), we obtain

$$\phi(x^2) \left[ g(x^2)\mathbf{DS}P_{n-1}(x^2) + P_{n-1}(x^2)\mathbf{DS}g(x^2) \right]$$
$$+ \psi(x^2) \left[ g(x^2)\mathbf{S}^2 P_{n-1}(x^2) + P_{n-1}(x^2)\mathbf{S}^2 g(x^2) \right]$$
$$= \left[ (\psi_1 + 2\phi_2)x^2 + \psi_0 + \psi_1 \left( \frac{B_n}{A_n} - \frac{B_{n-1}}{A_{n-1}} \right) + \phi_2 \left( n\frac{B_n}{A_n} + (2-n)\frac{B_{n-1}}{A_{n-1}} \right) \right]$$
$$\times g(x^2)P_{n-1}(x^2) + \frac{1}{A_{n-1}}(\psi_1 + (2n-3)\phi_2)(2P_n(x)g(x) + g(x)^2). \quad (4.12)$$

We compare the coefficients of $x^{2n+2r}$ in (4.12) and consider two cases:
1. If the degree of $\phi$ is less than two, then

$$2\psi_1 cb_{n-1} = \psi_1 cb_{n-1} + 2\frac{b_n}{A_{n-1}}c\psi_1$$

which is equivalent to

$$2\psi_1 cb_{n-1} = \psi_1 cb_{n-1} + 2c\psi_1 b_{n-1}.$$

Then, the fact that $\psi_1 b_{n-1} \neq 0$ implies that this is impossible if $c \neq 0$.
2. If the degree of $\phi$ is equal to two, then

$$\phi_2 \left( c(n-1)b_{n-1} + rcb_{n-1} \right) + 2c\psi_1 b_{n-1}$$
$$= (\psi_1 + 2\phi_2)cb_{n-1} + \frac{2}{A_{n-1}}(\psi_1 + (2n-3)\phi_2) cb_n$$

which is equivalent to

$$\phi_2 \left( c(n-1)b_{n-1} + rcb_{n-1} \right) + 2c\psi_1 b_{n-1}$$
$$= (\psi_1 + 2\phi_2)cb_{n-1} + 2(\psi_1 + (2n-3)\phi_2) cb_{n-1}.$$

The regularity of the corresponding linear functional with respect to the sequence $(P_n)_{n \geq 0}$ implies that $\psi_1 + (3n-3-r)\phi_2 \neq 0$ and the previous equality is impossible if $c \neq 0$.

□

**Corollary 4.2.** *The Wilson polynomials are characterized by the following non-linear difference equation*

$$\phi(x^2)\left[P_n(x^2)\mathbf{DS}P_{n-1}(x^2) + P_{n-1}(x^2)\mathbf{DS}P_n(x^2)\right]$$
$$+ \psi(x^2)\left[P_n(x^2)\mathbf{S}^2 P_{n-1}(x^2) + P_{n-1}(x^2)\mathbf{S}^2 P_n(x^2)\right]$$
$$= -\left[\frac{n(b+c+n-1)(b+d+n-1)(c+d+n-1)}{a+b+c+d+2n-2}\right]P_{n-1}^2(x^2)$$
$$+ \left[\frac{(a+b+c+d+n-2)(a+b+n-1)(a+c+n-1)(a+d+n-1)}{a+b+c+d+2n-2}\right]P_n^2(x^2)$$
$$+ \left[(a+b+c+d+2)x^2 + D_n\right]P_n(x^2)P_{n-1}(x^2),$$

*where $D_n$ depends on $n$, $a$, $b$, $c$ and $d$.*

**Corollary 4.3.** *The Continuous Dual Hahn polynomials are characterized by the following non-linear difference equation*

$$(-(a+b+c)x^2 + abc)\left[P_n(x^2)\mathbf{DS}P_{n-1}(x^2) + P_{n-1}(x^2)\mathbf{DS}P_n(x^2)\right]$$
$$+ (x^2 - ab - ac - bc)\left[P_n(x^2)\mathbf{S}^2 P_{n-1}(x^2) + P_{n-1}(x^2)\mathbf{S}^2 P_n(x^2)\right]$$
$$= \left(x^2 + D_n\right)P_n(x^2)P_{n-1}(x^2) - (a+b+n-1)(a+c+n-1)P_n^2(x^2)$$
$$+ n(b+c+n-1)P_{n-1}^2(x^2),$$

*where $D_n$ depends on $n$, $a$, $b$, $c$ and $d$.*

## 5. Non-linear Characterization for Orthogonal Polynomials on $q$-Quadratic Lattices

A family $p_n(x)$ of polynomials of degree $n$ is a family of classical $q$-quadratic orthogonal polynomials (also known as orthogonal polynomials on non-uniform lattices) if it is the solution of a divided-difference equation of the type (see [8,9])

$$\phi(x(s))\mathbb{D}_x^2 y(x(s)) + \psi(x(s))\mathbb{S}_x \mathbb{D}_x y(x(s)) + \lambda_n y(x(s)) = 0, \qquad (5.1)$$

where $\phi$ is a polynomial of maximal degree two and $\psi$ is a polynomial of exact degree one, $\lambda_n$ is a constant depending on the integer $n$ and the leading coefficients $\phi_2$ and $\psi_1$ of $\phi$ and $\psi$:

$$\lambda_n = -\gamma_n(\gamma_{n-1}\phi_2 + \alpha_{n-1}\psi_1)$$

and $x(s)$ is a non-uniform lattice defined by

$$x(s) = c_1 q^s + c_2 q^{-s} + c_3, \quad c_1 c_2 \neq 0, \qquad (5.2)$$

and the sequences $(\alpha_n)$ and $(\gamma_n)$ are given explicitly by :

$$\alpha_n = \frac{1}{2}\left(q^{\frac{n}{2}} + q^{-\frac{n}{2}}\right), \gamma_n = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}.$$

### 5.1. General Theorem

In this section, we state and prove a non-linear characterization result for classical orthogonal polynomials on non-uniform lattices. The result is stated in the following theorem.

**Theorem 5.1.** *Let $(P_n)_{n\geq 0}$ be a sequence of classical orthogonal polynomials on a non-uniform lattice. Then, for $n \geq 1$, $P_n(x(s))$ and $P_{n-1}(x(s))$ satisfy*

$$
\psi(x(s))\Big[P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s))\Big]
$$
$$
+\phi(x(s))\Big[P_n(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s))\Big]
$$
$$
= \Big[(D_{n-1} + D_n - G_{n-1}A_{n-1})x(s) + E_{n-1}
$$
$$
-G_{n-1}B_{n-1} + E_n\Big]P_n(x(s))P_{n-1}(x(s))
$$
$$
+G_{n-1}\left(P_n(x(s))\right)^2 - C_n G_n \left(P_{n-1}(x(s))\right)^2, \tag{5.3}
$$

*where*

$$
D_n = \frac{1}{2}(\lambda_n - \lambda_{n+1} + \psi_1), \quad E_n = \frac{1}{2}\left((\lambda_n - \lambda_{n+1})\frac{B_n}{A_n} + \psi_0\right),
$$
$$
G_n = \frac{1}{2A_n}\left(\lambda_{n-1} - \lambda_{n+1}\right). \tag{5.4}
$$

*Furthermore, if $(Q_n)_{n\in\mathbb{N}}$ is a sequence of polynomials a on non-uniform lattice such that $Q_0(x) = P_0(x)$ and, for $n \geq 1$, $Q_n(x(s))$ and $Q_{n-1}(x(s))$ satisfy (5.3). Then $Q_n(x(s)) = P_n(x(s))$, for all $n \geq 0$.*

*Proof.* Using the fact that $(P_n(x(s))_{n\geq 0}$ is a classical $q$-orthogonal polynomial sequence on non-uniform lattice, substituting $n$ by $n+1$ in (5.1) we obtain

$$
\phi(x(s))\mathbb{D}_x^2 P_{n+1}(x(s)) + \psi(x(s))\mathbb{S}_x\mathbb{D}_x P_{n+1}(x(s)) + \lambda_{n+1}P_{n+1}(x(s)) = 0. \tag{5.5}
$$

In (1.1), using the product rules given in [12, page 407], in [11, pages 741-742] or in [10, page 4], we obtain:

$$
\mathbb{D}_x^2 P_{n+1}(x(s)) = \left[A_n\alpha^2 x(s) + A_n\beta(\alpha + 1) + B_n\right]\mathbb{D}_x^2 P_n(x(s))
$$
$$
+2A_n\mathbb{D}_x\mathbb{S}_x P_n(x(s)) - A_n U_1(x(s))\mathbb{D}_x^2 P_n(x(s))
$$
$$
-C_n\mathbb{D}_x^2 P_{n-1}(x(s)) \tag{5.6}
$$

and

$$
\mathbb{S}_x\mathbb{D}_x P_{n+1}(x(s)) = \left[A_n\alpha^2 x(s) + A_n\beta(\alpha + 1) + B_n\right]\mathbb{S}_x\mathbb{D}_x P_n(x(s))
$$
$$
+2A_n\mathbb{S}_x^2 P_n(x(s)) - A_n U_1(x(s))\mathbb{S}_x\mathbb{D}_x P_n(x(s))
$$
$$
-A_n P_n(x(s)) - C_n\mathbb{S}_x\mathbb{D}_x P_{n-1}(x(s)). \tag{5.7}
$$

Using (1.1), (5.6) and (5.7) to replace $\mathbb{D}_x^2 P_{n+1}(x(s))$ , $\mathbb{S}_x \mathbb{D}_x P_{n+1}(x(s))$ and $P_{n+1}(x(s))$ in (5.5), we obtain:

$$\psi(x(s))\mathbb{S}_x^2 P_n(x(s)) + \phi(x(s))\mathbb{D}_x \mathbb{S}_x P_n(x(s))$$
$$= -\frac{C_n}{2A_n}\left[\lambda_{n-1} - \lambda_{n+1}\right] P_{n-1}(x(s))$$
$$+ \frac{1}{2}\left[(\lambda_n - \lambda_{n+1} + \psi_1)x(s) + (\lambda_n - \lambda_{n+1})\frac{B_n}{A_n} + \psi_0\right] P_n(x(s)), \ \forall n \geq 1,$$

which is equivalent to

$$\psi(x(s))\mathbb{S}_x^2 P_n(x(s)) + \phi(x(s))\mathbb{D}_x \mathbb{S}_x P_n(x(s))$$
$$= \left[D_n x(s) + E_n\right] P_n(x(s)) - C_n G_n P_{n-1}(x(s)), \ \forall n \geq 1, \qquad (5.8)$$

where $D_n$, $E_n$ and $G_n$ are defined in (5.4). For $n \geq 2$, we replace $n$ by $n-1$ in (5.8) and we obtain:

$$\psi(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + \phi(x(s))\mathbb{D}_x \mathbb{S}_x P_{n-1}(x(s))$$
$$= \left[D_{n-1} x(s) + E_{n-1}\right] P_{n-1}(x(s)) - C_{n-1} G_{n-1} P_{n-2}(x(s)), \ \forall n \geq 2. \tag{5.9}$$

We also replace $n$ by $n-1$ in (1.1) and use the resulting relation to replace $P_{n-2}(x(s))$ in (5.9) to obtain:

$$\psi(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + \phi(x(s))\mathbb{D}_x \mathbb{S}_x P_{n-1}(x(s)) = G_{n-1} P_n(x(s))$$
$$+ \left[(D_{n-1} - G_{n-1} A_{n-1})x(s) + E_{n-1} - G_{n-1} B_{n-1}\right] P_{n-1}(x(s)), \ \forall n \geq 1. \tag{5.10}$$

If we multiply (5.8) by $P_{n-1}(x(s))$ and (5.10) by $P_n(x(s))$ and add the resulting expressions, we obtain:

$$\psi(x(s))\left[P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s))\right]$$
$$+ \phi(x(s))\left[P_n(x(s))\mathbb{D}_x \mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x \mathbb{S}_x P_n(x(s))\right]$$
$$= \left[(D_{n-1} - G_{n-1} A_{n-1} + D_n)x(s) + E_{n-1}\right.$$
$$\left. - G_{n-1} B_{n-1} + E_n\right] P_n(x(s))P_{n-1}(x(s))$$
$$+ G_{n-1}\left(P_n(x(s))\right)^2 - C_n G_n\left(P_{n-1}(x(s))\right)^2.$$

This proves the first part of Theorem 5.1.

Now, we prove the second part.

Let $(Q_n(x(s)))_{n\in\mathbb{N}}$ be a sequence of polynomials of a $q$-quadratic variable such that $Q_0(x(s)) = P_0(x(s))$ and, for $n \geq 1$, $Q_n(x(s))$ and $Q_{n-1}(x(s))$ satisfy

$$\psi(x(s))\left[Q_n(x(s))\mathbb{S}_x^2 Q_{n-1}(x(s)) + Q_{n-1}(x(s))\mathbb{S}_x^2 Q_n(x(s))\right]$$
$$+ \phi(x(s))\left[Q_n(x(s))\mathbb{D}_x \mathbb{S}_x Q_{n-1}(x(s)) + Q_{n-1}(x(s))\mathbb{D}_x \mathbb{S}_x Q_n(x(s))\right]$$
$$= \left[(D_{n-1} - G_{n-1} A_{n-1} + D_n)x(s) + E_{n-1} - G_{n-1} B_{n-1} + E_n\right]$$
$$\times Q_n(x(s))Q_{n-1}(x(s)) + G_{n-1}\left(Q_n(x(s))\right)^2 - C_n G_n\left(Q_{n-1}(x(s))\right)^2. \tag{5.11}$$

Let $a_n$ be the leading coefficient of $Q_n(x(s))$. We shall firstly show by induction that $k_n = a_n$ for all $n \geq 0$. We have $a_0 = k_0$ and we assume that $n \geq 1$ and $a_{n-1} = k_{n-1}$. If we compare the coefficients of $F_{2n}(x(s))$ in (5.11), we find that we must consider two cases whether the degree of $\phi$ is less than two or equal to two.

- If the degree of $\phi(x(s))$ is less than two then, we have

$$\psi_1((\alpha_{n-1})^2 + (\alpha_n)^2)a_n k_{n-1} = (D_n + D_{n-1} - A_{n-1}G_{n-1})a_n k_{n-1} + G_{n-1}(a_n)^2$$

  and $a_n \neq 0$ implies that for the $q$-quadratic variable, we have $a_n = A_{n-1}k_{n-1} = k_n$.

- If the degree of $\phi(x(s))$ is equal to two then we get

$$\psi_1((\alpha_{n-1})^2 + (\alpha_n)^2)a_n k_{n-1} + \phi_2(\alpha_{n-1}\gamma_{n-1} + \alpha_n\gamma_n)a_n k_{n-1}$$
$$= (D_n + D_{n-1} - A_{n-1}G_{n-1})a_n k_{n-1} + G_{n-1}(a_n)^2$$

  and $a_n \neq 0$ implies that for the quadratic case or the $q$-quadratic case, we have $a_n = A_{n-1}k_{n-1} = k_n$.

We have by assumption $Q_0(x(s)) = P_0(x(s))$. Assume further that $n \geq 1$ and we $Q_{n-1}(x(s)) = P_{n-1}(x(s))$ but $Q_n(x(s)) \neq P_n(x(s))$.

Then $Q_n(x(s)) = P_n(x(s)) + g(x(s))$, where $g(x(s)) = c(F_r(x(s)) + \cdots)$, $c \neq 0$. Since $Q_n(x(s))$ and $P_n(x(s))$ have the same degree and the same leading coefficient, we must have $r < n$. From (5.11), we get

$$\psi(x(s)) \left[ (P_n(x(s)) + g(x(s)))\mathbb{S}_x^2 P_{n-1}(x(s)) \right.$$
$$+ P_{n-1}(x(s))(\mathbb{S}_x^2 P_n(x(s)) + \mathbb{S}_x^2 g(x(s)))]$$
$$+ \phi(x(s)) \left[ (P_n(x(s)) + g(x(s)))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) \right.$$
$$+ P_{n-1}(x(s))(\mathbb{D}_x\mathbb{S}_x P_n(x(s)) + \mathbb{D}_x\mathbb{S}_x g(x(s)))]$$
$$= [(D_{n-1} - G_{n-1}A_{n-1} + D_n)x(s) + E_{n-1} - G_{n-1}B_{n-1} + E_n]$$
$$\times (P_n(x(s))P_{n-1}(x(s)) + g(x(s))P_{n-1}(x(s))) + G_{n-1}\left( (P_n(x(s)))^2 \right.$$
$$+ 2g(x(s))P_n(x(s)) + (g(x(s)))^2) - C_n G_n (P_{n-1}(x(s)))^2 .$$

Using the fact that $P_n(x(s))$ and $P_{n-1}(x(s))$ satisfy (5.3), we obtain

$$\psi(x(s)) \left[ g(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 g(x(s)) \right]$$
$$+ \phi(x(s)) \left[ g(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x g(x(s)) \right]$$
$$= [(D_{n-1} - G_{n-1}A_{n-1} + D_n)x(s) + E_{n-1}$$
$$- G_{n-1}B_{n-1} + E_n] g(x(s))P_{n-1}(x(s)))$$
$$+ G_{n-1} \left( 2g(x(s))P_n(x(s)) + (g(x(s)))^2 \right) . \tag{5.12}$$

We compare the coefficients of $F_{n+r}(x(s))$ in (5.12). Two cases arise:

1. If the degree of $\phi(x(s))$ is less than two, then we get

$$\psi_1 \left( (\alpha_{n-1})^2 + (\alpha_r)^2 \right) ck_{n-1} = (D_n + D_{n-1} - G_{n-1}A_{n-1})ck_{n-1} + 2ck_n G_{n-1},$$

  which is equivalent to

$$\psi_1 \left( (\alpha_{n-1})^2 + (\alpha_r)^2 \right) ck_{n-1} = (D_n + D_{n-1} - G_{n-1}A_{n-1})ck_{n-1} + 2ck_{n-1}G_{n-1}A_{n-1}.$$

  Then, for the $q$-quadratic variable, this is impossible if $c \neq 0$.

2. If the degree of $\phi(x(s))$ is equal to two, then we get

$$
\psi_1 \left((\alpha_{n-1})^2 + (\alpha_r)^2\right) ck_{n-1} + \phi_2 \left(\alpha_{n-1}\gamma_{n-1} + \alpha_r\gamma_r\right) ck_{n-1}
$$
$$
= (D_n + D_{n-1} - G_{n-1}A_{n-1})ck_{n-1} + 2ck_nG_{n-1},
$$

which is equivalent to

$$
\psi_1 \left((\alpha_{n-1})^2 + (\alpha_r)^2\right) ck_{n-1} + \phi_2 \left(\alpha_{n-1}\gamma_{n-1} + \alpha_r\gamma_r\right) ck_{n-1}
$$
$$
= (D_n + D_{n-1} - G_{n-1}A_{n-1})ck_{n-1} + 2ck_{n-1}G_{n-1}A_{n-1}.
$$

Again this is impossible if $c \neq 0$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

## 5.2. Special Cases

We can specialize the above result to the various classical orthogonal polynomials on non-uniform lattice, namely Askey–Wilson, $q$-Racah, Continuous dual $q$-Hahn, Continuous $q$-Hahn, Dual $q$-Hahn, Al-Salam Chihara, $q$-Meixner–Pollaczek, Continuous $q$-Jacobi, Dual $q$-Krawtchouk, Continuous big $q$-Hermite, Continuous $q$-Laguerre and Continuous $q$-Hermite polynomials. Note that the results for the Askey–Wilson and the $q$-Racah polynomials would be enough since the other families can be obtained by some limit transitions. But here, we would like to provide a complete database for all these polynomials orthogonal on a $q$-quadratic lattices.

### 5.2.1. Askey–Wilson Polynomials.
The Askey–Wilson polynomials have the $q$-hypergeometric representation [18, P. 415]

$$
\frac{a^n p_n(x; a, b, c, d|q)}{(ab, ac, ad; q)_n} = {}_4\phi_3 \left( \left. \begin{array}{c} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{array} \right| q; q \right), \quad x = \cos\theta.
$$

They satisfy the divided-difference equation (5.1) with

$$
\phi(x(s)) = 2 \left(dcba + 1\right) x^2(s) - (a + b + c + d + abc + abd + acd + bcd) x(s)
$$
$$
+ ab + ac + ad + bc + bd + cd - abcd - 1,
$$
$$
\psi(x(s)) = \frac{4 \left(abcd - 1\right) q^{\frac{1}{2}} x(s)}{q - 1} + \frac{2 \left(a + b + c + d - abc - abd - acd - bcd\right) q^{\frac{1}{2}}}{q - 1}.
$$

The monic Askey–Wilson polynomials are characterized by the following non-linear recurrence relation

$$
\psi(x(s)) \left[ P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s)) \right]
$$
$$
+ \phi(x(s)) \left[ P_n(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s)) \right]
$$
$$
= \left[ K_n x(s) + M_n \right] P_n(x(s)) P_{n-1}(x(s))
$$
$$
- \frac{2(q + 1)(abcdq^{n-\frac{3}{2}} - q^{\frac{3}{2}-n})}{q - 1} \left( P_n(x(s)) \right)^2
$$
$$
- \frac{2(q + 1)(abcdq^{n-\frac{1}{2}} - q^{\frac{1}{2}-n})}{q - 1} C_n \left( P_{n-1}(x(s)) \right)^2,
$$

where

$$K_n = \frac{2\sqrt{q}[abcd\left(q^{2n-2}(q+1)^2 + 2q^n\right) - \left(2q^n + (q+1)^2\right)]}{(q-1)q^n},$$

$$C_n = \frac{1}{4}\frac{(1-q^n)(1-abq^{n-1})(1-acq^{n-1})(1-adq^{n-1})(1-abcdq^{n-1})}{(1-abcdq^{2n-3})(1-abcdq^{2n-1})}$$

$$\times \frac{(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})^2},$$

$$M_n = \frac{2\sqrt{q}(q^{2n}-abcd)}{q^n(q-1)}\left(a + a^{-1} - (\widetilde{A_n} + \widetilde{C_n})\right)$$

$$+ \left\{\frac{2\sqrt{q}[q^{2n-1}-abcdq+(q+1)(q^{2n-2}-abcdq)]}{q^n(q-1)}\right\}$$

$$\times \left(a + a^{-1} - (\widetilde{A}_{n-1} + \widetilde{C}_{n-1})\right)$$

$$+ \frac{2\left(a+b+c+d-abc-abd-acd-bcd\right)q^{\frac{1}{2}}}{q-1},$$

with

$$\widetilde{A_n} = \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})},$$

$$\widetilde{C_n} = \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n-2})}.$$

**5.2.2. $q$-Racah Polynomials.** The $q$-Racah polynomials have the $q$-hypergeometric representation [18, P. 422]

$$R_n(\mu(x); \alpha, \beta, \gamma, \delta|q) = {}_4\phi_3\left(\begin{array}{c} q^{-n}, \alpha\beta q^{n+1}, q^{-x}, \delta\gamma q^{x+1} \\ \alpha q, \beta\delta q, \gamma q \end{array}\middle| q; q\right), n = 0, 1, 2, \ldots, N$$

where

$$\mu(x) := q^{-x} + \delta\gamma q^{x+1}$$

and

$$\alpha q = q^{-N} \quad \text{or} \quad \beta\delta q = q^{-N} \quad \text{or} \quad \gamma q = q^{-N},$$

with $N$ a non-negative integer. They satisfy (5.1) with

$$\phi(x(s)) = (\beta\alpha q^2 + 1)x(s)^2 - q\left(\gamma q\alpha + \gamma q\beta\delta + q\alpha\beta\delta + q\alpha\beta + \beta\delta + \gamma\delta + \gamma + \alpha\right)x(s)$$

$$+ 2q\left(-q^2\alpha\beta\delta\gamma + \gamma q\alpha + \gamma^2 q\delta + \gamma q\delta^2\beta + q\alpha\beta\delta + q\gamma\delta\alpha + \gamma q\beta\delta - \gamma\delta\right),$$

$$\psi(x(s)) = 2\sqrt{q}\left(\frac{(\beta\alpha q^2 - 1)}{q-1}\right)x(s)$$

$$- 2\frac{q^{3/2}\left(\gamma q\alpha + \gamma q\beta\delta - \gamma\delta - \gamma + q\alpha\beta\delta + q\alpha\beta - \alpha - \beta\delta\right)}{q-1}.$$

The monic $q$-Racah polynomials are characterized by the following relation

$$\psi(x(s)) \left[ P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s)) \right]$$

$$+ \phi(x(s)) \left[ P_n(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s)) \right]$$

$$= \left[ K_n x(s) + M_n \right] P_n(x(s)) P_{n-1}(x(s)) - \frac{(q+1)\sqrt{q}(\alpha\beta q^{2n} - q)}{(q-1)q^n} \left( P_n(x(s)) \right)^2$$

$$- \frac{(q+1)\sqrt{q}(\alpha\beta q^{2n+1} - 1)}{(q-1)q^n} C_n \left( P_{n-1}(x(s)) \right)^2,$$

where

$$K_n = \frac{\alpha\beta\sqrt{q}\left(q^n(q+1)^2 + 2q^2\right)}{q-1} - \frac{\sqrt{q}\left((q+1)^2 + 2q^n\right)}{q^n(q-1)},$$

$$C_n = \frac{q(1-\alpha q^n)(1-\alpha\beta q^n)(1-\beta\delta q^n)(1-\gamma q^n)(1-q^n)(1-\beta q^n)(\gamma-\alpha\beta q^n)(\delta-\alpha q^n)}{(1-\alpha\beta q^{2n-1})(1-\alpha\beta q^{2n+1})(1-\alpha\beta q^{2n})^2},$$

$$M_n = \frac{\sqrt{q}(\alpha\beta q^{2n} - 1)}{q^n(q-1)} \left( \widetilde{A}_n + \widetilde{C}_n - q\gamma\delta - 1 \right)$$

$$+ \left( \frac{q^{\frac{3}{2}}(\alpha\beta q^{2n} - 1) + \sqrt{q}(q+1)(\alpha\beta q^{2n} - q)}{q^n(q-1)} \right) \left( \widetilde{A}_{n-1} + \widetilde{C}_{n-1} - q\gamma\delta - 1 \right)$$

$$- 2\frac{(q\gamma\alpha + q\gamma\beta\delta - \gamma\delta - \gamma + q\alpha\beta\delta + q\alpha\beta - \alpha - \beta\delta) q^{\frac{3}{2}}}{q-1}$$

with

$$\widetilde{A}_n = \frac{(1-\alpha q^{n+1})(1-\alpha\beta q^{n+1})(1-\beta\delta q^{n+1})(1-\gamma q^{n+1})}{(1-\alpha\beta q^{2n+1})(1-\alpha\beta q^{2n+2})},$$

$$\widetilde{C}_n = \frac{q(1-q^n)(1-\beta q^n)(\gamma-\alpha\beta q^n)(\delta-\alpha q^n)}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+1})}.$$

**5.2.3. Continuous Dual $q$-Hahn Polynomials.** The Continuous Dual $q$-Hahn polynomials have the $q$-hypergeometric representation [18, P. 429]

$$\frac{a^n p_n(x; a, b, c|q)}{(ab, ac; q)_n} = {}_3\phi_2 \left( \begin{array}{c} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac \end{array} \middle| q, q \right), \quad x = \cos\theta.$$

They satisfy (5.1) with

$$\phi(x(s)) = 2(x(s))^2 - (a+b+c+abc)x(s) - 1 + bc + ab + ac,$$

$$\psi(s(s)) = -\frac{4\sqrt{q}}{q-1}x(s) + \frac{2(a+b+c-abc)\sqrt{q}}{q-1}.$$

The monic continuous dual $q$-Hahn polynomials are characterized by the following non-linear recurrence relation

$$\left( -\frac{4\sqrt{q}}{q-1}x(s) + \frac{2(a+b+c-abc)\sqrt{q}}{q-1} \right) \left[ P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s)) \right]$$

$$+ \left( 2(x(s))^2 - (a+b+c+abc)x(s) - 1 + bc + ab + ac \right)$$

$$\times \left[ P_n(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s)) \right]$$

$$= \left[ -\left( \frac{2\left((q+1)^2 + 2q^n\right)\sqrt{q}}{q^n(q-1)} \right) x(s) - \frac{2q^{\frac{1}{2}}(B_n + (q^2 + 2q)B_{n-1})}{q^n(q-1)} + \frac{2(a+b+c-abc)\sqrt{q}}{q-1} \right]$$

$$\times P_n(s)P_{n-1}(s) + \frac{2(q+1)q^{\frac{1}{2}}}{q^n(q-1)} \left( P_n(x(s)) \right)^2 + \frac{2(q+1)q^{\frac{1}{2}}}{q^n(q-1)} C_n \left( P_{n-1}(x(s)) \right)^2,$$

where

$$B_n = -\frac{1}{2}\left(a + a^{-1} - a^{-1}(1 - abq^n)(1 - acq^n) - a(1 - q^n)(1 - bcq^{n-1})\right)$$

$$C_n = \frac{1}{4}(1 - q^n)(1 - abq^{n-1})(1 - acq^{n-1})(1 - bcq^{n-1}).$$

**5.2.4. Continuous $q$-Hahn Polynomials.** The Continuous $q$-Hahn polynomials have the $q$-hypergeometric representation [18, P. 415] or [8, P. 75]

$$\frac{(ae^{i\varphi})^n P_n(x; a, b, c, d; q)}{(ab, ac, ad; q)_n} = {}_4\phi_3\left(\begin{array}{c} q^{-n}, abcdq^{n-1}, ae^{i(\theta+2\varphi)}, ae^{-i\theta} \\ abe^{2i\varphi}, ac, ad \end{array}\middle| q; q\right),$$

here $x = \cos(\theta + \varphi)$. They satisfy the divided-difference equation (5.1) with

$$\phi\left(x(s)\right) = 2\left(dcba + 1\right)x^2\left(s\right) - \frac{\left(d + dcb + at^2 + bt^2 ad + abct^2 + c + acd + bt^2\right)x\left(s\right)}{t}$$
$$+ \frac{cat^2 + bt^2 d - t^2 cbad + cbt^2 + cd + t^2 + bt^4 a + t^2 ad}{t^2},$$

$$\psi\left(x(s)\right) = \frac{4\left(abcd - 1\right)q^{\frac{1}{2}}x\left(s\right)}{q - 1} - 2\sqrt{q}\frac{\left(-c - d + cda - bt^2 - at^2 + dcb + cbat^2 + bt^2 q\right)}{(q - 1)t},$$

where $t = e^{i\varphi}$.

The monic Continuous $q$-Hahn polynomials are characterized by the following non-linear recurrence relation

$$\psi(x(s))\left[P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s))\right]$$
$$+ \phi(x(s))\left[P_n(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s))\right]$$
$$= \left[K_n x(s) + M_n\right]P_n(x(s))P_{n-1}(x(s))$$
$$+ \frac{2(q + 1)(-abcdq^{2n-1} + q^2)}{\sqrt{q}(q - 1)q^n}\left(P_n(x(s))\right)^2$$
$$+ \frac{2(q + 1)(abcdq^{2n} - q)}{\sqrt{q}(q - 1)q^n}C_n\left(P_{n-1}(x(s))\right)^2,$$

where

$$K_n = \frac{2abcd[\left(q^{2n}(q + 1)^2 + 2q^{n+2}\right)]}{q^{\frac{3}{2}}(q - 1)q^n} - \frac{2\sqrt{q}\left((q + 1)^2 + 2q^n\right)}{(q - 1)q^n},$$

$$C_n = \frac{1}{4}\frac{(1 - q^n)(1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1}e^{-2i\varphi})(1 - abcdq^{n-2})}{(1 - abcdq^{2n-1})(1 - abcdq^{2n-3})}$$
$$\times \frac{(1 - abq^{n-1}e^{2i\varphi})(1 - acq^{n-1})(1 - adq^{n-1})}{(1 - abcdq^{2n-2})^2},$$

$$M_n = \frac{\sqrt{q}(1 - abcdq^{2n})}{q^n(q - 1)}\left(ae^{i\varphi} + a^{-1}e^{i\varphi} - (\widetilde{A}_n + \widetilde{C}_n)\right)$$
$$- \left\{(q + 2)\frac{(q^2 - q^{2n-1}abcd)}{\sqrt{q}(q - 1)q^n}\right\}\left(ae^{i\varphi} + a^{-1}e^{i\varphi} - (\widetilde{A}_{n-1} + \widetilde{C}_{n-1})\right)$$
$$- 2\sqrt{q}\frac{(-c - d + cda - bt^2 - at^2 + dcb + cbat^2 + bt^2 q)}{(q - 1)t}$$

with $t = e^{i\varphi}$,

$$\widetilde{A_n} = \frac{(1 - abq^n e^{2i\varphi})(1 - acq^n)(1 - adq^n)(1 - abcdq^{n-1})}{ae^{i\varphi}(1 - abcdq^{2n-1})(1 - abcdq^{2n})},$$

$$\widetilde{C_n} = \frac{ae^{i\varphi}(1 - q^n)(1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1}e^{-2i\varphi})}{(1 - abcdq^{2n-1})(1 - abcdq^{2n-2})}.$$

**5.2.5. Dual $q$-Hahn Polynomials.** The Dual $q$-Hahn polynomials have the $q$-hypergeometric representation ([18, P. 450] or [8, P. 76]

$$R_n(x(s); \gamma, \delta, N|q) = {}_3\phi_2\left( \begin{array}{c} q^{-n}, q^{-s}, \gamma\delta q^{s+1} \\ \gamma q, q^{-N} \end{array} \bigg| q; q \right), \quad n = 0, 1, \dots, N,$$

where $x(s) = q^{-s} + \gamma\delta q^{s+1}$ and $N$ a non-negative integer. They satisfy (5.1) with

$$\phi(x(s)) = (x(s))^2 - \frac{(\gamma q + q^{N+1}\gamma\delta + \gamma q^{N+1} + 1)x(s)}{2q^N}$$
$$+ \frac{\gamma(q^{N+1}\gamma\delta - \delta q^N + \delta + 1)q}{2q^N},$$

$$\psi(s(s)) = -\frac{2\sqrt{q}}{q-1}x(s) + \frac{(q^{N+1}\gamma\delta + \gamma q^{N+1} - \gamma q + 1)\sqrt{q}}{(q-1)q^N}.$$

The monic Dual $q$-Hahn polynomials are characterized by the following non-linear recurrence relation

$$\psi(x(s))\left[ P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s)) \right]$$
$$+ \phi(x(s))\left[ P_n(x(s))\mathbb{D}_x \mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x \mathbb{S}_x P_n(x(s)) \right]$$
$$= [K_n x(s) + M_n] P_n(x(s)) P_{n-1}(x(s)) + \frac{(q+1)q^{\frac{3}{2}}}{(1-q)q^n} (P_n(x(s)))^2$$
$$+ \frac{(1+q)\sqrt{q}}{(1-q)q^n} C_n (P_{n-1}(x(s)))^2,$$

where

$$K_n = \frac{\sqrt{q}(2q^n - q^2 + 1)}{q^n(1-q)},$$

$$M_n = \frac{\sqrt{q}(-q^2\gamma\delta + 2\gamma\delta q - q^2\gamma + 2\gamma q - 2q^{-N+1}\gamma + q^{-N}\gamma - q^{N+1} + 2q^{-N})}{q-1},$$

$$C_n = \gamma q(1 - q^{n-N-1})(1 - \gamma q^n)(1 - q^n)(\delta - q^{n-N-1}).$$

**5.2.6. Al-Salam–Chihara Polynomials.** The Al-Salam–Chihara polynomials have the $q$-hypergeometric representation [18, P. 455] or [8, P. 77]

$$Q_n(x; a, b|q) = \frac{(ab; q)_n}{a^n} {}_3\phi_2\left( \begin{array}{c} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0 \end{array} \bigg| q; q \right), \, x = \cos\theta.$$

They satisfy the divided-difference equation (5.1) with

$$\phi(x(s)) = 2(x(s))^2 - (a+b)x(s) + ab - 1,$$
$$\psi(s(s)) = -\frac{4\sqrt{q}x(s)}{q-1} + \frac{2(a+b)\sqrt{q}}{q-1}.$$

The monic Al-Salam–Chihara polynomials are characterized by the following non-linear recurrence relation

$$\psi(x(s)) \left[ P_n(x(s)) \mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s)) \mathbb{S}_x^2 P_n(x(s)) \right]$$
$$+ \phi(x(s)) \left[ P_n(x(s)) \mathbb{D}_x \mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s)) \mathbb{D}_x \mathbb{S}_x P_n(x(s)) \right]$$
$$= \left[ K_n x(s) + M_n \right] P_n(x(s)) P_{n-1}(x(s)) + \frac{2(q+1)q^{\frac{3}{2}}}{(1-q)q^n} \left( P_n(x(s)) \right)^2$$
$$+ \frac{(1+q)(1-q^n)(1-abq^{n-1})}{2(1-q)q^{n-\frac{1}{2}}} \left( P_{n-1}(x(s)) \right)^2,$$

where

$$K_n = \frac{2\sqrt{q} \left( 2q^n - q^2 + 1 \right)}{q^n (1-q)},$$
$$M_n = \frac{(a+b)\sqrt{q}(q-3)}{1-q}.$$

### 5.2.7. $q$-Meixner–Pollaczek Polynomials.

The $q$-Meixner–Pollaczek polynomials [18, P. 460] or [8, P. 78]

$$P_n(x; a|q) = a^{-n} e^{-in\varphi} \frac{(a^2; q)_n}{(q; q)_n} {}_3\phi_2 \left( \begin{matrix} q^{-n}, ae^{i(\theta+2\varphi)}, ae^{-i\theta} \\ a^2, 0 \end{matrix} \; \middle| \; q, q \right), x = \cos(\theta + \varphi).$$

They satisfy (5.1) with

$$\phi(x(s)) = 2(x(s))^2 - 2a \cos\varphi\, x(s) + a^2 - 1,$$
$$\psi(s(s)) = -\frac{4\sqrt{q}}{q-1} x(s) + \frac{4a\sqrt{q}\cos\varphi}{q-1}.$$

The monic $q$-Meixner–Pollaczek polynomials are characterized by the following non-linear recurrence relation

$$\left( -\frac{4\sqrt{q}}{q-1} x(s) + \frac{4a\sqrt{q}\cos\varphi}{q-1} \right) \left[ P_n(x(s)) \mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s)) \mathbb{S}_x^2 P_n(x(s)) \right]$$
$$+ \left( 2(x(s))^2 - 2a\cos\varphi\, x(s) + a^2 - 1 \right)$$
$$\times \left[ P_n(x(s)) \mathbb{D}_x \mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s)) \mathbb{D}_x \mathbb{S}_x P_n(x(s)) \right]$$
$$= \left[ -\left( \frac{2 \left( (q+1)^2 + 2q^n \right) \sqrt{q}}{q^n (q-1)} \right) x(s) + \frac{2a\sqrt{q}\cos\varphi(q+5)}{q-1} \right] P_n(x(s)) P_{n-1}(x(s))$$
$$+ \frac{2(q+1)q^{\frac{3}{2}}}{q^n(q-1)} \left( P_n(x(s)) \right)^2 + \left( \frac{\sqrt{q}(q+1)(1-q^n)(1-a^2q^{n-1})}{2q^n(q-1)} \right) \left( P_{n-1}(x(s)) \right)^2.$$

### 5.2.8. Continuous $q$-Jacobi Polynomials.

The Continuous $q$-Jacobi polynomials have the $q$-hypergeometric representation [18, P. 463] or [8, P. 78]

$$P_n^{(\alpha,\beta)}(x|q)$$
$$= \frac{(q^{\alpha+1}; q)_n}{(q; q)_n} {}_4\phi_3 \left( \begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta}, q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta} \\ q^{\alpha+1}, -q^{\frac{1}{2}(\alpha+\beta+1)}, -q^{\frac{1}{2}(\alpha+\beta+2)} \end{matrix} \; \middle| \; q; q \right), x = \cos\theta.$$

They satisfy the divided-difference equation (5.1) with

$$\phi(x(s)) = (p^{2\alpha+2\beta+4} + 1)\, x^2(s) + \frac{1}{2}(p+1)p^{\frac{1}{2}}\left(p^{2\alpha+2\beta+2} - p^\alpha - p^{\alpha+2\beta+2} + p^\beta\right) x(s)$$

$$-\frac{1}{2}\left(p^{2\alpha+2\beta+4} + p^{\alpha+\beta+3} - p^{2\alpha+2} + p^{\alpha+\beta+2} - p^{2\beta+2} + p^{\alpha+\beta+1} + 1\right),$$

$$\psi(x(s)) = \frac{4p\left(p^{2\alpha+2\beta+4} - 1\right) x(s)}{(p-1)(p+1)} - \frac{\left(-p^{2\alpha+\beta+2} - p^\alpha + p^{\alpha+2\beta+2} + p^\beta\right) p^{\frac{3}{2}}}{p-1},$$

with $p = q^2$.

The monic Continuous $q$-Jacobi polynomials are characterized by the following non-linear recurrence relation

$$\psi(x(s))\left[P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s))\right]$$
$$+ \phi(x(s))\left[P_n(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s))\right]$$
$$= \left[K_n x(s) + M_n\right] P_n(x(s))P_{n-1}(x(s))$$
$$- \frac{q^{\frac{3}{2}}(q+1)(q^{\alpha+\beta+2n-1} - 1)}{(q-1)q^n}\left(P_n(x(s))\right)^2$$
$$- \frac{\sqrt{q}(q+1)(q^{\alpha+\beta+2n+1} - 1)}{(q-1)q^n}C_n\left(P_{n-1}(x(s))\right)^2,$$

where

$$K_n = \frac{\sqrt{q}[((q+1)^2\left(-1 + q^{\alpha+\beta+2}\right)) + \left(2q^n\left(-1 + q^{\alpha+\beta+2}\right)\right)]}{(q-1)q^n},$$

$$C_n = \frac{1}{4}\frac{(1-q^n)(1-q^{n+\alpha})(1-q^{n+\beta})(1-q^{n+\alpha+\beta})\left(1 - q^{n+\frac{1}{2}(\alpha+\beta-1)}\right)}{(1-q^{2n-1+\alpha+\beta})(1-q^{2n+1+\alpha+\beta})}$$
$$\times \frac{\left(1 + q^{n+\frac{1}{2}(\alpha+\beta+1)}\right)\left(1 - q^{n+\frac{1}{2}(\alpha+\beta)}\right)^2}{(1-q^{2n+\alpha+\beta})^2},$$

$$M_n = -\frac{\sqrt{q}(q+1)(q^{\alpha+\beta+2n+2} - 1)}{2(q-1)q^n}\left(q^{\frac{1}{2}\alpha+\frac{1}{4}} + q^{-\frac{1}{2}\alpha-\frac{1}{4}} - (\widetilde{A_n} + \widetilde{C_n})\right)$$
$$-\left\{\frac{q^{\frac{3}{2}}[q^{\alpha+\beta+2n} - 1 + (q+1)(q^{\alpha+\beta+2n-1} - 1)]}{2q^n(q-1)}\right\}$$
$$\times \left(q^{\frac{1}{2}\alpha+\frac{1}{4}} + q^{-\frac{1}{2}\alpha-\frac{1}{4}} - (\widetilde{A}_{n-1} + \widetilde{C}_{n-1})\right)$$
$$+\frac{\left(-p^{2\alpha+\beta+2} - p^\alpha + p^{\alpha+2\beta+2} + p^\beta\right) p^{\frac{3}{2}}}{p-1},$$

with

$$\widetilde{A_n} = \frac{(1-q^{n+\alpha+1})(1-q^{\alpha+\beta+n+1})(1-q^{n+\frac{1}{2}(\alpha+\beta+1)})(1-q^{n+\frac{1}{2}(\alpha+\beta+2)})}{q^{\frac{1}{2}\alpha+\frac{1}{4}}(1-q^{\alpha+\beta+2n+1})(1-q^{\alpha+\beta+2n+2})},$$

$$\widetilde{C_n} = \frac{q^{\frac{1}{2}\alpha+\frac{1}{4}}(1-q^n)(1-q^{n+\beta})\left(1 + q^{n+\frac{1}{2}(\alpha+\beta)}\right)\left(1 + q^{n+\frac{1}{2}(\alpha+\beta+1)}\right)}{(1-q^{\alpha+\beta+2n})(1-q^{\alpha+\beta+2n+1})}.$$

**5.2.9. Dual $q$-Krawtchouk Polynomials.** The Dual $q$-Krawtchouk polynomials [18, P. 505] or [8, P. 80]

$$K_n(x(s); c, N|q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{-s}, cq^{s-N} \\ q^{-N}, 0 \end{matrix}\middle|\, q, q\right), \quad n = 0, 1, \ldots, N,$$

where $x(s) = q^{-s} + cq^{s-N}$. They satisfy (5.1) with

$$\phi(x(s)) = (x(s))^2 - (c+1)q^{-N}x(s) - 2c(q^{-N} - q^{-2N}),$$
$$\psi(s(s)) = -\frac{2\sqrt{q}}{q-1}x(s) + \frac{2(c+1)\sqrt{q}}{(q-1)q^N}.$$

The monic Dual $q$-Krawtchouk polynomials are characterized by the following non-linear recurrence relation

$$\left(-\frac{2\sqrt{q}}{q-1}x(s) + \frac{2(c+1)\sqrt{q}}{(q-1)q^N}\right)\left[P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s))\right]$$
$$+ \left((x(s))^2 - (c+1)q^{-N}x(s) - 2c(q^{-N} - q^{-2N})\right)$$
$$\times \left[P_n(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s))\right]$$
$$= \left[-\left(\frac{(2q^n + (1+q)^2)\sqrt{q}}{q^n(q-1)}\right)x(s) + \frac{(c+1)\sqrt{q}(q+5)}{(q-1)q^N}\right]P_n(x(s))P_{n-1}(x(s))$$
$$+ \frac{(q+1)q^{\frac{3}{2}}}{q^n(q-1)}\left(P_n(x(s))\right)^2 + \left(\frac{c\sqrt{q}(q+1)(1-q^n)(1-q^{n-N-1})}{q^{n+N}(q-1)}\right)(P_{n-1}(x(s)))^2.$$

### 5.2.10. Continuous Big $q$-Hermite Polynomials.

The continuous big $q$-Hermite polynomials [18, P. 509]

$$H_n(x; a, |q) = a^{-n}{}_3\phi_2\left(\begin{array}{c}q^{-n}, ae^{i\theta}, ae^{-i\theta}\\0, 0\end{array}\bigg| q, q\right), \quad x = \cos\theta.$$

They satisfy (5.1) with

$$\phi(x(s)) = 2(x(s))^2 - ax(s) - 1,$$
$$\psi(s(s)) = -\frac{4\sqrt{q}}{q-1}x(s) + \frac{2a\sqrt{q}}{q-1}.$$

The Continuous big $q$-Hermite polynomials are characterized by the following non-linear recurrence relation

$$\left(-\frac{4\sqrt{q}}{q-1}x(s) + \frac{2a\sqrt{q}}{q-1}\right)\left[P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s))\right]$$
$$+ \left(2(x(s))^2 - ax(s) - 1\right)\left[P_n(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s))\right]$$
$$= \left[-\left(\frac{2((q+1)^2 + 2q^n)\sqrt{q}}{q^n(q-1)}\right)x(s) + \frac{a\sqrt{q}(q+5)}{q-1}\right]P_n(x(s))$$
$$\times P_{n-1}(x(s)) + \frac{2(q+1)q^{\frac{3}{2}}}{q^n(q-1)}\left(P_n(x(s))\right)^2 - \frac{(q+1)(q^n-1)\sqrt{q}}{2q^n(q-1)}(P_{n-1}(x(s)))^2.$$

### 5.2.11. Continuous $q$-Laguerre Polynomials.

The Continuous $q$-Laguerre polynomials have the $q$-hypergeometric representation [18, P. 514] or [8, P. 81]

$$P_n^{(\alpha)}(x|q) = \frac{(q^{\alpha+1}; q)_n}{(q; q)_n}{}_3\phi_2\left(\begin{array}{c}q^{-n}, q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta}, q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta}\\q^{\alpha+1}, 0\end{array}\bigg| q; q\right), x = \cos\theta.$$

They satisfy the divided-difference equation (5.1) with

$$\phi(x(s)) = 2(x(s))^2 - \frac{p^{\alpha+\frac{1}{2}}(p+1)x(s)}{2q^N} + p^{2\alpha+2} - 1,$$

$$\psi(s(s)) = -\frac{4p}{p^2-1}x(s) + \frac{p^{\alpha+\frac{3}{2}}}{(p-1)},$$

with $p = q^2$.

The monic Continuous $q$-Laguerre polynomials are characterized by the following non-linear recurrence relation

$$\psi(x(s))\left[P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s))\right]$$
$$+ \phi(x(s))\left[P_n(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s))\right]$$
$$= [K_n x(s) + M_n] P_n(x(s))P_{n-1}(x(s))$$
$$- G_{n-1}(P_n(x(s)))^2 - G_n C_n (P_{n-1}(x(s)))^2,$$

where

$$\xi_n K_n = q^{n^2+2} - q^{11} + q^{10} + q^{n^2+5} + q^9 - pq^{\frac{15}{2}} + 2pq^{\frac{13}{2}} - q^{n^2+4} - q^8 - 2pq^{\frac{9}{2}}$$
$$- pq^{n^2+\frac{3}{2}} - q^{n^2+3} - q^{n^2+6} + pq^{\frac{7}{2}} + q^7 - q^5 - q^6 + q^4 - q^{n^2+9}$$
$$+ q^{n^2+8} + pq^{n^2+\frac{11}{2}} + 4pq^{n+\frac{11}{2}} - 2pq^{n^2+\frac{9}{2}} - 8pq^{n+\frac{9}{2}}$$
$$+ q^{n^2+7} + 4pq^{n+\frac{7}{2}} + 2pq^{n^2+\frac{5}{2}},$$

with $\xi_n = -(q-1)^2(p^2-1)q^{n+\frac{7}{2}}$,

$$\nu_n M_n = q^{2n+\frac{1}{2}\alpha+\frac{1}{2}} + q^{2n+\frac{1}{2}\alpha+1} + q^{2n+\frac{1}{2}\alpha+\frac{3}{2}} + q^{2n+\frac{1}{2}\alpha+2} + q^{\frac{1}{2}\alpha+\frac{7}{2}} + q^{\frac{1}{2}\alpha+4}$$
$$+ q^{2n+\frac{1}{2}\alpha+3} + q^{2n+\frac{1}{2}\alpha+\frac{5}{2}}4q^{\alpha+\frac{3}{2}}p^{\frac{11}{4}} + 4q^{\alpha+\frac{5}{2}}p^{\frac{11}{4}} - q^{2n+\frac{1}{2}\alpha+3}p^2 - q^{2n+\frac{1}{2}\alpha+\frac{5}{2}}p^2$$
$$- q^{2n+\frac{1}{2}\alpha+\frac{1}{2}}p^2 - q^{2n+\frac{1}{2}\alpha+\frac{1}{2}}p + q^{2n+\frac{1}{2}\alpha+\frac{7}{2}}p + q^{2n+\frac{1}{2}\alpha+3}p - q^{2n+\frac{1}{2}\alpha+2}p^2$$
$$- q^{2n+\frac{1}{2}\alpha+\frac{3}{2}}p^2 - q^{2n+\frac{1}{2}\alpha+1}p^2 - q^{2n+\frac{1}{2}\alpha}p - q^{\frac{1}{2}\alpha+\frac{9}{2}}p - q^{\frac{1}{2}\alpha+4}p^2$$
$$- q^{\frac{1}{2}\alpha+4}p - q^{\frac{1}{2}\alpha+\frac{7}{2}}p^2 + q^{\frac{1}{2}\alpha+\frac{7}{2}}p + q^{\frac{1}{2}\alpha+3}p,$$

with $\nu_n = 2(p^2-1)q^{\frac{11}{4}}$,

$$\tau_n G_n = q^{n^2+\frac{7}{2}} + q^{\frac{9}{2}}p - pq^{n^2+\frac{5}{2}} - pq^{\frac{7}{2}} - q^{n^2+7} - pq^{n^2+\frac{3}{2}} - pq^{\frac{5}{2}} + q^8 + pq^{n^2+5}$$
$$+ q^{n^2+3} + pq^{n^2+\frac{1}{2}} + pq^{\frac{3}{2}} - q^6 - q^4 + q^2 - q^{n^2+1},$$

where $\tau_n = -(q-1)^2(p^2-1)q^{n+\frac{3}{2}}$,

$$C_n = \frac{1}{4}(1-q^n)(1-q^{n+\alpha}).$$

### 5.2.12. Continuous $q$-Hermite Polynomials.

The Continuous $q$-Hermite polynomials have the $q$-hypergeometric representation [18, P. 540] or [8, P. 82]

$$H_n(x|q) = e^{in\theta}{}_2\phi_0\left(\begin{array}{c} q^{-n}, 0 \\ - \end{array}\bigg| q; q^n e^{-2in\theta}\right), x = \cos\theta.$$

They satisfy the divided-difference equation (5.1) with

$$\phi(x(s)) = 2(x(s))^2 - 1,$$
$$\psi(s(s)) = -\frac{4\sqrt{q}}{q-1}.$$

The monic Continuous $q$-Hermite polynomials are characterized by the following non-linear recurrence relation

$$\psi(x(s)) \left[ P_n(x(s))\mathbb{S}_x^2 P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{S}_x^2 P_n(x(s)) \right]$$
$$+ \phi(x(s)) \left[ P_n(x(s))\mathbb{D}_x\mathbb{S}_x P_{n-1}(x(s)) + P_{n-1}(x(s))\mathbb{D}_x\mathbb{S}_x P_n(x(s)) \right]$$
$$= \left[ \frac{\sqrt{q}\left(2q^n - q^2 + 1\right) x(s)}{q^n(1-q)} \right] P_n(x(s))P_{n-1}(x(s)) + \frac{(1+q)q^{\frac{3}{2}}}{(1-q)q^n} \left( P_n(x(s)) \right)^2$$
$$- \frac{(1+q)\sqrt{q}}{2(1-q)q^n} \left( P_{n-1}(x(s)) \right)^2.$$

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# References

[1] Al-Salam, W.A.: Characterization theorems for orthogonal polynomials. In: Nevai, P. (ed.) Orthogonal Polynomials: Theory and Practice, NATO ASI Series C: Math. and Phys. Sciences, vol. 294. Kluwer, Dordrecht, pp. 1–24 (1990)

[2] Al-Salam, W.A.: On the Bessel polynomials. Boll. Un. Mat. Ital. (3) **12**, 227–229 (1957)

[3] Al-Salam, W.A.: The Bessel polynomials. Duke Math. J. **24**, 529–545 (1957)

[4] Al-Salam, W.A., Chihara, T.S.: Another characterization of the classical orthogonal polynomials. SIAM J. Math. Anal. **3**, 65–70 (1972)

[5] Atakishiyev, N.M., Rahman, M., Suslov, S.K.: On classical orthogonal polynomials. Constr. Approx. **11**, 181–226 (1995)

[6] Brezinski, C.: Padé-Type Approximation and General Orthogonal Polynomials, International Series of Numerical Mathematics, vol. 50. Birkhäuser, Basel (1980)

[7] Chihara, T.S.: An Introduction to Orthogonal Polynomials. Gordon and Breach, New York (1978)

[8] Foupouagnigni, M.: On difference and differential equations for modifications of classical orthogonal polynomials. Habilitation Thesis, Fachbereich Mathematik der Universität Kassel (2006)

[9] Foupouagnigni, M.: On difference equations for orthogonal polynomials on non-uniform lattices. J. Differ. Equ. Appl. **14**, 127–174 (2008)

[10] Foupouagnigni, M., Mboutngam, S.: On the polynomial solution of divided-difference equations of hypergeometric type on nonuniform lattices. Axioms, **8** (2019)

[11] Foupouagnigni, M., Kenfack-Nangho, M., Mboutngam, S.: Characterization theorem of classical orthogonal polynomials on non-uniform lattices: the functional approach. Integr. Transf. Spec. Funct. **22**(10), 739–758 (2011)

[12] Foupouagnigni, M., Koepf, W., Kenfack-Nangho, M., Mboutngam, S.: On solutions of holonomic divided-difference equations on non-uniform lattices. Axioms **3**, 404–434 (2013)

[13] García, A.G., Marcellán, F., Salto, L.: A distributional study of discrete classical orthogonal polynomials. J. Comput. Appl. Math. **5**(7), 147–162 (1995)

[14] Gautschi, W.: Orthogonal Polynomials: Computation and Approximation. Oxford University Press, New York (2004)

[15] Gautschi, W.: Orthogonal polynomials, quadrature, and approximation: computational methods and software (in Matlab). In: Orthogonal Polynomials and Special Functions. Lecture Notes in Mathematics, vol. 1883, pp. 1–77. Springer, Berlin (2006)

[16] Kenfack-Nangho, M.: On Laguerre-Hahn orthogonal polynomials on non-uniform lattices. PhD Thesis, Université de Yaoundé I (2014)

[17] Koekoek, R., Swarttouw, R.: The Askey-scheme of hypergeometric orthogonal polynomials and its $q$-analogue, Report no. 98-17 (1998). Faculty of Information technology and Systems, Delft University of Technology

[18] Koekoek, R., Lesky, P.A., Swarttouw, R.F.: Hypergeometric Orthogonal Polynomials and their $q$-Analogues. Springer Monographs in Mathematics. Springer, Berlin (2010)

[19] Koepf, W., Schmersau, D.: On a structure formula for classical q-orthogonal polynomials. J. Comput. Appl. Math. **136**, 99–107 (2001)

[20] Krall, H.L., Frink, O.: A new class of orthogonal polynomials: the Bessel polynomials. Trans. Am. Math. Soc. **65**, 100–115 (1949)

[21] Kwon, K.H., Lee, D.W., Park, S.B., Yoo, B.H.: Discrete classical orthogonal polynomials. J. Differ. Equ. Appl. **4**, 145–162 (1998)

[22] Kwon, K.H., Lee, D.W., Park, S.B., Yoo, B.H.: Hahn class orthogonal polynomials. Kyungpook Math. J. **38**, 259–281 (1998)

[23] Marcellán, F., Branquinho, A.P., Petronilho, J.: Classical orthogonal polynomials: a functional approach. Acta Appl. Math. **34**(3), 283–303 (1994)

[24] Mboutngam, S.: On semi-classical orthogonal polynomials on non-uniform lattices. PhD Thesis, Université de Yaoundé I (2014)

[25] McCarthy, P.J.: Characterizations of classical orthogonal polynomials. Port. Math. **20**, 47–52 (1961)

[26] Njionou Sadjang, P.: Moments of classical orthogonal polynomials. PhD thesis. Universität Kassel (2013). http://nbn-resolving.de/urn:nbn:de:hebis:34-2013102244291

[27] Njionou Sadjang, P., Koepf, W., Foupouagnigni, M.: On structure formulas for Wilson polynomials. Integr. Transf. Spec. Funct. **26**(12), 1000–1014 (2015)

[28] Njionou Sadjang, P., Koepf, W., Foupouagnigni, M.: On moments of classical orthogonal polynomials. J. Math. Anal. Appl. **424**, 122–151 (2015)

[29] Suslov, S.K.: The theory of difference analogues of special functions of hypergeometric type. Russ. Math. Surv. **44**, 227–278 (1989)

[30] Tcheutia, D.D., Njionou Sadjang, P., Koepf, W., Foupouagnigni, M.: Divided-difference equation, inversion, connection, multiplication and linearization formulae of the Continuous Hahn and the Meixner-Pollaczek polynomials. Ramanujan J. **45**, 33–56 (2018)

[31] Trefethen, L.N.: Approximation Theory and Approximation Practice, Extended Society for Industrial and Applied Mathematics (SIAM), Philadelphia (2020)

[32] Tricomi, F.G.: Vorlesungen über Orthogonalreihen, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band LXXVI. Springer, Berlin (1955)

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