On Families of Iterated Derivatives

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Abstract:

We give an overview of an approach on special functions due to Truesdell, and show how it can be used to develop certain type of identities for special functions. Once obtained, these identities may be verified by an independent algorithmic method for which we give some examples.

1 The F-equation and F-functions

Truesdell [11] studied solutions of the functional equation

$$\frac{\partial}{\partial z} F(z,\alpha) = F(z,\alpha+1) \tag{1}$$

satisfying the initial condition

$$F(z_0,\alpha) = \Phi(\alpha) \tag{2}$$

where F is a function of the two variables z and α. Here z is assumed to be a real or complex variable, α is such that

$$\begin{array}{llcll} \text{either} & \alpha & = & \alpha_0 + k & (k \in \mathbb{N}_0)\ , \\ \text{or} & \alpha & = & \alpha_0 + k & (k \in \mathbb{Z})\ , \\ \text{or} & \alpha & \geq & \alpha_0 & (\alpha \in \mathbb{R})\ , \\ \text{or} & \operatorname{Re}\alpha & \geq & \alpha_0 & (\alpha \in \mathbb{C})\ , \end{array} \tag{3}$$

(α_0 may equal $-\infty$), and Φ is a given function of α. Equation (1) is called the *F-equation*, and we call a solution of the F-equation that satisfies the initial condition (2) an *F-function* corresponding to the initial function Φ.

Truesdell showed how a functional equation of the form

$$\frac{\partial}{\partial z} f(z,\alpha) = A(z,\alpha)\, f(z,\alpha) + B(z,\alpha)\, f(z,\alpha+1) \tag{4}$$

can be transformed to the F-equation (see [11], § 3, and § 7). As almost all special functions satisfy an equation of type (4) (see e. g. [8]), these can be brought in connection to F-functions, see [11], § 5.

As furthermore F-functions have nice properties that are easily established, these properties can be used to develop identities for special functions almost algorithmically. This is demonstrated in the given paper.

The simplest F-function is the exponential function $F(z, \alpha) = e^z$ which is independent of α, and satisfies the F-equation with initial function ($z_0 := 0$)

$$\Phi(\alpha) = F(0, \alpha) = 1 \ .$$

As another example solution of the F-equation—Truesdell presented 48 example functions in [11], §5, including hypergeometric functions, associated Legendre functions, classical orthogonal polynomials, as well as Bessel and Hankel functions—we consider the function

$$e^{-z^2} H_\alpha(-z) \tag{5}$$

([11], §5, Example 33) where H_n ($n \in \mathbb{N}_0$) denote the Hermite polynomials, corresponding to the initial function ($z_0 := 0$)

$$\Phi(n) = F(0, n) = H_n(0) = \begin{cases} (-1)^j \frac{(2j)!}{j!} & \text{if } n = 2j \ (j \in \mathbb{N}_0) \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

(see e.g. [1], (22.4)). We will use this example function in later sections to demonstrate the algorithmic use of Truesdell's approach.

Note that we use the standard notation for special functions of [1] rather that Truesdell's which is slightly different for the associated Legendre functions (see § 8).

2 Existence and uniqueness theorem

In this section we present the main result on analytic F-functions, namely an existence and uniqueness theorem, that is covered by a representation of an F-function as a power series (see [11], Theorem 11.1).

Theorem 1 (Existence, uniqueness and power series representation of F-functions) If Φ is bounded for values α given by (3), i. e. there is a constant $M > 0$ such that

$$|\Phi(\alpha)| \leq M \qquad (\alpha = \alpha_0 + k \ (k \in \mathbb{N}_0(\mathbb{Z}))\ , \quad \alpha \geq \alpha_0 \quad \text{or} \quad \operatorname{Re} \alpha \geq \alpha_0\ , \quad \text{respectively})$$

then there is one and only one solution of the F-equation

$$\frac{\partial}{\partial z} F(z, \alpha) = F(z, \alpha + 1)$$

which satisfies the initial condition

$$F(z_0, \alpha) = \Phi(\alpha) \qquad (\alpha = \alpha_0 + k \ (k \in \mathbb{N}_0(\mathbb{Z}))\ , \quad \alpha \geq \alpha_0 \quad \text{or} \quad \operatorname{Re} \alpha \geq \alpha_0\ , \quad \text{respectively})$$

and forms an analytic function for $z \in \mathbb{C}$ in the entire plane. For this F-function the power series representation

$$F(z, \alpha) = \sum_{k=0}^{\infty} \Phi(\alpha + k) \, \frac{(z - z_0)^k}{k!} \tag{6}$$

is valid for all $z \in \mathbb{C}$ and all $\alpha = \alpha_0 + k$ ($k \in \mathbb{N}_0(\mathbb{Z})$) (or $\alpha \geq \alpha_0$, or $\operatorname{Re} \alpha \geq \alpha_0$, respectively).

Independently of the boundedness of Φ, if the power series representation (6) has a positive radius of convergence, then it represents a unique analytic solution of (1) and (2) in its disk of convergence.

Proof: By comparison with the exponential function and by the boundedness of Φ we see that the function F given by (6) forms an analytic function for all $z \in \mathbb{C}$. Therefore we may differentiate termwise, and get

$$\frac{\partial}{\partial z} F(z, \alpha) = \sum_{k=1}^{\infty} \Phi(\alpha + k) \, \frac{(z - z_0)^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \Phi(\alpha + k + 1) \, \frac{(z - z_0)^k}{k!} = F(z, \alpha + 1) \; .$$

For $z = z_0$ we have the initial condition statement

$$F(z_0, \alpha) = \Phi(\alpha) \; .$$

Therefore F is an F-function corresponding to Φ.

To prove uniqueness, assume a second entire function G satisfies (1) and (2). Then obviously the function $H := F - G$ is entire and satisfies the F-equation together with the initial condition

$$H(z_0, \alpha) = 0$$

for all values α that are involved, i. e. H vanishes identically at the point $z = z_0$. Differentiating H, by the F-equation we see that the same conclusion holds for the derivative of H, and by induction we see that

$$\left. \frac{\partial^k}{\partial z^k} H(z, \alpha) \right|_{z = z_0} = 0$$

for all $k \in \mathbb{N}_0$, and so $H \equiv 0$ for all values α involved.

If Φ is unbounded, but the power series representation (6) has a positive radius of convergence, then in its disk of convergence the same argumentation may be applied, and the same conclusion follows. □

We remark that the theorem does not show the series representation (6) for each F-function: The usual counterexample ($z \in \mathbb{R}$)

$$G(z, 0) := \begin{cases} e^{-\frac{1}{z^2}} & \text{if } z \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

representing a real function whose Taylor polynomials at the origin of any order equal the zero function generates the F-function

$$G(z,n) := \begin{cases} G^{(n)}(z,0) & \text{if } z \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

corresponding to the initial function

$$\Phi(n) = G(0,n) = 0 \; .$$

Similarly to the fact that $G(z,0)$ grows so slowly in a neighborhood of the origin that the Taylor approximations of any order are identically zero, i. e. the function cannot be distiguished from the zero function by polynomial approximation, the function $G(z,n)$ in not distinguishable from the zero function by polynomial approximation around $z=0$. The theorem tells that the zero function ($n \in \mathbb{N}_0$)

$$F(z,n) = \sum_{k=0}^{\infty} \Phi(n+k) \, \frac{z^k}{k!} = \sum_{k=0}^{\infty} 0 \, \frac{z^k}{k!} \equiv 0$$

is the unique *analytic* solution of the F-equation with initial function $\Phi(n) = 0$. Nevertheless, other non-analytic solutions may exist, as $G(z,n)$ shows.

On the other hand, Truesdell's list of F-functions contains only locally analytic functions, and most special functions are analytic in certain regions, so that (6) is valid if we choose z_0 such that F does not have a singularity there.

3 Power series representations of F-functions

Theorem 1 can be used to obtain power series representations. For the exponential function $F(z,\alpha) = e^z$, Equation (6) reads

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!} \tag{7}$$

i. e., we have the usual power series representation.

Next we consider the example function $F(z,n) := e^{-z^2} H_n(-z)$ corresponding to the initial function

$$\Phi(n) = F(0,n) = H_n(0) = \begin{cases} (-1)^j \, \frac{(2j)!}{j!} & \text{if } n = 2j \;\, (j \in \mathbb{N}_0) \\ 0 & \text{if } n \; \text{ is odd} \end{cases} \; .$$

By Theorem 1 we have the representation

$$\begin{aligned} F(z,n) = e^{-z^2} H_n(-z) &= \sum_{k=0}^{\infty} \Phi(n+k) \, \frac{z^k}{k!} = \sum_{k=n}^{\infty} \Phi(k) \, \frac{z^{k-n}}{(k-n)!} \\ &= \sum_{j=[n/2]}^{\infty} (-1)^j \, \frac{(2j)!}{j!} \, \frac{z^{2j-n}}{(2j-n)!} = \sum_{j=0}^{\infty} (-1)^j \binom{2j}{n} \frac{n!}{j!} \, z^{2j-n} \; , \end{aligned}$$

or equivalently

$$e^{-z^2} H_n(z) = (-1)^n\, n! \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \binom{2j}{n} z^{2j-n} \; .$$

Note that the given representation is valid for the even functions H_{2m} as well as for the odd functions H_{2m+1} ($m \in \mathbb{N}_0$).

The given procedure leads to power series representations for F-functions, in our case for $e^{-z^2} H_n(z)$, but not for arbitrary special functions, e. g. not for $H_n(z)$ itself. We note that for the purpose of getting closed form series representations there is a better procedure, see [3]–[9]. For example, our MATHEMATICA [13] implementation [7] yields the power series representation

```
In[1]:= PowerSeries[E^(-z^2)*HermiteH[n,z],z]

                k   n            2 k            1 + n
           (-4)   2   Sqrt[Pi] z     Pochhammer[-----, k]
                                                  2
Out[1]= Sum[-------------------------------------------, {k, 0, Infinity}] +
                                          1 - n
                      (2 k)! Gamma[-----]
                                            2

              k   n              1 + 2 k            2 + n
         (-4)   2   n Sqrt[Pi] z          Pochhammer[-----, k]
                                                        2
>     Sum[-----------------------------------------------, {k, 0, Infinity}]
                                               n
                      (1 + 2 k)! Gamma[1 - -]
                                               2
```

The package `SpecialFunctions.m` can be obtained from the author.

4 F-functions as iterated derivatives

By the result of the last section, we can easily calculate the derivatives of analytic F-functions (see [11], Theorem 18.1).

Theorem 2 (Derivatives of F-functions) Any analytic F-function is infinitely often differentiable in its disk of convergence, and

$$F^{(k)}(z,\alpha) = \frac{\partial^k}{\partial z^k} F(z,\alpha) = F(z, \alpha + k) \tag{8}$$

for all $k \in \mathbb{N}_0$.

Proof: The F-equation gives the statement for $k = 1$. Assume, (8) is valid for the index k. Then an application of the F-equation yields

$$\frac{\partial^{k+1}}{\partial z^{k+1}} F(z,\alpha) = \frac{\partial}{\partial z} \left(\frac{\partial^k}{\partial z^k} F(z,\alpha) \right) = \frac{\partial}{\partial z} F(z, \alpha + k) = F(z, \alpha + k + 1) \; ,$$

and the induction is complete. □

Note that the theorem tells that F-functions $F(z,\alpha)$ form families of iterated derivatives $F(z,\alpha_0+n) = F^{(n)}(z,\alpha_0)$ that are numbered by the variable $\alpha = n$, with starting values α_0 (or $\alpha \in [\alpha_0, \alpha_0+1)$, or $\alpha_0 \leq \operatorname{Re}\alpha < \alpha_0+1$, respectively), and obviously each family of iterated derivatives forms an F-function. The theory of analytic F-functions therefore is the theory of iterated derivatives, a fact indicating their importance. From this point of view, Theorem 1 is obvious: As each analytic function is uniquely determined by the values of its derivatives at a single point (Taylor representation), given $\Phi(\alpha_0+n)$ $(n \in \mathbb{N}_0)$ determines $F(z,\alpha_0)$, and therefore $F(z,\alpha_0)^{(n)} = F(z,\alpha_0+n)$, uniquely.

As a further consequence derivatives of F-functions inherit this property.

Theorem 3 If F is an analytic F-function, then $F^{(k)}$ is an F-function for all $k \in \mathbb{N}$. If F has initial value function $\Phi(\alpha)$, then $F^{(k)}$ has initial value function $\Phi(\alpha+k)$.

Proof: By definition F satisfies (1) and (2), and so for $G(z,\alpha) := \frac{\partial}{\partial z}F(z,\alpha)$ we have

$$G(z,\alpha) = \frac{\partial}{\partial z}F(z,\alpha) = F(z,\alpha+1) \; ,$$

and therefore

$$\frac{\partial}{\partial z}G(z,\alpha) = \frac{\partial}{\partial z}F(z,\alpha+1) = G(z,\alpha+1) \; .$$

The initial value function for G is

$$G(z_0,\alpha) = F(z_0,\alpha+1) = \Phi(\alpha+1) \; .$$

The theorem follows then by induction. □

5 Generalized Rodrigues formulas

The use of Theorem 2 generates generalized Rodrigues formulas. For the exponential function $F(z,\alpha) = e^z$, Equation (8) reads

$$(e^z)^{(k)} = e^z$$

i. e., the formula for the symbolic order derivative of the exponential function.

We study the situation further with our general example function $F(z,n) := e^{-z^2}H_n(-z)$. In this case the conclusion of Theorem 2 reads as follows

$$\frac{\partial^k}{\partial z^k}\left(e^{-z^2}H_n(-z)\right) = e^{-z^2}H_{n+k}(-z) \qquad (k \in \mathbb{N}_0) \; ,$$

or after replacing z by $-z$, and multiplying by e^{z^2}, we have

$$H_{n+k}(z) = (-1)^k\, e^{z^2}\, \frac{\partial^k}{\partial z^k}\left(e^{-z^2}H_n(z)\right) \qquad (k \in \mathbb{N}_0) \; , \tag{9}$$

which is a Rodrigues type equation for $H_{n+k}(z)$ $(k \in \mathbb{N}_0)$ in terms of $H_n(z)$. Indeed, this identity generalizes the well-known Rodrigues formula for the Hermite polynomials which is the special case $n = 0$

$$H_k(z) = (-1)^k \, e^{z^2} \frac{\partial^k}{\partial z^k} e^{-z^2} \qquad (k \in \mathbb{N}_0) \tag{10}$$

as $H_0 \equiv 1$. Note that (9) is a higher order symbolic identity compared with (10).

Those identities, once obtained, may be verified by a method developed in [8]. Our Mathematica implementation generates a common differential equation for the left and right hand sides of identity (9)

```
In[2]:= SimpleDE[HermiteH[n+k,z],z]

Out[2]= 2 (k + n) F[z] - 2 z F'[z] + F''[z] == 0

In[3]:= SimpleDE[Exp[z^2]*D[Exp[-z^2]*HermiteH[n,z],{z,k}],z]

Out[3]= 2 (k + n) F[z] - 2 z F'[z] + F''[z] == 0
```

so that its validity is equivalent to two initial value statements that easily may be established (see [8], § 10).

6 Generating functions of F-functions

The generating function of an analytic F-function considered as a function of the second variable α has a simple integral representation (see [11], Theorem 14.3, who proved the theorem by another method).

Theorem 4 (Generating function of F-functions) For an analytic F-function we have in its disk of convergence

$$\sum_{k=0}^{\infty} F(z, \alpha + k) \, x^k = \int_0^{\infty} e^{-t} \, F(z + t\,x, \alpha) \, dt \tag{11}$$

for all x in the disk of convergence of the left hand expression.

Proof: We calculate the Taylor coefficients of the right hand side corresponding to the variable x with the aid of Theorem 2. As the F-function is analytic we may differentiate under the integral sign, and exchange the limit procedure with integration, and get as an application of Theorem 2 with the chain rule

$$\frac{1}{k!} \frac{\partial^k}{\partial x^k} \left(\int_0^{\infty} e^{-t} \, F(z + t\,x, \alpha) \, dt \right) \Bigg|_{x=0} = \lim_{x \to 0} \frac{1}{k!} \frac{\partial^k}{\partial x^k} \left(\int_0^{\infty} e^{-t} \, F(z + t\,x, \alpha) \, dt \right)$$

$$= \lim_{x\to 0} \frac{1}{k!} \int\limits_0^\infty e^{-t}\, t^k\, F(z+t\,x, \alpha+k)\, dt$$

$$= \frac{1}{k!} \int\limits_0^\infty e^{-t}\, t^k\, F(z, \alpha+k)\, dt = F(z, \alpha+k) \ ,$$

since $\int\limits_0^\infty e^{-t}\, t^k\, dt = k!$, and the result follows from Taylor's theorem. □

A much simpler result is available for the exponential generating function (see [12], § 2.3) of F with respect to α (see [11], Theorem 14.1).

Theorem 5 (Exponential generating function of F-functions) For an analytic F-function we have in its disk of convergence

$$\sum_{k=0}^\infty F(z, \alpha+k)\, \frac{x^k}{k!} = F(z+x, \alpha) \ . \tag{12}$$

Proof: For an analytic F-function it follows from Theorem 2, and the chain rule

$$\left.\frac{\partial^k}{\partial x^k} F(z+x, \alpha)\right|_{x=0} = \left. F(z+x, \alpha+k)\right|_{x=0} = F(z, \alpha+k) \ ,$$

and the result follows from Taylor's theorem. □

7 Generating functions of special functions

Theorem 4 can be used to obtain generating functions of special functions. As we shall see, in this context the exponential generating function is the more natural object.

For the exponential function $F(z,\alpha) = e^z$, e.g., (11) reads

$$e^z \sum_{k=0}^\infty x^k = \sum_{k=0}^\infty F(z, \alpha+k)\, x^k = \int\limits_0^\infty e^{-t}\, F(z+t\,x, \alpha)\, dt = \int\limits_0^\infty e^{-t(1-x)}\, e^z\, dt = \frac{e^z}{1-x}$$

or

$$\sum_{k=0}^\infty x^k = \frac{1}{1-x} \ .$$

Note that this statement at first glance seems to be completely independent of the given F-function $F(z,\alpha) = e^z$. This is not true, however, as in the first place the initial value function $\Phi(\alpha) = F(0,\alpha) \equiv 1$ generates this result.

On the other hand, (12) is equivalent to

$$e^z \sum_{k=0}^\infty \frac{x^k}{k!} = \sum_{k=0}^\infty F(z, \alpha+k)\, \frac{x^k}{k!} = F(z+x, \alpha) = e^{z+x} \ ,$$

or by (7)

$$e^z\, e^x = e^{z+x}\ ,$$

the addition formula for the exponential function. This is not astonishing as the use of (12), and (6) for $z_0 := 0$ generally creates formula (6) for the sum:

$$\begin{aligned} F(z+x,\alpha) &= \sum_{k=0}^{\infty} F(z,\alpha+k)\,\frac{x^k}{k!} = \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Phi(\alpha+j+k)\frac{z^j}{j!}\,\frac{x^k}{k!} \\ &= \sum_{n=0}^{\infty}\Phi(\alpha+n)\sum_{k=0}^{n}\frac{z^{n-k}}{(n-k)!}\,\frac{x^k}{k!} = \sum_{n=0}^{\infty}\Phi(\alpha+n)\frac{(z+x)^n}{n!}\ . \end{aligned}$$

Next, we study the situation with our general example function $F(z,n) := e^{-z^2}H_n(-z)$. In this case the conclusion of Theorem 4 reads as follows

$$\sum_{k=0}^{\infty} e^{-z^2}H_{n+k}(-z)\,x^k = \int\limits_0^{\infty} e^{-t}\,e^{-(z+tx)^2}H_n\Big(-(z+t\,x)\Big)\,dt\ .$$

Dividing both sides by e^{-z^2}, and replacing z by $-z$, we obtain in the special case $n=0$, using $H_0 \equiv 1$, the following integral representation of the generating function for the Hermite polynomials

$$\sum_{k=0}^{\infty} H_k(z)\,x^k = \int\limits_0^{\infty} e^{-t(1-2zx+tx^2)}\,dt\ .$$

Using MATHEMATICA (version 2.2), e. g., we get the representation

$$\int\limits_0^{\infty} e^{-t(1-2zx+tx^2)}\,dt = \frac{\sqrt{\pi}}{2\,|x|}\,\exp\left(\frac{(1-2\,x\,z)^2}{4\,x^2}\right)\left(1-\operatorname{erf}\left(\frac{1-2\,x\,z}{2\,|x|}\right)\right)$$

of the right hand integral, and therefore we are led to the representation

$$\sum_{k=0}^{\infty} H_k(z)\,x^k = \frac{\sqrt{\pi}}{2\,|x|}\,\exp\left(\frac{(1-2\,x\,z)^2}{4\,x^2}\right)\left(1-\operatorname{erf}\left(\frac{1-2\,x\,z}{2\,|x|}\right)\right)$$

for the generating function of the Hermite polynomials (see also [11], (14.35)).

Similarly, Theorem 5 allows us to obtain exponential generating functions of special functions. In our case we get

$$\sum_{k=0}^{\infty} e^{-z^2}H_{n+k}(-z)\,\frac{x^k}{k!} = e^{-(z+x)^2}H_n\Big(-(z+x)\Big)\ ,$$

in particular for $n=0$ the well-known (see e.g. [1], (22.9.17))

$$\sum_{k=0}^{\infty} H_k(z)\,\frac{x^k}{k!} = e^{2xz-x^2}\ .$$

8 Further results obtained by Truesdell's method

In this section we list some further results that can be obtained by the given method.

For the F-function

$$F(z,\alpha) := e^{i\alpha\pi}\, e^{-z}\, L_n^{(\alpha)}(z)$$

([11], § 5, Example 26) where $L_n^{(\alpha)}$ $(n \in \mathbb{N}_0, \alpha \in \mathbb{R})$ denote the generalized Laguerre polynomials, Theorem 2 generates the identity

$$L_n^{(\alpha+k)}(z) = e^z\, (-1)^k\, \frac{\partial^k}{\partial z^k} \left(e^{-z}\, L_n^{(\alpha)}(z)\right) \;, \tag{13}$$

and for $\alpha = 0$ we have in particular $(L_n = L_n^{(0)})$

$$L_n^{(k)}(z) = e^z\, (-1)^k\, \frac{\partial^k}{\partial z^k}\left(e^{-z}\, L_n(z)\right)$$

(see [11], (18.6); the result is a combination of [10], (5.2.1), and [2], 10.12 (28)). Note that with our MATHEMATICA implementation, we may prove (13) essentially by calculating the common differential equation

```
In[4]:= SimpleDE[LaguerreL[n,alpha+k,z],z]

Out[4]= n F[z] + (1 + alpha + k - z) F'[z] + z F''[z] == 0

In[5]:= SimpleDE[Exp[z]*(-1)^k*D[Exp[-z]*LaguerreL[n,alpha,z],{z,k}],z]

Out[5]= n F[z] + (1 + alpha + k - z) F'[z] + z F''[z] == 0
```

for its left and right hand sides, and checking two initial values.

Further Theorem 5 generates

$$L_n^{(\alpha)}(x+z) = e^x\, \sum_{k=0}^{\infty} (-1)^k\, L_n^{(\alpha+k)}(z)\, \frac{x^k}{k!} \;,$$

which is type of an addition formula for the Laguerre polynomials. For $\alpha = 0$, we have

$$L_n(x+z) = e^x\, \sum_{k=0}^{\infty} (-1)^k\, L_n^{(k)}(z)\, \frac{x^k}{k!} \;.$$

For the F-function

$$F(z,\alpha) := e^{i\alpha\pi}\, (1-z^2)^{-\alpha/2}\, P_n^{\alpha}(z)$$

(compare [11], § 5, Example 19) where P_n^{α} $(n \in \mathbb{N}_0,\ \alpha \in \mathbb{R})$ denote the associated Legendre functions, Theorem 2 generates the identity

$$P_n^{\alpha+k}(z) = (-1)^k\, (1-z^2)^{(\alpha+k)/2}\, \frac{\partial^k}{\partial z^k} \left((1-z^2)^{-\alpha/2}\, P_n^{\alpha}(z)\right) \;, \tag{14}$$

and in particular for $\alpha = 0$ ($P_n = P_n^0$)

$$P_n^k(z) = (-1)^k \, (1-z^2)^{k/2} \, \frac{\partial^k}{\partial z^k} P_n(z) \; ,$$

([1], (8.6.6)), a representation of the associated Legendre functions of positive integer order in terms of the Legendre polynomials.

Again, with MATHEMATICA, we can prove identity (14), calculating the differential equations

```
In[6]:= SimpleDE[LegendreP[n,alpha+k,z],z]

                         2         2       2    2  2                      2
Out[6]= (-(alpha + k)  + n + n  - n z  - n  z ) F[z] + 2 z (-1 + z ) F'[z] +

                2 2
>     (-1 + z )  F''[z] == 0

In[7]:= SimpleDE[(-1)^k*(1-z^2)^((alpha+k)/2)*
        D[(1-z^2)^(-alpha/2)*LegendreP[n,alpha,z],{z,k}],z]

                2                 2         2       2    2  2
Out[7]= (alpha  + 2 alpha k + k  - n - n  + n z  + n  z ) F[z] +

                                        2         2
>     2 (1 - z) z (1 + z) F'[z] - (-1 + z)  (1 + z)  F''[z] == 0
```

Theorem 5 leads to

$$P_n^\alpha(x+z) = (-1)^n \, (1-(x+z)^2)^{\alpha/2} \sum_{k=0}^{\infty} (1-z^2)^{-(\alpha+k)/2} \, P_n^{\alpha+k}(z) \, \frac{x^k}{k!} \; ,$$

in particular for $\alpha = 0$

$$P_n(x+z) = (-1)^n \, \sum_{k=0}^{\infty} (1-z^2)^{-k/2} \, P_n^k(z) \, \frac{x^k}{k!} \; .$$

Similar results are valid for the Legendre function Q_n^α as $F(z,\alpha) := e^{i\alpha\pi} \, (1-z^2)^{-\alpha/2} \, Q_n^\alpha(z)$ forms an F-function, too ([11], § 5, Example 20).

Finally we consider the Bessel and Hankel functions $J_\alpha(z), Y_\alpha(z), I_\alpha(z), K_\alpha(z), H_\alpha^{(1)}(z)$, and $H_\alpha^{(2)}(z)$ which generate the F-functions

$$\begin{aligned}
F(z,\alpha) &:= e^{i\alpha\pi} \, z^{-\alpha/2} \, J_\alpha(2\sqrt{z}) \; , \\
F(z,\alpha) &:= e^{i\alpha\pi} \, z^{-\alpha/2} \, Y_\alpha(2\sqrt{z}) \; , \\
F(z,\alpha) &:= z^{-\alpha/2} \, I_\alpha(2\sqrt{z}) \; , \\
F(z,\alpha) &:= e^{i\alpha\pi} \, z^{-\alpha/2} \, K_\alpha(2\sqrt{z}) \; , \\
F(z,\alpha) &:= e^{i\alpha\pi} \, z^{-\alpha/2} \, H_\alpha^{(1)}(2\sqrt{z}) \; ,
\end{aligned}$$

and

$$F(z,\alpha) \quad := \quad e^{i\alpha\pi}\, z^{-\alpha/2}\, H_\alpha^{(2)}(2\sqrt{z})$$

(compare [11], §5, Examples 37–40).

Theorem 2 generates for $J_\alpha(z)$

$$J_{\alpha+k}(2\sqrt{z}) = (-1)^k\, z^{(\alpha+k)/2}\, \frac{\partial^k}{\partial z^k}\left(z^{-\alpha/2}\, J_\alpha(2\sqrt{z})\right)\ , \tag{15}$$

and analogous results follow for $Y_\alpha(z), I_\alpha(z), K_\alpha(z), H_\alpha^{(1)}(z)$, and $H_\alpha^{(2)}(z)$.

Using the representations

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\,\sin z\ , \qquad \text{and} \qquad J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\,\cos z\ ,$$

we are led to

$$J_{k+1/2}(2\sqrt{z}) = \frac{(-1)^k\, z^{(k+1/2)/2}}{\sqrt{\pi}}\,\frac{\partial^k}{\partial z^k}\left(\frac{\sin(2\sqrt{z})}{\sqrt{z}}\right)\ , \tag{16}$$

and

$$J_{k-1/2}(2\sqrt{z}) = \frac{(-1)^k\, z^{(k-1/2)/2}}{\sqrt{\pi}}\,\frac{\partial^k}{\partial z^k}\cos(2\sqrt{z})\ . \tag{17}$$

Mathematica obtains the differential equations

```
In[8]:= SimpleDE[BesselJ[alpha+k,2*Sqrt[z]],z]

                2               2                               2
Out[8]= (-alpha  - 2 alpha k - k  + 4 z) F[z] + 4 z F'[z] + 4 z  F''[z] == 0

In[9]:= SimpleDE[(-1)^k*z^((alpha+k)/2)*
        D[z^(-alpha/2)*BesselJ[alpha,2*Sqrt[z]],{z,k}],z]

                 2               2                               2
Out[54]= (-alpha  - 2 alpha k - k  + 4 z) F[z] + 4 z F'[z] + 4 z  F''[z] == 0

In[10]:= SimpleDE[BesselJ[k+1/2,2*Sqrt[z]],z]

                        2                             2
Out[10]= (-1 - 4 k - 4 k  + 16 z) F[z] + 16 z F'[z] + 16 z  F''[z] == 0

In[11]:= SimpleDE[(-1)^k*z^((k+1/2)/2)*D[Sin[2*Sqrt[z]]/Sqrt[z],{z,k}],z]

                        2                             2
Out[11]= (-1 - 4 k - 4 k  + 16 z) F[z] + 16 z F'[z] + 16 z  F''[z] == 0
```

```
In[12]:= SimpleDE[BesselJ[k-1/2,2*Sqrt[z]],z]

                          2                                 2
Out[12]= (-1 + 4 k - 4 k   + 16 z) F[z] + 16 z F'[z] + 16 z   F''[z] == 0

In[13]:= SimpleDE[(-1)^k*z^((k-1/2)/2)*D[Cos[2*Sqrt[z]],{z,k}],z]

                          2                                 2
Out[13]= (-1 + 4 k - 4 k   + 16 z) F[z] + 16 z F'[z] + 16 z   F''[z] == 0
```

from which (15)–(17) follow.

Finally we like to consider the family of iterated derivatives of the Airy functions (see e. g. [1], § 10.4), defined by ($n \in \mathbb{N}_0$)

$$F(z,n) := \mathrm{Ai}^{(n)}\,(z)\ .$$

Obviously $F(z,n)$ forms an F-function, and from the initial values

$$\mathrm{Ai}\,(0) = \frac{1}{\sqrt[3]{9}\,\Gamma\,(2/3)} \qquad \text{and} \qquad \mathrm{Ai}'\,(0) = -\frac{1}{\sqrt[3]{3}\,\Gamma\,(1/3)}$$

(see e. g. [1], (10.4.4) and (10.4.5)) by the method of [4] we are led to the power series representation

$$\mathrm{Ai}\,(z) = \sum_{k=0}^{\infty} \frac{1}{9^{k+1/3}\,\Gamma(k+2/3)} \frac{z^{3k}}{k!} - \sum_{k=0}^{\infty} \frac{1}{9^{k+2/3}\,\Gamma(k+1/3)} \frac{z^{3k+1}}{k!} \tag{18}$$

for the Airy function (see [8], §4). This generates the initial function

$$\Phi(n) = F(0,n) = \mathrm{Ai}^{(n)}\,(0) = \begin{cases} \frac{1}{9^{k+1/3}\,\Gamma(k+2/3)} & \text{if } n = 3k\ (k \in \mathbb{N}_0) \\ \frac{1}{9^{k+2/3}\,\Gamma(k+1/3)} & \text{if } n = 3k+1\ (k \in \mathbb{N}_0) \\ 0 & \text{if } n = 3k+2\ (k \in \mathbb{N}_0) \end{cases} \ . \tag{19}$$

Note that one can also use the recurrence equation $\mathrm{Ai}_{n+1}\,(z) = z\ \mathrm{Ai}_{n-1}\,(z) + (n-1)\ \mathrm{Ai}_{n-2}\,(z)$ to derive (19) which is easily proved (see [8]).

Theorem 1 is then a restatement of (18), and Theorem 2 of the definition of $\mathrm{Ai}_n\,(z)$. On the other hand, from Theorem 4 ($n = 0$) we obtain the integral representation for the generating function of the Airy functions

$$\sum_{k=0}^{\infty} \mathrm{Ai}_k\,(z)\,x^k = \sum_{k=0}^{\infty} \mathrm{Ai}^{(k)}\,(z)\,x^k = \int_0^{\infty} e^{-t}\ \mathrm{Ai}\,(tx+z)\,dt\ ,$$

whereas Theorem 5 yields the relation

$$\mathrm{Ai}_n\,(x+z) = \sum_{k=0}^{\infty} \mathrm{Ai}_{k+n}\,(z)\,\frac{x^k}{k!}\ .$$

9 Families of iterated antiderivatives

Obviously there is a quite analogous theory for families of iterated antiderivatives. We call the corresponding functional equation

$$\frac{\partial}{\partial z} G(z,\alpha) = G(z,\alpha-1) \tag{20}$$

the *G-equation*, and a solution that satisfies the initial condition

$$G(z_0,\alpha) = \Phi(\alpha) \tag{21}$$

a *G-function*. Here z is assumed to be a real or complex variable, α is such that

$$\begin{array}{lrcll} \text{either} & \alpha & = & \alpha_0 - k & (k \in \mathbb{N}_0)\ , \\ \text{or} & \alpha & = & \alpha_0 + k & (k \in \mathbb{Z})\ , \\ \text{or} & \alpha & \le & \alpha_0 & (\alpha \in \mathbb{R})\ , \\ \text{or} & \operatorname{Re}\alpha & \le & \alpha_0 & (\alpha \in \mathbb{C})\ , \end{array} \tag{22}$$

and Φ is a given function of α. Note, however, that even though the function

$$G(z,n) := \sum_{k=0}^{n} \frac{z^k}{k!}$$

is a solution of the G-equation, it does not form a G-function as it is defined for $n \in D := \mathbb{N}_0$ and the defining set D does not have the necessary property $n \in D \Rightarrow n-1 \in D$.

We state the corresponding theorems without proof since the ones we presented are easily adapted.

Theorem 6 (Power series representation of G-functions) If Φ is bounded for values α given by (22), then there is one and only one analytic G-function corresponding to Φ that has the power series representation

$$G(z,\alpha) = \sum_{k=0}^{\infty} \Phi(\alpha-k)\,\frac{(z-z_0)^k}{k!}\ . \tag{23}$$

Independently of the boundedness of Φ, if the power series representation (23) has a positive radius of convergence, then it represents a unique analytic solution of (20) and (21) in its disk of convergence. □

Theorem 7 (Derivatives of G-functions) An analytic G-function satisfies

$$\frac{\partial^k}{\partial z^k} G(z,\alpha) = G(z,\alpha-k) \tag{24}$$

for all $k \in \mathbb{N}_0$. □

Note that, again, we get a theorem about iterated derivatives rather than iterated antiderivatives since the main "propagation direction" of the α-values is now decreasing. On the other hand, if D satisfies furthermore the relation $\alpha \in D \Rightarrow \alpha + 1 \in D$, we have

Theorem 8 (Antiderivatives of G-functions) If the defining set D for the α-values of an analytic G-function satisfies the relation $\alpha \in D \Rightarrow \alpha+1 \in D$, and if $G(z_0, \alpha+k) = 0$ ($k \in \mathbb{N}$), then

$$I^k\Big(G(z,\alpha)\Big) = G(z, \alpha+k)$$

for all $k \in \mathbb{N}$, where I denotes antidifferentiation $I\,\Big(G(z,\alpha)\Big) := \int\limits_{z_0}^{z} G(t,\alpha)\,dt$. □

Further, we have

Theorem 9 (Generating function of G-functions) For an analytic G-function we have in its disk of convergence

$$\sum_{k=0}^{\infty} G(z, \alpha-k)\,x^k = \int\limits_0^{\infty} e^{-t}\,G(z+t\,x, \alpha)\,dt \tag{25}$$

for all x in the disk of convergence of the left hand expression. □

Theorem 10 (Exponential generating function of G-functions) For an analytic G-function we have

$$\sum_{k=0}^{\infty} G(z, \alpha-k)\,\frac{x^k}{k!} = G(z+x, \alpha)\ . \tag{26}$$

in its disk of convergence. □

As an example function we consider the iterated antiderivatives of the complimentary error function

$$\begin{aligned} \mathrm{erfc}_n(z) &:= \int\limits_z^{\infty} \mathrm{erfc}_{n-1}(t)\,dt \qquad (27)\\ \mathrm{erfc}_{-1}(z) &:= \frac{2}{\sqrt{\pi}}\,e^{-z^2} \end{aligned}$$

(see e. g. [1] (7.2)). To fit in our program, we must consider this function for all integer n-values which is accomplished by using (27) in the differentiated form

$$\mathrm{erfc}_{n-1}(x) = -\frac{\partial}{\partial x}\,\mathrm{erfc}_n(x)$$

to define $\mathrm{erfc}_n(x)$ for negative n-values. Note that, for negative n-values, we have essentially a family of iterated derivatives, again.

Then obviously $G(z,n) := \mathrm{erfc}_n(-z)$ $(n \in \mathbb{Z})$ is a G-function with the initial values

$$G(0,n) = \mathrm{erfc}_n(0) = \frac{1}{2^n\,\Gamma\left(\frac{n}{2}+1\right)}$$

(see e. g. [1] (7.2.7)). Theorem 6 now gives the power series representation

$$\mathrm{erfc}_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{n-k}\,\Gamma\left(\frac{n-k}{2}+1\right)} \frac{z^k}{k!}$$

(see [1] (7.2.4)), whereas Theorem 7 gives the Rodrigues type equation

$$(-1)^k \frac{\partial^k}{\partial z^k} \mathrm{erfc}_n(z) = \mathrm{erfc}_{n-k}(z) \;.$$

by (10), which is established by

```
In[14]:= SimpleDE[Erfc[n-k,z],z]

Out[14]= -2 (-k + n) F[z] + 2 z F'[z] + F''[z] == 0

In[15]:= SimpleDE[D[Erfc[n,z],{z,k}],z]

Out[15]= 2 (k - n) F[z] + 2 z F'[z] + F''[z] == 0
```

In particular, for $n = -1$ we have the representation (compare (10))

$$\mathrm{erfc}_{-(k+1)}(z) = \frac{2\,(-1)^k}{\sqrt{\pi}} \frac{\partial^k}{\partial z^k} e^{-z^2} = \frac{2}{\sqrt{\pi}} e^{-z^2}\, H_k(z)$$

Note that our theory does only give Rodrigues type identities for decreasing rather than increasing values of α. For example the Rodrigues type formula ($\mathrm{erfc}\, z = \mathrm{erfc}_0(z)$)

$$\mathrm{erfc}_k(z) = \frac{(-1)^k\, e^{-z^2}}{2^k\, k!} \frac{\partial^k}{\partial z^k} \left(e^{z^2}\,\mathrm{erfc}\, z\right)$$

(see e. g. [8], § 10) cannot be obtained by Truesdell's method but can be proven by our general method ([8], § 10):

```
In[16]:= SimpleDE[Erfc[k,z],z]

Out[16]= -2 k F[z] + 2 z F'[z] + F''[z] == 0

In[17]:= SimpleDE[(-1)^k*Exp[-z^2]*2^k/k!*D[Exp[z^2]*Erfc[z],{z,k}],z]

Out[17]= 2 k F[z] - 2 z F'[z] - F''[z] == 0
```

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