KLAUSUR

Mathematische Methoden der Signalverarbeitung

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Bitte lassen Sie genügend Platz zwischen den Aufgaben und beschreiben Sie nur die Vorderseite der Blätter!

Zum Bestehen der Klausur sollten 9 Punkte erreicht werden.

1)	2)	3)

Punkte:	Note:

1. By using the two-sided z-transform a so-called LTI system is described in the z-domain through: $Y(z) = H(z)\,X(z)$, where

$$H(z) = \sum_{n=-\infty}^{\infty} h_n\, z^{-n}\,, \quad 0 \le r < |z| < R\,,$$

is known as the system function. What is the output sequence y_n of the system if the input is given by the impuls $x_n = \delta_{n,0} = \begin{cases} 1 & , \quad n = 0\,, \\ 0 & , \quad \text{otherwise}\,, \end{cases}$ and the complex sinusoid $x_n = e^{i\omega\, n}$ respectively. **(6P)**

2. The Haar wavelet is de£ned as: $\psi(t) = \begin{cases} 1 & , \quad 0 \le t < \frac{1}{2}\,, \\ -1 & , \quad \frac{1}{2} \le t < 1\,, \\ 0 & , \quad \text{sonst}\,. \end{cases}$ Show that for any $a < 0$ and f the wavelet transform is given by:

$$\mathcal{W}(f(t))(a,b) = \frac{1}{\sqrt{-a}} \left(-\int_{b+a}^{b+\frac{a}{2}} f(t)\,dt + \int_{b+\frac{a}{2}}^{b} f(t)\,dt \right)\,.$$

(4P)

3. Let f be square integrable $\int_{-\infty}^{\infty} |f(t)|^2\,dt < \infty$. Suppose that for any integer k there holds:

$$\int_{-\infty}^{\infty} f(t)\,\overline{f(t-k)}\,dt = \delta_{k,0}\,.$$

Show that the Fourier coeff£cients $c_k = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} g(\omega) e^{-i\,k\,\omega} d\omega$ of the 2π periodic function

$$g(\omega) = \sum_{j=-\infty}^{\infty} |\mathcal{F}(f(t))(\omega + 2\,\pi\,j)|^2$$

satisfy the relation: $c_{-k} = \dfrac{1}{2\,\pi}\,\delta_{k,0}$. Hint: use the Parseval-Plancherel identity and note that: $\mathcal{F}(f(t))(\omega) = \dfrac{1}{\sqrt{2\,\pi}} \displaystyle\int_{-\infty}^{\infty} f(t)\, e^{-i\,\omega\,t}\,dt$. **(8P)**

Solutions

1.) By using the convolution theorem of the z-transform we obtain the impulse response:

$$y_n = h_n * x_n = \sum_{\nu=-\infty}^{\infty} h_\nu \, \delta_{n-\nu,0} h_n$$

and the frequency response:

$$\begin{aligned} y_n &= h_n * x_n = \sum_{\nu=-\infty}^{\infty} h_\nu \, x_{n-\nu} \\ &= \sum_{\nu=-\infty}^{\infty} h_\nu \, e^{i\,\omega\,(n-\nu)} = \left(\sum_{\nu=-\infty}^{\infty} h_\nu \, \left(e^{i\,\omega}\right)^{-\nu} \right) e^{i\,\omega\,n} \\ &= H\left(e^{i\,\omega}\right) \, x_n \, . \end{aligned}$$

2.) Since $a < 0$ we have:

$$\begin{aligned} 0 \leq \frac{t-b}{a} < \frac{1}{2} \quad &\Longleftrightarrow \quad b \geq t > b + \frac{a}{2} \, , \\ \frac{1}{2} \leq \frac{t-b}{a} \leq 1 \quad &\Longleftrightarrow \quad b + \frac{a}{2} \geq t \geq b + a \, . \end{aligned}$$

After scaling and translation the Haar wavelet takes the form:

$$\psi\left(\frac{t-b}{a}\right) = \begin{cases} 1 & , \quad b \geq t > b + \frac{a}{2} \, , \\ -1 & , \quad b + \frac{a}{2} \geq t \geq b + a \, , \\ 0 & , \quad \text{otherwise} \, , \end{cases}$$

and the wavelet transform becomes

$$\mathcal{W}(f(t))(a,b) = \frac{1}{\sqrt{-a}} \left(- \int_{b+a}^{b+\frac{a}{2}} f(t) \, dt + \int_{b+\frac{a}{2}}^{b} f(t) \, dt \right) \, .$$

3.) The Parseval-Plancherel identity gives:

$$\begin{aligned}
\int_{-\infty}^{\infty} f(t)\,\overline{f(t-k)}\,dt &= \int_{-\infty}^{\infty} \mathcal{F}(f(t))(\omega)\,\overline{e^{-i\,k\,\omega}\,\mathcal{F}(f(t))(\omega)}\,d\omega \\
&= \int_{-\infty}^{\infty} e^{i\,k\,\omega}\,\left|\mathcal{F}(f(t))(\omega)\right|^2\,d\omega \\
&= \sum_{j=-\infty}^{\infty} \int_{2\,\pi\,j}^{2\,\pi\,(j+1)} e^{i\,k\,\omega}\,\left|\mathcal{F}(f(t))(\omega)\right|^2\,d\omega \\
&= \sum_{j=-\infty}^{\infty} \int_{0}^{2\,\pi} e^{i\,k\,\omega}\,\left|\mathcal{F}(f(t))(\omega+2\,\pi\,j)\right|^2\,d\omega \\
&= \int_{0}^{2\,\pi} e^{i\,k\,\omega} \sum_{j=-\infty}^{\infty} \left|\mathcal{F}(f(t))(\omega+2\,\pi\,j)\right|^2\,d\omega\,.
\end{aligned}$$

(In the last step we interchanged integration and summation. This needs a careful justi£cation by Levi's theorem). Finally, the following identity holds:

$$\int_{0}^{2\,\pi} \sum_{j=-\infty}^{\infty} \left|\mathcal{F}(f(t))(\omega+2\,\pi\,j)\right|^2\,e^{i\,k\,\omega}\,d\omega = \delta_{k,0}$$

Therefore, for the fourier coef£cients c_k of the 2π-periodic funktion

$$\sum_{j=-\infty}^{\infty} \left|\mathcal{F}(f(t))(\omega+2\,\pi\,j)\right|^2$$

we obtain:

$$c_{-k} = \frac{1}{2\,\pi}\,\delta_{k,0}\,.$$

Hieraus folgt dann die Behauptung.