KLAUSUR

Partielle Differenzialgleichungen für Ingenieure

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Bitte lassen Sie genügend Platz zwischen den Aufgaben und beschreiben Sie nur die Vorderseite der Blätter!

Zum Bestehen der Klausur sollten 10 Punkte erreicht werden.

1)	2)	3)	4)

Punkte:	Note:

1. Find the general solutions of the following £rst order equations:

$$y\,\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0\,, \qquad \frac{1}{x}\,\frac{\partial u}{\partial x} + y^3\,\frac{\partial u}{\partial y} = 0\,.$$

(6P)

2. The following problem describes a vibrating string:

$$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}\,, \qquad c > 0\,,$$

$$u(0,t) = u(l,t) = 0\,, \qquad u(x,0) = f(x)\,, \qquad \frac{\partial u}{\partial t}(x,0) = g(x)\,.$$

Give the solution of the problem in terms of Fourier series if $g(x) \equiv 0$ and

$$f(x) = \begin{cases} \frac{h}{q}\,x & , \quad 0 \le x \le q\,, \\ \frac{h}{l-q}\,(l-x) & , \quad q \le x \le l\,, \end{cases}$$

(8P)

3. Let $u(x,t)$ solve the equation

$$\frac{\partial^2 u}{\partial t^2} + k\,\frac{\partial u}{\partial t} = c^2\,\frac{\partial^2 u}{\partial x^2}\,, \qquad c, k = const.\,,$$

and set

$$u(x,t) = e^{\phi(x,t)}\,v(x,t)\,.$$

Determine ϕ such that v satis£es an equation of the form

$$\frac{\partial^2 v}{\partial t^2} + K\,v = c^2\,\frac{\partial^2 v}{\partial x^2}\,, \qquad c, K = const.\,.$$

(4P)

4. Show that for any £xed ξ the function

$$u(x,t) = \frac{1}{2\,c\,\sqrt{\pi\,t}}\,e^{-\frac{(x-\xi)^2}{4\,c^2\,t}}$$

satis£es the heat equation

$$\frac{\partial u}{\partial t} = c^2\,\frac{\partial^2 u}{\partial x^2}\,, \qquad c > 0\,.$$

(4P)

Solutions

1.) The characteristic curves of the equation

$$y\,\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

are given as solutions of the system:

$$\frac{dx}{d\epsilon} = y\,, \qquad \frac{dy}{d\epsilon} = 1\,.$$

Through elimination of the parameter ϵ we obtain

$$\frac{dy}{dx} = \frac{1}{y}$$

whose solutions are obtained by separation of variables

$$\int y\,dy = \int dx \qquad \Longleftrightarrow \qquad \frac{1}{2}\,y^2 = x + c\,.$$

Solving for c gives the following £rst integral to the characteristic equations:

$$c(x,y) = \frac{1}{2}\,y^2 - x\,.$$

The general solution reads as

$$u(x,y) = f(c(x,y))\,.$$

The characteristic curves of the equation

$$\frac{1}{x}\,\frac{\partial u}{\partial x} + y^3\,\frac{\partial u}{\partial y} = 0$$

are given as solutions of the system:

$$\frac{dx}{d\epsilon} = \frac{1}{x}\,, \qquad \frac{dy}{d\epsilon} = y^3\,.$$

Through elimination of the parameter ϵ we obtain

$$\frac{dy}{dx} = x\,y^3$$

whose solutions are obtained by separation of variables

$$\int \frac{1}{y^3}\, dy = \int x\, dx \quad \Longleftrightarrow \quad -\frac{1}{2}\,\frac{1}{y^2} = \frac{1}{2}\, x^2 + c\,.$$

Solving for c gives the following £rst integral to the characteristic equations:

$$c(x, y) = -\frac{1}{2}\, \left(x^2 + \frac{1}{y^2} \right)\,.$$

The general solution reads as

$$u(x, y) = f(c(x, y))\,.$$

Obviously, the given equation has to be considered either in the domain $x > 0$ or $x < 0$. Furthermore, $y = 0$ is a characteristic curve, where it is not possible to prescribe any initial condition.

2.) The solution can be described in the following form:

$$u(x, t) = \sum_{n=1}^{\infty} (C_n\, \cos(\lambda_n\, t) + D_n\, \sin(\lambda_n\, t))\, \sin\left(\frac{n\,\pi}{l} x\right)\,,$$

where

$$\lambda_n = c\,\frac{n\,\pi}{l}\,.$$

The coef£cients C_n, D_n are determined according to the Fourier-sine expansions of odd functions:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} C_n\, \sin\left(\frac{n\,\pi}{l} x\right)\,, \\ C_n &= \frac{2}{l} \int_0^l f(x)\, \sin\left(\frac{n\,\pi}{l} x\right)\, dx\,, \\ g(x) &= \sum_{n=1}^{\infty} D_n\, \lambda_n\, \sin\left(\frac{n\,\pi}{l} x\right)\,, \\ D_n &= \frac{2}{C_n\, pi} \int_0^l g(x)\, \sin\left(\frac{n\,\pi}{l} x\right)\, dx\,. \end{aligned}$$

Since $g(x) \equiv 0$ we obtain

$$D_n = 0\,, \quad n \geq 1\,.$$

The coef£cients C_n assume the following form:

$$\begin{aligned}
C_n &= \frac{2}{l} \int\limits_0^q \frac{h}{q}\, x\, \sin\left(\frac{n\,\pi}{l} x\right)\, dx + \frac{2}{l} \int\limits_q^l \frac{h}{l-q}\,(l-x)\, \sin\left(\frac{n\,\pi}{l} x\right)\, dx \\
&= \frac{2}{l}\,\frac{h}{q} \int\limits_0^q x\, \sin\left(\frac{n\,\pi}{l} x\right)\, dx \\
&\quad -\frac{2}{l}\,\frac{h}{l-q} \int\limits_l^q x\, \sin\left(\frac{n\,\pi}{l} x\right)\, dx + 2\,\frac{h}{l-q} \int\limits_l^q \sin\left(\frac{n\,\pi}{l} x\right)\, dx \\
&= \frac{2\,h}{q\,(l-q)} \left(\frac{l}{n\,\pi}\right)^2 \sin\left(\frac{n\,\pi}{l} q\right)\,.
\end{aligned}$$

3.) Let us calculate £rst the derivatives:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial e^{\phi}}{\partial t}\, v + e^{\phi}\, \frac{\partial v}{\partial t} \\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 e^{\phi}}{\partial t^2}\, v + 2\,\frac{\partial e^{\phi}}{\partial t}\,\frac{\partial v}{\partial t} + e^{\phi}\,\frac{\partial^2 v}{\partial t^2}\,, \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 e^{\phi}}{\partial x^2}\, v + 2\,\frac{\partial e^{\phi}}{\partial x}\,\frac{\partial v}{\partial x} + e^{\phi}\,\frac{\partial^2 v}{\partial x^2}\,.
\end{aligned}$$

Inserting into the given equation yields:

$$\begin{aligned}
&\frac{\partial^2 e^{\phi}}{\partial t^2}\, v + 2\,\frac{\partial e^{\phi}}{\partial t}\,\frac{\partial v}{\partial t} \\
&+ e^{\phi}\,\frac{\partial^2 v}{\partial t^2} + k\,\left(\frac{\partial e^{\phi}}{\partial t}\, v + e^{\phi}\,\frac{\partial v}{\partial t}\right) \\
&= c^2\,\left(\frac{\partial^2 e^{\phi}}{\partial x^2}\, v + 2\,\frac{\partial e^{\phi}}{\partial x}\,\frac{\partial v}{\partial x} + e^{\phi}\,\frac{\partial^2 v}{\partial x^2}\right)\,.
\end{aligned}$$

If ϕ depends solely upon t we obtain

$$\begin{aligned}
&\frac{\partial^2 e^{\phi}}{\partial t^2}\, v + 2\,\frac{\partial e^{\phi}}{\partial t}\,\frac{\partial v}{\partial t} \\
&+ e^{\phi}\,\frac{\partial^2 v}{\partial t^2} + k\,\left(\frac{\partial e^{\phi}}{\partial t}\, v + e^{\phi}\,\frac{\partial v}{\partial t}\right) \\
&= c^2\, e^{\phi}\,\frac{\partial^2 v}{\partial x^2}\,.
\end{aligned}$$

The coef£cient of the £rst derivative of v with respect to t disappears if

$$2\,\frac{\partial e^{\phi}}{\partial t} = -k\, e^{\phi} \quad \longleftrightarrow \quad \frac{\partial \phi}{\partial t} = -k\,.$$

Choosing

$$\phi(t) = -\frac{k}{2}\, t$$

we obtain the following equation for v:

$$\frac{\partial^2 v}{\partial t^2} + K\, v = c^2\,\frac{\partial^2 v}{\partial x^2}\,, \quad K = -\frac{k^2}{4}\,.$$

4.) Through differentiating we obtain:

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,t) &= \frac{1}{2\,c\,\sqrt{\pi}}\,\left(-\frac{1}{2}\,\frac{1}{t^{\frac{3}{2}}}\right)\, e^{-\frac{(x-\xi)^2}{4\,c^2\,t}} \\
&\quad + \frac{1}{2\,c\,\sqrt{\pi\, t}}\, e^{-\frac{(x-\xi)^2}{4\,c^2\,t}}\,\frac{(x-\xi)^2}{4\,c^2\,t^2}\,, \\
\frac{\partial u}{\partial x}(x,t) &= \frac{1}{2\,c\,\sqrt{\pi\, t}}\, e^{-\frac{(x-\xi)^2}{4\,c^2\,t}}\,\left(-\frac{1}{2}\,\frac{(x-\xi)}{c^2\, t}\right)\,, \\
\frac{\partial^2 u}{\partial x^2}(x,t) &= \frac{1}{2\,c\,\sqrt{\pi\, t}}\, e^{-\frac{(x-\xi)^2}{4\,c^2\,t}}\,\left(-\frac{1}{2}\,\frac{(x-\xi)}{c^2\, t}\right)^2 \\
&\quad \frac{1}{2\,c\,\sqrt{\pi\, t}}\, e^{-\frac{(x-\xi)^2}{4\,c^2\,t}}\,\left(-\frac{1}{2}\,\frac{1}{c^2\, t}\right)\,.
\end{aligned}$$

The assertion follows immediately.