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Automatic Computation of Laurent-Puiseux Series of Hypergeometric Type

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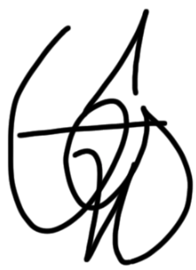
Abstract

A power series $f(z) = \sum_{k=k_0}^{\infty} a_k z^{k/n}$ is completely determined by the formula of the general coefficient a_k. A sequence of instructions allowing to find this kind of general coefficient is an algorithm of computation of formal power series. Such an algorithm is not yet implemented in all general computer algebra systems (CAS). In particular, the CAS Maxima misses such an implementation. Consequently, this essay studies the algorithm given by Wolfram Koepf in [6] in order to adapt it for Maxima with a restriction to one of the steps described to the special case of functions of hypergeometric type which covers a wide family of analytic functions.

Declaration

I, the undersigned, hereby declare that the work contained in this essay is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly.



Bertrand TEGUIA TABUGUIA, 18 May 2018.

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1. Introduction

The generalized form of formal power series (FPS) is given by the Laurent-Puiseux series (LPS) of the form $\sum_{k=k_0}^{\infty} a_k(z-z_0)^{k/n}$ $(a_k \in \mathbb{C}, n \in \mathbb{Z})$. Many areas of mathematics are spanned by the study of formal power series, for instance in combinatorics they are used to represent generating functions (see [11]), in dynamical systems their algebraic properties involve most of the constructions (see [9]), we can also enumerate analysis and commutative algebra [1]. There is a one-to-one correspondence between FPS with positive radius of convergence and corresponding analytic functions [6]. Thus an algorithmic approach which allows the computation of FPS for a given analytic function appears essential for mathematics.

Koepf has presented in [6] an algorithm for the computation of FPS for a very rich family. This algorithm has been implemented in the computer algebra systems (CASs) Mathematica and Maple. However the CAS Maxima of the same class is still using a pattern-based approach with the command *powerseries()*. And this is rather too limited since there is ignorance of many cases. As example we have at $z_0 = 0$

(%i1) powerseries(asin(z)^2/z^2,z,0);

$$\frac{\left(\sum_{i1=0}^{\infty} \frac{\mathrm{genfact}(2i1-1,i1,2)\, z^{2i1+1}}{(2i1+1)\,\mathrm{genfact}(2i1,i1,2)}\right)^2}{z^2} \tag{\%o1}$$

instead of

$$\sum_{k=0}^{\infty} \frac{4^k\, k!^2\, z^{2k}}{(k+1)\,(2k+1)!}$$

which clearly shows that the implementation behind does not use an algorithmic approach and fails to compute a power series expansion of this non-classical analytic function but writes the result as the square of a power series. Therefore as one of the main goals of CASs is to keep alive mathematical objects in the computer, one has to find a better Maxima implementation that will be able to extend Maxima to the same formal power series computing status as Maple and Mathematica.

Of course for this purpose we will use the Koepf algorithm and this essay is more focused in those special functions called functions of hypergeometric type defined in [6]. As we will see later, these functions cover a large family of analytic functions. For a given expression of an analytic function, we want to compute its power series by using the following steps:

1. Compute a holonomic differential equation DE for an expression.
2. Convert a holonomic DE into a holonomic recurrence equation RE for the coefficients of its Taylor series.
3. Solve the corresponding RE in particular in the hypergeometric type case.

After placing some mathematical backgrounds behind power series in computer algebra, our work will be mainly based on the well understanding of the above three steps in order to carry out a Maxima's implementation for each of them so that gathering all in all will make us end up with a program able to compute Laurent-Puiseux series of hypergeometric type and more.

2. Mathematical Foundations

For our work, the first thing is to understand how it makes sense to use those three steps for the computation and to go through each step and understand how they work so that it will be easier to write the corresponding algorithm. Nevertheless, before going into that we need some recalls to make sure we are in agreement with the mathematical foundations.

2.1 Power Series and Convergence

The results of this section are taken from the book [8](see chapters VII-IX).

2.1.1 Definition. A power series of the variable z (complex in general) is a series whose the general term is of the form $a_n z^n,\ n \in \mathbb{N}$ where (a_n) denotes a given sequence of complex numbers. Precisely a_n is the $(n+1)^{\text{th}}$ coefficient, or coefficient with order n. The first term a_0 is usually called constant term.

Now having a power series we may look for its convergence.

2.1.2 Lemma (Abel's lemma)**.** Let $z_0 \in \mathbb{C}$ such that the sequence $(a_n z_0^n)$ is bounded (which is the case when in particular the series $\sum a_n z_0^n$ is convergent). Then, for all $z \in \mathbb{C}$ such that $|z| < |z_0|$, the series $\sum a_n z^n$ is absolutely convergent; and this series is normally convergent[1] in the closed disc $D(0, k|z_0|)$, $0 \leqslant k < 1$.

Thus we can talk about the set of values where we have the convergence for any power series.

2.1.3 Definition (Radius of convergence)**.** The radius of convergence of a power series $\sum a_n z^n$ is the smallest upper bound in $\overline{\mathbb{R}_+}$ of the set of positive real numbers r satisfying that $(a_n r^n)$ is bounded.

2.1.4 Theorem. *Let R be the radius of convergence of the power series $\sum a_n z^n$, $(0 \leqslant R \leqslant +\infty)$*

1. *If $R = 0$, this series converges only for $z = 0$.*
2. *If $R = +\infty$, this series converges absolutely for any $z \in \mathbb{C}$. And this convergence is normal, so uniform in any bounded subset of $\mathbb{C}$.*
3. *If $0 < R < +\infty$, the series is absolutely convergent for $|z| < R$, and divergent for $|z| > R$. Moreover this series converges normally (so uniformly) in any disc $\overline{D}(0, r)$, for any $r < R$.*

For $R \neq 0$, the open disc $D(0, R)$ is called disc of convergence of the series.

2.1.5 Proposition (Hadamard formula)**.** The radius of convergence of the power series $\sum a_n z^n$ is the real number R defined by

$$\frac{1}{R} = \limsup_{n \longrightarrow +\infty} |a_n|^{1/n}. \tag{2.1.1}$$

Nevertheless in practice, the following approach called D'Alembert approach is more useful.

[1] A series $\sum u_n(z)$ is said to be normally convergent if there exist a real series $\sum v_n$ such that $\forall n \in \mathbb{N}, \forall z \in \mathbb{C}$, $\|u_n(z)\| \leqslant v_n$

2.1.6 Proposition. Given a power series $\sum a_n z^n$ and assuming that the sequence $\left|\dfrac{a_{n+1}}{a_n}\right|$ is convergent, then we have the radius of covergence R verifying

$$\frac{1}{R} = \lim_n \left|\frac{a_{n+1}}{a_n}\right|. \tag{2.1.2}$$

2.1.7 Example. Whatever the polynomial $P \in \mathbb{C}[z] \setminus \{0\}$, the radius of convergence of the power series $\sum P(n) z^n$ is equal to 1 since $\frac{P(n+1)}{P(n)}$ tends to 1 when $n \to \infty$.

It is easy to compute the derivative of a power series in its disc of convergence termwise. Moreover its derivative is also a power series.

2.1.8 Definition. Let $\sum a_n z^n$ be a power series whose the radius of convergence R is not 0. Then the sum $\sum_{n=0}^{\infty} a_n z^n$ is a holomorphic function (differentiability in $\mathbb{C}$) of z in its disc of convergence, and in that disc, we have

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}. \tag{2.1.3}$$

Proof. If we denote by R, R' the radius of convergence of the series $\sum a_n z^n$ and $\sum n a_n z^{n-1}$, then we have

$$\frac{1}{R'} = \limsup |n a_n|^{1/n} = \limsup |a_n|^{1/n} = \frac{1}{R},$$

since $(n^{1/n}) \longrightarrow 1$ as $n \to \infty$. □

The sum and the product of two power series A and B gives a power series with radius of convergence at least equal to the smallest of their radius of convergence. Moreover, one can construct the ring of power series (hence the computation of the inverse of some power series). Other used operations for power series is the composition and the integration of power series.

The case of product of power series leads to some important results. Considering two power series $\sum a_n z^n$ and $\sum b_n z^n$, the product series resulting is the power series with general term

$$c_n z^n = \left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n. \tag{2.1.4}$$

This result helps to compute many cases of power series. For example, for $|z| < 1$ we know that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \tag{2.1.5}$$

we can deduce

$$\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} 1\right) z^n = \sum_{n=0}^{\infty}(n+1) z^n = \sum_{n=0}^{\infty}\binom{n+1}{1} z^n,$$
$$\frac{1}{(1-z)^3} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}(k+1)\right) z^n = \sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2} z^n = \sum_{n=0}^{\infty}\binom{n+2}{2} z^n.$$

We can even generalize for any power $p \in \mathbb{N}$ by using Pascal's triangle. And then we obtain:

$$\frac{1}{(1-z)^p} = \sum_{n=0}^{\infty}\binom{n+p-1}{p-1} z^n. \tag{2.1.6}$$

2.1.9 Definition (Analytic function)**.** Let $U \subset \mathbb{C}$ [resp. $U \subset \mathbb{R}$] be a open set. A map $f\colon U \longrightarrow \mathbb{C}$ is said to be analytic in U if for any point $z_0 \in U$, the map $u \mapsto f(z_0+u)$ can be expressed as a power series in a neighborhood of the origin in $\mathbb{C}$ [resp. $\mathbb{R}$].

In other words : f is analytic in U if any point $z_0 \in U$ has a neighborhood on which $f(z)$ can be expressed as a power series of the variable $u = z - z_0$. This expansion coincides with the Taylor series of f at z_0. Moreover we can see that a power series can be identified by its coefficients as a unique analytic function defined in its disc of convergence.

2.1.10 Proposition. Let U be an open set of $\mathbb{C}$ [resp. $\mathbb{R}$] and f an analytic function in U. Then f is indefinitely differentiable on U, and around any point $z_0 \in U$ the representation

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(z_0)(z-z_0)^n \tag{2.1.7}$$

converges.

Because of this proposition for analytic functions Taylor expansions are convergent power series expansions.

2.1.11 Example. (Some power series expansions) Around $z_0 = 0$ we have the following, where R denotes the radius of convergence:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}, \quad (R=\infty) \tag{2.1.8}$$

$$\cos z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}, \quad (R=\infty) \tag{2.1.9}$$

$$\sin z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!}, \quad (R=\infty) \tag{2.1.10}$$

$$\cosh z = \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!}, \quad (R=\infty) \tag{2.1.11}$$

$$\sinh z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}, \quad (R=\infty) \tag{2.1.12}$$

$$\frac{1}{(1-z)^\alpha} = (1-z)^{-\alpha} = \sum_{k=0}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+k-1)}{k!} z^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k, \quad (R=1) \tag{2.1.13}$$

$$\ln(1+z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{k+1}}{k+1}, \quad (R=1) \tag{2.1.14}$$

$$\arctan z = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{2k+1}, \quad (R=1) \tag{2.1.15}$$

$$\operatorname{arctanh} z = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1}, \quad (R=1) \tag{2.1.16}$$

$$\arcsin z = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} z^{2k+1}, \quad (R=1), \tag{2.1.17}$$

$$\operatorname{arcsinh} z = \sum_{k=0}^{\infty}(-1)^k \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} z^{2k+1}, \quad (R=1). \tag{2.1.18}$$

We end here for the recalls in power series, we are now able to search algorithmically an approach to get the formal power series of an analytic function given. Any others mathematical tools of the work will be explained directly in the next section.

2.2 Formal Power Series (FPS) in Computer Algebra

We consider the Formal Power Series (FPS) of the form

$$f(z) = \sum_{k=0}^{\infty} a_k z^n, \ a_k \in \mathbb{C}. \tag{2.2.1}$$

To deal with many special functions [3], it is important to know the generalized hypergeometric series

$$ {}_pF_q\left(\begin{matrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{matrix}\middle| z\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdot (a_2)_k \cdots (a_p)_k}{(b_1)_k \cdot (b_2)_k \cdots (b_q)_k k!} z^k = \sum_{k=0}^{\infty} A_k z^k. \tag{2.2.2}$$

Here $(a)_k$ denotes the *shifted factorial of* a also called *Pochhammer symbol of* a is defined by

$$(a)_k := \begin{cases} 1 & \text{if } k=0 \\ a\cdot(a+1)\cdots(a+k-1) & \text{if } k \in \mathbb{N}^* \end{cases}. \tag{2.2.3}$$

The coefficients are

$$A_k := \frac{(a_1)_k \cdot (a_2)_k \cdots (a_p)_k}{(b_1)_k \cdot (b_2)_k \cdots (b_q)_k k!}, \ k = 0, 1, 2, \cdots, \tag{2.2.4}$$

where the denominator factors are chosen in such a way that they can never be zero. We have

$$A_0 = \frac{(a_1)_0 \cdot (a_2)_0 \cdots (a_p)_0}{(b_1)_0 \cdot (b_2)_0 \cdots (b_q)_0 0!} = 1, \tag{2.2.5}$$

and for $k \geqslant 0$, using the fact that $(a)_{k+1} = (a)_k(a+k)$ and $(k+1)! = (k+1)k!$ we have

$$\frac{A_{k+1}}{A_k} = \frac{(a_1)_k(k+a_1)\cdot(a_2)_k(k+a_2)\cdots(a_p)_k(k+a_p)}{(b_1)_k(k+b_1)\cdot(b_2)_k(k+b_2)\cdots(b_q)_k(k+b_q)(k+1)k!} \times \frac{(b_1)_k\cdot(b_2)_k\cdots(b_q)_k k!}{(a_1)_k\cdot(a_2)_k\cdots(a_p)_k},$$

which gives

$$\frac{A_{k+1}}{A_k} = \frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_q)(k+1)}. \tag{2.2.6}$$

When one of the numerator parameters a_i is a negative integer(1.4 in [4]), there exist $k \in \mathbb{N}, a_i + k = 0$, so the generalized hypergeometric function will have a finite number of coefficients which is nothing but a polynomial in z. Otherwise, the radius of convergence R of the generalized hypergeometric series is given by

$$\frac{1}{R} = \lim_{k\to+\infty} \frac{A_{k+1}}{A_k} = \lim_{k\to+\infty} \frac{k^p}{k^{q+1}} = \begin{cases} 0 & \text{if } \ p < q+1 \\ 1 & \text{if } \ p = q+1 \\ \infty & \text{if } \ p > q+1 \end{cases},$$

so

$$R = \begin{cases} \infty & \text{if} \quad p < q+1 \\ 1 & \text{if} \quad p = q+1 \\ 0 & \text{if} \quad p > q+1 \end{cases} . \tag{2.2.7}$$

Of course the most interesting cases are the cases where $R \neq 0$.

For the case where $R = 1$ we have:

$$\begin{aligned} {}_1F_0\left(\begin{matrix} -a \\ - \end{matrix} \middle| -z\right) = {}_2F_1\left(\begin{matrix} -a & b \\ b & \end{matrix} \middle| -z\right) &= \sum_{k=0}^{\infty} \frac{(-a)_k (b)_k}{(b)_k k!} (-z)^k \\ &= \sum_{k=0}^{\infty} (-1)^{2k} \frac{a \cdot (a-1) \cdots (a-k+1)}{k!} z^k \\ &= \sum_{k=0}^{\infty} \frac{a \cdot (a-1) \cdots (a-k+1)}{k!} z^k \\ &= (1+z)^a. \end{aligned} \tag{2.2.8}$$

By remarking that $(1)_k = 1(1+1)\cdots(1+k-1) = k!$ and $(2)_k = 2 \cdot 3 \cdots (2+k-1) = (k+1)!$, we have

$$\begin{aligned} z\ {}_2F_1\left(\begin{matrix} 1 & 1 \\ 2 & \end{matrix} \middle| -z\right) &= z \cdot \sum_{k=0}^{\infty} \frac{(1)_k (1)_k}{(2)_k k!} (-z)^k \\ &= \sum_{k=0}^{\infty} \frac{k! k!}{(k+1)! k!} (-1)^k z^{k+1} \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{z^{k+1}}{k+1} \\ &= \ln(1+z). \end{aligned} \tag{2.2.9}$$

Also, we have

$$\frac{\left(\frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k} = \frac{\frac{1}{2} \cdot \left(\frac{1}{2}+1\right) \cdots \left(\frac{1}{2}+k-1\right)}{\frac{3}{2} \cdot \left(\frac{3}{2}+1\right) \cdots \left(\frac{3}{2}+k-1\right)} = \frac{\frac{1}{2} \cdot \left(\frac{3}{2}\right) \left(\frac{3}{2}+1\right) \cdots \left(\frac{3}{2}+k-2\right)}{\frac{3}{2} \cdot \left(\frac{3}{2}+1\right) \cdots \left(\frac{3}{2}+k-2\right) \cdot \left(\frac{3}{2}+k-1\right)} = \frac{1}{2k+1} \tag{2.2.10}$$

and

$$\left(\frac{1}{2}\right)_k = \frac{1}{2} \cdot \left(\frac{1}{2}+1\right) \cdots \left(\frac{1}{2}+k-1\right) = \frac{\prod_{j=0}^{k-1} (2j+1)}{2^k} = \frac{\prod_{j=0}^{k-1} (2j+1)(2(j+1))}{2^k \prod_{j=0}^{k-1} 2(j+1)} = \frac{(2k)!}{2^{2k} k!} \tag{2.2.11}$$

which lead to

$$\begin{aligned} z\ {}_2F_1\left(\begin{matrix} 1/2 & 1/2 \\ 3/2 & \end{matrix} \middle| z^2\right) &= z \cdot \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k k!} z^{2k} \\ &= \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k} (k!)^2 (2k+1)} z^{2k+1} \\ &= \arcsin z. \end{aligned} \tag{2.2.12}$$

We can also show that

$$z\ {}_2F_1\left(\begin{matrix}1/2 & 1\\ 3/2 & \end{matrix}\ \middle|\ -z^2\right) = \arctan z. \tag{2.2.13}$$

For $R=\infty$, we have:

$${}_0F_0\left(\begin{matrix}-\\-\end{matrix}\ \middle|\ z\right) = e^z, \tag{2.2.14}$$

$$z\cdot\ {}_0F_1\left(\begin{matrix}-\\3/2\end{matrix}\ \middle|\ -\frac{z^2}{4}\right) = z\cdot\sum_{k=0}^{\infty}\frac{1}{\left(\frac{3}{2}\right)_k k!}\left(-\frac{z^2}{4}\right)^k = \sum_{k=0}^{\infty}\frac{(-1)^k}{4^k\left(\frac{3}{2}\right)_k k!}z^{2k+1}$$

and using the same reasoning as we did for $\left(\frac{1}{2}\right)_k$, we find that

$$\left(\frac{3}{2}\right)_k = \frac{(2k+1)!}{4^k k!}, \tag{2.2.15}$$

which leads to

$$z\cdot\ {}_0F_1\left(\begin{matrix}-\\3/2\end{matrix}\ \middle|\ -\frac{z^2}{4}\right) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}z^{2k+1} = \sin z. \tag{2.2.16}$$

In the same way

$${}_0F_1\left(\begin{matrix}-\\1/2\end{matrix}\ \middle|\ -\frac{z^2}{4}\right) = \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}z^{2k} = \cos z, \tag{2.2.17}$$

and

$${}_0F_1\left(\begin{matrix}-\\1/2\end{matrix}\ \middle|\ \frac{z^2}{4}\right) = \sum_{k=0}^{\infty}\frac{1}{(2k)!}z^{2k} = \cosh z. \tag{2.2.18}$$

It follows clearly that all the examples in (2.1.11) can be expressed in generalized hypergeometric form. Thus one can see that many analytic functions can be written as a generalized hypergeometric series.

The recurrence relation (2.2.6) is the Recurrence Equation (RE) that characterizes the generalized hypergeometric series $f(z)=\sum_{k=0}^{\infty}A_k z^k$. Note that $\frac{A_{k+1}}{A_k}$ is a rational function in k. Generally having a rational function $R(k)=\frac{A_{k+1}}{A_k}$ connects the corresponding function with a hypergeometric series. Indeed $R(k)$ can be factorized as

$$\begin{aligned} R(k) &= \frac{a_m(k+\alpha_1)(k+\alpha_2)\cdots(k+\alpha_p)}{b_n(k+\beta_1)(k+\beta_2)\cdots(k+\beta_q)(k+\beta_{q+1})}\\ &= c\frac{(k+\alpha_1)(k+\alpha_2)\cdots(k+\alpha_p)}{(k+\beta_1)(k+\beta_2)\cdots(k+\beta_q)(k+\beta_{q+1})},\quad c=\frac{a_m}{b_n}, \end{aligned}$$

where the complex numbers a_m and b_n $(a_m b_n \neq 0)$ are respectively the leading coefficients of the numerator and the denominator. $\beta_i, i\in[\![1,q+1]\!]$[2] are poles of R and $\alpha_i, i\in[\![1,p]\!]$ are zeros of R. If there is some $\beta_i=-1(\beta_{q+1}=-1)$, then the function corresponds to a hypergeometric series evaluated at some point cz (c being the quotient of the leading coefficient of the numerator and the denominator of R). Whereas if there is no such β_i, the extra factor $(k+1)$ can be compensated by one of the factors $(k+\alpha_i)$ in the numerator by taking $\alpha_{p+1}=-1$.

[2] For $m,n\in\mathbb{N}, m<n$ we define $[\![m,n]\!] := \{m,m+1,\ldots,n\}$

Let

$$f(z) = {}_pF_q \left(\begin{matrix} a_1 & a_2 & \cdots & a_p \\ b_1 & b_2 & \cdots & b_q \end{matrix} \middle| z \right) = \sum_{k=0}^{\infty} A_k z^k,$$

and the differential operators $D = \frac{d}{dz}$ and $\theta = z\frac{d}{dz} = zD$. Then from [3] we know that f satisfies the differential equation

$$\theta(\theta + b_1 - 1)\cdots(\theta + b_q - 1)f = z(\theta + a_1)\cdots(\theta + a_p)f. \tag{2.2.19}$$

To see how we obtain this equation we first remark that

$$\theta(f(z)) = zD\left(\sum_{k=0}^{\infty} A_k z^k\right) = z\sum_{k=1}^{\infty} kA_k z^{k-1} = \sum_{k=0}^{\infty} kA_k z^k$$

and for $p \geqslant 1$, assuming $\theta^p(f(z)) = \sum_{k=0}^{\infty} k^p A_k z^k$ implies

$$\theta^{p+1}(f(z)) = zD\left(\sum_{k\geqslant 0} k^p A_k z^k\right) = z\sum_{k=1}^{\infty} k^{p+1} A_k z^{k-1} = \sum_{k=0}^{\infty} k^{p+1} A_k z^k.$$

So by induction we have

$$\theta^p(f(z)) = \sum_{k=0}^{\infty} k^p A_k z^k, \; p \in \mathbb{N}. \tag{2.2.20}$$

Thus for any polynomial P we can state by linearity that

$$P(\theta)(f(z)) = \sum_{k=0}^{\infty} P(k) A_k z^k. \tag{2.2.21}$$

From the recurrence relation of the generalized hypergeometric series (2.2.6) we have

$$A_{k+1}(b_1 + k)\cdots(b_q + k)(k + 1) = (a_1 + k)\cdots(a_p + k)A_k,$$

and therefore

$$\sum_{k=0}^{\infty} A_{k+1}(b_1 + k)\cdots(b_q + k)(k + 1)z^{k+1} = \sum_{k=0}^{\infty}(a_1 + k)\cdots(a_p + k)A_k z^{k+1},$$

setting $j = k + 1$ this is equivalent to

$$\sum_{j=1}^{\infty} A_j(j + b_1 - 1)\cdots(j + b_q - 1)jz^j = z\left(\sum_{k=0}^{\infty}(k + a_1)\cdots(k + a_p)A_k z^k\right).$$

On the left hand side each coefficient gives a polynomial of degree $q+1$ in the variable j, so expanding it leads to an expression on the form

$$A_j(j + b_1 - 1)\cdots(j + b_q - 1)j = P(j)A_j z^j,$$

where P is a polynomial of degree $q+1$. Now, taking the summation we obtain

$$\begin{aligned}\sum_{j=1}^{\infty} A_j(j+b_1-1)\cdots(j+b_q-1)j &= \sum_{j=1}^{\infty} P(j)A_j z^j \\ &= P(\theta)f \text{ from } (2.2.21) \\ &= \theta(\theta+b_1-1)\cdots(\theta+b_q-1)f(z),\end{aligned}$$

where the last line is due to the fact that the numbers $0, b_1-1, \ldots, b_q-1$ are the roots of P.

With the same reasoning for the right hand side we get

$$z\left(\sum_{k=0}^{\infty}(k+a_1)\cdots(k+a_p)A_k z^k\right) = z(\theta+a_1)\cdots(\theta+a_p)f(z).$$

Hence, we have found the differential equation

$$\theta(\theta+b_1-1)\cdots(\theta+b_q-1)f = z(\theta+a_1)\cdots(\theta+a_p)f,$$

as expected.

Nevertheless, if we develop this differential equation in such a way to have the derivatives $D^j f = \dfrac{d^j f}{dz^j}$ of f, then we will obtain the differential equation $(Q := \max(p,q)+1,\ c_{j,l} \in \mathbb{C})$

$$\sum_{j=0}^{Q} P_j(z)D^j f = \sum_{j=0}^{Q}\sum_{l=0}^{Q} c_{j,l} z^l D^j f = 0,. \tag{2.2.22}$$

2.2.1 Definition (Holonomic differential equation [12]). A holonomic differential equation is a linear homogeneous ordinary differential equation with polynomial coefficients

$$P_n(z)D^n f(z) + \cdots + P_1(z)Df(z) + P_0(z)f(z) = 0, \tag{2.2.23}$$

$P_n, \ldots, P_1, P_0 \in \mathbb{C}[z]$.

A function satisfying a holonomic differential equation is called holonomic function. One can see that the generalized hypergeometric function is holonomic.

Based on the fact that(see [2] page 233), when we consider a function $g(z) = (z-z_0)^{k_0} f(z)$ with $1/f$ analytic near z_0, we have $1/g$ analytic near z_0 too, and around z_0

$$\frac{1}{g(z)} = \frac{1}{(z-z_0)^{k_0}}\sum_{k=0}^{\infty} a_k(z-z_0)^k = \sum_{k=0}^{\infty} a_k(z-z_0)^{k-k_0} = \sum_{k=-k_0}^{\infty} c_k(z-z_0)^k,$$

we will later consider mostly *formal Laurent series* (FLS) at $z_0 = 0$ with a representation

$$\sum_{k=k_0}^{\infty} a_k z^k = z^{k_0}\sum_{k=0}^{\infty} a_{k+k_0} z^k = \ (a_{k_0} \neq 0) \tag{2.2.24}$$

for $k_0 \in \mathbb{Z}$. As we can see in the previous illustration the FLS are shifted FPS, and they correspond to meromorphic[3] functions with a pole of order $-k_0$. Note that an analytic function of the form $f(z^m)$ is called m-fold symmetric. We can see that even functions are 2-fold symmetric and odd functions are shifted 2-fold symmetric. One is then led to the following more general definition.

2.2.2 Definition (Functions of hypergeometric type [6]). An FLS $f := \sum_{k=k_0}^{\infty} a_k z^k$ is called to be of hypergeometric type if its coefficients a_k satisfy a RE of the form

$$\begin{aligned} a_{k+m} &= R(k)a_k \quad \text{for } k \geqslant k_0 \\ a_k &= I_k \quad \text{for } k = k_0, k_0+1, \cdots, k_0+m-1 \end{aligned} \tag{2.2.25}$$

For some $m \in \mathbb{N}$, $I_k \in \mathbb{C}(k = k_0, k_0+1, \cdots, k_0+m-1)$, $I_{k_0} \in \mathbb{C} \setminus \{0\}$, and some rational function R. The number m is then called symmetric number of (the given representation) of f. A RE of this type is also called to be of hypergeometric type.

This definition is more general than the definition of generalized hypergeometric functions. For example the power series of $\sin z$ is

$$\sin z = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!},$$

hence for its coefficients we have $a_{2k} = 0$ and $a_{2k+1} = \dfrac{(-1)^k}{(2k+1)!}$. We don't have a RE for $m = 1$, hence the sine function is not a generalized hypergeometric function! But for $m = 2$ we find the relation we want. Thus $\sin z$ is of hypergeometric type with symmetry number 2.

2.2.3 Remark. Each FLS with symmetry number m can be represented as the sum of $m-$fold symmetry functions as follows

$$f(z) = \sum_{j=0}^{m-1} \sum_{k=0}^{\infty} I_j R(k_0+j+m) R(k_0+j+2m) \cdots R(k_0+j+mk) z^{mk+j}. \tag{2.2.26}$$

The following lemma gives a list of transformations on FLS that preserve the hypergeometric type.

2.2.4 Lemma. [Lemma 2.1 in [6]] Let f be an FLS of hypergeometric type with representation (2.2.24). Then

(a) $z^s f$ $(s \in \mathbb{N})$, (b) f/z^s $(s \in \mathbb{N})$, (c) $f(Cz)$ $(C \in \mathbb{C})$,

(d) $f(z^n)$ $(n \in \mathbb{N})$, (e) $\dfrac{f(z) \pm f(-z)}{2}$, (f) f'

are of hypergeometric type, too. If f has symmetry number m, then $f(z^n)$ has symmetry number nm, and $\dfrac{f(z) \pm f(-z)}{2}$ has symmetry number $2m$ as odd or even part of f. If f is an FPS, then so is

(g) $\int f$.

[3] holomorphic function where the only singularities are poles.

To deal with function of hypergeometric type in our development, it is essential that it satisfies a holonomic DE.

2.2.5 Theorem (Theorem 2.1 in [6])**.** *Each FLS of hypergeometric type satisfies a holonomic DE.*

Thus we are able to do our development for many FLS and in particular FPS. However, as it is our first goal to work with FPS, in the sequel we will work with them in order to extend the algorithm by using lemma (2.2.4) above for the research of FLS and further for a Laurent Puiseux Series (LPS) that will be defined later.

Now, we consider a function representing an FPS given. In order to find the coefficient formula, it is a reasonable approach to search for its DE, to transfer this DE into its RE, and finish by an adaption of the coefficient formula for the hypergeometric function corresponding to the transformation of f which preserves its hypergeometric type. In the next chapter, we will discuss these steps in detail in some examples and end by a general algorithm.

3. Examples and General Algorithm

We denote by $\mathbb{Q}(z)$ the field of rational functions in z with rational coefficients.

3.1 Analysis Examples

In this section, we will treat some examples in order to progressively highlight the algorithmic process.

First case: $f(z) = \arcsin z$.

Searching for the holonomic DE: $f'(z) = \dfrac{1}{\sqrt{1-z^2}}$, and therefore there is no $A_0(z) \in \mathbb{Q}(z)$ such that $f'(z) + A_0(z)f(z) = 0$ because $A_0(z)$ should be $-\dfrac{1}{\sqrt{1-z^2}\arcsin z}$ which is not rational. However, for the second order, the relation $f''(z) + A_1(z)f'(z) + A_0(z)f(z) = 0$ with $f''(z) = \dfrac{z}{(1-z^2)\sqrt{1-z^2}}$, leads to a soluble system. We get by writing the sum in terms of linear independent parts as

$$\left(\frac{z}{(1-z^2)\sqrt{1-z^2}} + \frac{A_1(z)}{\sqrt{1-z^2}}\right) + A_0(z)\arcsin z = 0, \tag{3.1.1}$$

so $\begin{cases} A_1(z) + \dfrac{z}{(1-z^2)} = 0 \\ A_0(z) = 0 \end{cases} \iff \left(A_1(z) = -\dfrac{z}{(1-z^2)} \text{ and } A_0(z) = 0\right)$. Hence the corresponding holonomic DE is obtained after multiplication with the common denominator

$$(1-z^2)f''(z) - zf'(z) = 0. \tag{3.1.2}$$

Transformation of the DE into its RE: We set

$$f(z) = \sum_{k=0}^{\infty} a_k z^k, \text{ so } f'(z) = \sum_{k=1}^{\infty} k a_k z^{k-1} = \sum_{k=0}^{\infty} (k+1)a_{k+1} z^k$$

and $f''(z) = \sum_{k=2}^{\infty} k(k-1)a_k z^{k-2} = \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}z^k$. Furthermore

$$zf'(z) = \sum_{k=1}^{\infty} k a_k z^k = \sum_{k=0}^{\infty} k a_k z^k \text{ and } z^2 f''(z) = \sum_{k=2}^{\infty} k(k-1)a_k z^k = \sum_{k=0}^{\infty} k(k-1)a_k z^k.$$

Thus from the DE we get

$$\begin{aligned} 0 = (1-z^2)f''(z) - zf'(z) &= \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}z^k - \sum_{k=0}^{\infty} k(k-1)a_k z^k - \sum_{k=0}^{\infty} k a_k z^k \\ &= \sum_{k=0}^{\infty} \left[(k+2)(k+1)a_{k+2} - k(k-1)a_k - k a_k\right] z^k \\ &= \sum_{k=0}^{\infty} \left[(k+2)(k+1)a_{k+2} - k^2 a_k\right] z^k = 0, \end{aligned}$$

hence by equating the coefficients we find the RE

$$(k+2)(k+1)a_{k+2} - k^2 a_k = 0,\ \ k = 0, 1, 2 \cdots . \tag{3.1.3}$$

Solving the RE: The recurrence relation obtained is of hypergeometric type as only two summands occur. And we can already see that the symmetry number is $m = 2$. Moreover $f(0) = 0$ implies that the first term in the Taylor expansion of f is 0, thus f is odd and then the RE is for odd index:

$$\begin{aligned} a_{2k+1+2} = a_{2(k+1)+1} &= \frac{(2k+1)^2}{(2k+1+2)(2k+1+1)} a_{2k+1} \\ &= \frac{\left(k+\frac{1}{2}\right)^2}{\left(k+\frac{3}{2}\right)(k+1)} a_{2k+1}, \end{aligned}$$

and since $a_1 = \frac{f'(0)}{1!} = 1$, one can use the hypergeometric coefficient formula (2.2.4) and gets

$$a_{2k+1} = \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k} = \frac{(2k)!}{2^{2k}(k!)^2(2k+1)}. \tag{3.1.4}$$

Hence

$$f(z) = \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} z^{2k+1}. \tag{3.1.5}$$

3.1.1 Remark (DEtoRE)**.** in general, we can find a map which allows us to go from each term of the expansion of the DE to the term of the RE. Firstly it is easy to see that

$$\left(a_k z^k\right)^{(j)} = k(k-1)\cdots(k-j+1)a_k z^{k-j} = (k+1-j)_j \cdot a_k z^{k-j},$$

so if we multiply by z^l, we obtain

$$z^l \left(a_k z^k\right)^{(j)} = (k+1-j)_j \cdot a_k z^{k-j+l}.$$

And shifting the index by setting $p = k - j + l \Rightarrow k = p + j - l$, then we obtain the correspondence

$$z^l \left(a_k z^k\right)^{(j)} \longrightarrow (p+1-l)_j \cdot a_{p+j-l} z^p,$$

which allows us to consider the map between a DE and its RE

$$z^l f^{(j)} \longrightarrow (p+1-l)_j \cdot a_{p+j-l}. \tag{3.1.6}$$

For example in our previous case, the equation

$$(1-z^2)f''(z) - zf'(z) = f''(z) - z^2 f''(z) - zf'(z) = 0$$

gives

$$f'' = z^0 f^{(2)} \longrightarrow (p+1-0)_2 \cdot a_{p+2-0} = (p+1)(p+2)a_{p+2},\ \ z^2 f^{(2)} \longrightarrow (p+1-2)_2 \cdot a_{p+2-2} = (p-1)pa_p,$$

and $zf^{(1)} = (p+1-1)_1 \cdot a_{p+1-1} = pa_p$. So we can see how to deduce the RE by a substitution in the DE.

Second case: $f(z) = \arctan z$.

Searching for the holonomic DE: $f'(z) = \dfrac{1}{1+z^2} \Rightarrow \dfrac{f'(z)}{f(z)} = \dfrac{1}{(1+z^2)\arctan z} \notin \mathbb{Q}(z)$, so there is no independent linear combination with rational coefficients like $f'(z) + A_0(z)f(z) = 0$ with $A_0(z) \in \mathbb{Q}(z)$. In the other hand $f''(z) = -\dfrac{2z}{(1+z^2)^2}$, and writing the sum in terms of linear independent parts as

$$\left(-\frac{2z}{(1+z^2)^2} + A_1(z)\frac{1}{1+z^2}\right) + A_0(z)\arctan z = 0 \tag{3.1.7}$$

gives $A_0(z) = 0$ and $A_1(z) = \dfrac{2z}{1+z^2}$. Therefore the holonomic DE is

$$(1+z^2)f''(z) + 2zf'(z) = 0. \tag{3.1.8}$$

Transformation of the DE into its RE: We have the DE

$$(1+z^2)f''(z) + 2zf'(z) = 0 \iff f''(z) + z^2 f''(z) + 2zf'(z) = 0,$$

and using the correspondence (3.1.6) we obtain

$$\begin{cases} f''(z) \longrightarrow (k+1-0)_2 \cdot a_{k+2-0} = (k+1)(k+2)a_{k+2} \\ z^2 f''(z) \longrightarrow (k+1-2)_2 \cdot a_{k+2-2} = (k-1)k a_k \\ zf'(z) \longrightarrow (k+1-1)_1 \cdot a_{k+1-1} = k a_k \end{cases},$$

hence the RE

$$(k+2)a_{k+2} + k a_k = 0, \;\; k = 0, 1, \cdots. \tag{3.1.9}$$

Solving the RE: the RE obtained gives $a_{k+2} = -\dfrac{k}{k+2}a_k$, we have the symmetry number $m = 2$ and moreover $f(0) = 0$, so f is odd and the RE is for odd coefficients

$$\begin{aligned} a_{2(k+1)+1} &= -\frac{2k+1}{2k+1+2}a_{2k+1}, \;\; k \geqslant 0 \\ &= -\frac{k+\frac{1}{2}}{k+\frac{3}{2}}a_{2k+1}. \end{aligned}$$

$a_1 = \dfrac{f'(0)}{1!} = 1$, so one can use the hypergeometric coefficient formula (2.2.4) and gets

$$a_{2k+1} = (-1)^k \frac{\left(\frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k} = \frac{(-1)^k}{2k+1}, \tag{3.1.10}$$

thus

$$f(z) = -\frac{(-1)^k}{2k+1}z^{2k+1}. \tag{3.1.11}$$

Third case: $f(z) = \ln(1+z)$.

Searching for the holonomic DE: $f'(z) = \frac{1}{1+z}$ and we have $\frac{f'(z)}{f(z)} = \frac{1}{(1+z)\ln(1+z)} \notin \mathbb{Q}(z)$. But as in the previous case we find as linear combination with independent term with rational coefficients for the second derivative, the rational functions are:

$$A_0(z) = 0, \quad \text{and} \quad A_1(z) = \frac{1}{1+z}.$$

Hence the holonomic DE is

$$(z+1)f''(z) + f'(z) = 0. \tag{3.1.12}$$

Transformation of the DE into its RE: we have the DE $zf''(z) + f''(z) + f'(z) = 0$, so substituting the correspondence of each term using (3.1.6) we obtain

$$k(k+1)a_{k+1} + (k+1)(k+2)a_{k+2} + (k+1)a_{k+1} = 0,$$

which gives the RE

$$(k+2)a_{k+2} + (k+1)a_{k+1} = 0, k = 0, 1, \cdots. \tag{3.1.13}$$

Solving the RE: the RE gives the relation

$$a_{k+2} = -\frac{k+1}{k+2}a_{k+1}, k \geqslant 0 \iff a_{k+1} = -\frac{k}{k+1}a_k, k \geqslant 1.$$

The symmetry number is 1, and the lowest derivative of f that does not vanish at 0 is the first derivative, so $a_1 = \frac{f'(0)}{1!} = 1$. Therefore one can use the hypergeometric coefficient formula (2.2.4) and gets

$$a_k = (-1)^{k+1}\frac{(1)_k}{(2)_k} = \frac{(-1)^{k+1}}{k+1}, \tag{3.1.14}$$

hence

$$f(z) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k+1} z^k. \tag{3.1.15}$$

Fourth case: $f(z) = \cos(z)$.

Searching for the holonomic DE: $f'(z) = -\sin(z)$, as $\frac{f'(z)}{f(z)} = -\tan(z) \notin \mathbb{Q}(z)$. We go to the next derivative and we find the rational functions $A_0(z) = 1$ and $A_1(z) = 0$ for the linear combination with independent parts $f''(z) + A_1(z)f'(z) + A_0(z)f(z) = 0$. Hence the holonomic DE is

$$f''(z) + f(z) = 0. \tag{3.1.16}$$

Transformation of the DE into its RE: the RE is easily deduced from all what we have done using the correspondence (3.1.6). It is

$$(k+1)(k+2)a_{k+2} + a_k = 0, \; k = 0, 1, \cdots. \tag{3.1.17}$$

Solving the RE: the symmetry number is $m = 2$ and $f'(0) = 0$ so f is even so the recurrence relation is for even coefficient

$$a_{2k+2} = -\frac{1}{(2k+1)(2k+2)}a_{2k} = -\frac{1}{4\left(k+\frac{1}{2}\right)(k+1)}a_{2k}.$$

As $f(0) = 1 = a_0$, we deduce by using the hypergeometric coefficient formula that

$$a_{2k} = \frac{(-1)^k}{4^k \left(\frac{1}{2}\right)_k k!} = \frac{(-1)^k}{(2k)!}, \tag{3.1.18}$$

hence

$$f(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} z^{2k}. \tag{3.1.19}$$

Fifth case: $f(z) = \operatorname{erf}(z) := \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$**.**

Now as we get used to the algorithm, we will go directly.

Here $f'(z) = \dfrac{2}{\sqrt{\pi}} e^{-z^2}$ and $f''(z) = -\dfrac{4}{\sqrt{\pi}} z e^{-z^2}$. The algorithm leads to the DE

$$f''(z) + 2zf'(z) = 0, \tag{3.1.20}$$

and so the RE

$$(k+1)(k+2)a_{k+2} + 2ka_k = 0, k = 0, 1, \ldots. \tag{3.1.21}$$

We have the symmetry number 2 and $f(0) = 0$, so f is odd and the RE used for odd coefficients

$$a_{2(k+1)+1} = \frac{k + \frac{1}{2}}{\left(k + \frac{3}{2}\right)(k+1)} a_{2k+1}$$

with initial value $a_1 = \dfrac{f'(0)}{1!} = \dfrac{2}{\sqrt{\pi}}$, so that finally

$$a_{2k+1} = \frac{2}{\sqrt{\pi}} \frac{(-1)^k \left(\frac{1}{2}\right)_k}{\left(\frac{3}{2}\right)_k k!} = \frac{2}{\sqrt{\pi}} \frac{(-1)^k}{(2k+1)k!}. \tag{3.1.22}$$

Therefore

$$f(z) = \sum_{k=0}^{\infty} \frac{2}{\sqrt{\pi}} \frac{(-1)^k}{(2k+1)k!} z^{2k+1}. \tag{3.1.23}$$

3.2 General Algorithm

From the examples, we are able to describe an algorithm which allows the computation of formal power series and formal Laurent series related to a given hypergeometric type function. This algorithm is a sub-case of the more general algorithm of computation of formal power series given by Koepf in [3]. We present first the general algorithm and after words in the next three chapters, we will gives details, implementation in Maxima and present some results for each step.

Algorithm 1 Computation of the Laurent series expansion for an analytic function f at $z_0 = 0$

1: **procedure** FPS

2: (1) **Find a holonomic DE:** fix an integer $N_{\max}$ and search the differential equation of lowest order $N \leqslant N_{\max}$ of the form

$$\text{DE} : D^N f(z) + \sum_{j=0}^{N-1} A_j(z) D^j f(z) = 0, \tag{3.2.1}$$

where $A_j(z) \in \mathbb{Q}(z)$. Deduce the holonomic DE by multiplying the equation obtained by the least common multiple of $(A_0(z), A_1(z), \ldots, A_{N-1}(z))$.

3: (2) **Find the corresponding RE:** use the correspondence (3.1.6) to convert the holonomic DE obtained into a recurrence equation of the form

$$\text{RE} : \sum_{j=0}^{M} P_j(k) a_{k+j} = 0. \tag{3.2.2}$$

for the coefficients a_k, where P_j are polynomials and $M \in \mathbb{N}$.

4: (3) **Solving the RE:** if the RE obtained has two summands, then the function is of hypergeometric type and it can be solved. Otherwise we are not in the case of hypergeometric type.

3.2.1 Remark. Searching for the differential equation in the form (3.2.1) of order N is equivalent to search a holonomic DE of order N since one can divide a holonomic DE of order N by the polynomial coefficient of the N^{th} derivative to obtain the form (3.2.1) or multiply the form (3.2.1) by the least common multiple of all the denominators of rational functions which occur.

4. Finding a Holonomic Differential Equation

4.1 Mathematical Algorithm

We have a holonomic function f given, and we search for a DE for f of the form (2.2.22). There are several equations of this type. Indeed, the fact of having a DE allows to have another one by differentiation. And adding two DE gives another one. For example, if we consider the case of the sine function $f(z) = \cos(z)$ we have the holonomic DE

$$f''(z) + f(z) = 0, \tag{4.1.1}$$

which corresponds to the RE $(k+2)(k+1)a_{k+2} + a_k = 0$. And by differentiation of (4.1.1) we also have $f^{(3)}(z) + f'(z) = 0$ which leads to the same series coefficients but adding the two equations we get $f^{(3)}(z) + f''(z) + f'(z) + f(z) = 0$ which is a DE essentially different as it is equivalent to the RE

$$(k+1)(k+2)(k+3)a_{k+3} + (k+1)(k+2)a_{k+2} + (k+1)a_{k+1} + a_k = 0 \tag{4.1.2}$$

which is not of hypergeometric type. The algorithm we present here finds the holonomic DE of lowest order with respect to all the holonomic DE satisfied by f.

Algorithm 2 Searching for a holonomic DE of a holonomic function f (see [6])

---

procedure HolonomicDE

(a) If $f = 0$ then the DE is found and we stop.

(b) $f \neq 0$, compute $A_0(z) = \dfrac{Df(z)}{f(z)}$,

  (b-1) if $A_0(z) \in \mathbb{Q}(z)$ i.e $A_0(z) = \dfrac{P(z)}{Q(z)}$ where P and Q are polynomials, then we have found a holonomic DE satisfied by f:

$$Q(z)Df(z) + P(z)f(z) = 0.$$

  (b-2) If $A_0(z) \notin \mathbb{Q}(z)$, then go to (b).

(c) Fix a number $N_{max} \in \mathbb{N}$, the maximal order of the DE searched for; a suitable value is $N_{max} := 4$.

  (c-1) set $N := 2$;

  (c-2) compute $D^N f$;

  (c-3) expand the expression

$$D^N f(z) + A_{N-1}D^{N-1}f(z) + \cdots + A_0 f(z) = \sum_{i=0}^{E} S_i,$$

Algorithm 2 Searching for a holonomic DE of a holonomic function f (continued)

2: (c-3) (continued)

in elementary summands with $A_N, A_{N-1}, \ldots, A_0$ as constants. $E \geqslant N$ is a total number of summands S_i obtained after expansion. For each pair of summands S_i and S_j $(0 \leqslant i \neq j \leqslant E)$, group them additively together if there exists $r(z) \in \mathbb{Q}(z)$ such that $S_i(z) = r(z)S_j(z)$. If the number of groups is N then we have N linearly independent expressions. In that case, there exists a solution which can be found by equating each group to zero. The resulting system is linear for the constants $A_0, A_1, \ldots, A_{N-1}$. Solving this system gives rational functions in z, and the solution is unique since we normalized $A_N = 1$. After multiplication by the common denominator of $A_0(z), A_1(z), \ldots A_{N-1}(z)$ you get the holonomic DE searched for. Finally cancel common factors of the polynomial coefficients. If otherwise the number of groups is larger than N, then there is no solution and the step is not successful.

(b-4) If (b-3) was not successful, then increment N, and go back to (b-1), until $N = N_{max}$.

Assuming that the first step of searching the holonomic DE has failed, we have to show for any integer $N \geqslant 2$ of the algorithm searching for a holonomic DE

$$D^N f(z) + A_{N-1}D^{N-1}f(z) + \cdots + A_0 f(z) = 0 \tag{4.1.3}$$

for f, that either

1. the number of linearly independent summands of (4.1.3) equals N and the linear equation system that we get by setting the coefficients of the linearly independent terms to zero, has a unique solution $(A_0(z), A_1(z), \ldots A_{N-1}(z)) \in \mathbb{Q}(z)^N$.

2. or the number of linearly independent summands of (4.1.3) is larger than N, and there is no solution.

Proof. Since the first step has failed, the number of linearly independent terms is at least 2, and we must proceed with $2 \leqslant N \leqslant N_{\max}$.

Now, for $2 \leqslant N \leqslant N_{\max}$, if we assume that the algorithm searching for a holonomic DE (4.1.3) has failed until N, then the number of linearly independent terms is at least $N+1$, and we must proceed with $N+1$. In that case, suppose now the number of linearly independent terms is less than or equal to $N+1$. Then we are able to find a solution vector $(A_0, A_1, \cdots, A_N) \in \mathbb{Q}(z)^N$, and it remains to show that the solution is unique. Indeed, if we have another solution $(B_0, B_1, \ldots, B_N)$ then f verifies

$$D^{N+1}f + A_N D^N f + \cdots + A_0 f(z) = D^{N+1}f + B_N D^N f + \cdots + B_0 f = 0$$

which implies that the DE

$$(A_N - B_N)D^N f + (A_{N-1} - B_{N-1})D^{N-1}f + \cdots (A_0 - B_0)f = 0$$

is also valid for f. This new DE is of order N but we know from our hypothesis that this is not possible. Hence we must have $A_N = B_N, A_{N-1} = B_{N-1}, \ldots A_0 = B_0$.

Therefore by induction we have shown our statement. □

4.2 Maxima Implementation

In this section we present the key lines of the procedure we used for the implementation of the algorithm (2) in the Computer Algebra System (CAS) Maxima [10].

1. We first write a function *ratfunp* that will be used extensively in the algorithm, it is a function that returns a Boolean value that is true if the expression is rational with respect to the variables in the input list and false otherwise.

```
/* test if an expression is rational with respect to the variable in the list L*/
ratfunp(expr,L):=block([r],
  r: ratsimp(expr),
  polynomialp(num(r),L) and polynomialp(denom(r),L)
)$
/*polynomialp is a Maxima function for the same test for polynomials*/
```

2. If the given function is zero then the holonomic DE of lowest order is $F = 0$.

3. The step (b) in the algorithm (2) can be done by one test with *ratfunp* and a use of Maxima's commands *ratsimp* to have the same denominator and *num* to take the numerator of the resulting expression which will represent the left hand side of the holonomic DE in the case of success with the rational test.

```
/* step (a) */
    A0:diff(f,z)/f,
    if(ratfunp(A0,[z])=true) then num(ratsimp('diff(F,z)-A0*F))=0
```

4. When otherwise the step (b) fails, we initialize a new boolean variable *flag* false to have the status for the searching of linearly independent summands. Starting at $N = 2$ until $N_{\max}$ (with default value 4), we expand completely the expression (3.2.1) as follows

```
/*expansion*/
algebraic:true,
Eq: expand(ratsimp(diff(f,z,N))+sum(ratsimp(Coef[i+1]*diff(f,z,i)),i,0,N-1)),
```

   Maxima's variable *algebraic* is set to true in order to make the complete simplification using *ratsimp*. That is why we are using *ratsimp* for each derivative.

5. After collecting all the terms in the variable *S* using Maxima's command *args(Eq)*, to group the linearly dependent terms together, we make a loop to add together all terms whose quotient is rational with respect to the variables in the list *var: append(Coef,[z])*. The principle is to remove in *S* all terms whose quotient with the first term is rational and put them in additive form in the list *Sumds* until *S* becomes empty.

```
 /* grouping of linear dependent terms */
 Sumds: [],
 while S#[] do (
    del: [S[1]],
```

```
    tmp: S[1],
    if(length(S)>1) then
        for j: 2 thru length(S) do
            if(ratfunp(S[j]/S[1],var)=true) then (
                tmp: tmp + S[j],
                del: endcons(S[j], del)
            ),
    Sumds: endcons(tmp, Sumds),
    S: sublist(S, lambda([v], not(member(v,del))))
),
```

6. Once the grouping is done, we solve the linear system with equations being what we obtained by equating each group to 0 with respect to the variables in the list *Coef*. If there is a solution we set *flag* to true to leave the loop and return the holonomic DE as in the step 2 by using *num* and *ratsimp*. If there is no solution we increment the current value of *N* and go back to the expansion in 4.

   However each equation of the system obtained has to be filtered of any multiplicative factor part depending only on the variable z, so that we will just have a linear equation part with respect to the variables in the list *Coef*. Otherwise one may find a current Maxima bug Quotient by a polynomial of higher degree or the quotient is not exact when the system contains some irrational functions for example.

4.2.1 Example. Let us consider $f(z) := \sqrt{\frac{1+\sqrt{1-z}}{z}}$, the research of its holonomic DE of lowest order leads to a linear system of two equations and two variables $A[0]$ and $A[1]$:

$$\frac{5\sqrt{\sqrt{1-z}+1}\,z^{\frac{5}{2}}}{16z^5-32z^4+16z^3}-\frac{11\sqrt{\sqrt{1-z}+1}\,z^{\frac{3}{2}}}{16z^5-32z^4+16z^3}+\frac{6\sqrt{\sqrt{1-z}+1}\,\sqrt{z}}{16z^5-32z^4+16z^3}-\frac{A_1\sqrt{\sqrt{1-z}+1}\,z^{\frac{3}{2}}}{4z^3-4z^2}+\frac{A_1\sqrt{\sqrt{1-z}+1}\,\sqrt{z}}{4z^3-4z^2}+\frac{A_0\sqrt{\sqrt{1-z}+1}}{\sqrt{z}}=0 \quad \text{(Eq1)}$$

$$-\frac{8\sqrt{\sqrt{1-z}+1}\,\sqrt{1-z}\,z^{\frac{3}{2}}}{16z^5-32z^4+16z^3}+\frac{6\sqrt{\sqrt{1-z}+1}\,\sqrt{1-z}\,\sqrt{z}}{16z^5-32z^4+16z^3}+\frac{A_1\sqrt{\sqrt{1-z}+1}\,\sqrt{1-z}\,\sqrt{z}}{4z^3-4z^2}=0 \quad \text{(Eq2)}$$

Trying to solve this equation with Maxima gives

(%i2) solve([Eq1,Eq2],[A[0],A[1]]);

Quotient by a polynomial of higher degree
– an error. To debug this try: debugmode(true);

To deal with that issue, we write each expression in form of a fraction in order to work with each numerator as the denominators are independent of the variables in *Coef*. Then we set Maxima's variable *logexpand* to all so that we can decompose products in sum and take the arguments of the $\log$ function which are polynomials with respect to the variables in the list *var*. And then reset *logexpand* to true. With the example (4.2.1) this approach leads to the linear system

$$16A_0\,z^3-4A_1\,z^2-16A_0\,z^2+4A_1z+5z-6=0 \quad \text{(Eq1)}$$
$$2A_1\,z^2-2A_1z-4z+3=0, \quad \text{(Eq2)}$$

which can be easily solved by Maxima

(%i2) solve([Eq1,Eq2],[A[0],A[1]]);

$$[[A_0 = \frac{3}{16z^2 - 16z}, A_1 = \frac{4z-3}{2z^2-2z}]] \quad (\%o2)$$

4.2.2 Remark. The right Maxima command to solve the system obtained in 6. should be *linsolve*, it is true that *solve* will actually do the same work but after identifying the system as a linear system and this is not optimal, therefore in our program we used *linsolve*.

4.3 Results

In this section we present some results obtained with our function HolonomicDE(f,F(z)) which computes the holonomic differential equation of lowest order $N \leqslant 4$ if it exists, of a given function f in the variable z. In our implementation, the maximum order of the DE searching for can be increased by changing the value of the variable *Nmax* that we have set to be an integer greater or equal to 4.

(%i1) f:sqrt((1+sqrt(1-z))/z)$ HolonomicDE(f,F(z));

$$\left(16z^2 - 16z\right)\left(\frac{d^2}{dz^2}\,\mathrm{F}(z)\right) + (32z - 24)\left(\frac{d}{dz}\,\mathrm{F}(z)\right) + 3\,\mathrm{F}(z) = 0 \quad (\%o1)$$

(%i3) f:cos(z)$ HolonomicDE(f,F(z));

$$\frac{d^2}{dz^2}\,\mathrm{F}(z) + \mathrm{F}(z) = 0 \quad (\%o3)$$

(%i5) f:erf(z)$ HolonomicDE(f,F(z));

$$\frac{d^2}{dz^2}\,\mathrm{F}(z) + 2z\left(\frac{d}{dz}\,\mathrm{F}(z)\right) = 0 \quad (\%o5)$$

(%i7) f:log(1+z)$ HolonomicDE(f,F(z));

$$(z+1)\left(\frac{d^2}{dz^2}\,\mathrm{F}(z)\right) + \frac{d}{dz}\,\mathrm{F}(z) = 0 \quad (\%o7)$$

(%i9) f: cos(z)^2+sin(z)^2$ HolonomicDE(f,F(z));

$$\frac{d}{dz}\,\mathrm{F}(z) = 0 \quad (\%o9)$$

(%i11) f:asin(z)^5$ HolonomicDE(f,F(z));

false (%o11)

(%i13) Nmax:6$ HolonomicDE(f,F(z));

$$\left(z^6 - 3z^4 + 3z^2 - 1\right)\left(\frac{d^6}{dz^6}\,\mathrm{F}(z)\right) + \left(15z^5 - 30z^3 + 15z\right)\left(\frac{d^5}{dz^5}\,\mathrm{F}(z)\right) + \left(65z^4 - 85z^2 + 20\right)\left(\frac{d^4}{dz^4}\,\mathrm{F}(z)\right) + \left(90z^3 - 75z\right)\left(\frac{d^3}{dz^3}\,\mathrm{F}(z)\right) + \left(31z^2 - 16\right)\left(\frac{d^2}{dz^2}\,\mathrm{F}(z)\right) + z\left(\frac{d}{dz}\,\mathrm{F}(z)\right) = 0 \quad (\%o13)$$

More results can be found in ($B.1$)

5. Getting the Recurrence Equation (RE)

We have shown in the remark (3.1.1) how to get the RE from the DE. Therefore the algorithm to deduce the RE from a DE is given below.

5.1 Mathematical Algorithm

Algorithm 3 From an holonomic DE to its Recurrence Equation.

procedure DEtoRE

(a) Expand the DE to have it in the form

$$\sum_{j=0}^{Q}\sum_{l=0}^{Q} c_{j,l} z^l D^j f = 0 \tag{5.1.1}$$

where $Q \in \mathbb{N}$ and $c_{j,l}$ are some constants.

(b) Use the correspondence

$$z^l D^j f \longrightarrow (k+1-l)_j \cdot a_{k+j-l}. \tag{5.1.2}$$

to substitute each term. Therefore we get

$$\sum_{j=0}^{Q}\sum_{j=0}^{Q} c_{j,l}(k+1-l)_j \cdot a_{k+j-l} = 0, \tag{5.1.3}$$

and get finally the RE after simplification.

In (5.1.3) it remains to observe that as $(k+1-l)_j$ is a polynomial in k for each fixed j and k, the calculated RE is then of the special type (3.2.2) where $P_j(j = 0, \ldots, M)$ are polynomial functions in k and $M \in \mathbb{N}$.

5.2 Maxima Implementation

In this section we present our implementation of the algorithm (3) in the CAS Maxima. We wrote a function DEtoRE(DE,F(z),k) which converts the holonomic differential equation DE depending on the variable z in its corresponding recurrence equation with the index k.

1. We first expand the DE it in the form (5.1.1).
2. Secondly we collect all derivative orders occurring in the DE.

```
Terms: args(lhs(DE)),

/* List of order of derivatives per term */

OrdDiffs: map(lambda([v], derivdegree(v, F, z)),Terms),
```

We notice that the command *args()* does not give terms in the particular case of the DEs $DF = 0$ and $F = 0$, these cases are checked and treated separately in the program with the same procedure below by working with simple variables instead of lists.

3. We collect constant coefficients and the powers of z in the DE.

```
/* List of powers of z and list of constant coefficients*/
    n: length(OrdDiffs),
    Zpows: [],
    Coefs: [],
    for i: 1 thru n do (
        s: subst(1,'diff(F,z,OrdDiffs[i]),Terms[i]),
        Zpows: endcons(hipow(s,z),Zpows),
        Coefs: endcons(subst(1,z,s), Coefs)
    ),
```

We notice that the derivatives should be removed in each term so that the procedure used can collect the powers as expected, that is the goal of the first substitution in the *for loop* above.

4. Thus we can apply the correspondence (5.1.2) by substitution in the expanded DE.

```
 /* Mapping to the terms of the RE*/
    l: [],
    aterm: [],
    for i: 1 thru n do (
        aterm: endcons(subst(k+OrdDiffs[i]-Zpows[i],k,a), aterm),
        l: endcons(Coefs[i]*pochhammer(k+1-Zpows[i],OrdDiffs[i])*aterm[i],l)
    ),
```

5. The left hand side of the corresponding RE is the sum of all terms in the list l occurring in the program of step 4.

5.3 Results

We wrote another function FindRE(f,z,a[k]) which called our functions HolonomicDE(f,F(z)) and DEtoRE(DE,z,k) to produce a recurrence equation of a given holonomic function f. We present here some result obtained using this function.

(%i1) FindRE(0,z,a[k]);

$$a_k = 0 \tag{\%o1}$$

(%i3) f:cos(z)$ FindRE(f,z,a[k]);

$$(k+1)\,(k+2)\,a_{k+2} + a_k = 0 \tag{\%o3}$$

(%i5) f:asin(z)$ FindRE(f,z,a[k]);

$$k^2\,a_k - (k+1)\,(k+2)\,a_{k+2} = 0 \tag{\%o5}$$

(%i7) f:z*atan(z)$ FindRE(f,z,a[k]);

$$(k-2)\ (k-1)\ a_k + a_{k-2}\ (k-3)\ (k-2) = 0 \tag{\%o7}$$

(%i10) declare(p,constant)$ f:1/(1-z)^p$ FindRE(f,z,a[k]);

$$(k+p)\ a_k - (k+1)\ a_{k+1} = 0 \tag{\%o10}$$

More examples can be found in $(B.2)$.

5.3.1 Remark. (see [6]) We mention that one cannot guarantee that for functions f or hypergeometric type the algorithm produces a DE corresponding to a RE of hypergeometric type. Indeed, for the function $f(z) := e^z \sin z$ the algorithm produces the RE

(%i11) FindRE(exp(z)*sin(z),z,a[k]);

$$(k+1)\ (k+2)\ a_{k+2} - 2\,(k+1)\ a_{k+1} + 2a_k = 0 \tag{\%o11}$$

which is not of hypergeometric type. However f also fulfills the fourth order DE $D^4 f + 4f = 0$

(%i13) f:exp(z)*sin(z)$ diff(f,z,4)+4*f=0,pred;

true (%o13)

corresponding to the RE of hypergeometric type

(%i15) DE: diff(F(z),z,4)+4*F(z)$ DEtoRE(DE,F(z),a[k]);

$$(k+1)\ (k+2)\ (k+3)\ (k+4)\ a_{k+4} + 4a_k = 0 \tag{\%o15}$$

Nevertheless this seems to be a rare situation. In all but very few cases hypergeometric type FLS are recognized by the given algorithm.

6. Solving the Recurrence Equation and Deducing of the FPS

We are interested in the case when the recurrence equation obtained is of hypergeometric type, i.e. the recurrence equation has two summands. We will consider the form

$$a_{k+m} = R(k)a_k \quad \text{or} \quad Q(k)a_{k+m} = P(k)a_k, \tag{6.0.1}$$

P, Q polynomials and R rational. a_k are Taylor coefficients of the function f. m is the symmetry number.

6.1 Mathematical Algorithms

We split the problem in two cases related to the number of summands.

First Case: RE with exactly one summand

In this case, either $P(k) = 0$ or $Q(k) = 0$. It is clear that an FLS of hypergeometric type with a RE which has exactly one summand leads to a power series expansion with at most one non-zero term. Therefore we are facing a monomial. Any monomial $c_n z^n$ ($c_n \in \mathbb{C}, n \in \mathbb{Z}$) has the RE

(%i1) declare(c_n, constant), declare(n,constant), declare(n,integer)$
(%i2) FindRE(c_n*z^n,z,a[k]);

$$(k-n)\ a_k = 0 \tag{\%o2}$$

We notice that this RE is valid for monomials with negative exponents since $n \in \mathbb{Z}$. Moreover, despite the singularity at zero, $\log(z)$ appears to be of hypergeometric type, we have the RE

(%i3) FindRE(c_n*log(z),z,a[k]);

$$(k+1)^2\ a_{k+1} = 0 \iff k^2 a_k = 0, k \neq 0 \tag{\%o3}$$

which is actually not essentially different from the RE of a constant. So we have to take $\log(z)$ into account, but not in a way to expand it, it is rather to return it as a monomial which appears isolated. This approach coincides with [6] in Definition 8.1. Indeed, $\log(z)$ will be considered as a particular "constant" multiplicative which allows to write independently the terms in the power series expansion as the way it occurs. This could be justified by the fact that

$$\lim_{n\to -1} \frac{z^{n+1}-1}{n+1} = \lim_{n\to -1} \int_1^z x^n dx = \log(z). \tag{6.1.1}$$

On the other hand, one may identify the presence of $\log(z)$ from the RE by looking for the multiplicity of the root of the polynomial factor of the single coefficient term. Moreover this factor allows to identify the monomial term.

(%i4) p:3$ FindRE(c_n*z^n*log(z)^p,z,a[k]);

$$(k-n)^4\ a_k = 0 \tag{\%o4}$$

(%i5) p:4$ FindRE(a_n*z^n*log(z)^p,z,a[k]);

$$(k-n)^5\,a_k = 0 \tag{\%o5}$$

Therefore one can state that the hypergeometric type function related to an FLS (or its antiderivative) which has a recurrence equation with only one summand leads to the special monomial

$$z^n(b+c\log(z)^p) \text{ where the RE is } (k-n)^{p+1}a_k = 0 \ \ (n,p\in\mathbb{Z}, b,c\in\mathbb{C}). \tag{6.1.2}$$

As the shape of the expansion searching for is well known in this case, it is enough for our work to recognize them from the given function although the approach that we have done is able to find monomials of the form described in (6.1.2).

6.1.1 Remark (More considerations)**.** Furthermore, this result mostly allows the computation of the powers series expansion of any linear combination of powers of $\log(z)$: $\sum_{i=1}^{p} f_i \log(z)^i$ where the f_i are hypergeometric type functions. However we mention that almost all of these linear combinations have REs which are not of hypergeometric type but as we see how to extend our work to find them, our Maxima implementation will be able to treat these cases. To fully use this extension of our work, let us move to the main case of hypergeometric type functions, corresponding to REs with two summands.

Second Case: RE with Exactly two summands

From (2.2.26) we know that the FLS will be given as a linear combination of m hypergeometric type functions with their initial values taken from the first terms of the Taylor expansion. Looking at lemma (2.2.4) in (a) and (b), in order to take into account FLS as shifted FPS we have two essential steps to consider for searching initial values and these cases lead to the representation of the given function.

First step: Find the shifted factor (if it exists)

In fact for a given FLS $f(z) := z^s g(z), s\in\mathbb{Z}$ where g is an FPS, either

$$\lim_{z\to 0} f(z) = \infty (s<0) \quad \text{or} \quad \lim_{z\to 0} f(z) = 0 (s>0).$$

So the search for s_0 will be related to one of the two possibilities above which lead to the formula

$$s_0 = \min_{j\in\mathbb{Z}} \left\{ j \mid \lim_{z\to 0} z^{j+1} f(z) = 0 \text{ and } z^j f(z) \text{ is defined at } 0 \right\}. \tag{6.1.3}$$

Proof. Indeed, if $\lim_{z\to 0} f(z) = \infty$, then we are looking for the minimal integer $s_0 > 0$ such that $\lim_{z\to 0} f(z) < \infty$. For such an s_0 we have $z^{s_0+1} f(z) \to 0$ as z goes to zero. Hence our statement is true.

If otherwise $\lim_{z\to 0} f(z) = 0$, then we are first looking for the maximal integer $s_1 > 0$ such that $\lim_{z\to 0} \frac{f(z)}{z^{s_1}} = \lim_{z\to 0} z^{-s_1} f(z) \neq 0$. However if $\frac{f}{z^{s_1}}$ is not defined at 0 then certainly the FPS involved vanish at zero and s_1 should be decremented. Therefore to gather the two cases together we have to look for s_1 in such a way that $\frac{f}{z^{s_1-1}} \to 0$ as z goes to zero and $\frac{f}{z^{s_1}}$ is defined at zero, which remain true when the corresponding FPS does not vanish at zero. Furthermore, finding such an s_1 is equivalent to find the minimal $s_0 = -s_1$ in $\mathbb{Z}$ such that $z^{s_0+1} f \to 0$ as z goes to zero and $z^{s_0} f$ is defined at 0. □

6.1.2 Example. $f(z) := \log(1+z)$, we have $\lim_{z\to 0} z f(z) = 0$ and $f(z)$ well defined at zero. Since $z^{-1} f$ is not defined at 0, therefore $s_0 = 0$.

$f(z) := \frac{\cos z}{z}$, we have $\lim_{z\to 0} z^2 f(z) = 0$ and zf is defined at zero. Since $z^0 f$ is not defined at zero therefore $s_0 = 1$.

Thus we will have $s_0 = -s$, and consider the function $g = z^{s_0} f$ with the RE

$$b_{k+m} = R_1(k) b_k, \text{ where } b_k = a_{k+s_0} \text{ and } R_1(k) = R(k+s_0) \tag{6.1.4}$$

since the shift induced by the factor z^s does not affect the symmetry number. So the powers of z in the expansion will be taken as z^{s_0+k}.

Second step: Find the first nonzero coefficient. In this step, we consider an FPS function [1] g, and we want to determined the k_0 corresponding to

$$k_0 = \min_{j\in\mathbb{N}} \left\{ j \mid \lim_{z\to 0} g^{(j)}(z) \in \mathbb{C} \setminus \{0\} \right\}. \tag{6.1.5}$$

We are sure that such k_0 exists because otherwise we come back in the first case, here the power series expansion has at least two terms.

Moreover, we can write k_0 as $k_0 = mj_0 + i_0$, $i_0 \in [\![0, m-1]\!]$, which connects k_0 to the $(i_0+1)^{th}$ hypergeometric type function term in the linear combination. Hence we can define the initial values as follows:

$$b_i = \lim_{z\to 0} \frac{g^{(mj_0+i)}(z)}{(mj_0+i)!} = \lim_{z\to 0} \frac{g^{(k_0+i-i_0)}(z)}{(k_0+i-i_0)!}, \; i = 0, 1, \cdots m-1. \tag{6.1.6}$$

It follows that the $(i+1)^{\text{th}}(i \leqslant m-1)$ hypergeometric type function term will be written in series with the powers $z^{m(k+j_0)+i+s_0} = z^{mk+k0+i-i_0+s_0}$, s_0 coming from the first step. Thus, it just remains to find how to write properly the general coefficient of each hypergeometric type function term in the linear combination.

For the $(i+1)^{\text{th}}(i \leqslant m-1)$ hypergeometric type function, the sequence of coefficients is

$$(b_{mk+k_0+i-i_0})_{k=0,1,\ldots} = \left(b_{m(k+j_0)+i}\right)_{k=0,1,\ldots} = \left(b_{mj_0+i}, b_{m(1+j_0)+i}, \cdots\right), \quad (b_{k+s_0} = a_k) \tag{6.1.7}$$

so by substitution the RE (6.1.4) will be taken in the form

$$b_{m(k+j_0)+i+m} = R_1\left(m(k+j_0)+i\right) b_{m(k+j_0)+i}, \quad (R_1(k) = R(k+s_0)). \tag{6.1.8}$$

We notice that the symmetry number does not change and our interest in (6.1.8) is the form of the rational function R. Hence according to relation (6.1.8) the general coefficient will be deduced from the rational function R taken as $R\left(mk + k_0 + i - i_0 + s_0\right)$.

6.1.3 Example. For the FPS $f(z) := \cos z$ we have $m = 2$, $s_0 = 0$, $k_0 = i_0 = 0$ which leads to $(b_{2k+1})_{k=0,1,\ldots}$ and $(b_{2k})_{k=0,1,\ldots}$ with $b_k = a_k$. For the FLS $f(z) := z\log(1+z)$ we find $s_0 = -1$, and $k_0 = 1, i_0 = 0$ hence the sequence of coefficients $(b_{k+1})_{k=0,1,\ldots}$ with $b_k = a_{k-1}$.

Finally gathering all the results, a mathematical algorithm to go from a RE of hypergeometric type to an FLS representation as in (2.2.24) can be written as follows

[1] function which has an FPS expansion

[2] The reminder in the euclidean division.

Algorithm 4 From a RE of hypergeometric type with two summands to the corresponding FPS

procedure RETOFPS

(a) Identify the symmetry number m.

(b) Write the RE in the form (6.0.1).

(c) Find s_0:

(c-1) set $g = f$ and $s_0 = 0$

(c-2) compute the Taylor expansion of order 0 of f and assign it to T,

(c-3) set p to be the degree of T

(c-4) if $p < 0$ then

$$s_0 = s_0 + 1 \text{ and } g = zg$$

(c-5) else

$$s_0 = s_0 - 1 \text{ and } g = g/z$$

(c-6) repeat (c-2) - (c-5) until $p = 0$ or $T \in \mathbb{C} \setminus \{0\}$.

(c-7) if g is defined at 0 then keep g and the value of s_0 and go to (d)

(c-8) otherwise $s_0 = s_0 - 1$ and $g = zg$ and go to (d).

(d) Find k_0:

(d-1) set $k_0 = 0$,

(d-2) evaluate

$$l := \lim_{z\to 0} g^{(k_0)}(z),$$

(d-3) if $l \neq 0$ then go to (f)

(d-4) else increment k_0 and go back to (d-2).

(e) Find i_0 so that $k_0 \equiv i_0[m]$[2]

(f) Compute the initial values

$$b_i = \lim_{z\to 0} \frac{g^{(k_0+i-i_0)}(z)}{(k_0+i-i_0)!},\ \ i = 0, 1, \cdots m-1.$$

(g) Substitute the variable k in R by $mk + k_0 + i - i_0 + s_0$.

(h) Use the algorithm of deduction of the general coefficient (will be seen in the next paragraph) to find the coefficient of each hypergeometric type function in the linear combination of f.

(i) return the series with general coefficient connected to $z^{mk+k_0+i-i_0+s_0}$.

Algorithm of Simplification for the General Coefficient

We will present in this paragraph an algorithm little bit adapted to the CAS Maxima for simplifications of Pochammer symbols involved in a rational function. We have a rational function $R(k) := \frac{P(k)}{Q(k)}$ and we want to compute the product of $R(k)$ for $k = 0, 1, \ldots n-1$, $n \in \mathbb{N}$. Factorizing R gives

$$R(k) = C\frac{(k+a_1)(k+a_2)\cdots(k+a_p)}{(k+b_1)(k+b_2)\cdots(k+b_q)}, \tag{6.1.9}$$

where p and q are respectively the degree of P and Q, C is a constant and the a_i, b_i are zeros and poles of R respectively. From the definition of the Pochhammer symbol, using (6.1.9) one can see that

$$\prod_{k=0}^{n-1} R(k) = C^n \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n}. \tag{6.1.10}$$

Now, we are going to use a property of the gamma function (see chapter 1 in [7]) that generalizes the factorial. The Pochhammer symbol of a number x can also be given as

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}, \tag{6.1.11}$$

and as $\Gamma(1) = 1$ we have $\Gamma(n) = (n-1)!$ for any positive integer. So in our Maxima algorithm, the Pochhammer symbols in (6.1.10) can be converted into factorial symbols by using the gamma transformation. However, the conversion can bring inconvenient results when we make it, because $(x)_n$ will be replaced by $\frac{\Gamma(x+n)}{\Gamma(x)}$ and $\Gamma(x)$ will be computed, for example

```
(%i6)   makefact(makegamma(pochhammer(3/2,n)));
```

$$\frac{2\left(n+\frac{1}{2}\right)!}{\sqrt{\pi}} \tag{\%o6}$$

cannot be used to replace $(\frac{3}{2})_n$ because of the computation of $\Gamma(\frac{3}{2})$ the function *makefact()* will just react on $\Gamma(k+\frac{3}{2})$. On the other hand, if we consider two numbers x and y such that $x - y = p \in \mathbb{N}$, then we have

$$\begin{aligned}
\frac{(y)_n}{(x)_n} = \frac{\frac{\Gamma(y+n)}{\Gamma(x+n)}}{\frac{\Gamma(y)}{\Gamma(x)}} &= \frac{\frac{(y-1+n)!}{(x-1+n)!}}{\frac{(y-1)!}{(x-1)!}} \\
&= \frac{\frac{(y-1+n)!}{(y+p-1+n)!}}{\frac{(y-1)}{(y+p-1)!}} \\
&= \frac{y(y+1)\cdots(y+p-1)}{(y+n)(y+n+1)\cdots(y+n+p-1)} \\
&= \frac{(y)_p}{(y+n)_p}.
\end{aligned} \tag{6.1.12}$$

6.1.4 Example. since $\frac{3}{2} - \frac{1}{2} = 1$, we have $\frac{\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n} = \frac{\left(\frac{1}{2}\right)_1}{\left(\frac{1}{2}+n\right)_1} = \frac{1}{2n+1}$.

Generally for any number x, y such that $x - y = p \in \mathbb{Z}$ we have

$$\frac{(y)_n}{(x)_n} = \begin{cases} \dfrac{(y)_p}{(y+n)_p} & \text{if } p > 0, \\ \dfrac{(x+n)_p}{(x)_p} & \text{if } p < 0. \end{cases} \tag{6.1.13}$$

Fortunately, this nice computation can be done by Maxima when we combine *makegamma()*, *makefact()*, *minfactorial()* and *factor()*. Therefore to deal with the simplification with Pochhammer symbol in (6.1.10), we proceed as follows.

(a) If there is a positive integer as zero of R ($\exists a_i, -a_i \in \mathbb{N}$) then we are in the case of a polynomial. If we denote by a^* the minimum of the positive zeros of R, then the polynomial looking for is given by

$$f(z) = \sum_{n=0}^{a^*} C^n \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} z^{mn+k_0+i-i_0+s_0}, \qquad (6.1.14)$$

where k_0, i_0, s_0 was found as explained in (6.1.5) and (6.1.3). We can see that this summation is finite. Let's take the example of the fourth Chebychev polynomial $f(z) := \cos(4 \arccos z)$

6.1.5 Example. We have the RE

(%7) FindRE(cos(4*acos(z)),z,a[k]);

$$(k-4)\ (k+4)\ a_k - (k+1)\ (k+2)\ a_{k+2} = 0 \qquad (\%o7)$$

so the symmetry number is $m = 2$ and we have $f(0) = 1$ and $f'(0) = 0$, so f is even an the rational function considered is

$$R(k) = \frac{(k-2)(k+2)}{(k+1)(k+\frac{1}{2})}, \qquad (6.1.15)$$

hence we have the zero $a^* = 2$ and therefore

$$f(z) = \sum_{n=0}^{2} \frac{(-2)_n (2)_n}{n!\,\left(\frac{1}{2}\right)_n} z^{2n} = 1 - 8z^2 + 8z^4. \qquad (6.1.16)$$

6.1.6 Remark. The result above shows that the algorithm implemented can be used as a simplifier for some expressions.

(b) For any $a_i, i \in [\![1,p]\!]$ if there is one $b_j, j \in [\![1,q]\!]$ such that $a_i - b_j \in \mathbb{Z}$ then replace $\frac{(a_i)_n}{(b_j)_n}$ by *factor(minfactorial(makefact(makegamma(*$\frac{(a_i)_n}{(b_j)_n}$*))))*, knowing that $(a)_n$ is done in Maxima by the command *pochhammer(a,k)*.

(c) For the remaining a_i and b_j after applied (a), continue as follows (we use c_i to denote a_i or b_j, since the procedure will be the same)

(c-1) if $c_i \notin \mathbb{Q}$ then we keep $(c_i)_n$,

(c-2) if $c_i \in \mathbb{Q}$ and $c_i > 0$ then

(c-2-1) if $c_i \in \mathbb{N}$ then replace $(c_i)_n$ by $\binom{n+c_i-1}{c_i} c_i!$,

(c-2-2) else if $c_i \in \mathbb{Q}$ with denominator equal to 2, then $c_i = \dfrac{r}{2}$, with $r \in \mathbb{N}$ an odd integer (otherwise c_i will be an integer). we set t the number such that $r = 2t + 1$. Therefore one can easily show that

$$(c_i)_n = \frac{[2(t+n)]!}{(2t)!4^n \binom{t+n}{t} n!} \qquad (6.1.17)$$

(c-2-3) else c_i is rational with denominator different to 2, we keep $(c_i)_n$.

(c-3) If $c_i \in \mathbb{Q}$ and $c_i < 0$ then we find the smallest integer $u \in \mathbb{N}$ such that $c_i + u > 0$. Therefore we have

$$(c_i)_n = (c_i)_u \cdot (c_i + u)_{n-u}, \qquad (6.1.18)$$

and we come back to the step (b-2) with $(c_i + u)_{n-u}$ thus the output will be multiplied by $(c_i)_u$.

Formal Laurent Puiseux Series (LPS)

We consider the formal Laurent Puiseux series of the form

$$f(z) := \sum_{k=k_0}^{\infty} a_k z^{k/n}, \tag{6.1.19}$$

with coefficient $a_k \in \mathbb{C}$. LPS are FLS evaluated at $\sqrt[n]{z}$ and n is called the Puiseux number of the given representation of f. We shall make sure now that LPS are of hypergeometric type.

6.1.7 Theorem (see Theorem 8.1 in [6]). *Let f correspond to an FLS of hypergeometric type with representation (2.2.24), then $f(\sqrt[n]{z}), n \in \mathbb{N}$ is a function of hypergeometric type.*

From this theorem we can deduce that the approach to find the power series expansion of a given LPS will affect the whole algorithm just in the resolution of the recurrence equation. Thus finding the power series expansion of a given hypergeometric type function f will be based on the search for the corresponding Puiseux number n from the RE, in order to look for the FLS $h(z) := f(z^n)$ and get the final result by substituting z by $z^{1/n}$ inside the expansion of h.

Let an FLS f with the representation (2.2.24), we have

$$f(z^{1/n}) = \sum_{k=k_0}^{\infty} a_k z^{k/n} = \sum_{\substack{k=(k_0+j)/n \\ j=0,1,2,\cdots}} a_{nk} z^k, \tag{6.1.20}$$

therefore the recurrence equation (6.0.1) can be seen as

$$a_{nk+m} = R(nk)a_{nk}, \ k \in \left\{0, \frac{1}{n}, \frac{2}{n}, \cdots\right\}. \tag{6.1.21}$$

So the presence of a factor n in front of the variable k in the factorization of R could show that we are in the case of LPS. This number is found by taking the least common multiple of the denominators of all rational zeros and poles of R from (6.0.1) (see [3] Sect.5).

6.2 Implementation in Maxima

This section is just to specify few commands used to success the implementation of the algorithm (4) in the CAS Maxima. We wrote a function REtoFPS(RE,k,z,f) to find the power series expansion of the holonomic function f in the variable z with a recurrence equation of hypergeometric type RE using the index variable k.

1. We wrote the function *finds_0(expr,z)* which modified an expression depending on z and return it with the value of s_0 as explain at the step (c) in the algorithm (4).

   (%i8) finds_0(z^3*cos(z),z);

$$[3, \cos(z)] \tag{\%o8}$$

(%i9) finds_0(exp(z)/z^4,z);

$$[-4, \%e^z] \quad (\%o9)$$

--> /* functions vanishing at 0*/
(%i10) finds_0(log(1+z)*z^2,z);

$$[2, \log(z+1)] \quad (\%o10)$$

(%i11) finds_0(sin(z)/z^3,z);

$$[-3, \sin(z)] \quad (\%o11)$$

2. For the simplification of Pochhammer symbols, we wrote a function *facgamsimp(R,k)* which make the simplification for a given rational function R in the variable k. This function use another function *transfone(r,k)* that we wrote for the case of single simplification (see part (b) in the algorithm of simplification of the general coefficient). That function gives according to the procedure described in the previous section the simplification (if possible) of $(r)_k$ where r is any constant.

(%i12) transfone(1/2,k);

$$\frac{(2k)!}{4^k k!} \quad (\%o12)$$

(%i13) facgamsimp((2*k+1)/(2*k+3),k);

$$\frac{1}{2k+1} \quad (\%o13)$$

3. Our main program for the computation of formal power series at $z_0 = 0$ is the function FPS(f,z,k) which generates if possible (RE of hypergeometric type) the power series expansion of a given expression f in the variable z around $z_0 = 0$ with the variable index k for the general term. This function use all the other functions and particularly it is the one which checks the Puiseux number and calls itself. Our program FPS is also able consider the effect of the function $\log(z)$ and adapt the output as explained in (6.1.1). On the other hand, there are four interesting cases for this function, let us assume that the input is (f, z, k):

   - if f is a polynomial in z, then there is no computation to be done and the output is *expand(f)*; nevertheless we notice that our implementation can found any polynomial which leads to an RE of hypergeometric type as the the work of this essay is set, we just avoid some useless computations.
   - if the number of summands on the left hand side of the RE is strictly greater than 2, then the output is false with a print of the RE;
   - the remaining case is the case of function where the RE is of hypergeometric type, this part of the program is done as follows

```
/*finding the roots of each summands*/
root1: solve(terms[1],k),
root2: solve(terms[2],k),
rootCoef: sublist(append(root1,root2),
     lambda([v], lhs(v)#recvar[1] and lhs(v)#recvar[2])),
```

```
/*denoinator of roots */
dnroot: map(lambda([v], denom(rhs(v))), rootCoef),
/*Puiseux number candidates*/
Puiseuxcan: sublist(dnroot, lambda([v], v#1)),
if(Puiseuxcan#[]) then Puiseuxnbr: Puiseuxcan[1]
else Puiseuxnbr:1,
/*to make sure that we are in the case of Puiseux series*/
checkf: taylor(f,z,0,Puiseuxnbr),
if(Puiseuxnbr=1) then REtoFPS(RE, k, z, f)
else (
    assume(z>0),
    h: ev(f,z=z^Puiseuxnbr),
    fls: hypfps(h,z,k),
    if(fls#false) then subst(z^(1/Puiseuxnbr),z,fls)
    else false
)
```

One may ask why the recursion with *hypfps* instead of FPS, indeed to extend the program with the case of explicit polynomial and linear combination of the power of $\log(z)$, we needed to add some special function which will work with the main part explained above. And in our implementation all are gathering in the main function FPS.

6.3 Results

Here we present some results obtained by using our function FPS(). Let us first recover all the power series presented in (2.1.11) with our program.

(%i1) FPS(exp(z),z,k);

$$\sum_{k=0}^{\infty} \frac{z^k}{k!} \tag{\%o1}$$

(%i2) FPS(cos(z),z,k);

$$\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \tag{\%o2}$$

(%i3) FPS(sin(z),z,k);

$$2\sum_{k=0}^{\infty} \frac{(k+1)\,(-1)^k z^{2k+1}}{(2\,(k+1))!} \tag{\%o3}$$

(%i4) FPS(cosh(z),z,k);

$$\sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} \tag{\%o4}$$

(%i5) FPS(sinh(z),z,k);

$$2\sum_{k=0}^{\infty} \frac{(k+1)\, z^{2k+1}}{(2\,(k+1))!} \tag{\%o5}$$

(%i7) declare(p,constant)$ FPS(1/(1-z)^p,z,k);

$$\sum_{k=0}^{\infty} \frac{(p)_k\, z^k}{k!} \tag{\%o7}$$

(%i8) FPS(log(1+z),z,k);

$$\sum_{k=0}^{\infty} \frac{(-1)^k\, z^{k+1}}{k+1} \tag{\%o8}$$

(%i9) FPS(atan(z),z,k);

$$\sum_{k=0}^{\infty} \frac{(-1)^k\, z^{2k+1}}{2k+1} \tag{\%o9}$$

(%i10) FPS(atanh(z),z,k);

$$\sum_{k=0}^{\infty} \frac{z^{2k+1}}{2k+1} \tag{\%o10}$$

(%i11) FPS(asin(z),z,k);

$$\sum_{k=0}^{\infty} \frac{(2k)!z^{2k+1}}{(2k+1)\; 4^k\, k!^2} \tag{\%o11}$$

(%i12) FPS(asinh(z),z,k);

$$\sum_{k=0}^{\infty} \frac{(-1)^k\,(2k)!z^{2k+1}}{(2k+1)\; 4^k\, k!^2} \tag{\%o12}$$

Some other examples, more can be found in $(B.3)$

(%i13) FPS(sin(sqrt(z))/z,z,k);

$$2\sum_{k=0}^{\infty} \frac{(k+1)\;(-1)^k\, z^{\frac{2k-1}{2}}}{(2\,(k+1))!} \tag{\%o13}$$

(%i14) FPS(z^2*atan(z),z,k);

$$\sum_{k=0}^{\infty} \frac{(-1)^k\, z^{2k+3}}{2k+1} \tag{\%o14}$$

(%i15) FPS(log(z)*sin(z)+exp(z^(1/3)),z,k);

$$2\log(z)\left(\sum_{k=0}^{\infty} \frac{(k+1)\;(-1)^k\, z^{2k+1}}{(2\,(k+1))!}\right) + \sum_{k=0}^{\infty} \frac{z^{\frac{k}{3}}}{k!} \tag{\%o15}$$

(%i16) FPS(erf(z^2),z,k);

$$\frac{2\sum_{k=0}^{\infty} \frac{(-1)^k\, z^{4k+2}}{(2k+1)k!}}{\sqrt{\pi}} \tag{\%o16}$$

7. Conclusion

We have reached the end of our work, our goal was to produce a Maxima program able to compute a power series expansion for any Laurent-Puiseux series of hypergeometric type. We have seen that functions of hypergeometric type cover a wide family of analytic functions and in order to find the power series expansion three main steps have to be considered: Finding a Holonomic Differential Equation, this step consists of the search for a linear homogeneous differential equation with polynomial coefficients fulfilled by the given analytic expression and as explained in [6] our implementation is able to find such a differential equation, mostly of lowest order. Moreover, our implementation approach is efficient enough to find a differential equation of order 14 corresponding to $\sin(z)^6 \arcsin(z)$ in about 16 seconds on a HP Intel core i5 (see (B.1)). The second step is Getting the Recurrence Equation consisting in the conversion of the DE obtained in the first step in terms of a recurrence equation for the sought Taylor coefficients, this step is based on (3.1.1). The last step is Solving the Recurrence Equation, this is where our implementation is restricted compared to Koepf's complete algorithm. Indeed our implementation is not taking into account recurrence equations with more than two summands. Nevertheless as the main case to be considered, the implementation for REs of hypergeometric type is enough to cover a very large family of functions and moreover our implementation is able to cover other cases of expressions close to cases of functions of hypergeometric type; for instance linear combinations of powers of $\log(z)$ with hypergeometric type functions as coefficients.

Thus we wrote a program named FPS(f, z, k) which computes the power series expansion at $z_0 = 0$ of a given analytic expression f of hypergeometric type depending on the variable z, and with k as index variable for the general coefficient. According to the steps of the algorithm, this program uses three other programs: HolonomicDE($f, F(z)$) which allows to find the holonomic differential equation of mostly (see (C)) lowest order of a given expression f depending on the variable the variable z which is the argument of F (in Maxima *z=first(F)*). The second program is DEtoRE($DE, F(z), a[k]$) which converts the holonomic differential equation DE to its corresponding recurrence equation where $a[k]$ will denote the general coefficient term. And the program REtoFPS(RE, z, k, f) which finds the general coefficient and gives the power series expansion looked for by solving the recurrence equation RE. We remind that the power series expansion at any $z_0 \in \mathbb{C}$ is obtained just by substituting z by $z - z_0$.

On the other hand, the general case of rational functions can not be treated by our implementation, indeed we always find a holonomic DE of order 1 but in most cases the polynomial numerator and denominator lead to an RE with more than two summands. However this case can be easily treated by using partial fraction decomposition in order to have a sum of hypergeometric type functions like (2.1.6) as explained in [6]. Another special case is the one which corresponds to REs so called *exp-like* type where all coefficients in the RE obtained are constant, this is solved using linear algebra with the first initial coefficients fulfilling the RE. However the most important project which could follow this work could be the combination of the algorithm implemented with the Petkovsek-van-Hoeij algorithm to compute hypergeometric term solutions of the holonomic recurrence equations, see [7], Chapter 9, and its implementation in Maxima. And this is in order to find a complete algorithmic approach for computing power series expansion of hypergeometric type functions.

Appendix A. Effect of Polynomial Solutions of a Holonomic DE

We want to point out that the result is not found just by taking a linear combination of hypergeometric type functions as stated in (2.2.26). In fact if we assume that f is a holonomic function with a DE

$$\sum_{j=0}^{Q} P_j f^{(j)} = 0, \tag{A.0.1}$$

P_j are polynomials, and t is a polynomial such that $f + t$ has the DE (A.0.1), then we have

$$0 = \sum_{j=0}^{Q} P_j (f+t)^{(j)} = \sum_{j=0}^{Q} P_j \left(f^{(j)} + t^{(j)}\right) = \sum_{j=0}^{Q} P_j t^{(j)},$$

therefore we can state that if there exist a polynomial t satisfying the same holonomic DE with f then $f+t$ is also holonomic with the same DE. The duty of this consideration is to draw attention to the fact that mostly the recurrence equation obtained from the DE of $f+t$ leads to the correct values of the coefficients just for the $(n+1)$ first terms, where n is the degree of t.

(%i1) declare(C,constant)$ HolonomicDE(C+exp(z),F(z));

$$\frac{d^2}{dz^2}\mathrm{F}(z) - \frac{d}{dz}\mathrm{F}(z) = 0 \tag{\%o1}$$

(%i2) HolonomicDE(1+exp(z),F(z));

$$\frac{d^2}{dz^2}\mathrm{F}(z) - \frac{d}{dz}\mathrm{F}(z) = 0 \tag{\%o2}$$

(%i3) HolonomicDE(exp(asinh(z)),F(z));

$$(z^2+1)\left(\frac{d^2}{dz^2}\mathrm{F}(z)\right) + z\left(\frac{d}{dz}\mathrm{F}(z)\right) - \mathrm{F}(z) = 0 \tag{\%o3}$$

(%i4) HolonomicDE(exp(asinh(z))-z,F(z));

$$(z^2+1)\left(\frac{d^2}{dz^2}\mathrm{F}(z)\right) + z\left(\frac{d}{dz}\mathrm{F}(z)\right) - \mathrm{F}(z) = 0 \tag{\%o4}$$

The RE obtained with $C + \exp(z)$ $(t = C \in \mathbb{C})$ leads to the same result as for $\exp(z)$, so the RE does not really take into account the constant C, thus if we applied directly (2.2.26) then the FPS will be

$$(C+1)\sum_{k=0}^{\infty} \frac{z^k}{k!} \tag{A.0.2}$$

which is wrong, just the first value is true ($t = C$ has the degree 0), the correct FPS is just

$$C + \sum_{k=0}^{\infty} \frac{z^k}{k!}. \tag{A.0.3}$$

Therefore we need to consider the cases of presence of an explicit polynomial summand that we have to extract, and find the FPS of the non-polynomial part (of course in the case of hypergeometric type functions) and return the sum of the FPS and the polynomial extracted.

The second case is the same situation with $f(z) := \exp(\operatorname{arcsinh}(z))$, one may say that this function is connected to another hypergeometric type function g so that $f = z + g$, this g is known and it is $\sqrt{(1+z^2)}$. (2.2.26) gives a summation of two hypergeometric type functions, one of them is exactly the power series expansion of $\sqrt{1+z^2}$ but the second one related to the polynomial $t = z$ is just correct for the computation of its first value z while the others values should not occur.

For a given hypergeometric function f with a symmetry number m, we suspect that an implicit polynomial can appear in its power series expansion only if the degree of this polynomial is less than m. Thus as the resolution of the recurrence equation gives the general coefficients for each hypergeometric summand, each monomial of the implicit polynomial can be detected just by checking if its corresponding next term exist in the Taylor expansion of order $2m-1$ (assuming $k_0 = s_0 = 0$ see (6.1.5) (6.1.3)) of f.

A.0.1 Example. The polynomial z is detected in the power series expansion of $f(z) := \exp(\operatorname{arcsinh}(z))$ by the fact that z^3 does not appear in the Taylor expansion of order 3 of f. Hence we found the right result

(%i5) FPS(exp(asinh(z)),z,k);

$$z - 2\sum_{k=0}^{\infty} \frac{(2\,(k-1))!(-1)^k\,z^{2k}}{(k-1)!^2 k\,4^k} \qquad \text{(\%o5)}$$

Appendix B. Examples

As the work of this essay should provide several examples, this appendix is made according to that purpose.

B.1 Holonomic DE Examples

(%i2) batchload(FPS)$
(%i5) declare(alpha,constant)$ f:(1+sqrt(1+z))^alpha$ HolonomicDE(f,F(z));

$$\left(4z^2+4z\right)\left(\frac{d^2}{dz^2}\,\mathrm{F}(z)\right)+\left(\left(6-4\alpha\right)z-4\alpha+4\right)\left(\frac{d}{dz}\,\mathrm{F}(z)\right)+\left(\alpha^2-\alpha\right)\,\mathrm{F}(z)=0 \qquad (\%o5)$$

(%i8) declare(p,constant)$ f:1/(1-z)^p$ HolonomicDE(f,F(z));

$$\left(z-1\right)\left(\frac{d}{dz}\,\mathrm{F}(z)\right)+p\,\mathrm{F}(z)=0 \qquad (\%o8)$$

(%i10) f: cos(z)^2+sin(z)^2$ HolonomicDE(f,F(z));

$$\frac{d}{dz}\,\mathrm{F}(z)=0 \qquad (\%o10)$$

(%i12) f: sqrt(1+z) + 1/sqrt(1+z)$ HolonomicDE(f,F(z));

$$\left(2z^2+6z+4\right)\left(\frac{d}{dz}\,\mathrm{F}(z)\right)-z\,\mathrm{F}(z)=0 \qquad (\%o12)$$

(%i14) f: atan(z)+sin(2*z)$ HolonomicDE(f,F(z));

$$\left(2z^6+9z^4+8z^2+1\right)\left(\frac{d^4}{dz^4}\,\mathrm{F}(z)\right)+\left(4z^5+20z^3-8z\right)\left(\frac{d^3}{dz^3}\,\mathrm{F}(z)\right)$$
$$+\left(8z^6+36z^4+32z^2+4\right)\left(\frac{d^2}{dz^2}\,\mathrm{F}(z)\right)+\left(16z^5+80z^3-32z\right)\left(\frac{d}{dz}\,\mathrm{F}(z)\right)=0 \qquad (\%o14)$$

--> /*DE of order 14*/;
(%i2) showtime: true;

Evaluation took 0.0000 seconds (0.0000 elapsed)

true (showtime)

(%i5) Nmax:15$ f:sin(z)^6*asin(z)$ BigDE: HolonomicDE(f,F(z))$

Evaluation took 0.0000 seconds (0.0000 elapsed)
Evaluation took 0.0000 seconds (0.0000 elapsed)
Evaluation took 15.2400 seconds (15.9100 elapsed)

B.2 DEtoRE Examples

(%i1) batchload(FPS)$
(%i3) f:log(1+z)$ FindRE(f,z,a[k]);

$$(k+1)\,(k+2)\,a_{k+2}+(k+1)^2\,a_{k+1}=0 \tag{\%o3}$$

(%i6) declare(alpha,constant)$ f:(1+sqrt(1+z))^alpha$ FindRE(f,z,a[k]);

$$4\,(k+1)\,(k-\alpha+1)\,a_{k+1}+(2k-\alpha)\,(2k-\alpha+1)\,a_k=0 \tag{\%o6}$$

(%i8) f:sin(z)+cos(z)$ FindRE(f,z,a[k]);

$$(k+1)\,(k+2)\,a_{k+2}+a_k=0 \tag{\%o8}$$

(%i10) f:exp(z)$ FindRE(f,z,a[k]);

$$(k+1)\,a_{k+1}-a_k=0 \tag{\%o10}$$

(%i12) f:z*log(1+z)$ FindRE(f,z,a[k]);

$$(k-2)\,(k-1)\,a_k+a_{k-1}\,(k-2)^2=0 \tag{\%o12}$$

(%i14) f:sin(sqrt(z))/z^2$ FindRE(f,z,a[k]);

$$2\,(k+2)\,(2k+3)\,a_k+a_{k-1}=0 \tag{\%o14}$$

B.3 FPS Examples

Some of the following results can be checked from [3] ,[5] or [6].

(%i1) FPS(cos(z)^2+sin(z)^2,z,k);

$$1 \tag{\%o1}$$

(%i2) FPS(cos(asin(z))-sqrt(1-z^2),z,k);

$$0 \tag{\%o2}$$

(%i3) FPS(atan((2*z^2-1)/(2*z^2+1)),z,k);

$$2\left(\sum_{k=0}^{\infty}\frac{(-4)^k\,z^{4k+2}}{2k+1}\right)-\frac{\pi}{4} \tag{\%o3}$$

(%i4) FPS(exp(asinh(z)),z,k);

$$z-2\sum_{k=0}^{\infty}\frac{(2\,(k-1))!(-1)^k\,z^{2k}}{(k-1)!^2 k\,4^k} \tag{\%o4}$$

(%i5) assume(z>0)$ FPS(atan(1/z),z,k);

$$\frac{\pi}{2}-\sum_{k=0}^{\infty}\frac{(-1)^k\,z^{2k+1}}{2k+1} \tag{\%o5}$$

(%i6) f:integrate(erf(t),t,0,z)$FPS(f,z,k);

$$-\frac{\sum_{k=0}^{\infty} \frac{(-1)^k\, z^{2k}}{(2k-1)k!}}{\sqrt{\pi}} - \frac{1}{\sqrt{\pi}} \tag{\%o6}$$

(%i7) FPS(asin(z)^2/z^2,z,k);

$$2\sum_{k=0}^{\infty} \frac{4^k\, k!^2\, z^{2k}}{(2\,(k+1))!} \tag{\%o7}$$

(%i8) declare(alpha,constant)$ FPS(z^3*(cos(alpha*z)+sin(alpha*z)),z,k);

$$2\left(\sum_{k=0}^{\infty} \frac{(k+1)\,(-1)^k\, \alpha^{2k+1}\, z^{2k+4}}{(2\,(k+1))!}\right) + \sum_{k=0}^{\infty} \frac{(-1)^k\, \alpha^{2k}\, z^{2k+3}}{(2k)!} \tag{\%o8}$$

(%i9) FPS(z^2*(cos(sqrt(z))+sin(sqrt(z))),z,k);

$$2\left(\sum_{k=0}^{\infty} \frac{(k+1)\,(-1)^k\, z^{\frac{2k+5}{2}}}{(2\,(k+1))!}\right) + \sum_{k=0}^{\infty} \frac{(-1)^k\, z^{\frac{2k+4}{2}}}{(2k)!} \tag{\%o9}$$

(%i10) FPS((1+sqrt(1+z))^alpha,z,k);

$$2^{\alpha} \sum_{k=0}^{\infty} \frac{\left(-\frac{\alpha-1}{2}\right)_k \left(-\frac{\alpha}{2}\right)_k (-1)^k\, z^k}{(1-\alpha)_k\, k!} \tag{\%o10}$$

(%i11) /*Chebychev polynomial*/ FPS(cos(8*acos(z)),z,k);

$$128z^8 - 256z^6 + 160z^4 - 32z^2 + 1 \tag{\%o11}$$

(%i12) FPS(exp(z)*(1-erf(sqrt(z))),z,k);

$$\left(\sum_{k=0}^{\infty} \frac{z^k}{k!}\right) - \frac{\sum_{k=0}^{\infty} \frac{(k+1)\,4^{k+1} k! z^{\frac{2k+1}{2}}}{(2(k+1))!}}{\sqrt{\pi}} \tag{\%o12}$$

(%i13) FPS(z*atan(z)-log(1+z^2)/2,z,k);

$$\frac{\sum_{k=0}^{\infty} \frac{(-1)^k\, z^{2k+2}}{(k+1)\,(2k+1)}}{2} \tag{\%o13}$$

(%i14) FPS(exp(z)-2*exp(-z/2)*cos(sqrt(3)*z/2 + %pi/3),z,k);

$$3\sum_{k=0}^{\infty} \frac{z^{3k+1}}{\left(\frac{2}{3}\right)_k \left(\frac{4}{3}\right)_k 27^k k!} \tag{\%o14}$$

(%i15) FPS(log((1+z)/(1-z))/2 - atan(z),z,k);

$$2\sum_{k=0}^{\infty} \frac{z^{4k+3}}{4k+3} \tag{\%o15}$$

(%i16) FPS(1/(1-z^(1/3)),z,k);

$$\sum_{k=0}^{\infty} z^{\frac{k}{3}} \tag{\%o16}$$

(%i17) f:diff(asech(z),z)$ FPS(f,z,k);

$$-\sum_{k=0}^{\infty} \frac{(2k)! z^{2k-1}}{4^k\, k!^2} \tag{\%o17}$$

Appendix C. Some Limitations

Our experience in this work made us discover some limitations. Indeed in (6.1.16) we have seen that our program (Koepf's algorithm in fact) can be used as a simplifier, however this result also shows that HolonomicDE fails to find the holonomic differential equation of lowest order for the Chebychev polynomials $f(z) := \cos(n \arccos(z)), n \in \mathbb{N}$, the correct order of the DE should be 1.

(%i2) declare(n,constant)$ HolonomicDE(cos(n*acos(z)),F(z));

$$\left(z^2-1\right)\left(\frac{d^2}{dz^2}\,\mathrm{F}(z)\right)+z\left(\frac{d}{dz}\,\mathrm{F}(z)\right)-n^2\,\mathrm{F}(z)=0 \tag{\%o2}$$

But this is the best the algorithm can do as long as now further knowledge is given. Moreover, the algorithm will find its polynomial representation at the end and is therefore a simplifier. Looking forward for other simplifications, especially for functions equivalent to the zero function, because of the lack of algebraic simplifications, if the algorithm is not able to recognize an expression equivalent to the zero function, then it may happen that either the search for the differential equation or the Laurent factor z^{s_0} in (6.1.5) is not successful. The following examples present these issues, in our program we defined a variable to fix the maximum power for the Laurent factor in order to avoid an infinite loop.

(%i2) f1:sqrt((1-sqrt(1-z))/z) - (sqrt(1+sqrt(z))-sqrt(1-sqrt(z)))/(sqrt(2*z));
f2:1+tan(z)^2-1/cos(z)^2;

$$\frac{\sqrt{1-\sqrt{1-z}}}{\sqrt{z}}-\frac{\sqrt{\sqrt{z}+1}-\sqrt{1-\sqrt{z}}}{\sqrt{2}\,\sqrt{z}} \tag{f1}$$

$$\tan{(z)}^2-\frac{1}{\cos{(z)}^2}+1 \tag{f2}$$

Differential equations found

(%i4) HolonomicDE(f1,F(z)); HolonomicDE(f2,F(z));

$$\left(16z^2-16z\right)\left(\frac{d^2}{dz^2}\,\mathrm{F}(z)\right)+(32z-24)\left(\frac{d}{dz}\,\mathrm{F}(z)\right)+3\,\mathrm{F}(z)=0 \tag{\%o3}$$

$$\text{false} \tag{\%o4}$$

Looking for the corresponding formal power series expansion of the case ($\%o3$) (since the other one is going to give false as we can see above) we obtain

(%i5) FPS(f1,z,k);
Infinite computations for the Laurent factor suspected

which is our output when the maximum Laurent factor is reached. We think that this problem could be considered in an extension of this work as a nice subproblem!

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