Concatenating Arithmetic Progressions

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Introduction

Let $(u(n))_{n\in\mathbb{N}} := (u(0) + d\,n)_{n\in\mathbb{N}}$ be an integral arithmetic progression of common difference $d \in \mathbb{N}\setminus\{0\}$. By $\overline{(.)}^b$, we mean the concatenation in base $b \geq 2$. We consider $(s(n))_{n\in\mathbb{N}}$, $(s_r(n))_{n\in\mathbb{N}}$, and $(s_*(n))_{n\in\mathbb{N}}$, $s_*(0) := u(0)$, such that:

1. $s(n) := \overline{u(0)u(1)\cdots u(n)}^b$;
2. $s_r(n) := \overline{u(n)u(n-1)\cdots u(0)}^b$;
3. $s_*(n) := \overline{s(n)s_r(n-1)}^b$, $n \geq 1$.

Special cases of these sequences include OEIS A007908, OEIS A000422, and OEIS A173426. This poster reports results from [3].

Guessed and Proved Recurrences

Let l be a positive integer. We call l-digit concatenations in $(s(n))_n$, $(s_r(n))_n$ or $(s_*(n))_n$, all their terms obtained by concatenating l digits in their precedents. For instance, in the decimal base, the terms 1, 12, 123, …, 123456789 correspond to 1-digit concatenations in A007908. We used the guess-and-prove paradigm with the GFUN package [4] (command `listtorec`) to find and prove that these sequences satisfy holonomic recurrence equations for every fixed-length concatenation.

Proposition 1. *Let l be a positive integer and $b \geq 5$ a natural number base. The recurrence equations (1), (2), and (3) are satisfied by all terms of l-digit concatenations in $(s(n))_n$, $(s_r(n))_n$ and $(s_*(n))_n$, respectively.*

$$a(n+3) - (b^l+2)\cdot a(n+2) + (2\cdot b^l+1)\cdot a(n+1) - b^l\cdot a(n) = 0. \tag{1}$$

$$a(n+3) - (2\cdot b^l+1)\cdot a(n+2) + (b^{2l}+2\cdot b^l)\cdot a(n+1) - b^{2l}\cdot a(n) = 0. \tag{2}$$

$$a(n+3) - \left(1+b^l+b^{2l}\right)\cdot a(n+2) + \left(b^l+b^{2l}+b^{3l}\right)\cdot a(n+1) - b^{3l}\cdot a(n) = 0. \tag{3}$$

We solve these recurrence equations using `LREtools:-mhypergeomsols` to obtain formulas for these sequences in every intervals $\{t_l, t_l+1, \ldots, t_{l+1}\}$, $t_l = \lceil (b^{l-1} - s(0))/d \rceil$.

Case of $(s(n))_{n\in\mathbb{N}}$

Theorem 1. *The general term of $(s(n))_n$, $b \geq 5$, can be computed as follows:*

$$s(n) = \alpha_l + \mu_l(n - t_l) + \theta_l\, b^{l(n-t_l)},$$

$$l = \lceil \log_b(n\,d + s(0) + 1)\rceil,\ t_l = \left\lceil \frac{b^{l-1} - s(0)}{d} \right\rceil,$$

$$\alpha_l = -\frac{\left(b^l - 1\right)\cdot u(t_l) + d\cdot b^l}{\left(b^l-1\right)^2},$$

$$\mu_l = -\frac{d}{b^l - 1},\ \theta_l = \frac{\kappa_2 - 2\cdot\kappa_1 + \kappa_0}{\left(b^l-1\right)^2},$$

$$\kappa_0 = s(t_l),\ \ \kappa_1 = s(t_l+1),\ \ \kappa_2 = s(t_l+2).$$

See also an equivalent formula in [1]. Theorem 1 applies to A007908, A019519, and A019520.

Case of $(s_r(n))_{n\in\mathbb{N}}$

Theorem 2. *$(s_r(n))_n$, $b \geq 5$, can be computed as follows (l, t_l as in Theorem 1 with $s_r(0)$):*

$$s_r(n) = \alpha_l + \mu_l\cdot b^{l(n-t_l)} + \theta_l\cdot(n-t_l)\cdot b^{l(n-t_l)}$$

$\nu_l \equiv$ digit length of $s_r(t_l)$,

$$\alpha_l = \frac{\kappa_2 - 2\cdot b^l\cdot\kappa_1 + b^{2l}\cdot\kappa_0}{(b^l-1)^2},$$

$$\mu_l = \frac{\left(\left(b^l-1\right)\cdot u(t_l) - d\right)\cdot b^{\nu_l}}{\left(b^l-1\right)^2},$$

$$\theta_l = \frac{d\cdot b^{\nu_l}}{b^l - 1},$$

$$\kappa_0 = s_r(t_l),\ \ \kappa_1 = s_r(t_l+1),\ \ \kappa_2 = s_r(t_l+2).$$

Theorem 2 applies to A000422, A038395, and A038396.

Case of $(s_*(n))_{n\in\mathbb{N}}$

Theorem 3. *$(s_*(n))_n$, $b \geq 5$ can be computed as follows (l, t_l as in Theorem 1 with $s_*(0)$):*

$$s_*(n) = \alpha_l + \mu_l\cdot b^{l(n-t_l)} + \theta_l\cdot b^{2l(n-t_l)},$$

$$\alpha_l = \frac{b^{3l}\cdot\kappa_0 - b^l\cdot\left(b^l+1\right)\cdot\kappa_1 + \cdot\kappa_2}{\left(b^l+1\right)\cdot\left(b^l-1\right)^2},$$

$$\mu_l = -\frac{b^{2l}\cdot\kappa_0 - \left(b^{2l}+1\right)\cdot\kappa_1 + \kappa_2}{b^l\cdot\left(b^l-1\right)^2},$$

$$\theta_l = \frac{b^l\cdot\kappa_0 - \left(b^l+1\right)\cdot\kappa_1 + \kappa_2}{b^l\cdot\left(b^l+1\right)\cdot\left(b^l-1\right)^2},$$

$$\kappa_0 = s_*(t_l),\ \ \kappa_1 = s_*(t_l+1),\ \ \kappa_2 = s_*(t_l+2).$$

Theorem 3 leads to an explicit formula for computing terms of A173426.

Asymptotic Computations

We implemented Theorem 1 for A007908 as `Smarandache:-Sm`. We compare existing Maple codes for computing terms of A007908 with our implementation. The best performance for naive approaches is given by the code `a007908` (see [3]).
Table 1 shows that our implementation `Smarandache:-Sm` is faster than `a007908` for asymptotic computations.

Table 1: `Smarandache:-Sm` vs `a007908`

l	5	6	7	8
CPUTime(Smarandache:-Sm(10^l-1))	0.046	0.125	1.766	31.532
CPUTime(a007908(10^l-1))	0.079	0.719	10.969	208.391

We obtain similar results for A000422 with our implementation `Smarandache:-Smr` as illustrated in Table 2 below.

Table 2: `Smarandache:-Smr` vs `a000422`

l	5	6	7	8
CPUTime(Smarandache:-Smr(10^l-1))	0.016	0.516	7.313	123.657
CPUTime(a000422(10^l-1))	0.047	1.047	12.921	215.765

Non-Holonomic Character

An annihilator is a linear operator defined by a holonomic recurrence equation.
Observe that the formulas for l-digit concatenations in $(s(n))_n$ and $(s_r(n))_n$ may be seen as linear combinations of two linearly independent hypergeometric terms. This shows that the minimal annihilating operators for fixed-length concatenations in $(s(n))_n$ and $(s_r(n))_n$ are second-order annihilators. For $(s_*(n))_n$, the minimal annihilating operator is the one defined by (3).

Theorem 4 (Theorem 4 in [3]). *The sequences $(s(n))_n, (s_r(n))_n$, and $(s_*(n))_n$ are not holonomic.*

Proof. Sketch of the proof: Let $(a(n))_n$ be any of the sequences $(s(n))_n$, $(s_r(n))_n$, and $(s_*(n))_n$, and q_l be their minimal annihilator for l-digit concatenations, where l is sufficiently large. The proof relies on the fact that the existence of a linear operator p that annihilates $a(n)$ for all n would imply that $\gcd(p, q_l)$ annihilates $a(n)$ for l-digit concatenations; a contradiction because $\gcd(p, q_l)$ is of lower order than q_l. For further details, see [3, Theorem 4], and the background in [2]. □

References

[1] Max A Alekseyev, Joseph Samuel Myers, Richard Schroeppel, Scott R Shannon, Neil James Alexander Sloane, and Paul Zimmermann. Three cousins of Recamán's sequence. *The Fibonacci Quarterly*, 60(3):201–219, 2022.

[2] Manuel Bronstein and Marko Petkovšek. An introduction to pseudo-linear algebra. *Theoretical Computer Science*, 157(1):3–33, 1996.

[3] Florian Luca and Bertrand Teguia Tabuguia. Concatenations of terms of an arithmetic progression. *https://arxiv.org/abs/2201.07127v2*, May 2024.

[4] Bruno Salvy and Paul Zimmermann. GFUN: A Maple package for the manipulation of generating and holonomic functions in one variable. *ACM Transactions on Mathematical Software*, 20(2):163–177, 1994.