Guessing and arithmetic of D-algebraic sequences

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A sequence is difference-algebraic (or D-algebraic) if finitely many shifts of its general term satisfy a polynomial relationship. We refer to their equations as algebraic difference equations (ADEs). A key motivation for considering nonlinear polynomial equations for sequences is to enable broader closure properties for their symbolic computations. It is well-known that reciprocals and ratios of D-finite sequences are "almost never" D-finite [5, Chapter 4], [2]. We recently proved that any D-finite recurrence can be converted into a non-trivial D-algebraic rational recursion (see (1)) using linear algebra [8, 6].[1]

This talk focuses on arithmetic operations of D-algebraic sequences, building upon ideas presented in [1, 7]. We aim to present a theoretical framework outlining the necessary hypotheses for constructing ADEs satisfied by sums, products, divisions, and various other operations with D-algebraic sequences. This framework primarily serves to establish the theoretical foundations for these operations, as it relies on computationally intensive elimination with Gröbner bases and is therefore not intended for practical use with generic sequences.

Consider, for instance, the sequence of general term $s_n = \frac{F_n}{C_n}$, where $(F_n)_{n\in\mathbb{N}}$ is the Fibonacci sequence ($F_0 = 0, F_1 = 1, F_{n+2} = F_{n+1} + F_n,\ n \geq 0$), and $(C_n)_{n\in\mathbb{N}}$ is the Catalan sequence ($C_0 = 1, (n+2)\,C_{n+1} = (4n+2)\,C_n,\ n \geq 0$). Using the algorithm from [8], we get the following D-algebraic representation for $(C_n)_{n\in\mathbb{N}}$.

$$C_{n+2} = \frac{2C_{n+1}\,(8C_n + C_{n+1})}{10C_n - C_{n+1}},\ C_0 = 1,\ C_1 = 1. \tag{1}$$

With this equation, we could employ the Gröbner bases framework to compute an equation satisfied by s_n. Unfortunately for this particular example, these computations did not complete even after an hour on our working computer.

An alternative approach is the guess-and-proof paradigm, which aims to construct the desired equations from the initial terms of $(s_n)_{n\in\mathbb{N}}$. In this case, the correctness is readily verifiable using the closed forms of F_n and C_n. This method yields a successful result within a second, despite our somewhat 'straightforward' implementation. We obtain a third-order ADE of (total) degree 4.

[1] The next version of [6] is being updated with the complete proof.

$$\begin{aligned}&2240s_{n+2}^3s_{n+1} - 140s_{n+2}^3s_n + 1176s_{n+2}^2s_{n+1}^2 + 52s_{n+2}^2s_{n+1}s_n - 6912s_{n+3}s_{n+2}^2s_{n+1}\\ &+ 2s_{n+2}^2s_n^2 + 544s_{n+3}s_{n+2}^2s_n - 140s_{n+2}s_{n+1}^3 - 27s_{n+2}s_{n+1}^2s_n - 832s_{n+3}s_{n+2}s_{n+1}^2\\ &- s_{n+2}s_{n+1}s_n^2 - 332s_{n+3}s_{n+2}s_{n+1}s_n + 4096s_{n+3}^2s_{n+2}s_{n+1} + 2s_{n+3}s_{n+2}s_n^2 - 512s_{n+3}^2s_{n+2}s_n\\ &- 140s_{n+3}s_{n+1}^3 - 34s_{n+3}s_{n+1}^2s_n + 512s_{n+3}^2s_{n+1}^2 - 2s_{n+3}s_{n+1}s_n^2 - 32s_{n+3}^2s_{n+1}s_n - 4s_{n+3}^2s_n^2 = 0.\end{aligned} \tag{2}$$

We will detail the underlying method. This is our first step toward finding more effective algorithms for the arithmetic of D-algebraic sequences. Similar approaches can be found in [3]. Future developments could exploit the more advanced guessing techniques described in [4].

Keywords: Algebraic difference equation, D-algebraic guessing, elimination with Gröbner bases

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