FPS For Interlaced Hypergeometric Sequences

Bertrand Teguia Tabuguia[a], Wolfram Koepf[b], ISSAC'22

[a] Max Planck Institute Leipzig (teguia@mis.mpg.de), [b] University of Kassel (koepf@mathematik.uni-kassel.de)

Introduction

The acronym FPS stands for Formal Power Series. It refers to an algorithm for computing coefficients of univariate hypergeometric type functions in a closed-form. The algorithm (see [3, 4]) is implemented in Maple (`convert/FormalPowerSeries` in Maple 2022) and Maxima, and the corresponding packages are accessible from http://www.mathematik.uni-kassel.de/~bteguia/FPS_webpage/FPS.htm. We present how combining FPS with an algorithm for guessing univariate holonomic functions enables one to find the explicit formula of a full sequence from its truncation.

Method

- Let $(a_n)_{n\in\mathbb{N}}$ be *C-finite*. There exists a rational function f such that $f(x)=\sum_{n=0}^{\infty} a_n\, x^n$.
- One can recover an explicit representation of f as a rational function from a truncation $\sum_{n=0}^{N} a_n\, x^n$, $N\in\mathbb{N}$, by means of Padé approximants.
- For sequences of rational numbers (therefore integers), one can use `convert/ratpoly` or `Gfun:-listtoratpoly` (see [1]).

Proposition 1 (Application of Theorem 13 in [4])**.** *For a rational function $f(x)$ whose denominator roots can be computed explicitly, FPS finds the following formula with **interlaced hypergeometric term coefficients***

$$f(x)=\sum_{i=1}^{M}\left(\sum_{n=0}^{\infty} a_{m_i\,n}\, x^{m_i\,n}+\cdots+\sum_{n=0}^{\infty} a_{m_i\,n+m_i-1}x^{m_i\,n+m_i-1}\right), \tag{1}$$

$0<m_i\in\mathbb{N}, 1\le i\le M\in\mathbb{N}$, *and thus gives an explicit formula for a_n, $n\in\mathbb{N}$.*

What FPS does

Given a *D*-finite function f,

1. compute a holonomic DE satisfied by $f(x)$;
2. convert the holonomic DE into a holonomic RE;
3. use `mfoldHyper` (see [4]) to compute a basis of all interlaced hypergeometric term solutions of the obtained RE;
4. use linear algebra and some initial coefficients to compute the linear combination that corresponds to the power series of $f(x)$.

Sequence 1: A307717

Let us consider the sequence **A307717** from `https://oeis.org/A307717`. We denote it by $(a_n)_{n\in\mathbb{N}}$. a_n counts the number of palindromic squares, k^2, of length $n+1$ such that k is also palindromic. For instance, there are only two palindromic squares of length 3 whose root is also palindromic. $11^2=121$ and $22^2=484$. Thus, $a_2=2$. We use the 33 first terms of the sequence to guess its ordinary generating function $f_a(x)$ given by

$$f_a(x)=-\frac{2x^{16}-x^{14}-5x^{12}+5x^{10}+12x^8-5x^6-11x^4+2x^2+4}{-x^{16}+4x^{12}-6x^8+4x^4-1}. \tag{2}$$

Sequence 2: A226782

Now let $(b_n)_{n\in\mathbb{N}}$ be **A226782** from `https://oeis.org/A226782`. $b_n=0$ if $n+1$ is even, and the inverse of 4 in the ring $\mathbb{Z}/(n+1)\mathbb{Z}^*$ if $n+1$ is odd. The first few terms $0,0,1,0,4,0,2,0,7,0,3,0,10,0,4,0,13$ suffice to determine its generating function

$$f_b(x)=-\frac{-x^8+4x^4+x^2}{-x^8+2x^4-1}. \tag{3}$$

Like for **A307717**, **A226782** has a zero subsequence at odd indices (we start sequences at index 0). These type of sequences are particularly addressed in [2]. Our method overcomes the presence of zeros in the present situations.

Results: Compute FPS(f,x,n) ($f=f_a$, and $f=f_b$) in Maple or Maxima

Theorem 2. *The sequence $(a_n)_{n\in\mathbb{N}}$ representing **A307717** from OEIS has the explicit formula*

$$a_0=4$$

$$a_{2n+1}=0, n\ge 0 \tag{4}$$

$$a_{2n}=\frac{((-1)^n+3)\,n^3-(9\,(-1)^n+15)\,n^2+(65\,(-1)^n+111)\,n+21\,(-1)^n+171}{96}, n\ge 1 \tag{5}$$

Theorem 3. *The sequence $(b_n)_{n\in\mathbb{N}}$ representing **A226782** from OEIS has the explicit formula*

$$b_0=0$$

$$b_{2n+1}=0, n\ge 0 \tag{6}$$

$$b_{2n}=\frac{(-1)^n\,n}{2}+\frac{(-1)^n}{4}+n+\frac{3}{4}, n\ge 1 \tag{7}$$

Conclusion

- We recommend our FPS implementation to the community of computer algebraists or scientists interested in finding explicit formulas for (integer) sequences. Combining it with guessing strategies enables one to discover or recover explicit formulas. Of course, questions and comments are welcome.
- It is not guaranteed that FPS always gives the best possible formula in the sense of compactness. Indeed, many linear combinations can be valid in its last step. We choose from the set of solutions to a linear system encoding the initial conditions. Currently, FPS prioritizes solutions involving only rational numbers and thus avoiding algebraic extensions.
- More on Guessing: for non-*P*-recursive (therefore non-hypergeometric type) sequences, it is interesting to think of how to find their recursions automatically from finitely many of their first terms. This research direction is a current investigation of the first author, and preliminary results appear at this same ISSAC'22 as a software demonstration.

References

[1] Stanley Cabay and Dong-Koo Choi. Algebraic computations of scaled padé fractions. *SIAM Journal on Computing*, 15(1):243–270, 1986.

[2] Manuel Kauers and Thibaut Verron. Why you should remove zeros from data before guessing. *ACM Communications in Computer Algebra*, 53(3):126–129, 2019.

[3] Bertrand Teguia Tabuguia. *Power Series Representations of Hypergeometric Types and Non-Holonomic Functions in Computer Algebra.* PhD thesis, University of Kassel, https://kobra.uni-kassel.de/handle/123456789/11598, May 2020.

[4] Bertrand Teguia Tabuguia and Wolfram Koepf. Symbolic conversion of holonomic functions to hypergeometric type power series. *Programming and Computer Software*, 48(2):125–146, 2022.