```
> restart
```

Guessing "not nice" sequences

Bertrand Teguia Tabuguia

November, 2021

This printed worksheet showcases my current research theme with δ_2-finite functions. I recently added the procedure delta2guess in my Maple FPS package. This procedure computes a quadratic differential equation satisfied by the generating function from finitely many coefficients of its power series given in a list. By default, the procedure searches a differential equation with polynomial coefficients of degree 2. The differential equation given in the output can easily be solved by the Maple dsolve command. Using some of the values given in the input list enables us to find the corresponding generating function. In each example below, I use the procedures QDE and FindQRE from my FPS package to show that the differential equations and the recurrence equations in the outputs of delta2guess coincide with those computed directly from the expected generating functions.

```
> with(FPS):
```

1- Exponential generating function of Bernoulli numbers

```
> L:=[seq(bernoulli(i)/i!,i=0..12)]
```

$$L := \left[1, -\frac{1}{2}, \frac{1}{12}, 0, -\frac{1}{720}, 0, \frac{1}{30240}, 0, -\frac{1}{1209600}, 0, \frac{1}{47900160}, 0, -\frac{691}{1307674368000}\right] \quad \textbf{(1)}$$

```
> G1:=delta2guess(L)
```

$$G1 := \left[\_C \left(\sum_{k=0}^{n} a(k)\, a(n-k)\right) + \_C0 \left(\sum_{k=0}^{n-1} a(k)\, a(n-1-k)\right) + (\_C\, n - \_C)\, a(n) + \_C0\, a(n-2) + (\_C0\, n + \_C - 2\,\_C0)\, a(n-1) = 0, \left(-\_C + (-\_C0 + \_C)\, z + \_C0\, z^2\right) y(z) + (\_C0\, z^2 + \_C\, z) \left(\frac{d}{dz}\, y(z)\right) + (\_C0\, z + \_C)\, y(z)^2 = 0\right] \quad \textbf{(2)}$$

```
> subs([_C=1,_C0=0],G1)
```

(3)

$$\left[\left(\sum_{k=0}^{n} a(k)\, a(n-k)\right) + (n-1)\, a(n) + a(n-1) = 0,\ (-1+z)\, y(z) + z\left(\frac{d}{dz}\, y(z)\right) + y(z)^2 = 0\right] \quad (3)$$

```
> FindQRE(z/(exp(z)-1),z,a(n))
```

$$\left(\sum_{\_k=0}^{n} a(\_k)\, a(n-\_k)\right) + (n-1)\, a(n) + a(n-1) = 0 \quad (4)$$

```
> QDE(z/(exp(z)-1),y(z))
```

$$(-1+z)\, y(z) + z\left(\frac{d}{dz}\, y(z)\right) + y(z)^2 = 0 \quad (5)$$

2- Exponential generating function of Bell numbers

```
> L:=[seq(combinat:-bell(i)/i!,i=0..21)]
```

$$L := \left[1, 1, 1, \frac{5}{6}, \frac{5}{8}, \frac{13}{30}, \frac{203}{720}, \frac{877}{5040}, \frac{23}{224}, \frac{1007}{17280}, \frac{4639}{145152}, \frac{22619}{1330560}, \frac{4213597}{479001600}, \frac{27644437}{6227020800}, \frac{95449661}{43589145600}, \frac{276591709}{261534873600}, \frac{10480142147}{20922789888000}, \frac{255755771}{1097800704000}, \frac{97439543737}{914624815104000}, \frac{5832742205057}{121645100408832000}, \frac{263898766507}{12412765347840000}, \frac{158289938718917}{17030314057236480000}\right] \quad (6)$$

```
> G2:=delta2guess(L)
```

$$G2 := \left[\_C0\left(\sum_{k=0}^{n-1} (k+1)\,(k+2)\, a(k+2)\, a(n-1-k)\right) - \_C\left(\sum_{k=0}^{n-2} (k+1)\,(k+2)\, a(k+2)\, a(n-2-k)\right) - \_C0\left(\sum_{k=0}^{n-1} (k+1)\, a(k+1)\,(n-k)\, a(n-k)\right) + \_C\left(\sum_{k=0}^{n-2} (k+1)\, a(k+1)\,(n-1-k)\, a(n-1-k)\right) - \_C0\left(\sum_{k=0}^{n-1} (k+1)\, a(k+1)\, a(n-1-k)\right) + \_C\left(\sum_{k=0}^{n-2} (k+1)\, a(k+1)\, a(n-2-k)\right) = 0,\ \left(-\_C z^2 + \_C0\, z\right)\left(\frac{d^2}{dz^2}\, y(z)\right) y(z) + \left(\_C z^2 - \_C0\, z\right) y(z)\left(\frac{d}{dz}\, y(z)\right) + \left(\_C z^2 - \_C0\, z\right)\left(\frac{d}{dz}\, y(z)\right)^2 = 0\right] \quad (7)$$

```
> simplify(subs([_C=1,_C0=0],G2))
```

(8)

$$\left[-\left(\sum_{k=0}^{n-2} (k+1)(k+2)\, a(k+2)\, a(n-2-k)\right) - \left(\sum_{k=0}^{n-2} (k+1)\, a(k+1)\, (-n+1+k)\, a(n-1-k)\right) + \left(\sum_{k=0}^{n-2} (k+1)\, a(k+1)\, a(n-2-k)\right) = 0, \left(-\left(\frac{d^2}{dz^2} y(z)\right) y(z) + \left(\frac{d}{dz} y(z)\right)\left(y(z) + \frac{d}{dz} y(z)\right)\right) z^2 = 0 \right] \quad \textbf{(8)}$$

```
> FindQRE(exp(exp(z)-1),z,a(n))
```

$$\left(\sum_{\_k=0}^{n} (\_k+1)(\_k+2)\, a(\_k+2)\, a(n-\_k)\right) - \left(\sum_{\_k=0}^{n} (\_k+1)\, a(\_k+1)\, a(n-\_k)\right) - \left(\sum_{\_k=0}^{n} (\_k+1)\, a(\_k+1)\, (n-\_k+1)\, a(n-\_k+1)\right) = 0 \quad \textbf{(9)}$$

```
> QDE(exp(exp(z)-1),y(z))
```

$$\left(\frac{d^2}{dz^2} y(z)\right) y(z) - y(z)\left(\frac{d}{dz} y(z)\right) - \left(\frac{d}{dz} y(z)\right)^2 = 0 \quad \textbf{(10)}$$

3- Tangent series coefficients

```
> T:=series(tan(z),z,18)
```

$$T := z + \frac{1}{3} z^3 + \frac{2}{15} z^5 + \frac{17}{315} z^7 + \frac{62}{2835} z^9 + \frac{1382}{155925} z^{11} + \frac{21844}{6081075} z^{13} + \frac{929569}{638512875} z^{15} + \frac{6404582}{10854718875} z^{17} + O(z^{19}) \quad \textbf{(11)}$$

```
> L:=[seq(coeff(T,z,i),i=0..18)]
```

$$L := \left[0, 1, 0, \frac{1}{3}, 0, \frac{2}{15}, 0, \frac{17}{315}, 0, \frac{62}{2835}, 0, \frac{1382}{155925}, 0, \frac{21844}{6081075}, 0, \frac{929569}{638512875}, 0, \frac{6404582}{10854718875}, 0\right] \quad \textbf{(12)}$$

```
> G3:=delta2guess(L)
```

$$G3 := \left[\_C0\, n\, (n+1)\, a(n+1) - \frac{\_C\,(n-1)\, n\, a(n)}{2} - 2\,\_C0 \left(\sum_{k=0}^{n-1} (k+1)\, a(k+1)\, a(n-1-k)\right) + \_C \left(\sum_{k=0}^{n-2} (k+1)\, a(k+1)\, a(n-2-k)\right) = 0, \left(\_C0\, z - \frac{1}{2}\,\_C\, z^2\right)\left(\frac{d^2}{dz^2} y(z)\right) + \left(\_C\, z^2 - 2\,\_C0\, z\right) y(z) \left(\frac{d}{dz} y(z)\right) = 0 \right] \quad \textbf{(13)}$$

```
> subs([_C=0,_C0=1],G3)
```

$$\left[n\,(n+1)\,a(n+1) - 2\left(\sum_{k=0}^{n-1} (k+1)\,a(k+1)\,a(n-1-k)\right) = 0,\ z\left(\frac{d^2}{dz^2}\,y(z)\right) - 2\,z\,y(z)\left(\frac{d}{dz}\,y(z)\right) = 0\right] \tag{14}$$

```
> FindQRE(tan(z),z,a(n))
```

$$(n+1)\,(n+2)\,a(n+2) - 2\left(\sum_{\_k=0}^{n} (\_k+1)\,a(\_k+1)\,a(n-\_k)\right) = 0 \tag{15}$$

```
> QDE(tan(z),y(z))
```

$$-2\,y(z)\left(\frac{d}{dz}\,y(z)\right) + \frac{d^2}{dz^2}\,y(z) = 0 \tag{16}$$

4- $\dfrac{\mathrm{sqrt}(1+z)}{1-\log(1+z)}$ series coefficients (from recent slides of Manuel Kauers (September 2021))

```
> T:=series(sqrt(1+z)/(1-log(1+z)),z,81):
> L:=[seq(coeff(T,z,i),i=0..80)]:
> G4:=delta2guess(L,degpoly=3)
```

$$G4 := \Bigg[(4\,\_C0 - 2\,\_C)\left(\sum_{k=0}^{n} (k+1)\,a(k+1)\,(n-k+1)\,a(n-k+1)\right) + (6\,\_C0 - 4\,\_C)\left(\sum_{k=0}^{n-1} (k+1)\,a(k+1)\,(n-k)\,a(n-k)\right) - 2\,\_C\left(\sum_{k=0}^{n-2} (k+1)\,a(k+1)\,(n-1-k)\,a(n-1-k)\right) - 2\,\_C0\left(\sum_{k=0}^{n-3} (k+1)\,a(k+1)\,(n-2-k)\,a(n-2-k)\right) + 2\,\_C0\left(\sum_{k=0}^{n-2} (k+1)\,a(k+1)\,a(n-2-k)\right) + (-4\,\_C0 + 2\,\_C)\left(\sum_{k=0}^{n} (k+1)\,a(k+1)\,a(n-k)\right) + (-2\,\_C0 + 2\,\_C)\left(\sum_{k=0}^{n-1} (k+1)\,a(k+1)\,a(n-1-k)\right) \tag{17}$$

$$-\frac{\_C0\left(\sum_{k=0}^{n-1} a(k)\,a(n-1-k)\right)}{4}+\left(\frac{\_C0}{2}-\frac{\_C}{4}\right)\left(\sum_{k=0}^{n} a(k)\,a(n-k)\right)+(-2\,\_C0+\_C)\left(\sum_{k=0}^{n}(k+1)\,(k+2)\,a(k+2)\,a(n-k)\right)+(-3\,\_C0+2\,\_C)\left(\sum_{k=0}^{n-1}(k+1)\,(k+2)\,a(k+2)\,a(n-1-k)\right)+\_C\left(\sum_{k=0}^{n-2}(k+1)\,(k+2)\,a(k+2)\,a(n-2-k)\right)+\_C0\left(\sum_{k=0}^{n-3}(k+1)\,(k+2)\,a(k+2)\,a(n-3-k)\right)=0,\ \left(-4\,\_C0+2\,\_C+(-2\,\_C0+2\,\_C)\,z+2\,\_C0\,z^2\right)y(z)\left(\frac{d}{dz}y(z)\right)+\left(-2\,\_C0+\_C+(-3\,\_C0+2\,\_C)\,z+\_C\,z^2+\_C0\,z^3\right)\left(\frac{d^2}{dz^2}y(z)\right)y(z)+\left(4\,\_C0-2\,\_C+(6\,\_C0-4\,\_C)\,z-2\,\_C\,z^2-2\,\_C0\,z^3\right)\left(\frac{d}{dz}y(z)\right)^2+\left(\frac{1}{2}\,\_C0-\frac{1}{4}\,\_C-\frac{1}{4}\,\_C0\,z\right)y(z)^2=0\Bigg]$$

```
> subs([_C=1,_C0=0],G4)
```

$$\Bigg[-2\left(\sum_{k=0}^{n}(k+1)\,a(k+1)\,(n-k+1)\,a(n-k+1)\right)-4\left(\sum_{k=0}^{n-1}(k+1)\,a(k+1)\,(n-k)\,a(n-k)\right)-2\left(\sum_{k=0}^{n-2}(k+1)\,a(k+1)\,(n-1-k)\,a(n-1-k)\right)+2\left(\sum_{k=0}^{n}(k+1)\,a(k+1)\,a(n-k)\right)+2\left(\sum_{k=0}^{n-1}(k+1)\,a(k+1)\,a(n-1-k)\right)-\frac{\left(\sum_{k=0}^{n} a(k)\,a(n-k)\right)}{4}+\left(\sum_{k=0}^{n}(k+1)\,(k+2)\,a(k+2)\,a(n-k)\right)+2\left(\sum_{k=0}^{n-1}(k+1)\,(k+2)\,a(k+2)\,a(n-1-k)\right)+\left(\sum_{k=0}^{n-2}(k+1)\,(k+2)\,a(k+2)\,a(n-2-k)\right) \qquad \textbf{(18)}$$

$$= 0,\ (2+2z)\,y(z)\left(\frac{d}{dz}\,y(z)\right) + \left(z^2+2z+1\right)\left(\frac{d^2}{dz^2}\,y(z)\right)y(z) + \left(-2z^2-4z-2\right)\left(\frac{d}{dz}\,y(z)\right)^2 - \frac{y(z)^2}{4} = 0\Big]$$

```
> FindQRE(sqrt(1+z)/(1-log(1+z)),z,a(n))
```

$$4\left(\sum_{\_k=0}^{n-2}(\_k+1)(\_k+2)\,a(\_k+2)\,a(n-2-\_k)\right) - 8\left(\sum_{\_k=0}^{n-2}(\_k+1)\,a(\_k+1)\,(n-1-\_k)\,a(n-1-\_k)\right) + 8\left(\sum_{\_k=0}^{n-1}(\_k+1)(\_k+2)\,a(\_k+2)\,a(n-1-\_k)\right) + 8\left(\sum_{\_k=0}^{n-1}(\_k+1)\,a(\_k+1)\,a(n-1-\_k)\right) - 16\left(\sum_{\_k=0}^{n-1}(\_k+1)\,a(\_k+1)\,(n-\_k)\,a(n-\_k)\right) + 4\left(\sum_{\_k=0}^{n}(\_k+1)(\_k+2)\,a(\_k+2)\,a(n-\_k)\right) - \left(\sum_{\_k=0}^{n}a(\_k)\,a(n-\_k)\right) + 8\left(\sum_{\_k=0}^{n}(\_k+1)\,a(\_k+1)\,a(n-\_k)\right) - 8\left(\sum_{\_k=0}^{n}(\_k+1)\,a(\_k+1)\,(n-\_k+1)\,a(n-\_k+1)\right) = 0 \quad \textbf{(19)}$$

```
> QDE(sqrt(1+z)/(1-log(1+z)),y(z))
```

$$-y(z)^2 + (8+8z)\,y(z)\left(\frac{d}{dz}\,y(z)\right) - 8(1+z)^2\left(\frac{d}{dz}\,y(z)\right)^2 + 4(1+z)^2\left(\frac{d^2}{dz^2}\,y(z)\right)y(z) = 0 \quad \textbf{(20)}$$

5- 2^{2^z} series coefficients

```
> T:=series(2^(2^z),z,81):
> L:=[seq(coeff(T,z,i),i=0..80)]:
> G5:=delta2guess(L)
```

$$G5 := \Bigg[\_C\left(\sum_{k=0}^{n}(k+1)(k+2)\,a(k+2)\,a(n-k)\right) + \_C0\left(\sum_{k=0}^{n-1}(k+1)(k+2)\,a(k+2)\,a(n-1-k)\right) + \_C1\left(\sum_{k=0}^{n-2}(k+1)(k+2)\,a(k+2)\,a(n-2-k)\right) - \_C\left(\sum_{k=0}^{n}(k \quad \textbf{(21)}\right.$$

$$+ 1)\, a(k+1)\,(n-k+1)\,a(n-k+1)\Bigg) - \_C0 \left(\sum_{k=0}^{n-1} (k+1)\,a(k+1)\,(n-k)\,a(n\right.$$

$$\left. - k)\right) - \_C1 \left(\sum_{k=0}^{n-2} (k+1)\,a(k+1)\,(n-1-k)\,a(n-1-k)\right) - \ln(2)\,\_C \left(\sum_{k=0}^{n} (k\right.$$

$$\left. + 1)\,a(k+1)\,a(n-k)\right) - \ln(2)\,\_C0 \left(\sum_{k=0}^{n-1} (k+1)\,a(k+1)\,a(n-1-k)\right)$$

$$- \_C1 \ln(2) \left(\sum_{k=0}^{n-2} (k+1)\,a(k+1)\,a(n-2-k)\right) = 0, \left(\_C1\, z^2 + \_C0\, z + \_C\right) \left(\frac{d^2}{dz^2}\right.$$

$$\left. y(z)\right) y(z) + \left(-\ln(2)\,\_C1\, z^2 - \ln(2)\,\_C0\, z - \ln(2)\,\_C\right) y(z) \left(\frac{d}{dz}\, y(z)\right) + \left(-\_C1\, z^2\right.$$

$$\left. - \_C0\, z - \_C\right) \left(\frac{d}{dz}\, y(z)\right)^2 = 0\Bigg]$$

```
> subs([_C=1,_C0=-1,_C1=0],G5)
```

$$\left[\left(\sum_{k=0}^{n} (k+1)\,(k+2)\,a(k+2)\,a(n-k)\right) - \left(\sum_{k=0}^{n-1} (k+1)\,(k+2)\,a(k+2)\,a(n-1-k)\right)\right. \tag{22}$$

$$- \left(\sum_{k=0}^{n} (k+1)\,a(k+1)\,(n-k+1)\,a(n-k+1)\right) + \left(\sum_{k=0}^{n-1} (k+1)\,a(k+1)\,(n\right.$$

$$\left. - k)\,a(n-k)\right) - \ln(2) \left(\sum_{k=0}^{n} (k+1)\,a(k+1)\,a(n-k)\right) + \ln(2) \left(\sum_{k=0}^{n-1} (k+1)\,a(k\right.$$

$$\left. + 1)\,a(n-1-k)\right) = 0, (-z+1) \left(\frac{d^2}{dz^2}\, y(z)\right) y(z) + (\ln(2)\, z - \ln(2))\, y(z) \left(\frac{d}{dz}\, y(z)\right)$$

$$\left. + (-1+z) \left(\frac{d}{dz}\, y(z)\right)^2 = 0\right]$$

```
> FindQRE(2^(2^z),z,a(n))
```

$$\left(\sum_{\_k=0}^{n} (\_k+1)\,(\_k+2)\,a(\_k+2)\,a(n-\_k)\right) - \ln(2) \left(\sum_{\_k=0}^{n} (\_k+1)\,a(\_k+1)\,a(n-\_k)\right) \tag{23}$$

$$- \left(\sum_{\_k=0}^{n} (\_k+1)\,a(\_k+1)\,(n-\_k+1)\,a(n-\_k+1)\right) = 0$$

```
> QDE(2^(2^z),y(z))
```

$$-\ln(2)\, y(z) \left(\frac{d}{dz}\, y(z)\right) - \left(\frac{d}{dz}\, y(z)\right)^2 + \left(\frac{d^2}{dz^2}\, y(z)\right) y(z) = 0 \tag{24}$$

6- $\frac{1}{\text{LambertW}(-x)+1}$

```
> f:=1/(LambertW(-x) + 1):
> T:=series(f,x,30):
> L:=[seq(coeff(T,x,i),i=0..30)]:
> G:=delta2guess(L)
```

$$G := \left[\_C \left(\sum_{k=0}^{n-1} (k+1)\,(k+2)\,a(k+2)\,a(n-1-k)\right) + \_C0 \left(\sum_{k=0}^{n-2} (k+1)\,(k+2)\,a(k+2)\,a(n-2-k)\right) - \frac{\_C\,n\,(n+1)\,a(n+1)}{3} - \frac{\_C0\,(n-1)\,n\,a(n)}{3} - 3\,\_C \left(\sum_{k=0}^{n-1} (k+1)\,a(k+1)\,(n-k)\,a(n-k)\right) - 3\,\_C0 \left(\sum_{k=0}^{n-2} (k+1)\,a(k+1)\,(n-1-k)\,a(n-1-k)\right) + \frac{\_C \left(\sum_{k=0}^{n} (k+1)\,a(k+1)\,a(n-k)\right)}{3} + \frac{\_C0 \left(\sum_{k=0}^{n-1} (k+1)\,a(k+1)\,a(n-1-k)\right)}{3} - \frac{n\,\_C0\,a(n)}{3} - \frac{\_C\,(n+1)\,a(n+1)}{3} = 0, \left(-\frac{1}{3}\,\_C\,z - \frac{1}{3}\,\_C0\,z^2\right)\left(\frac{d^2}{dz^2}\,y(z)\right) + \left(-\frac{\_C}{3} - \frac{\_C0\,z}{3}\right)\left(\frac{d}{dz}\,y(z)\right) + \left(-3\,\_C0\,z^2 - 3\,\_C\,z\right)\left(\frac{d}{dz}\,y(z)\right)^2 + \left(\_C0\,z^2 + \_C\,z\right)\left(\frac{d^2}{dz^2}\,y(z)\right) y(z) + \left(\frac{\_C}{3} + \frac{\_C0\,z}{3}\right)\left(\frac{d}{dz}\,y(z)\right) y(z) = 0\right] \quad \textbf{(25)}$$

```
> subs([_C=0,_C0=1],G)
```

$$\left[\left(\sum_{k=0}^{n-2} (k+1)\,(k+2)\,a(k+2)\,a(n-2-k)\right) - \frac{(n-1)\,n\,a(n)}{3} - 3\left(\sum_{k=0}^{n-2} (k+1)\,a(k \right. \quad \textbf{(26)}$$

$$+ 1) (n - 1 - k) a(n - 1 - k)\Bigg) + \frac{\left(\sum_{k=0}^{n-1} (k+1) a(k+1) a(n-1-k)\right)}{3} - \frac{n a(n)}{3}$$

$$= 0, -\frac{z^2 \left(\frac{d^2}{dz^2} y(z)\right)}{3} - \frac{z \left(\frac{d}{dz} y(z)\right)}{3} - 3 z^2 \left(\frac{d}{dz} y(z)\right)^2 + z^2 \left(\frac{d^2}{dz^2} y(z)\right) y(z)$$

$$+ \frac{z \left(\frac{d}{dz} y(z)\right) y(z)}{3} = 0\Bigg]$$

```
> FindQRE(f,x,a(n))
```

$$\left(\sum_{\_k=0}^{n} (\_k+1) a(\_k+1) a(n-\_k)\right) + 3 \left(\sum_{\_k=0}^{n-1} (\_k+1) (\_k+2) a(\_k+2) a(n-1-\_k)\right) - 9 \left(\sum_{\_k=0}^{n-1} (\_k+1) a(\_k+1) (n-\_k) a(n-\_k)\right) - (n+1)^2 a(n+1) = 0 \quad \textbf{(27)}$$

```
> QDE(f,y(x))
```

$$-\frac{d}{dx} y(x) + \left(\frac{d}{dx} y(x)\right) y(x) - 9 x \left(\frac{d}{dx} y(x)\right)^2 - x \left(\frac{d^2}{dx^2} y(x)\right) + 3 x \left(\frac{d^2}{dx^2} y(x)\right) y(x) = 0 \quad \textbf{(28)}$$