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Global stress regularity of convex and some nonconvex variational problems

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Abstract We derive a global regularity theorem for stress fields which correspond to minimizers of convex and some special nonconvex variational problems with mixed boundary conditions on admissible domains. These are Lipschitz domains satisfying additional geometric conditions near those points, where the type of the boundary conditions changes. In the first part it is assumed that the energy densities defining the variational problem are convex but not necessarily strictly convex and satisfy a convexity inequality. The regularity result for this case is derived with a difference quotient technique. In the second part the regularity results are carried over from the convex case to special nonconvex variational problems taking advantage of the relation between nonconvex variational problems and the corresponding (quasi-) convexified problems. The results are applied amongst others to the variational problems for linear elasticity, the p-Laplace operator, Hencky elasto-plasticity with linear hardening and for scalar and vectorial two-well potentials (compatible case).

Keywords Global stress regularity · Convex variational problem · Nonconvex variational problem · Nonsmooth domain · Difference quotient technique

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1 Introduction

We investigate the global regularity of stress fields which are related to minimizers of convex, but not necessarily strictly convex, variational problems with mixed boundary conditions on domains with Lipschitz boundary. Furthermore, the results are carried over to stress fields of special nonconvex variational problems. The variational

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problems under consideration are of the following type: for a bounded domain $\Omega \subset \mathbb{R}^d$ we denote by $\Gamma_D \subset \partial\Omega$ the Dirichlet boundary; furthermore, let $g \in W^{1,p}(\Omega)$, $f \in \left(W^{1,p}(\Omega)\right)'$ and $V := \{v \in W^{1,p}(\Omega) : v_{|\Gamma_D} = 0\}$. The variational problem is

$$\text{Find } u : \Omega \to \mathbb{R}^m, u \in g + V \text{ such that for every } v \in g + V$$
$$I(u) \le I(v) = \int_\Omega W(\nabla v(x))\,\mathrm{d}x - \langle f, v\rangle. \tag{1.1}$$

Here, $W : \mathbb{R}^{m\times d} \to \mathbb{R}$ is a given energy density. If u is a minimizer of I then the corresponding stress field σ is defined as

$$\sigma(x) := \mathrm{DW}(\nabla u(x)), \quad x \in \Omega,$$

where the notation $\mathrm{DW}(A)_{ik} = \frac{\partial W(A)}{\partial A_{ik}}, 1 \le i \le m, 1 \le k \le d$, is used for the derivative of W. The goal of this paper is to derive a global regularity result for the stress σ on nonsmooth domains.

In the first part of this paper (Sect. 2) we study the convex case. Here, the main assumption is that the energy density W is a $\mathcal{C}^1$ function and satisfies the following convexity inequality for every $A, B \in \mathbb{R}^{m\times d}$:

$$(W(A) - W(B) - \mathrm{DW}(B) : (A - B))\left(1 + |A|^s + |B|^s\right) \ge c\,|\mathrm{DW}(A) - \mathrm{DW}(B)|^r \tag{1.2}$$

for some constants $c > 0$, $s \ge 0$ and $r > 1$. In addition, we assume that W is of p-growth for some $p \in (1, \infty)$. Inequality (1.2) implies that W is convex but not necessarily strictly convex. Examples for functions W with (1.2) include the energy densities of linear elastic materials, a model of Hencky elasto-plasticity with linear hardening and energy densities, which correspond to quasilinear elliptic systems of p-structure.

It is well known for linear and quasilinear elliptic equations that the global regularity of weak solutions does not only depend on the smoothness of the right hand sides but also on the smoothness of the boundary of Ω, we refer to [14,15,25,31,33,36] and the references therein for the linear case. Global regularity results on nonsmooth domains for weak solutions u of quasilinear elliptic systems of p-structure were derived by Ebmeyer and Frehse [16,19] (mixed boundary conditions, polyhedral domains) and Savaré [45] (pure Dirichlet or pure Neumann conditions on Lipschitz domains). The essential assumptions in these papers imply the following convexity inequality for the energy densities which define the boundary value problems [16,19,28,45]:

$$W(A) - W(B) - \mathrm{DW}(B) : (A - B) \ge c(\kappa + |A| + |B|)^{p-2}\,|A - B|^2 \tag{1.3}$$

for $A, B \in \mathbb{R}^{m\times d}$ and some $\kappa \in \{0, 1\}$. The regularity results in [16,19,45] are derived with a difference quotient technique based on (1.3). Due to the similarity between (1.3) and our main assumption (1.2), it is possible to adapt the techniques in [16,19,45] to our case. We combine the geometrical assumptions of the above mentioned references and introduce the notion of *admissible domains* in Definition 2.3. Admissible domains are Lipschitz domains which satisfy an additional geometrical assumption near points with changing boundary conditions. We then prove the following regularity result for the stress field σ on admissible domains (Theorem 2.2): For every $\delta > 0$ we have

$$\sigma = \mathrm{DW}(\nabla u) \in W^{\frac{1}{r}-\delta, \frac{pr}{p+s}}(\Omega). \tag{1.4}$$

Here, r and s are the exponents from (1.2). This result is proved with a difference quotient technique on the basis of convexity inequality (1.2). A comparison of (1.4) with well known results for linear elasticity ($p = r = 2, s = 0$) and for the p-Laplacian indicates the optimality of (1.4) within the class of considered domains, see Sect. 2.5.

In the second part of the paper (Sect. 3) we discuss the regularity properties of the stress fields of a special class of nonconvex variational problems. Nonconvex problems need not have minimizers and in that case it is reasonable to study the relaxed variational problem

$$\text{Find } u \in g + V \text{ such that for every } v \in g + V$$
$$I^R(u) \leq I^R(v) = \int_\Omega W^R(\nabla v(x))\,\mathrm{d}x - \langle f, v\rangle. \tag{1.5}$$

Here, W^R is in general given by the quasiconvex envelope W^{qc} of the original energy density W [1,13,42]. It follows from the relaxation theory in the calculus of variations that if the original problem has a solution u then it is also a solution of the relaxed problem. Moreover, the stresses of both problems coincide: $\sigma = \mathrm{DW}(\nabla u) = \mathrm{DW}^{qc}(\nabla u)$. We use this relation in combination with the additional strong assumption that the quasiconvex envelope W^{qc} is equal to the convex envelope W^c of the original energy density W in order to carry over regularity results for stresses of convex problems to special nonconvex problems. Examples for such problems are scalar or vectorial two-well potentials (compatible case) and a special case of the Ericksen–James energy density.

This paper and the examples herein are highly motivated by an article by Carstensen and Müller, where local and global stress regularity results for smooth domains are proved [9]. There, the main assumption is that the energy density W satisfies the following monotonicity inequality

$$\big((\mathrm{DW}(A) - \mathrm{DW}(B)) : (A - B)\big)\left(1 + |A|^s + |B|^s\right) \geq c\,|\mathrm{DW}(A) - \mathrm{DW}(B)|^r \tag{1.6}$$

with $c > 0$, $s \geq 0$ and $r > 1$. Inequality (1.2) is motivated by (1.3) and (1.6) and is a stronger assumption than (1.6). In Lemma 2.2 we describe sufficient conditions on W for which the monotonicity inequality (1.6) and the convexity inequality (1.2) are equivalent. The local results in [9] are proved with a difference quotient technique. There, the main idea is to use double differences $\tilde{v}(x) = \eta^2(x)(u(x+h) - 2u(x) + u(x-h))$ as test functions for the weak formulation and to apply the monotonicity inequality (1.6). The function η is a cut off function with $\operatorname{supp}\eta \subset \Omega$ and thus the function $\tilde{v}$ is indeed an admissible test function for the weak formulation. In the proof of our main result for Lipschitz domains (Theorem 2.2), we use single differences $v(x) = \eta^2(u(x+h) - u(x))$ as test functions and apply the convexity inequality (1.2). Here, η is a cut off function with $\operatorname{supp}\eta \cap \Omega \neq \emptyset$ and $\operatorname{supp}\eta \cap (\mathbb{R}^d \backslash \overline{\Omega}) \neq \emptyset$, $h \in \mathbb{R}^d \backslash \{0\}$. The vectors h may cross the boundary and thus the weak solution u has to be extended to the exterior domain in such a way that v is still an admissible test function for the weak formulation. Due to the assumed conditions on the geometry of $\partial\Omega$, it is possible to find suitable extensions. In general, double differences of these extended functions are not admissible test functions.

The paper is organized as follows: After a description of the assumptions on the energy density W and the geometry of the domain Ω, we formulate in Sect. 2 the main result on the global regularity of stress fields of convex variational problems (Theorem 2.2). The proof is based on the difference quotient technique. These results are then applied to convex examples from continuum mechanics. In Sect. 3 we formulate a regularity theorem for the nonconvex case and illustrate it with further examples. The paper closes with an appendix where we give some technical proofs concerning examples for admissible domains.

2 Regularity in the convex case

2.1 Notation

Let us first introduce some notation and general assumptions. For $m \times d$-matrices $A, B \in \mathbb{R}^{m\times d}$ the inner product is defined by $A : B = \operatorname{tr}\left(A^\top B\right) = \operatorname{tr}\left(B^\top A\right) = \sum_{i=1}^m \sum_{k=1}^d A_{ik}B_{ik}$ and $|A| = \sqrt{A : A}$ is the corresponding Frobenius norm.

If not otherwise stated it is assumed that $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is a bounded domain with Lipschitz boundary $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$. Γ_D and Γ_N are open and disjoint and denote the Dirichlet and Neumann boundary, respectively. Throughout the whole paper a domain with Lipschitz boundary is a domain $\Omega \subset \mathbb{R}^d$ with a boundary which can locally be described as the graph of a Lipschitz continuous function (after a suitable rotation). We refer to [25] for a precise definition.

For $p \in (1,\infty)$ and $s > 0$ the spaces $W^{s,p}(\Omega)$ are the usual Sobolev–Slobodeckij spaces, see, e.g. [3,25]. Furthermore,

$$V = \{v \in W^{1,p}(\Omega) : v_{|\Gamma_D} = 0\}. \tag{2.1}$$

For the formulation of the boundary conditions we need the following trace space and its dual for an open subset $\Gamma \subset \partial\Omega, p \in (1,\infty)$:

$$W^{1-\frac{1}{p},p}(\Gamma) = \left\{u \in L^p(\Gamma) : \exists \hat{u} \in W^{1,p}(\Omega) \text{ such that } \hat{u}_{|\Gamma} = u\right\}, \tag{2.2}$$

$$\tilde{W}^{-\frac{1}{p'},p'}(\Gamma) = \left(W^{1-\frac{1}{p},p}(\Gamma)\right)'. \tag{2.3}$$

Throughout the whole paper p' is the conjugate exponent of p, $\frac{1}{p} + \frac{1}{p'} = 1$. Furthermore, the dual pairing for elements u of a Banach space X and elements f of its dual X' is denoted by $\langle f,u\rangle = \langle f,u\rangle_X$. Besides the usual Sobolev spaces we deal also with Nikolskii spaces. Nikolskii spaces are very useful for proving regularity results with a difference quotient technique since their norms are based on difference quotients. For convenience we cite here the definition of Nikolskii spaces and an embedding theorem.

Definition 2.1 (Nikolskii space) [3,39] Let $s = m + \delta$, where $m \geq 0$ is an integer and $0 < \delta < 1$. For $1 < p < \infty$ the Nikolskii spaces are defined as

$$\mathcal{N}^{s,p}(\Omega) := \{\, u \in L^p(\Omega)\,;\ \|u\|_{\mathcal{N}^{s,p}(\Omega)} < \infty \,\}$$

with

$$\|u\|^p_{\mathcal{N}^{s,p}(\Omega)} = \|u\|^p_{L^p(\Omega)} + \sum_{|\alpha|=m} \sup_{\substack{\eta>0\\ h\in\mathbb{R}^d\\ 0<|h|<\eta}} \int_{\Omega_\eta} \frac{|D^\alpha u(x+h) - D^\alpha u(x)|^p}{|h|^{\delta p}}\,\mathrm{d}x \tag{2.4}$$

and $\Omega_\eta = \{x \in \Omega :\ \operatorname{dist}(x,\partial\Omega) > \eta\}$.

Lemma 2.1 [3,39,49,50] *Let s, p be as in Definition 2.1 and let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. The following embeddings are continuous for every $\epsilon > 0$:*

$$\mathcal{N}^{s+\epsilon,p}(\Omega) \subset W^{s,p}(\Omega) \subset \mathcal{N}^{s,p}(\Omega).$$

An equivalent norm is generated if the supremum in (2.4) is replaced by a supremum over a basis of $\mathbb{R}^d$, [34,39].

2.2 The convex minimization problem

We study minimization problems where the energy density $W : \mathbb{R}^{m\times d} \to \mathbb{R}$, $m, d \geq 1$, has the following properties:

H1 $W \in \mathcal{C}^1(\mathbb{R}^{m\times d}, \mathbb{R})$.

H2 There exist constants $p \in (1, \infty)$, $c_0, c_1, c_2, c_{21}, c_3 > 0$ such that for every $A \in \mathbb{R}^{m\times d}$

$$c_0 |A|^p - c_1 \leq W(A) \leq c_2 |A|^p + c_{21}, \tag{2.5}$$

$$|\mathrm{DW}(A)| \leq c_3(1 + |A|^{p-1}). \tag{2.6}$$

H3 There exist constants $c > 0$, $r > 1$, $s \geq 0$ such that we have for every $A, B \in \mathbb{R}^{m\times d}$

$$\big(W(A) - W(B) - \mathrm{DW}(B) : (A - B)\big)\left(1 + |A|^s + |B|^s\right) \geq c\,|\mathrm{DW}(A) - \mathrm{DW}(B)|^r. \tag{2.7}$$

Condition **H3** implies that the energy density W is convex but not necessarily strictly convex. Let us remark that **H3** is also an essential assumption in [8], where the convergence of adaptive FEM for convex problems is studied. The following lemma describes sufficient conditions on W for which convexity inequality (2.7) and monotonicity inequality (1.6) are equivalent.

Lemma 2.2 *Let $W \in \mathcal{C}^1(\mathbb{R}^{m\times d}, \mathbb{R})$ satisfy (2.5) with $p > 1$ and let monotonicity inequality (1.6) be valid for $s \geq 0$ and $r > 1$. We denote by W^* the conjugate function of W and by $\partial W^*(\sigma)$ the subdifferential of W^* at $\sigma \in \mathbb{R}^{m\times d}$.*

1. *If $s = 0$ then we have for every $\sigma_1, \sigma_2 \in \mathbb{R}^{m\times d}$ and $A \in \partial W^*(\sigma_2)$ with the constant c from monotonicity inequality (1.6)*

$$W^*(\sigma_1) - W^*(\sigma_2) - A : (\sigma_1 - \sigma_2) \geq c r^{-1} |\sigma_1 - \sigma_2|^r. \tag{2.8}$$

Furthermore, it holds for every $A, B \in \mathbb{R}^{m\times d}$ with c from (1.6)

$$W(A) - W(B) - \mathrm{DW}(B) : (A - B) \geq \frac{c}{r} |\mathrm{DW}(A) - \mathrm{DW}(B)|^r. \tag{2.9}$$

2. *Let $s \neq 0$ and assume in addition that (2.6) is satisfied. Then there exist constants $\kappa, \delta > 0$ such that we have for every $\sigma_1, \sigma_2 \in \mathbb{R}^{m\times d}$ and $A_i \in \partial W^*(\sigma_i)$*

$$W^*(\sigma_1) - W^*(\sigma_2) - A_2 : (\sigma_1 - \sigma_2) \geq \frac{\kappa\,|\sigma_1 - \sigma_2|^r}{1 + |A_1|^s + |A_2|^s + \delta(|A_1|^{ps} + |A_2|^{ps})}. \tag{2.10}$$

Moreover, it holds for every $A_1, A_2 \in \mathbb{R}^{m\times d}$

$$W(A_1) - W(A_2) - \mathrm{DW}(A_2) : (A_1 - A_2) \geq \frac{\kappa\,|\mathrm{DW}(A_1) - \mathrm{DW}(A_2)|^r}{1 + |A_1|^s + |A_2|^s + \delta(|A_1|^{ps} + |A_2|^{ps})}. \tag{2.11}$$

If $c_0 = c_2$ *in* (2.5) *or if* $|A| \leq c(1 + |\mathrm{DW}(A)|^{\frac{1}{p-1}})$ *for some* $c > 0$ *and every* $A \in \mathbb{R}^{m\times d}$*, then* (2.10) *and* (2.11) *hold with* $\delta = 0$.

Remark 1 The case $r = 2, s = 0$ is treated in [26, Exercise 1.7, p. 21].

Proof Inequalities (2.9) and (2.11) follow from (2.8) and (2.10) via the relation $\sigma = \mathrm{DW}(A) \Leftrightarrow W^*(\sigma) + W(A) = A : \sigma$ [41, Thm. 23.5]. For the proof of (2.8) and (2.10) let $\sigma_1 \neq \sigma_2 \in \mathbb{R}^{m\times d}$. We define $f(t) = W^*(\sigma_2 + t(\sigma_1 - \sigma_2))$, $t \in \mathbb{R}$ and

$$f'_+(t) := \lim_{\lambda\searrow 0} \lambda^{-1} \left(f(t+\lambda) - f(t)\right).$$

Theorem 23.1 in [41] guarantees that $f'_+(t) : \mathbb{R} \to \mathbb{R}$ is well defined. Moreover, it follows from [41, Thm. 23.4] that

$$f'_+(t) = \sup\{A : (\sigma_1 - \sigma_2);\ A \in \partial W^*(\sigma_2 + t(\sigma_1 - \sigma_2))\}. \tag{2.12}$$

Since $\partial W^*(\sigma)$ is compact for every $\sigma \in \mathbb{R}^{m\times d}$, there exists for every t an element $A_+(t) \in \partial W^*(\sigma_2 + t(\sigma_1 - \sigma_2))$ for which the supremum in (2.12) is attained. Taylor's expansion [41, Cor. 24.2.1] and monotonicity inequality (1.6) yield for every $A_2 \in \partial W^*(\sigma_2)$

$$\begin{aligned} W^*(\sigma_1) - W^*(\sigma_2) - A_2 : (\sigma_1 - \sigma_2) &= \int_0^1 f'_+(t) - (A_2 : (\sigma_1 - \sigma_2))\,\mathrm{d}t \\ &= \int_0^1 \frac{1}{t}(A_+(t) - A_2) : t(\sigma_1 - \sigma_2)\,\mathrm{d}t \\ &\overset{(1.6)}{\geq} c\int_0^1 t^{-1}(1+|A_+(t)|^s+|A_2|^s)^{-1}\,|t(\sigma_1-\sigma_2)|^r\,\mathrm{d}t. \end{aligned} \tag{2.13}$$

This proves (2.8) if $s = 0$. Assume now that $s > 0$. The next task is to find an upper bound for $|A_+(t)|$. If the estimate $|A| \leq c(1 + |\mathrm{DW}(A)|^{\frac{1}{p-1}})$ is valid for every A, then analogous arguments as subsequent to (2.17) here below imply (2.10) with $\delta = 0$. If this estimate does not hold, then direct calculations show [use (2.5) and the definition of W^*] that for every $\sigma \in \mathbb{R}^{m\times d}$:

$$q^{-1}(c_2p)^{\frac{-1}{p-1}}\,|\sigma|^q - c_{21} \leq W^*(\sigma) \leq c_1 + q^{-1}(c_0p)^{\frac{-1}{p-1}}\,|\sigma|^q\,, \tag{2.14}$$

where c_0, c_1, c_2 are the constants from (2.5) and $\frac{1}{q} + \frac{1}{p} = 1$. The convexity of W^* and (2.14) imply for every $\sigma \in \mathbb{R}^{m\times d}$ and $A \in \partial W^*(\sigma)$, $A \neq 0$,

$$\begin{aligned} |A| = A : (|A|^{-1}A) &\leq W^*(\sigma + |A|^{-1}A) - W^*(\sigma) \\ &\overset{(2.14)}{\leq} d_1 + d_0\left|\sigma + |A|^{-1}A\right|^q - d_2\,|\sigma|^q\,, \end{aligned} \tag{2.15}$$

where $d_0 = q^{-1}(c_0p)^{\frac{-1}{p-1}}$, $d_1 = c_1 + c_{21}$ and $d_2 = q^{-1}(c_2p)^{\frac{-1}{p-1}}$. Furthermore, Taylor's expansion yields for $\sigma, \tau \in \mathbb{R}^{m\times d}$

$$|\sigma + \tau|^q - |\sigma|^q \leq q\int_0^1 (|\sigma| + t\,|\tau|)^{q-1}\,|\tau|\,\mathrm{d}t \leq q(|\sigma| + |\tau|)^{q-1}\,|\tau|\,. \tag{2.16}$$

Combining inequalities (2.15) and (2.16) leads to

$$|A| \leq d_1 + (d_0 - d_2)\,|\sigma|^q + d_0 q(|\sigma| + 1)^{q-1} \tag{2.17}$$

for every $A \in \partial W^*(\sigma)$. Thus, it follows for $t \in (0,1)$ and $\sigma(t) = \sigma_2 + t(\sigma_1 - \sigma_2)$ together with (2.6) that

$$\begin{aligned} |A_+(t)| &\leq d_1 + (d_0 - d_2)\,|\sigma(t)|^q + d_0 q(|\sigma(t)| + 1)^{q-1} \\ &\leq d_1 + (d_0 - d_2)(|\sigma_1| + |\sigma_2|)^q + d_0 q(|\sigma_1| + |\sigma_2| + 1)^{q-1} \\ &\overset{(2.6)}{\leq} d_1 + c(d_0 - d_2)(1 + |A_1|^p + |A_2|^p) + c(1 + |A_1| + |A_2|). \end{aligned} \tag{2.18}$$

Here, $c > 0$ is a constant and $A_1 \in \partial W^*(\sigma_1)$ and $A_2 \in \partial W^*(\sigma_2)$ are arbitrary. Furthermore, we have used that $(|A| + |B|)^\alpha \leq c_\alpha(|A|^\alpha + |B|^\alpha)$ for $\alpha > 0$, see, e.g., [32]. Together with (2.13) we obtain finally

$$W^*(\sigma_1) - W^*\ (\sigma_2)\ - A_2 : (\sigma_1 - \sigma_2) \geq \frac{\kappa\,|\sigma_1 - \sigma_2|^r}{1 + (d_0 - d_2)(|A_1|^{ps} + |A_2|^{ps}) + |A_1|^s + |A_2|^s} \tag{2.19}$$

for every $\sigma_1, \sigma_2 \in \mathbb{R}^{m \times d}$ and every $A_i \in \partial W^*(\sigma_i)$ with a constant $\kappa > 0$ which is independent of σ_i and A_i. This proves (2.10) with $\delta = d_0 - d_2 \geq 0$. If $c_0 = c_2$, then $\delta = d_0 - d_2 = 0$. □

The existence of minimizers of problem (1.1) follows with standard arguments from the direct method in the calculus of variations, see, e.g., [13].

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and assume that the energy density $W : \mathbb{R}^{m \times d} \to \mathbb{R}$ satisfies* **H1–H3** *with $p \in (1, \infty)$. Furthermore, let $g \in W^{1,p}(\Omega)$, $f \in L^{p'}(\Omega)$ and $h \in \tilde{W}^{-\frac{1}{p'},p'}(\Gamma_N)$. If $\Gamma_D = \emptyset$, we require in addition that f and h satisfy the solvability condition $\int_\Omega fv\,\mathrm{d}x + \langle h, v\rangle_{W^{1-\frac{1}{p},p}(\partial\Omega)} = 0$ for every constant $v \in \mathbb{R}^m$. Then there exists $u \in g + V$ such that for every $v \in g + V$ we have*

$$I(u) \leq I(v) = \int_\Omega W(\nabla v(x))\,\mathrm{d}x - \int_\Omega fv\,\mathrm{d}x - \langle h, v\rangle_{W^{1-\frac{1}{p},p}(\Gamma_N)}. \tag{2.20}$$

Minimizers fulfill the weak Euler–Lagrange equations: For every $v \in V$ we have

$$\int_\Omega \mathrm{DW}(\nabla u(x)) : \nabla v(x)\,\mathrm{d}x = \int_\Omega fv\,\mathrm{d}x + \langle h, v\rangle_{W^{1-\frac{1}{p},p}(\Gamma_N)}. \tag{2.21}$$

The lower bound $|\mathrm{DW}(A) - \mathrm{DW}(B)|^r$ in **H3** implies that the stress field σ is unique. With obvious changes the theorem remains true if the energy I in (2.20) is defined via the linearized strain tensor $\varepsilon(v) = \frac{1}{2}(\nabla v + (\nabla v)^\top)$ for $v : \mathbb{R}^d \to \mathbb{R}^d$. Now, $\sigma = \mathrm{DW}(\varepsilon(v))$.

2.3 Admissible domains

It is known from the regularity theory for weak solutions of linear elliptic equations that the global regularity does not only depend on the smoothness of the data but also on the geometry of the domain Ω. In this section we describe geometrical assumptions on Ω which enable us to apply the difference quotient technique for the derivation of global regularity results for σ. The geometrical assumptions depend on the boundary conditions and are a slight generalization of the assumptions in [16]. We first give an abstract definition of admissible domains and describe two and three dimensional examples afterwards.

Definition 2.2 (Cone) A set $\mathcal{K} \subset \mathbb{R}^d$ is a cone with vertex in $x_0 \in \mathbb{R}^d$ if there exists a simply connected, open and nonempty set $\mathcal{C} \subset \partial B_1(0) = \{ x \in \mathbb{R}^d\,;\, |x| = 1 \}$ such that $\mathcal{K} = \{ x \in \mathbb{R}^d \backslash \{x_0\}\,;\ (x - x_0)/|x - x_0| \in \mathcal{C} \}$.

Definition 2.3 (Admissible domain) Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ where Γ_D and Γ_N are open (possibly empty) and disjoint.

case $\overline{\Gamma_D} \cap \overline{\Gamma_N} = \emptyset$: Ω is an admissible domain if it has a Lipschitz boundary.
case $\overline{\Gamma_D} \cap \overline{\Gamma_N} \neq \emptyset$: Ω is an admissible domain if it has a Lipschitz boundary and if in addition there exists a finite number of open balls $B_{R_j}(x_j)$ with radius R_j and center $x_j \in \overline{\Gamma_D} \cap \overline{\Gamma_N}$ and a finite number of cones $\mathcal{K}_j \subset \mathbb{R}^d$ with vertex in 0 such that $\overline{\Gamma_D} \cap \overline{\Gamma_N} \subset \cup_{j=1}^J B_{R_j}(x_j)$. Furthermore, for every j there exist open domains $\Omega_D^j, \Omega_N^j \subset B_{R_j}(x_j)$ with $\Omega_D^j \cap \Omega_N^j = \emptyset$, $\overline{B_{R_j}(x_j) \backslash \Omega} = \overline{\Omega_D^j \cup \Omega_N^j}$ and

$$\Gamma_D \cap B_{R_j}(x_j) \subset \partial\Omega_D^j, \quad \Gamma_N \cap B_{R_j}(x_j) \subset \partial\Omega_N^j, \tag{2.22}$$

$$\left((B_{R_j}(x_j) \backslash \overline{\Omega_N^j}) + \mathcal{K}_j \right) \cap \Omega_N^j = \emptyset, \tag{2.23}$$

$$(\Omega_D^j + \mathcal{K}_j) \cap \left(B_{R_j}(x_j) \backslash \overline{\Omega_D^j} \right) = \emptyset, \tag{2.24}$$

see also Fig. 1 (the index j is omitted). Here, the notation $\Omega + \mathcal{K} = \{ y \in \mathbb{R}^d\,;\, y = x + h, x \in \Omega, h \in \mathcal{K} \}$ is used.

The cones $\mathcal{K}_j$ from Definition 2.3 play an important role in the proof of the stress regularity. They determine the vectors h with respect to which we extend weak solutions across the boundary and with respect to which we then define the difference quotients. It follows from (2.22) to (2.24) that $x + h \in (\Omega \cap B_{R_j}(x_j)) \cup \Omega_D^j$ for every $x \in \Omega \cap B_{R_j}(x_j)$ and $h \in \mathcal{K}_j$. This means that translations with $h \in \mathcal{K}_j$ could go across the Dirichlet boundary but not across the Neumann boundary.

Example 2.1 The domain in Fig. 2 is an admissible domain. In the neighborhood of the point S, the domains Ω_D, Ω_N and the cone $\mathcal{K}$ can be chosen as follows: $\Omega_D = \{ x \in \mathbb{R}^3\,;\, x_3 > 0 \}$, $\Omega_N = \{ x \in \mathbb{R}^3\,;\, x_i < 0, 1 \le i \le 3 \}$ and $\mathcal{K} = \{ h \in \mathbb{R}^3;\ h = \sum_{i=1}^3 r_i v_i,\ r_i > 0 \}$.

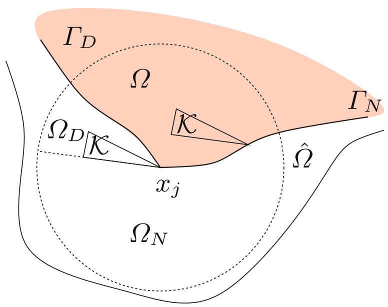
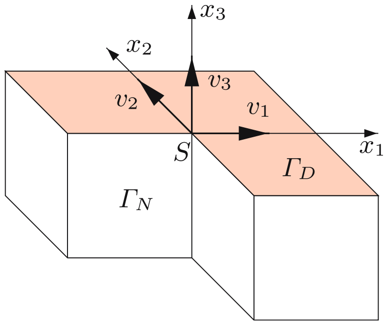


Fig. 1 Example for conditions (2.22)–(2.24)

Fig. 2 Example for an admissible domain



The next lemma describes further examples of admissible domains for $d = 2, 3$. The proof of this lemma is technical and is given in the appendix.

Lemma 2.3 1. *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz-polygon. Ω is admissible if and only if the interior opening angle at those points, where $\overline{\Gamma_D}$ and $\overline{\Gamma_N}$ intersect, is strictly less than π.*
2. [16] *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz-polyhedron where at most three faces intersect in the neighborhood of those points, where the type of the boundary conditions changes. Assume in addition that the interior opening angle between the Dirichlet and Neumann boundary is strictly less than π. Then Ω is an admissible domain.*

2.4 Stress regularity in the convex case

We are now ready to formulate the main result on the global regularity of stress fields of convex minimization problems with mixed boundary conditions on admissible domains. In the following, we denote by $\mathfrak{n}$ the exterior normal vector on $\partial\Omega$.

Theorem 2.2 *Let $\Omega \subset \mathbb{R}^d$ be an admissible domain and assume that $W : \mathbb{R}^{m\times d} \to \mathbb{R}$ satisfies* **H1–H3** *for $r, p > 1$, $s \geq 0$ with $\frac{rp}{p+s} > 1$. Let $\hat{\Omega} \supset\supset \Omega$ be an arbitrary domain and assume further that $f \in L^{p'}(\Omega)$, $g \in W^{2,p}(\hat{\Omega})$, $\nabla g \in L^\infty(\hat{\Omega})$ and $H \in W^{1,p'}(\hat{\Omega}, \mathbb{R}^{m\times d}) \cap L^\infty(\hat{\Omega})$. Let $u \in W^{1,p}(\Omega)$ be a minimizer of problem (2.20) with $u_{|\Gamma_D} = g_{|\Gamma_D}$ and $h = H_{|\Gamma_N}\mathfrak{n}$ on Γ_N.*

If $\nabla u \in L^\alpha(\Omega)$ for some $\alpha \geq p$ and if $\sigma = \mathrm{DW}(\nabla u) \in L^\gamma(\Omega)$ with $\gamma = \max\{p', \frac{\alpha r}{\alpha+s}\}$, then σ has the following global regularity for every $\delta > 0$:

$$\sigma = \mathrm{DW}(\nabla u) \in \mathcal{N}^{\frac{1}{r},\tau}(\Omega) \subset W^{\frac{1}{r}-\delta,\tau}(\Omega). \tag{2.25}$$

Here, $\tau = \frac{\alpha r}{\alpha+s} \geq \frac{pr}{p+s} > 1$.

Remark 2 The theorem remains true if ∇u is replaced by $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top)$ in the minimization problem (2.20).

In [9], Carstensen and Müller proved the local regularity result $\sigma \in W^{1,\tau}_{\mathrm{loc}}(\Omega)$ with $\tau = \frac{pr}{p+s}$ on the basis of monotonicity inequality (1.6). For scalar problems ($m = 1$) with $p = 2$, $r = 2$, $s = 0$, they obtained $\sigma \in W^{1,2}(\Omega)$ globally under the assumptions that Ω is a $C^{2,1}$-domain and that W depends on $|\nabla u|$, only. It is an interesting open problem whether the local result from [9] can be extended to domains with smooth boundaries for arbitrary energy densities satisfying (1.6) or (1.2).

A comparison of the result in Theorem 2.2 with well known results for linear elasticity and for systems of p-structure indicates that (2.25) is optimal within the class of admissible domains (see Sect. 2.5).

Proof We apply a difference quotient technique to deduce estimates for the stress fields in Nikolskii norms. For the derivation of these estimates the domain Ω is covered by a finite number of balls and the estimates are proved for each of these balls separately. The estimates are obtained by inserting suitable differences of weak solutions and shifted weak solutions into the weak formulation and by applying the convexity inequality. The main difficulty is that weak solutions have to be extended across the boundary of Ω in such a way that differences of weak solutions and shifted weak solutions are admissible test functions for the weak formulation. Due to the assumptions on the domain Ω it is possible to define such extensions. We partially take up the ideas from [17] in the proof.

Let $\Omega \subset \mathbb{R}^d$ be an admissible domain. In particular, Ω is a Lipschitz domain and satisfies therefore the uniform interior and exterior cone condition [25]. In view of Definition 2.3 it follows that there exists a finite number of balls $B_{R_j}(x_j)$ and cones $\mathcal{K}_j$ with vertices in 0 such that $\overline{\Omega} \subset \cup_{j=1}^{J} B_{R_j}(x_j)$ and each of the pairs $(B_{R_j}(x_j), \mathcal{K}_j)$ satisfies one of the following four cases:

1. $\overline{B_{R_j}(x_j)} \subset \Omega$.
2. $\overline{\left(B_{R_j}(x_j) \cap \partial\Omega\right)} \subset \Gamma_D$ and $((x+\mathcal{K}_j) \cap B_{R_j}(x_j)) \cap \overline{\Omega} = \emptyset$ for every $x \in B_{R_j}(x_j) \cap \Gamma_D$.
3. $\overline{\left(B_{R_j}(x_j) \cap \partial\Omega\right)} \subset \Gamma_N$ and $((x+\mathcal{K}_j) \cap B_{R_j}(x_j)) \subset \Omega$ for every $x \in B_{R_j}(x_j) \cap \overline{\Omega}$.
4. $x_j \in \overline{\Gamma_D} \cap \overline{\Gamma_N}$ and the pair $(B_{R_j}(x_j), \mathcal{K}_j)$ satisfies (2.22)–(2.24) of Definition 2.3 with suitable domains Ω_D^j and Ω_N^j.

Note that there exists $\theta > 0$ such that the balls $B_{R_j-\theta}(x_j)$ still cover $\overline{\Omega}$. We prove now that

$$\sigma\big|_{\Omega \cap B_{R_j-\theta}(x_j)} \in \mathcal{N}^{\frac{1}{r},\tau}(\Omega \cap B_{R_j-\theta}(x_j))$$

for every j and consider the fourth case in detail. The remaining cases can be treated similarly. In order to simplify the notation we omit the index j in the following.

Let B_R be a ball, $\mathcal{K}$ a cone with vertex in 0 and $\Omega_D, \Omega_N \subset B_R$ domains such that (2.22)–(2.24) of Definition 2.3 hold. Let $u \in W^{1,p}(\Omega)$ be a weak solution of minimization problem (2.20) with $f \in L^{p'}(\Omega)$, $g \in W^{2,p}(\hat{\Omega})$, $\nabla g \in L^\infty(\hat{\Omega})$ and $H \in W^{1,p'}(\hat{\Omega}, \mathbb{R}^{m\times d}) \cap L^\infty(\hat{\Omega})$. Note that the Neumann term in (2.20) can be rewritten as

$$\langle h, v\rangle_{W^{1-\frac{1}{p},p}(\Gamma_N)} = \langle H\mathfrak{n}, v\rangle_{W^{1-\frac{1}{p},p}(\Gamma_N)} = \int_\Omega v \operatorname{div} H \,\mathrm{d}x + \int_\Omega H : \nabla v \,\mathrm{d}x \qquad (2.26)$$

for $v \in V$. Let $\Omega_0 = \operatorname{int}\left(\overline{\Omega \cap B_R} \cup \overline{\Omega}_D\right) = B_R \backslash \overline{\Omega}_N$ and assume that $\Omega_0 \subset \hat{\Omega}$, see Fig. 1. We extend u to Ω_D as follows:

$$\tilde{u}(x) = \begin{cases} u(x), & x \in \Omega, \\ g(x), & x \in \Omega_0 \backslash \Omega. \end{cases} \qquad (2.27)$$

Since $u_{|\Gamma_D} = g_{|\Gamma_D}$ it follows that $\tilde{u} \in W^{1,p}(\Omega \cup \Omega_0)$. Choose $\eta \in C_0^\infty(B_R)$ with $\eta = 1$ on $B_{R-\theta}$ and define for $x \in \Omega$ and $h \in \mathcal{K}$ with $|h| < h_0 = \frac{1}{2}\,\mathrm{dist}(\mathrm{supp}\,\eta, \partial B_R)$:

$$\begin{aligned} v(x) &= \eta^2(x)\left(\tilde{u}(x+h) - g(x+h) - (\tilde{u}(x) - g(x))\right) \\ &= \eta^2(x)\triangle_h(\tilde{u}(x) - g(x)). \end{aligned} \tag{2.28}$$

Here, we use the notation $\triangle_h w(x) = w(x+h) - w(x)$ for $h \in \mathbb{R}^d$. Note that $v \in W_0^{1,p}(\Omega)$ and therefore v is an admissible test function for the weak formulation (2.21). Assume that $\nabla u \in L^\alpha(\Omega)$ for some $\alpha \geq p$ and let $\tau = \frac{\alpha r}{\alpha + s}$. It follows from convexity inequality (2.7) with $A = \nabla\tilde{u}(x+h)$, $B = \nabla\tilde{u}(x)$ and Hölder's inequality with $\frac{r}{\tau} \geq 1$ that

$$\begin{aligned} \int_\Omega \eta^{\frac{4\tau}{r}} \left|\triangle_h \mathrm{DW}(\nabla\tilde{u})\right|^\tau \mathrm{d}x &\overset{(2.7)}{\leq} c\int_\Omega \eta^{\frac{4\tau}{r}} \left(1 + \left|\nabla\tilde{u}\right|^s + \left|\nabla\tilde{u}(x+h)\right|^s\right)^{\frac{\tau}{r}} \\ &\quad \times \left(\triangle_h W(\nabla\tilde{u}) - \mathrm{DW}(\nabla\tilde{u}) : \triangle_h\nabla\tilde{u}\right)^{\frac{\tau}{r}} \mathrm{d}x \\ &\leq c\left(\int_\Omega \eta^{\frac{2\tau}{r-\tau}} \left(1 + \left|\nabla\tilde{u}\right|^s + \left|\nabla\tilde{u}(x+h)\right|^s\right)^{\frac{\tau}{r-\tau}} \mathrm{d}x\right)^{\frac{r-\tau}{r}} \\ &\quad \times \left(\int_\Omega \eta^2 \left(\triangle_h W(\nabla\tilde{u}) - \mathrm{DW}(\nabla\tilde{u}) : \triangle_h\nabla\tilde{u}\right) \mathrm{d}x\right)^{\frac{\tau}{r}} \\ &= cI_1 I_2. \end{aligned} \tag{2.29}$$

It is $s\tau(r-\tau)^{-1} = \alpha$ and since $\nabla u \in L^\alpha(\Omega)$ and $\nabla g \in L^\infty(\hat{\Omega})$, the factor I_1 is bounded independently of $h \in \mathcal{K}$. Therefore, there exists a constant $c > 0$ such that for every $h \in \mathcal{K}$ with $|h| < h_0$

$$\begin{aligned} c\left\|\eta^{\frac{4}{r}}\left|\triangle_h \mathrm{DW}(\nabla\tilde{u})\right|\right\|^r_{L^\tau(\Omega)} &\leq \int_\Omega \eta^2 \triangle_h W(\nabla\tilde{u})\mathrm{d}x - \int_\Omega \eta^2 \mathrm{DW}(\nabla\tilde{u}) : \triangle_h\nabla\tilde{u}\,\mathrm{d}x \\ &= I_{21} + I_{22}. \end{aligned} \tag{2.30}$$

We now prove that $|I_{21}| + |I_{22}| \leq c\,|h|$ for a constant $c > 0$ which is independent of $h \in \mathcal{K}$. Due to the product rule for differences, $\triangle_h(f(x)g(x)) = f(x)\triangle_h g(x) + g(x+h)\triangle_h f(x)$, we obtain for I_{21}

$$\begin{aligned} I_{21} &= \int_\Omega \triangle_h\left(\eta^2 W(\nabla\tilde{u})\right) \mathrm{d}x - \int_\Omega W(\nabla\tilde{u}(x+h))\triangle_h\eta^2\,\mathrm{d}x. \\ &= I_{211} + I_{212}. \end{aligned} \tag{2.31}$$

Note that $[\operatorname{supp}\eta \cup \operatorname{supp}\eta(\cdot + h)] \subset B_R$ for $h \in \mathcal{K}$ with $|h| < h_0$ and therefore we get taking into account the definition of $\tilde{u}$

$$\begin{aligned} I_{211} &= \int\limits_{\Omega\cap B_R} \eta^2(x+h)W(\nabla\tilde{u}(x+h))\,\mathrm{d}x - \int\limits_{\Omega} \eta^2 W(\nabla\tilde{u})\,\mathrm{d}x \\ &= \int\limits_{(\Omega+h)\cap B_R} \eta^2 W(\nabla\tilde{u})\,\mathrm{d}x - \int\limits_{\Omega\cap B_R} \eta^2 W(\nabla\tilde{u})\,\mathrm{d}x \\ &= \int\limits_{((\Omega+h)\backslash\Omega)\cap B_R} \eta^2 W(\nabla g)\,\mathrm{d}x - \int\limits_{((\Omega\backslash(\Omega+h))\cap B_R} \eta^2 W(\nabla u)\,\mathrm{d}x. \end{aligned} \tag{2.32}$$

From $\nabla g \in L^\infty(\hat{\Omega})$ and assumption **H2**, inequality (2.5), we obtain

$$\begin{aligned} I_{211} &\le |((\Omega+h)\backslash\Omega)\cap B_R| \left\|\eta^2 W(\nabla g)\right\|_{L^\infty(B_R)} - \int\limits_{(\Omega\backslash(\Omega+h))\cap B_R} \eta^2 \left(c_0 |\nabla u|^p - c_1\right)\,\mathrm{d}x \\ &\le c\,|h|(\|\eta^2 W(\nabla g)\|_{L^\infty(B_R)} + c_1) - c_0 \int\limits_{(\Omega\backslash(\Omega+h))\cap B_R} \eta^2\,|\nabla u|^p\,\mathrm{d}x \end{aligned} \tag{2.33}$$

and the constant c is independent of $h \in \mathcal{K}$. Since $\eta \in C_0^\infty(B_R)$, there exists $c(\eta) > 0$ such that

$$\left|\eta^2(x+h) - \eta^2(x)\right| \le c(\eta)\,|h|$$

for every $h \in \mathbb{R}^d$ and $x \in B_R$. Thus the term I_{212} can be estimated as follows:

$$|I_{212}| \le c(\eta)\,|h| \int\limits_{(\Omega\cup\Omega_D)\cap B_R} \left|W(\nabla\tilde{u})\right|\,\mathrm{d}x \le c\,|h|\,. \tag{2.34}$$

We obtain finally from (2.33) and (2.34) that

$$I_{21} = I_{211} + I_{212} \le c\,|h| - c_0 \int\limits_{(\Omega\backslash(\Omega+h))\cap B_R} \eta^2\,|\nabla u|^p\,\mathrm{d}x, \tag{2.35}$$

where c is independent of $h \in \mathcal{K}$ and c_0 is the constant from (2.5). Applying the product rule to I_{22} and taking into account (2.26) and that $v = \eta^2\triangle_h(\tilde{u} - g)$ is an admissible test function for the weak formulation (2.21) we obtain

$$
\begin{aligned}
I_{22} &= -\int_\Omega \eta^2 \mathrm{DW}(\nabla\tilde{u}) : \triangle_h \nabla\tilde{u}\,\mathrm{d}x \\
&= -\int_\Omega \mathrm{DW}(\nabla\tilde{u}) : \nabla\left(\eta^2\triangle_h(\tilde{u}-g)\right)\mathrm{d}x - \int_\Omega \mathrm{DW}(\nabla\tilde{u}) : \nabla(\eta^2\triangle_h g)\mathrm{d}x \\
&\quad + \int_\Omega \mathrm{DW}(\nabla\tilde{u}) : (\triangle_h\tilde{u}\otimes\nabla\eta^2)\,\mathrm{d}x \\
&= -\int_\Omega \eta^2(f+\operatorname{div} H)\triangle_h(\tilde{u}-g)\,\mathrm{d}x - \int_\Omega H : \nabla(\eta^2\triangle_h\tilde{u})\,\mathrm{d}x \\
&\quad + \int_\Omega H : \nabla(\eta^2\triangle_h g)\,\mathrm{d}x \\
&\quad + \int_\Omega \mathrm{DW}(\nabla\tilde{u}) : (\triangle_h\tilde{u}\otimes\nabla\eta^2)\,\mathrm{d}x - \int_\Omega \mathrm{DW}(\nabla\tilde{u}) : \nabla(\eta^2\triangle_h g)\,\mathrm{d}x \\
&= I_{221} + \ldots + I_{225}.
\end{aligned}
\tag{2.36}
$$

Here, $(\triangle_h\tilde{u}\otimes\nabla\eta^2)_{ij} = (\partial_i\eta)(\triangle_h\tilde{u}_j)$ and $\tilde{u}_j$ is the j-th component of $\tilde{u}$. Lemma 7.23 in [23] implies for $\tilde{u}, g \in W^{1,p}(\Omega\cup\Omega_0)$ and $h \in \mathcal{K}$ that

$$
\left\|\eta\triangle_h(\tilde{u}-g)\right\|_{L^p((\Omega\cup\Omega_0)\cap B_R)} \le |h|\,\|u-g\|_{L^p(\Omega\cap B_R)}\,\|\eta\|_{L^\infty(B_R)}.
$$

Therefore, there exists a constant $c > 0$ which is independent of $h \in \mathcal{K}$ such that

$$
\begin{aligned}
|I_{221}| + |I_{224}| &\le \|f+\operatorname{div} H\|_{L^{p'}(\Omega)} \left\|\eta\triangle_h(\tilde{u}-g)\right\|_{L^p((\Omega\cup\Omega_0)\cap B_R)} \\
&\quad + \|\mathrm{DW}(\nabla u)\|_{L^{p'}(\Omega)}\, c(\eta)\left\|\eta\triangle_h\tilde{u}\right\|_{L^p((\Omega\cup\Omega_0)\cap B_R)} \\
&\le c\,|h|.
\end{aligned}
\tag{2.37}
$$

Since $g \in W^{2,p}(\hat{\Omega})$, similar arguments show that

$$
|I_{223}| + |I_{225}| \le c\,|h|. \tag{2.38}
$$

In order to estimate I_{222} we apply again the product rules for differences and derivatives:

$$
\begin{aligned}
I_{222} &= -\int_\Omega H : \nabla(\eta^2\triangle_h\tilde{u})\,\mathrm{d}x \\
&= -\int_\Omega H : \left(\triangle_h\tilde{u}\otimes\nabla\eta^2\right)\mathrm{d}x - \int_\Omega \triangle_h\left(\eta^2 H : \nabla\tilde{u}\right)\mathrm{d}x \\
&\quad + \int_\Omega (\triangle_h(\eta^2 H)) : \nabla\tilde{u}(x+h)\,\mathrm{d}x.
\end{aligned}
\tag{2.39}
$$

The first term can be treated similarly to I_{224}, the third term similarly to I_{221}. The second term can be transformed as follows [compare also (2.32)]:

$$
\int_\Omega \triangle_h(\eta^2 H:\nabla\tilde{u})\mathrm{d}x = \int_{(\Omega+h\setminus\Omega)\cap B_R} \eta^2 H:\nabla g\,\mathrm{d}x - \int_{(\Omega\setminus(\Omega+h))\cap B_R} \eta^2 H:\nabla u\,\mathrm{d}x. \tag{2.40}
$$

Since $\nabla g, H \in L^\infty(\hat{\Omega}, \mathbb{R}^{m\times d})$, we obtain

$$\left| \int\limits_{(\Omega+h\backslash\Omega)\cap B_R} \eta^2 H : \nabla g \,\mathrm{d}x \right| \le \left| \left((\Omega+h)\backslash\Omega\right)\cap B_R \right| \left\| \eta^2 H : \nabla g \right\|_{L^\infty(\hat{\Omega})} \le c\,|h| \quad (2.41)$$

and c is independent of $h \in \mathcal{K}$. By Hölder's and Young's inequality we get for the second term in (2.40) for every $\epsilon > 0$

$$\begin{aligned} &\left| \int\limits_{(\Omega\backslash(\Omega+h))\cap B_R} \left(\epsilon^{-1}\eta^{\frac{2}{p'}} H\right) : \left(\epsilon\eta^{\frac{2}{p}} \nabla u\right) \mathrm{d}x \right| \\ &\quad \le \int\limits_{(\Omega\backslash(\Omega+h))\cap B_R} \tfrac{1}{p'} \left| \epsilon^{-1}\eta^{\frac{2}{p'}} H \right|^{p'} \mathrm{d}x + \int\limits_{(\Omega\backslash(\Omega+h))\cap B_R} \tfrac{\epsilon^p}{p}\eta^2 \left|\nabla u\right|^p \mathrm{d}x \qquad (2.42) \end{aligned}$$

and the first term is bounded by $c\,|h|$, where c depends on $\epsilon > 0$. Estimates (2.39)–(2.42) imply that

$$|I_{222}| \le c\,|h| + \frac{\epsilon^p}{p} \int\limits_{(\Omega\backslash\Omega+h)\cap B_R} \eta^2 \left|\nabla u\right|^p \,\mathrm{d}x. \qquad (2.43)$$

Collecting the above estimates [inequalities (2.35), (2.37), (2.38), (2.43)] we obtain finally that there exists a constant $c > 0$ such that it holds for every $h \in \mathcal{K}$, $|h| < h_0$

$$\begin{aligned} \left\| \eta^{\frac{4}{r}} \left|\triangle_h \mathrm{DW}(\nabla\tilde{u})\right| \right\|^r_{L^\tau(\Omega)} &\le I_{21} + I_{221} + I_{222} + I_{223} + I_{224} + I_{225} \\ &\le c\,|h| + \left(p^{-1}\epsilon^p - c_0\right) \int\limits_{(\Omega\backslash(\Omega+h))\cap B_R} \eta^2 \left|\nabla u\right|^p \,\mathrm{d}x. \end{aligned}$$

Choosing $0 < \epsilon < (pc_0)^{\frac{1}{p}}$ and taking into account that $\eta\big|_{B_{R-\theta}} = 1$ we get

$$\|\triangle_h \mathrm{DW}(\nabla u)\|^r_{L^\tau(\Omega\cap B_{R-\theta})} \le c\,|h|$$

for every $h \in \mathcal{K}$, $|h| < h_0$, with a constant c which is independent of h. This implies that $\sigma = \mathrm{DW}(\nabla u) \in \mathcal{N}^{\frac{1}{r},\tau}(\Omega \cap B_{R-\theta}) \subset W^{\frac{1}{r}-\delta,\tau}(\Omega\cap B_{R-\theta})$ for every $\delta > 0$, see the definition of the Nikolskii norm and Lemma 2.1. This finishes the proof. □

2.5 Convex examples

As examples for energy densities which satisfy the convexity inequality (2.7) we consider the energy densities of linear elastic materials, of a variational functional from the deformation theory of plasticity and the energy densities describing equations or systems of p-structure. A typical example here is the p-Laplace equation.

2.5.1 Linear elasticity

The energy density for linear elastic materials with elasticity tensor $\mathbf{C} \in \mathbb{R}^{(d\times d)\times(d\times d)}$, symmetric and positive definite, is given by $W(\varepsilon) = \frac{1}{2}\mathbf{C}\varepsilon : \varepsilon$ for $\varepsilon \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. Obviously, it holds due to the positive definiteness of $\mathbf{C}$ that

$$W(\varepsilon_1) - W(\varepsilon_2) - \mathrm{DW}(\varepsilon_2) : (\varepsilon_1 - \varepsilon_2) \ge c\,|\mathbf{C}\varepsilon_1 - \mathbf{C}\varepsilon_2|^2$$

and thus $p = 2, s = 0, r = 2$ in (2.7).

Corollary 2.1 *Let $\Omega \subset \mathbb{R}^d$ be an admissible domain and let u be a solution of the minimization problem for linear elasticity. It follows for every $\delta > 0$ that $\sigma = \mathbf{C}\varepsilon(u) \in \mathcal{N}^{\frac{1}{2},2}(\Omega) \subset W^{\frac{1}{2}-\delta,2}(\Omega)$.*

This result is well known and optimal for boundary value problems with pure Dirichlet or pure Neumann conditions [15]. For polyhedral domains Ω, the behavior of displacement and stress fields near corners and edges can be characterized completely by asymptotic expansions [15,31,35]. Let $\Omega \subset \mathbb{R}^2$ be a polygon with mixed boundary conditions and suppose that $\mathbf{C}$ describes an isotropic material. It is shown in [38,43] by a careful study of the asymptotic expansions that $\sigma \in W^{\frac{1}{2},2}(\Omega)$ if Ω is an admissible domain, i.e., if $\measuredangle(\Gamma_D, \Gamma_N) < \pi$ at every point $S \in \overline{\Gamma_D} \cap \overline{\Gamma_N}$. Moreover, if $S \in \overline{\Gamma_D} \cap \overline{\Gamma_N}$ with $\measuredangle(\Gamma_D, \Gamma_N) > \pi$, then weak solutions exist with $\sigma \in W^{\alpha-\delta,2}(\Omega)$ for an appropriate $0 < \alpha < \frac{1}{2}$ and every $\delta > 0$ but not for $\delta = 0$. The parameter α depends on the material parameters and the opening angle at S and $\alpha \to \frac{1}{2}$ as $\measuredangle(\Gamma_D, \Gamma_N) \to \pi$. This example shows the optimality of Corollary 2.1 for admissible domains.

2.5.2 Hencky elasto-plasticity with linear hardening

For $\varepsilon \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ we define as in [22]

$$W(\varepsilon) = \frac{1}{2}\kappa_0(\operatorname{tr}\varepsilon)^2 + g_0(|\varepsilon^D|), \tag{2.44}$$

where $\kappa_0 > 0$ is a constant and $\varepsilon^D = \varepsilon - \frac{1}{d}\operatorname{tr}\varepsilon I$ is the deviatoric part of ε. It is assumed that $g_0 \in \mathcal{C}^1(\mathbb{R}) \cap \mathcal{C}^2(\mathbb{R}\backslash\{t_0\})$ for some $t_0 > 0$ and the left and right limits of g_0'' exist at t_0. The quantity $g_0'(t_0)$ may be interpreted in this context as yield stress. Furthermore, we suppose that there exist constants $\kappa_1, \kappa_2 > 0$ such that for every $t \in \mathbb{R}$

$$\kappa_1 \le \min\left\{g_0''(t), t^{-1}g_0'(t)\right\} \le \max\left\{g_0''(t), t^{-1}g_0'(t)\right\} \le \kappa_2. \tag{2.45}$$

It follows with Taylor's expansion that $c_0t^2 - c_1 \le g_0(t) \le c_2(1+t^2)$ for every t and some constants $c_i > 0$. The variational problem related to energy density (2.44) is

Find $u \in W^{1,2}(\Omega)$ with $u_{|\Gamma_D} = g_{|\Gamma_D}$ such that for every $v \in W^{1,2}(\Omega)$ with $v_{|\Gamma_D} = g_{|\Gamma_D}$ we have $I(u) \le I(v)$, where

$$I(v) = \int_\Omega \frac{1}{2}\kappa_0(\operatorname{tr}\varepsilon(v))^2 + g_0(|\varepsilon^D(v)|)\,\mathrm{d}x - \int_\Omega fv\,\mathrm{d}x - \langle h, v\rangle_{W^{\frac{1}{2},2}(\Gamma_N)}. \tag{2.46}$$

Functionals of this type describe in the framework of deformation theory of plasticity the behavior of materials with linear hardening. The local regularity of stress fields corresponding to minimizers of (2.46) is studied in [22,46].

Lemma 2.4 *Energy density W from (2.44) satisfies the convexity inequality (2.7) on $\mathbb{R}^{d\times d}_{sym}$ with $s = 0$ and $r = 2$.*

Proof Let $\varepsilon_1, \varepsilon_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$, $\varepsilon_1 \ne \varepsilon_2$ and $\theta(s) = \varepsilon_2 + s(\varepsilon_1 - \varepsilon_2)$, $s \in [0,1]$. Note that there are at most two elements $s_i \in [0,1]$ with $|\theta(s_i)| = t_0$. Therefore, we may apply Taylor's expansion at least piecewise on $[0,1]$ and obtain

$$\begin{aligned}
&W(\varepsilon_1) - W(\varepsilon_2) - \mathrm{D}W(\varepsilon_2) : (\varepsilon_1 - \varepsilon_2) \\
&= \frac{\kappa_0}{2} |\mathrm{tr}(\varepsilon_1 - \varepsilon_2)|^2 + \int_0^1 (1-s) \frac{g_0''(|\theta^D(s)|)}{|\theta^D(s)|^2} \left(\theta^D(s) : (\varepsilon_1 - \varepsilon_2)^D\right)^2 \mathrm{d}s \\
&\quad + \int_0^1 (1-s) \frac{g_0'(|\theta^D(s)|)}{|\theta^D(s)|} \left(\left|\varepsilon_1^D - \varepsilon_2^D\right|^2 - \frac{(\theta^D(s) : (\varepsilon_1^D - \varepsilon_2^D))^2}{|\theta^D(s)|^2} \right) \mathrm{d}s \\
&\overset{(2.45)}{\geq} \frac{\kappa_0}{2} |\mathrm{tr}(\varepsilon_1 - \varepsilon_2)|^2 + \int_0^1 (1-s)\kappa_1 (\theta^D(s) : (\varepsilon_1^D - \varepsilon_2^D))^2 |\theta^D(s)|^{-2} \mathrm{d}s \\
&\quad + \int_0^1 (1-s)\kappa_1 \left(\left|\varepsilon_1^D - \varepsilon_2^D\right|^2 - (\theta^D(s) : (\varepsilon_1^D - \varepsilon_2^D))^2 |\theta^D(s)|^{-2} \right) \mathrm{d}s \\
&= \frac{1}{2} \left(\kappa_0 |\mathrm{tr}(\varepsilon_1 - \varepsilon_2)|^2 + \kappa_1 \left|\varepsilon_1^D - \varepsilon_2^D\right|^2 \right).
\end{aligned} \tag{2.47}$$

In a similar way it follows again by (2.45) that there exists a constant $c > 0$ with

$$\begin{aligned}
|\mathrm{D}W(\varepsilon_1) - \mathrm{D}W(\varepsilon_2)| &\leq \int_0^1 \left| D^2 W(\theta(s))(\varepsilon_1 - \varepsilon_2) \right| \mathrm{d}s \\
&\leq c \left(|\mathrm{tr}(\varepsilon_1 - \varepsilon_2)| + \left|\varepsilon_1^D - \varepsilon_2^D\right| \right).
\end{aligned} \tag{2.48}$$

Combining (2.47) and (2.48) finishes the proof. □

Corollary 2.2 *Let $\Omega \subset \mathbb{R}^d$ be an admissible domain and let $u \in W^{1,2}(\Omega)$ be a minimizer of (2.46) with data $f, g, h = H\mathfrak{n}$ as in Theorem 2.2 ($p = 2$). Then $\sigma = \mathrm{D}W(\varepsilon(u)) \in \mathcal{N}^{\frac{1}{2},2}(\Omega) \cap W^{\frac{1}{2}-\delta,2}(\Omega)$ for every $\delta > 0$.*

Remark 3 The function

$$W_\eta(\varepsilon) = \tfrac{1}{2}\, \mathbf{C}\varepsilon : \varepsilon - (4\mu(1+\eta))^{-1} \max\left\{0, \left|(\mathbf{C}\varepsilon)^D\right| - \sigma_y\right\}^2$$

from [4,48] fits into this framework. Here, $\mathbf{C}$ is the elasticity tensor for isotropic materials, μ a Lamé constant, $\sigma_y > 0$ the yield stress and $\eta > 0$ a hardening parameter. It is shown in [48, Chap. III.1.3] that the stresses σ_η converge for $\eta \to 0$ to a stress field σ_H which corresponds to the elastic, perfect-plastic Hencky model. One could now try to carry over the regularity results for σ_η from Corollary 2.2 to σ_H. By a different approximation of the Hencky stress σ_H, namely the Norton/Hoff or Ramberg/Osgood approximation, it is shown that $\sigma_H \in W^{\frac{1}{2}-\delta,2}(\Omega) \cap W^{1,2}_{\mathrm{loc}}(\Omega)$ for every $\delta > 0$ on admissible domains, see [6] for the local and [29] for the global result. This correlates well with Corollary 2.2. Interestingly, a global result for σ_H on smooth domains is not proved yet, see the discussion in [47].

2.5.3 Systems of p-structure

Let $p \in (1, \infty)$ and assume that $W : \mathbb{R}^{m\times d} \to \mathbb{R}$ satisfies **H1**, **H2** and in addition **H4** here below:

H4 $W \in \mathcal{C}^2(\mathbb{R}^{m\times d}\backslash\{0\}, \mathbb{R})$ and there exist constants $c_1, c_2 > 0$, $\kappa \in \{0, 1\}$ such that we have for every $A, B \in \mathbb{R}^{m\times d}$, $A \neq 0$,

$$\left|D^2W(A)\right| \leq c_1(\kappa + |A|)^{p-2}, \tag{2.49}$$

$$D^2W(A)[B, B] = \sum_{k,j=1}^{m}\sum_{r,s=1}^{d} \frac{\partial^2 W(A)}{\partial A_{ks}\partial A_{jr}} B_{ks}B_{jr} \geq c_2(\kappa + |A|)^{p-2}\,|B|^2\,. \tag{2.50}$$

Here, the notation$(D^2W(A))_{ksjr} = \frac{\partial^2 W(A)}{\partial A_{ks}\partial A_{jr}}$ with $D^2W(A) \in \mathbb{R}^{(m\times d)\times(m\times d)}$ is used. Energy densities of this type lead to quasilinear elliptic systems of p-structure. Note that the function $W(A) = \frac{1}{p}\,|A|^p$, $A \in \mathbb{R}^d$, which corresponds to the p-Laplace equation, satisfies **H4** with $\kappa = 0$.

Lemma 2.5 *Assume that W satisfies* **H1**, **H2** *and* **H4** *for some* $p \in (1, \infty)$ *and let* $q = p' = \frac{p}{p-1}$. *If* $p \geq 2$, *then convexity inequality* (2.7) *holds with* $r = 2$ *and* $s = p - 2$. *If* $p \in (1, 2]$, *then the convexity inequality holds with* $r = p' = q$ *and* $s = 0$.

Proof Let $p \in (1, \infty)$ be arbitrary. Due to [2,24] there exist for every $\beta > -1$ and every $\kappa \geq 0$ constants $c_1, c_2 > 0$ such that

$$c_1(\kappa + |A| + |B|)^{\beta} \leq \int_0^1 (\kappa + |B + t(A - B)|)^{\beta}\,\mathrm{d}t \leq c_2(\kappa + |A| + |B|)^{\beta} \tag{2.51}$$

for every $A, B \in \mathbb{R}^{m\times d}$ with $|A|+|B| > 0$ if $\kappa = 0$. Thus it follows by Taylor's expansion together with (2.50) that

$$\begin{aligned} &W(A) - W(B) - \mathrm{D}W(B) : (A - B) \\ &\quad = \int_0^1 (1 - t)D^2W(B + t(A - B))[A - B, A - B]\,\mathrm{d}t \\ &\quad \geq c(\kappa + |A| + |B|)^{p-2}\,|A - B|^2\,. \end{aligned} \tag{2.52}$$

Furthermore, Taylor's expansion, inequality (2.51) and assumption (2.49) imply

$$\begin{aligned} |\mathrm{D}W(A) - \mathrm{D}W(B)| &\leq \int_0^1 |D^2W(B + t(A - B))|\,\mathrm{d}t\,|A - B| \\ &\leq c(\kappa + |A| + |B|)^{p-2}\,|A - B|\,. \end{aligned} \tag{2.53}$$

Combining (2.52) and (2.53) finishes the proof for $p \geq 2$. If $p \leq 2$, it holds for $q = p' \geq 2$:

$$|A - B|^q \leq (\kappa + |A| + |B|)^{q-2}\,|A - B|^2 \tag{2.54}$$

and thus, together with (2.53),

$$|\mathrm{DW}(A) - \mathrm{DW}(B)|^q \leq (\kappa + |A| + |B|)^{q(p-2)+q-2}\,|A - B|^2\,. \tag{2.55}$$

This finishes the proof for $p \leq 2$ since $q(p-2) + q - 2 = p - 2$. □

The following global regularity results are available for weak solutions $u \in W^{1,p}(\Omega)$ of problem (2.20) on admissible domains with energy densities W satisfying **H1**, **H2** and **H4** [17,18,45]:

$$u \in W^{1+\frac{1}{p}-\delta,p}(\Omega) \qquad \text{if } p \in [2,\infty),$$
$$u \in W^{\frac{3}{2}-\delta,\frac{2dp}{2d-2+p}}(\Omega) \quad \text{if } p \in (1,2]$$

for every $\delta > 0$. In both cases, the Sobolev embedding theorems yield $\nabla u \in L^{\frac{dp}{d-1}-\delta}(\Omega)$. From **H2** and lemma 2.5 we conclude that $\sigma \in L^{\frac{dq}{d-1}-\delta}(\Omega)$ for $q = p'$ and for every $\delta > 0$.

Corollary 2.3 *Let $p \in (1,\infty)$, $q = p' = \frac{p}{p-1}$ and let $u \in W^{1,p}(\Omega)$ be a minimizer of* (2.20) *with an energy density satisfying* **H1**, **H2** *and* **H4** *on an admissible domain $\Omega \subset \mathbb{R}^d$. Let the data $f, g, h = H\mathfrak{n}$ be given according to Theorem* 2.2 *with $\alpha = \frac{dp}{d-1}$. Then it holds for the stress $\sigma = \mathrm{DW}(\nabla u)$ and every $\delta > 0$*

$$\sigma = \mathrm{DW}(\nabla u) \in W^{\frac{1}{2}-\delta,\frac{2dq}{2d-2+q}}(\Omega) \qquad \text{if } p \geq 2, \tag{2.56}$$

$$\sigma = \mathrm{DW}(\nabla u) \in \mathcal{N}^{\frac{1}{q},q}(\Omega) \subset W^{\frac{1}{q}-\delta,q}(\Omega) \quad \text{if } p \in (1,2]. \tag{2.57}$$

Let $\Omega \subset \mathbb{R}^2$ be an admissible polygon and assume that the stress $\sigma = |\nabla u|^{p-2}\,\nabla u$ corresponds to a weak solution of the p-Laplace equation

$$\operatorname{div} |\nabla u|^{p-2}\,\nabla u + f = 0$$

with $p \in (1,\infty)$, $q = p'$. Assume furthermore that σ is of the form $\sigma = r^\gamma \sigma_0(\varphi)$, where (r,φ) are polar coordinates with respect to a corner point S and $\sigma_0 \not\equiv 0$. By [44, Lemma 2.3.1] we obtain that σ is an element of the spaces in Corollary 2.3 if and only if $\gamma \geq -1/q$. In [5] a weak solution u of the p-Laplace equation is constructed for a domain with a crack and vanishing Neumann conditions on both crack faces, where $\gamma = -1/q$. This indicates the optimality of Theorem 2.2 also for nonlinear elliptic equations of p-structure. We finally remark that global regularity results for u on smooth domains are derived, e.g., in [20,37,40].

3 Regularity for stresses of nonconvex variational problems

Nonconvex variational problems may fail to have minimizers and a relaxed problem is studied instead. This relaxed problem is in general defined through an energy density which is the quasi-convex envelope of the nonconvex energy density. Weak cluster points of infimizing sequences of the nonconvex problem are minimizers of the relaxed problem [2,13,21]. Moreover, if the nonconvex problem has a minimizer, then this minimizer is also a minimizer of the relaxed problem and the corresponding stress fields coincide under suitable assumptions on the energy densities. This relation is the key for carrying over regularity results from the convex case to minimizers of nonconvex problems. After a short description of these relations we formulate the regularity theorem and illustrate it with some examples.

3.1 Regularity for stress fields of nonconvex variational problems

Let $W \in \mathcal{C}(\mathbb{R}^{m\times d}, \mathbb{R})$ be an energy density satisfying growth condition (2.5) for some $p > 1$ and let I be the energy functional related to W, see (2.20). By W^{qc} and W^c we denote the quasi-convex and convex envelope of W, respectively, i.e., for $A \in \mathbb{R}^{m\times d}$

$$W^{qc}(A) = \sup\{g(A) : g \leq W \text{ and } g \text{ is quasi-convex}\}$$

and similar for W^c. For a definition of quasi-convexity we refer to Dacorogna's book [13]. Furthermore, we define for $v \in W^{1,p}(\Omega)$

$$I^{qc}(v) = \int_\Omega W^{qc}(\nabla v)\,\mathrm{d}x - \int_\Omega fv\,\mathrm{d}x - \langle h, v\rangle_{W^{1-\frac{1}{p},p}(\Gamma_N)},$$

where f, h are given as in Theorem 2.1; I^c is analogously defined. The following well known theorem describes the relation between minimizers of I^{qc} and infimizing sequences of I. For convenience we reformulate it here for our situation.

Theorem 3.1 [2,13,21] *Let $W \in \mathcal{C}(\mathbb{R}^{m\times d}, \mathbb{R})$ satisfy (2.5) for $p > 1$, $g \in W^{1,p}(\Omega)$, $f \in L^{p'}(\Omega)$, $h \in W^{-\frac{1}{p'},p'}(\Gamma_N)$ and assume that f, h satisfy the solvability condition if $\Gamma_D = \emptyset$. Then the minimization problem for I^{qc} on $g+V$, where V is the space defined in (2.1), has a minimizer $u^{qc} \in g+V$ and it holds $\inf_{v\in g+V} I(v) = I^{qc}(u^{qc})$. Furthermore, every weak cluster point of infimizing sequences of I is a minimizer of I^{qc}.*

Due to Theorem 3.1 we have

Lemma 3.1 *Let the assumptions of Theorem 3.1 be satisfied and suppose that $u \in g+V$ is a minimizer of I. Then $W(\nabla u) = W^{qc}(\nabla u)$ almost everywhere in Ω. Furthermore, let $\mathcal{M} = \{A \in \mathbb{R}^{m\times d};\, W(A) = W^{qc}(A)\}$ and assume that W and W^{qc} are differentiable on an open neighborhood of $\mathcal{M}$. Then $\mathrm{DW}(\nabla u) = \mathrm{DW}^{qc}(\nabla u)$ a.e. in Ω.*

Proof The first assertion of Lemma 3.1 follows from the definition of W^{qc} and Theorem 3.1. The second assertion can be shown as follows: Let $A \in \mathcal{M}$, $H \in \mathbb{R}^{m\times d}$ be arbitrary. Then

$$\mathrm{DW}(A) : H \geq \lim_{t\searrow 0} t^{-1}(W^{qc}(A+tH) - W^{qc}(A)) = \mathrm{DW}^{qc}(A) : H,$$

$$\mathrm{DW}(A) : H = \lim_{t\nearrow 0} t^{-1}(W(A+tH) - W(A)) \leq \mathrm{DW}^{qc}(A) : H.$$

Since H is arbitrary this implies $\mathrm{DW}(A) = \mathrm{DW}^{qc}(A)$. □

Lemma 3.1 and Theorem 2.2 imply the following regularity theorem for stress fields in the nonconvex case:

Theorem 3.2 *Let $\Omega \subset \mathbb{R}^d$ be an admissible domain and let $W \in \mathcal{C}(\mathbb{R}^{m\times d}, \mathbb{R})$ satisfy (2.5) for $p > 1$. Moreover, let W be differentiable on a neighborhood of $\mathcal{M}$ with $\mathcal{M}$ as in Lemma 3.1. Assume that the data f, g, H is given as in regularity theorem 2.2. Furthermore, we suppose that the convex envelope and the quasi-convex envelope of W coincide, $W^{qc} = W^c$, and that W^c satisfies* **H1**, **H2** *and convexity inequality* **H3** *with*

$s \geq 0$, $r > 1$ *and* $\tau = \frac{rp}{p+s} > 1$. *Let* u *be a minimizer of the minimization problem for* I *and assume that* $\mathrm{DW}^c(\nabla u) \in L^\gamma(\Omega)$ *with* $\gamma = \max\{p', \tau\}$. *Then* $\mathrm{DW}(\nabla u) = \mathrm{DW}^c(\nabla u)$ *and for every* $\delta > 0$

$$\sigma = \mathrm{DW}(\nabla u) \in \mathcal{N}^{\frac{1}{r},\tau}(\Omega) \subset W^{\frac{1}{r}-\delta,\tau}(\Omega).$$

The assumption $W^{qc} = W^c$ is automatically satisfied if $m = 1$ or $d = 1$. For $\min\{m, d\} \geq 2$, only a few examples with $W^{qc} = W^c$ are known and the equality does not hold in general. Thus, the assumption $W^{qc} = W^c$ is rather restrictive in the vectorial case. Note that Theorem 3.2 holds also if ∇u is replaced with $\varepsilon(u)$ in the definition of I.

3.2 Nonconvex examples

Typical examples for nonconvex energy densities with $W^{qc} = W^c$ are the scalar and vectorial two-well potentials (compatible case). It is shown in [9] that the convexified energy densities satisfy the monotonicity inequality. We prove here that convexity inequality (2.7) holds as well and, as a consequence, regularity Theorem 3.2 is applicable. Finally, we discuss a special case of the Ericksen–James energy density.

3.2.1 Scalar two-well potential, $m = 1$

The energy density of the scalar two-well potential reads for $A \in \mathbb{R}^d$ and fixed $A_1 \neq A_2 \in \mathbb{R}^d$

$$W(A) = |A - A_1|^2 \, |A - A_2|^2 \, .$$

Since $m = 1$, the convex and the quasi-convex envelopes W^c and W^{qc} coincide and

$$W^c(A) = \max\left\{|A - F|^2 - |G|^2, 0\right\}^2 + 4\big(|G|^2 \, |A - F|^2 - (G \cdot (A - F))^2\big),$$

where $G = (A_2 - A_1)/2$ and $F = (A_1 + A_2)/2$, see [10]. Furthermore, it is shown in [10] that W^c satisfies monotonicity inequality (1.6) with $p = 4$ and $r = s = 2$.

Lemma 3.2 *There exists a constant* $c > 0$ *such that it holds for every* $A \in \mathbb{R}^d$

$$|A| \leq c\left(1 + \left|\mathrm{DW}^c(A)\right|^{\frac{1}{3}}\right). \tag{3.1}$$

Therefore, W^c *satisfies convexity inequality* (2.7) *with* $p = 4$ *and* $r = s = 2$ *due to Lemma* 2.2.

Proof For $A \neq F$ it holds

$$\begin{aligned}\left|\mathrm{DW}^c(A)\right| &= \sup_{H \in \mathbb{R}^d \setminus \{0\}} \mathrm{DW}^c(A) : H \, |H|^{-1} \geq \mathrm{DW}^c(A) : (A - F) \, |A - F|^{-1} \\ &\geq 4 \max\left\{|A - F|^2 - |G|^2, 0\right\} |A - F| \, .\end{aligned}$$

Assume now that $|A - F| \geq |G|$. Young's inequality yields for every $\delta > 0$

$$\begin{aligned}\left|\mathrm{DW}^c(A)\right| &\geq 4 \, |A - F|^3 - \delta^{-1} 4 \, |G|^2 \, \delta \, |A - F| \\ &\geq \left(4 - \frac{\delta^3}{3}\right) |A - F|^3 - \frac{16}{3} \, |G|^3 \, \delta^{-\frac{3}{2}} .\end{aligned}$$

For $\delta_0 = 2^{\frac{2}{3}}$ it holds $c_1 := 4 - \frac{\delta_0^3}{3} > 0$. Moreover, it follows for $|A - F| \leq |G|$ that

$$\left(4 - \frac{\delta_0^3}{3}\right) |A - F|^3 - \frac{16}{3}\delta_0^{-\frac{3}{2}} |G|^3 \leq 0.$$

Therefore, we have $\left|\mathrm{DW}^c(A)\right| \geq c_1 |A - F|^3 - c_1$ for every $A \in \mathbb{R}^d$. With $|A - F| \geq |A| - |F|$ and applying once more Young's inequality we obtain finally (3.1). □

Corollary 3.1 *Let $\Omega \subset \mathbb{R}^d$ be an admissible domain and let u be a minimizer of I or I^c. Then it follows for the corresponding stress field for every $\delta > 0$: $\sigma \in \mathcal{N}^{\frac{1}{2},\frac{4}{3}}(\Omega) \subset W^{\frac{1}{2}-\delta,\frac{4}{3}}(\Omega)$.*

3.2.2 A vectorial two-well potential, $m = d$

For $\varepsilon_1 \neq \varepsilon_2 \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$ we consider the following energy densities

$$W_i(\varepsilon) = \frac{1}{2}\mathbf{C}(\varepsilon - \varepsilon_i) : (\varepsilon - \varepsilon_i) + W_i^0, \quad \varepsilon \in \mathbb{R}^{d\times d}_{\mathrm{sym}},\ i = 1, 2, \tag{3.2}$$

where $\mathbf{C}$ is the elasticity tensor for linear elastic materials and $W_i^0 \in \mathbb{R}$. Let

$$W(\varepsilon) = \min\{W_1(\varepsilon), W_2(\varepsilon)\}, \quad \varepsilon \in \mathbb{R}^{d\times d}_{\mathrm{sym}}. \tag{3.3}$$

The nonconvex function W describes in a geometrically linear framework the elastic strain energy density of a two-phase material with stress-free strains ε_i, see, e.g., [27, 30]. It is assumed that both phases have identical elasticity tensors. If the strains ε_1 and ε_2 are compatible, i.e., if there exist $a, b \in \mathbb{R}^d$ with $\varepsilon_1 - \varepsilon_2 = \frac{1}{2}(a \otimes b + b \otimes a)$, then the convex and quasi-convex envelopes of W coincide and are given by [30]

$$W^c(\varepsilon) = \begin{cases} W_2(\varepsilon) & \text{if } W_2(\varepsilon) + \gamma \leq W_1(\varepsilon), \\ W_3(\varepsilon) & \text{if } |W_1(\varepsilon) - W_2(\varepsilon)| \leq \gamma, \\ W_1(\varepsilon) & \text{if } W_1(\varepsilon) + \gamma \leq W_2(\varepsilon), \end{cases} \tag{3.4}$$

where $\gamma = \frac{1}{2}\mathbf{C}(\varepsilon_1 - \varepsilon_2) : (\varepsilon_1 - \varepsilon_2)$ and

$$W_3(\varepsilon) = \frac{1}{2}(W_2(\varepsilon) + W_1(\varepsilon)) - \frac{1}{4\gamma}(W_2(\varepsilon) - W_1(\varepsilon))^2 - \frac{\gamma}{4}. \tag{3.5}$$

It is shown in [11] that W^c satisfies the monotonicity inequality with $p = r = 2$ and $s = 0$. From Lemma 2.2 we obtain immediately that W^c satisfies also the convexity inequality (2.7) with $p = r = 2$ and $s = 0$.

Corollary 3.2 *Let $\Omega \subset \mathbb{R}^d$ be an admissible domain and u a minimizer of I or I^c. Then $\sigma = \mathrm{DW}(\varepsilon(u)) \in \mathcal{N}^{\frac{1}{2},2}(\Omega) \subset W^{\frac{1}{2}-\delta,2}(\Omega)$ for every $\delta > 0$.*

3.2.3 A special case of the Ericksen–James energy

The last example deals with a special case of the two dimensional Ericksen–James energy function [12]. Let $\kappa_1, \kappa_2 > 0$. For $A \in \mathbb{R}^{2\times 2}$ and $C = A^\top A$ we consider the function

$$W(A) = \kappa_1(\operatorname{tr} C - 2)^2 + \kappa_2 c_{12}^2 = \kappa_1(|A|^2 - 2)^2 + \frac{\kappa_2}{4}(a(A, A))^2, \tag{3.6}$$

where the bilinear form $a(\cdot,\cdot)$ is defined as

$$a(A,B) = a_{11}b_{12} + a_{12}b_{11} + a_{21}b_{22} + a_{22}b_{21}, \quad A,B \in \mathbb{R}^{2\times 2}. \tag{3.7}$$

Note that $2c_{12} = a(A,A)$. The complete Ericksen–James energy has the additional term

$$\kappa_3 \left(\frac{1}{4} \left(a_{11}^2 + a_{21}^2 - a_{12}^2 - a_{22}^2 \right)^2 - \varepsilon^2 \right)^2, \quad \kappa_3 > 0$$

and is applied to model crystalline microstructure, see [12] and the references therein. In this context, $u : \Omega \to \mathbb{R}^2$ is the deformation field, $W(\nabla u)$ the stored energy function of a two dimensional crystal and $C = \nabla u^\top \nabla u$ the right Cauchy–Green strain tensor. Let us emphasize that we consider here only the case $\kappa_3 = 0$ since the quasiconvex envelope is known only for that case. It is shown by Bousselsal and Brighi in [7] that the convex and the quasiconvex envelopes W^{qc} and W^c of W from (3.6) coincide for $\kappa_3 = 0$ and have the form

$$W^c(A) = \Phi_i(A) \quad \text{for } A \in \mathcal{M}_i,\ 1 \le i \le 4, \tag{3.8}$$

where $\cup_{i=1}^4 \mathcal{M}_i = \mathbb{R}^{2\times 2}$,

$$\begin{aligned}
\mathcal{M}_1 &= \{A \in \mathbb{R}^{2\times 2} : |a(A,A)| \le 2 - |A|^2\},\\
\mathcal{M}_2 &= \{A \in \mathbb{R}^{2\times 2} : \kappa_2\, |a(A,A)| \le 4\kappa_1(|A|^2 - 2)\},\\
\mathcal{M}_3 &= \{A \in \mathbb{R}^{2\times 2} : \kappa_2 a(A,A) \ge 4\kappa_1(|A|^2 - 2) \ge 0\\
&\qquad\qquad\qquad\qquad \text{or } a(A,A) \ge 2 - |A|^2 \ge 0\},\\
\mathcal{M}_4 &= \{A \in \mathbb{R}^{2\times 2} : -\kappa_2 a(A,A) \ge 4\kappa_1(|A|^2 - 2) \ge 0\\
&\qquad\qquad\qquad\qquad \text{or } -a(A,A) \ge 2 - |A|^2 \ge 0\}
\end{aligned}$$

and $\Phi_1(A) = 0$, $\Phi_2(A) = W(A)$, $\Phi_3(A) = \Phi_4(A)$ with

$$\Phi_3(A) = \frac{\kappa_1\kappa_2}{4\kappa_1 + \kappa_2}\left(|A|^2 - 2 + |a(A,A)| \right)^2.$$

Lemma 3.3 *W^c from (3.8) satisfies convexity inequality (2.7) with $p = 4$, $r = 2$, $s = 2$.*

Corollary 3.3 *Let $\Omega \subset \mathbb{R}^2$ be an admissible domain and u a minimizer of I or I^c with energy density W from (3.6) and W^c from (3.8), respectively. Assume that the data f, g, H is given according to theorem 2.2 with $p = 4$. Then we have $\sigma = \mathrm{DW}(\nabla u) \in \mathcal{N}^{\frac{1}{2},\frac{4}{3}}(\Omega) \subset W^{\frac{1}{2}-\delta,\frac{4}{3}}(\Omega)$ for every $\delta > 0$.*

Proof of Lemma 3.3 The proof of Lemma 3.3 is quite technical and we split it into two parts. In the first step we show that Φ_i satisfies the convexity inequality for every $A, B \in \mathcal{M}_i$, $1 \le i \le 4$. Putting these estimates together we show in the second step that W^c satisfies the convexity inequality for arbitrary $A, B \in \mathbb{R}^{2\times 2} = \cup_{i=1}^4 \mathcal{M}_i$.

Let $i = 2$ and $A, B \in \mathcal{M}_2$, $A \ne B$. It follows

$$\begin{aligned}
&\Phi_2(A) - \Phi_2(B) - D\Phi_2(B) : (A - B)\\
&\quad = \kappa_1(|A|^2 - |B|^2)^2 + \frac{\kappa_2}{4}(a(A,A) - a(B,B))^2\\
&\qquad + 2\kappa_1(|B|^2 - 2)\,|A - B|^2 + \frac{\kappa_2}{2} a(B,B) a(A - B, A - B)\\
&\quad = s_1 + \ldots + s_4,
\end{aligned} \tag{3.9}$$

where $a(\cdot,\cdot)$ is defined in (3.7). Let $T(A) = \left(\begin{smallmatrix} a_{12} & a_{11} \\ a_{22} & a_{21}\end{smallmatrix}\right)$ and note that $T(A) : B = a(A, B)$ and $D_A a(A, A) = 2T(A)$. Young's inequality yields

$$\begin{aligned}
|D\Phi_2(A) - D\Phi_2(B)|^2 &= \big|4\kappa_1(|A|^2 - |B|^2)A + \kappa_2(a(A,A) - a(B,B))T(A) \\
&\quad +4\kappa_1(|B|^2 - 2)(A - B) + \kappa_2 a(B,B)T(A - B)\big|^2 \\
&\le c\big((|A|^2 - |B|^2)^2\,|A|^2 + (a(A,A) - a(B,B))^2\,|A|^2\big) \\
&\quad +c\big|4\kappa_1(|B|^2 - 2)(A - B) + \kappa_2 a(B,B)T(A - B)\big|^2 \\
&= c(t_1 + t_2) + ct_3.
\end{aligned} \tag{3.10}$$

Obviously, there exists a constant $c > 0$ such that $t_1 + t_2 \le c(1 + |A|^2 + |B|^2)(s_1 + s_2)$. It remains to show that

$$t_3 \le c(1 + |A|^2 + |B|^2)(s_3 + s_4). \tag{3.11}$$

If $a(B,B)a(A - B, A - B) \ge 0$, then

$$\begin{aligned}
&(1 + |A|^2 + |B|^2)a(B,B)a(A - B, A - B) \\
&\quad \ge (|B|^2 - 2)a(B,B)a(A - B, A - B),
\end{aligned} \tag{3.12}$$

$$\begin{aligned}
&(1 + |A|^2 + |B|^2)(|B|^2 - 2)\,|A - B|^2 \\
&\quad \ge (|B|^2 - 2)^2\,|A - B|^2 \overset{B\in\mathcal{M}_2}{\ge} \frac{\kappa_2^2}{16\kappa_1^2} a^2(B,B)\,|A - B|^2 .
\end{aligned} \tag{3.13}$$

Evaluating t_3 and taking into account estimates (3.12) and (3.13) finally implies (3.11).

If $a(B,B)a(A - B, A - B) < 0$, then

$$\begin{aligned}
s_3 + s_4 &= \Big(2\kappa_1(|B|^2 - 2) - \frac{\kappa_2}{2}\,|a(B,B)|\Big)\,|A - B|^2 \\
&\quad +\frac{\kappa_2}{2}\,|a(B,B)|\,\big(\,|A - B|^2 - |a(A - B, A - B)|\,\big)
\end{aligned} \tag{3.14}$$

and both terms are nonnegative. On the other hand,

$$\begin{aligned}
t_3 &= 4\Big(2\kappa_1(|B|^2 - 2) - \frac{\kappa_2}{2}\,|a(B,B)|\Big)^2\,|A - B|^2 \\
&\quad +8\kappa_1\kappa_2(|B|^2 - 2)\,|a(B,B)|\,\big(\,|A - B|^2 - |a(A - B, A - B)|\,\big)
\end{aligned} \tag{3.15}$$

and since $B \in \mathcal{M}_2$, we have

$$\begin{aligned}
&\Big(2\kappa_1(|B|^2 - 2) - \frac{\kappa_2}{2}\,|a(B,B)|\Big)^2 \\
&\quad \le c\,\Big(2\kappa_1(|B|^2 - 2) - \frac{\kappa_2}{2}\,|a(B,B)|\Big)\,(1 + |A|^2 + |B|^2)
\end{aligned} \tag{3.16}$$

for a constant $c > 0$ which is independent of A, B. Combining (3.14)–(3.16) results in (3.11) and convexity inequality (2.7) is proved for Φ_2 on $\mathcal{M}_2$ with $r = s = 2$.

Let $i = 3$. For $A, B \in \mathcal{M}_3$, $A \ne B$, it holds

$$\begin{aligned}
&(4\kappa_1 + \kappa_2)(\kappa_1\kappa_2)^{-1}\,(\Phi_3(A) - \Phi_3(B) - D\Phi_3(B) : (A - B)) \\
&\quad = (|A|^2 - |B|^2 + a(A,A) - a(B,B))^2 \\
&\qquad +2\big(\,|B|^2 - 2 + a(B,B)\big)\big(\,|A - B|^2 + a(A - B, A - B)\big)
\end{aligned} \tag{3.17}$$

and both summands are nonnegative. On the other hand, by Young's inequality, there exists a constant $c > 0$ such that

$$
\begin{aligned}
& c\big|D\Phi_3(A) - D\Phi_3(B)\big|^2 \\
& \quad \leq \big(|A|^2 - |B|^2 + a(A,A) - a(B,B)\big)^2 \,|A + T(A)|^2 \\
& \qquad + \big(|B|^2 - 2 + a(B,B)\big)^2 \big(2\,|A - B|^2 + a(A - B, A - B)\big).
\end{aligned} \tag{3.18}
$$

$B \in \mathcal{M}_3$ implies $0 \leq |B|^2 - 2 + a(B,B) \leq 2(1 + |A|^2 + |B|^2)$ and therefore, combining (3.17) and (3.18), it follows that Φ_3 satisfies the convexity inequality on $\mathcal{M}_3$. The case $i = 4$ can be treated in the same way.

In order show that the convexity inequality is valid for every $A, B \in \mathbb{R}^{2\times 2}$ note first that there exists a $J_0 \in \mathbb{N}$ such that it holds for every $A, B \in \mathbb{R}^{2\times 2}$: There exist real numbers $0 = t_0 < t_1 \ldots < t_J = 1$, $J \leq J_0$, and numbers $i_0, \ldots, i_{J-1} \in \{1, \ldots, 4\}$ such that $F(t) = B + t(A - B) \in \mathcal{M}_{i_j}$ for $t \in [t_j, t_{j+1}]$, $0 \leq j \leq J - 1$. We obtain

$$
\begin{aligned}
& W^c(A) - W^c(B) - \mathrm{DW}^c(B) : (A - B) \\
& \quad = \sum_{j=1}^{J} W^c(F(t_j)) - W^c(F(t_{j-1})) - \mathrm{DW}^c(F(t_{j-1})) : (F(t_j) - F(t_{j-1})) \\
& \qquad + \sum_{j=1}^{J} \big(\mathrm{DW}^c(F(t_{j-1})) - \mathrm{DW}^c(F(0))\big) : (F(t_j) - F(t_{j-1})) \\
& \quad = s_1 + s_2.
\end{aligned} \tag{3.19}
$$

Since W^c is convex, the derivative DW^c is a monotone function and thus

$$
s_2 = \sum_{j=2}^{J} \frac{t_j - t_{j-1}}{t_{j-1}} \big(\mathrm{DW}^c(F(t_{j-1}) - \mathrm{DW}^c(F(0))\big) : (F(t_{j-1}) - F(0)) \geq 0.
$$

Moreover, $F(t_{j-1}), F(t_j) \in \mathcal{M}_{i_{j-1}}$ and therefore the convexity inequality may be applied to every summand of s_1 separately due to the first part of this proof:

$$
s_1 \geq c \sum_{j=1}^{J} \Big(1 + \big|F(t_j)\big|^2 + \big|F(t_{j-1})\big|^2\Big)^{-1} \big|\mathrm{DW}^c(F(t_j)) - \mathrm{DW}^c(F(t_{j-1}))\big|^2.
$$

Note that $(1 + \big|F(t_j)\big|^2 + \big|F(t_{j-1})\big|^2)^{-1} \geq \frac{1}{4}(1 + |A|^2 + |B|^2)^{-1}$ and that $\sum_{j=1}^{J} \big|B_j\big|^2 \geq J^{-1}\big|\sum_{j=1}^{J} B_j\big|^2$ for $B_j \in \mathbb{R}^{2\times 2}$ and thus, since $J \leq J_0$,

$$
s_1 \geq \frac{c}{4J_0}(1 + |A|^2 + |B|^2)^{-1} \big|\mathrm{DW}^c(A) - \mathrm{DW}^c(B)\big|^2.
$$

This finishes the proof. □

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A Appendix: Proof of Lemma 2.3

A.1 The two dimensional case

Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz-polygon and assume that $0 \in \overline{\Gamma_D} \cap \overline{\Gamma_N}$. Then there exists $R > 0$ such that $B_R(0) \cap \partial\Omega$ does not contain any further corner point of $\partial\Omega$. Assume further that $\Gamma_N \cap B_R(0)$ is a subset of the positive x_1-axis and that there exists $\Phi > 0$ such that $\Omega \cap B_R(0) = \{x \in B_R(0) : 0 < \varphi < \Phi\}$ [polar coordinates, $x = |x|\,(\cos\varphi, \sin\varphi)^\top$].

Case 1 Let $\measuredangle(\Gamma_D, \Gamma_N) < \pi$, i.e., $\Phi < \pi$. Choose $\Omega_N = \{x \in B_R(0) : \pi < \varphi < 2\pi\}$, $\Omega_D = \{x \in B_R(0) : \Phi < x < \pi\}$ and $\mathcal{K} = \{x \in \mathbb{R}^2 : \Phi < x < \pi\}$. Then conditions (2.23)–(2.24) of Definition 2.3 are satisfied and thus Ω is an admissible domain.

Case 2 Assume that Ω is admissible. We have to show that $\Phi < \pi$. Let $\mathcal{K}, \Omega_D$ and Ω_N be the cone and domains of (2.23)–(2.24) in Definition 2.3 corresponding to the corner $0 \in \overline{\Gamma_D} \cap \overline{\Gamma_N}$. It follows from $\Gamma_N \cap B_R(0) \subset (\partial\Omega_N \cap B_R(0)) \subset$ positive x_1-axis together with (2.23) that $\mathcal{K}$ is completely contained in the upper half plane, i.e., $\mathcal{K} = \{x \in \mathbb{R}^2 : \Phi_1 < \varphi < \Phi_2\}$ and $0 \leq \Phi_1 < \Phi_2 \leq \pi$. Furthermore, (2.24) together with $\Gamma_D \cap B_R(0) \subset \{x \in \mathbb{R}^2 : \varphi = \Phi\}$ implies $\Phi \leq \Phi_1$ and thus $\Phi < \pi$.

A.2 The three dimensional case

Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz-polyhedron according to Part 2 of Lemma 2.3 and let $x_0 \in \overline{\Gamma_D} \cap \overline{\Gamma_N}$. There exists $R > 0$ and a polyhedral cone $\tilde{\mathcal{K}}$ with vertex in x_0 such that Ω coincides with $\tilde{\mathcal{K}}$ on $B_R(x_0)$:

$$\Omega \cap B_R(x_0) = \tilde{\mathcal{K}} \cap B_R(x_0).$$

We assume that $\tilde{K}$ has exactly three faces $\Gamma_i, 1 \leq i \leq 3$, which intersect at x_0 and which satisfy $(\Gamma_1 \cup \Gamma_2) \cap B_R(x_0) \subset \Gamma_N$ and $\Gamma_3 \cap B_R(x_0) \subset \Gamma_D$. Furthermore, we assume that $\measuredangle(\Gamma_1, \Gamma_2) \neq \pi$. The remaining cases can be treated similarly. Let n_i be the exterior unit normal vector on Γ_i and denote by $H_i = \{x \in \mathbb{R}^3 : (x - x_0)n_i < 0\}$ the "interior" half space with respect to Γ_i and n_i. Due to the assumption $\measuredangle(\Gamma_D, \Gamma_N) < \pi$ it follows that $(\Omega \cup \Gamma_N) \cap B_R(x_0) \subset H_3$. Therefore, we have exactly the following two cases for $\tilde{\mathcal{K}}$:

$$\tilde{\mathcal{K}} = H_1 \cap H_2 \cap H_3 \;\text{ or }\; \tilde{\mathcal{K}} = (H_1 \cup H_2) \cap H_3,$$

depending on whether $\measuredangle(\Gamma_1, \Gamma_2) < \pi$ or $> \pi$. In order to show that Ω is an admissible domain we have to construct domains Ω_D, Ω_N and a cone $\mathcal{K}$ according to (2.23)–(2.24) of Definition 2.3. We define

$$\Omega_D = \{x \in B_R(x_0) : (x - x_0)n_3 > 0\}, \quad \Omega_N = B_R(x_0) \backslash (\overline{\Omega_D} \cup \overline{\Omega}).$$

Since $\measuredangle(\Gamma_D, \Gamma_N) < \pi$ it follows that $\Omega_D \cap \Omega = \emptyset$ and $\Omega_N \neq \emptyset$. Let e_1 be tangential to $\overline{\Gamma_1} \cap \overline{\Gamma_3}$, e_2 be tangential to $\overline{\Gamma_2} \cap \overline{\Gamma_3}$ and e_3 tangential to $\overline{\Gamma_1} \cap \overline{\Gamma_2}$. The orientation of the vectors e_i is chosen in such a way that

$$e_1 n_2 < 0, \quad e_2 n_1 < 0, \quad e_3 n_3 > 0. \tag{A.1}$$

This choice is always possible since $\measuredangle(\Gamma_1, \Gamma_2) \neq \pi$ and $\measuredangle(\Gamma_N, \Gamma_D) < \pi$. Note that $\{e_1, e_2, e_3\}$ is a basis of $\mathbb{R}^3$. We define the cone $\mathcal{K}$ by

$$\mathcal{K} = \left\{ v \in \mathbb{R}^3 : v = \sum_{i=1}^{3} \lambda_i e_i,\ \lambda_i \geq 0 \right\}.$$

Then Ω_D, Ω_N and $\mathcal{K}$ satisfy (2.23)–(2.24) of Definition 2.3, which can be seen as follows.

Choose $x \in \Omega_D$ and $v = \sum_i \lambda_i e_i \in \mathcal{K}$ such that $x + v \in B_R(x_0)$. Since $e_1 n_3 = 0$, $e_2 n_3 = 0$ we get from the definition of Ω_D and (A.1) that

$$(x + v - x_0) n_3 = (x - x_0) n_3 + \lambda_3 e_3 n_3 > 0$$

and therefore $x + v \in \Omega_D$ and (2.24) is proved. For the proof of (2.23) choose $x \in \Omega \cap B_R(x_0) = \tilde{\mathcal{K}} \cap B_R(x_0)$ and $v = \sum_i \lambda_i e_i \in \mathcal{K}$ such that $x + v \in B_R(x_0)$. If $\lambda_3 \geq -((x - x_0)n_3)/(e_3 n_3)$, then $(x + v - x_0) n_3 \geq 0$ which yields $x + v \in \overline{\Omega_D}$ and (2.23) holds for this case. If $\lambda_3 < -((x - x_0)n_3)/(e_3 n_3)$, then $x + v \in H_3$ and we have to show that $x + v \in \tilde{\mathcal{K}}$ in order to verify (2.23).

Case 1 $\tilde{K} = H_1 \cap H_2 \cap H_3$. It follows for $i, j \in \{1, 2\}$ with $i \neq j$ from the definitions of $\mathcal{K}$, H_i and from (A.1) that

$$(x + v - x_0) n_i = (x - x_0) n_i + \lambda_j e_j n_i < 0$$

and therefore $x + v \in H_1 \cap H_2 \cap H_3$.

Case 2 $\tilde{K} = (H_1 \cup H_2) \cap H_3$. It follows for $i, j \in \{1, 2\}$, $i \neq j$ as before that

$$(x + v - x_0) n_i = (x - x_0) n_i + \lambda_j e_j n_i.$$

Since $x \in H_1 \cup H_2$ we have $(x - x_0) n_1 < 0$ or $(x - x_0) n_2 < 0$. Together with $\lambda_j e_j n_i \leq 0$ we obtain finally $(x + v - x_0) n_1 < 0$ or $(x + v - x_0) n_2 < 0$ which shows that $x + v \in H_1 \cup H_2$.

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