STRESS BEHAVIOUR IN A POWER-LAW HARDENING MATERIAL

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Dedicated to Alois Kufner on the occasion of his 70th birthday

ABSTRACT. We consider a class of hyper-elastic materials with constitutive equations of power-law type. The power exponent is a hardening parameter, which is characteristic for metals with extended strain hardening capability and low proportionality limits.

If the elastic body has reentrant corners and edges or the material parameters are jumping, then high stresses appear near these geometrical or structural peculiarities [13], [12], [10], [24], [20].

We give a variational formulation of the nonlinear field equations under consideration and describe its solvability and uniqueness. The main focus lies on the investigation of the smoothness of the stress and displacement fields in Lipschitz domains. Furthermore, we discuss how our regularity results are connected with the well-known HRR-fields [10], [24]. Finally, we present some regularity results for elastic fields in composites with varying hardening exponents.

1. INTRODUCTION

The behaviour of displacement and stress fields of linear elastic bodies with geometrical and structural singularities is well investigated. Geometric singularities can be corners, edges, V-notches or cracks; structural singularities are characterised by discontinuities in the material parameters across interfaces. Near such zones high stress concentrations appear. The mathematical modelling leads to linear elliptic systems of partial differential equations with piecewise constant coefficients equipped with boundary and transmission conditions. There is a large number of papers in mathematics and mechanics where stress singularities are analysed and computed by different

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methods. We recall two mathematical methods to derive regularity results for linear problems. One is to use layer potential methods for boundary value problems on Lipschitz domains [28], [3], [4], [18]. These regularity results are formulated in Sobolev-Slobodeckij, Besov and Hölder spaces and describe the worst possible regularity in the considered class of problems. Another possibility to find more detailed regularity results is to apply the Mellin technique [14], [15], [5], [19] in polyhedral domains or composites. In particular, the smoothness of the solutions is described by asymptotic expansions with respect to the distance to the geometrical and structural peculiarities. In the two-dimensional case the expansion of the displacement field u reads in polar coordinates as follows:

$$\eta^S u = \eta^S \sum_{0<\operatorname{Re}\beta<1} c_\beta r^\beta v_\beta(\ln r, \varphi) + \eta^S u_{\mathrm{reg}}, \tag{1}$$

where r is the distance to a corner point S. The singular exponents β are eigenvalues of a corresponding nonlinear eigenvalue problem and the functions v_β consist of generalised eigenfunctions and of powers of $\ln r$. The constants c_β denote the stress intensity factors and are calculated from the given loading. The smoothness of the solution u is determined by the singular exponent β with the smallest positive real part and can be calculated explicitly for a fixed geometry and material parameters. We underline that every weak solution of linear elliptic boundary value problems admits an expansion of the above type.

Much less is known on the global regularity of weak solutions for nonlinear elliptic boundary value and transmission problems in non-smooth domains. In this paper we concentrate to quasilinear elliptic problems of power-law growth. For boundary value problems in Lipschitz domains a difference quotient technique was adapted and developed in [25], [7], [8]. The obtained regularity results are formulated in Nikol'skij and Sobolev-Slobodeckij spaces and guarantee a minimum smoothness. It is an open question whether asymptotic expansions near geometrical and structural singularities can be expected as in the linear case. J. W. Hutchinson [10], and J. K. Rice and G. F. Rosengren [24] have used an ansatz analogously to the linear case in order to investigate crack tip singularities for nonlinear elastic Ramberg/Osgood materials of power-law type. This approach leads to fully nonlinear eigenvalue problems for the determination of the singular terms. Numerical investigations for crack and V-notch problems [30], [29] show the dependence of the actual regularity of weak solutions on the geometry of the domain and indicate that the regularity may be covered by asymptotic expansions in the nonlinear case, too.

This paper is split into two parts. The first part is concerned with a mixed boundary value problem for nonlinear elastic Ramberg/Osgood materials in Lipschitz domains. Starting from the physical model we give a weak formulation in appropriate anisotropic Sobolev spaces. The existence and uniqueness of weak solutions is obtained from a minimization problem for the complementary energy by standard arguments from the calculus of variations. We present local and global regularity results for the displacement and stress fields on Lipschitz domains. These results are proved with a difference quotient technique. It turns out that weak solutions are smoother in the interior, stress singularities are situated at the boundary. Furthermore, our results are in coincidence with the stress singularities predicted by the HRR-approach.

The second part deals with boundary transmission problems in a body which is composed of different nonlinear elastic materials of power-law type. It is known from the linear theory that the stress singularities near cross points are strongly influenced by the number and the geometry of the subdomains and the jumps of the material parameters. In general, the arising singularities are stronger than those of boundary value problems with smooth coefficients in Lipschitz domains. We formulate here sufficient conditions on the distribution of the subdomains and the corresponding elastic energies in order to assure the same regularity results as for one domain. This new condition is called *quasi-monotone covering condition* and is a generalization of the quasi-monotonicity condition which was introduced by [6] for the Laplace equation with piecewise constant coefficients.

2. The material model

We consider physically nonlinear elastic materials where the corresponding constitutive law is described by a power-law like relationship which was first suggested by W. Ramberg and W. R. Osgood in 1943 for aluminium alloys [21]. Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded domain with Lipschitz boundary which is split into a Dirichlet part Γ_D and a Neumann part Γ_N. The field equations for the determination of the displacement field $u\colon \Omega \to \mathbb{R}^d$ and stress field $\sigma\colon \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}}$ read as follows:

$$\operatorname{div}\sigma + f = 0 \quad \text{in } \Omega, \tag{2}$$

$$\varepsilon(u) - A\sigma - \frac{3\tilde{\alpha}}{2E}\left(\frac{\sigma_e}{\sigma_y}\right)^{q-2}\sigma^D = 0 \quad \text{in } \Omega, \tag{3}$$

$$u = g \quad \text{on } \Gamma_D, \tag{4}$$

$$\sigma\vec{n} = h \quad \text{on } \Gamma_N. \tag{5}$$

Here, f denotes the volume force density, h the surface force density and g the displacement on Γ_D. Furthermore $\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the linearised strain tensor, $\sigma^D = \sigma - (1/d)\,\mathrm{tr}\,\sigma I$ the deviatoric part of the stress tensor and $\sigma_e = \sqrt{3/2}\,|\sigma^D| = \sqrt{(3/2)\sum_{i,j=1}^d (\sigma_{ij}^D)^2}$ the von Mises effective stress. The material constants in the constitutive law (3) have the following meaning: $A \in \mathrm{Lin}(\mathbb{R}^{d\times d}, \mathbb{R}^{d\times d})$ is a symmetric and positive definite elasticity tensor with $(A\sigma)_{ij} = \sum_{k,l=1}^d A_{ijkl}\sigma_{kl}$; E is the Young modulus, σ_y the yield stress and $\tilde{\alpha}$ a further material parameter. The exponent q is called strain hardening coefficient and describes the hardening behaviour of the material. Assuming that A corresponds to an isotropic material and that $\sigma = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, relation (3) reduces to

$$\varepsilon_{11} = \frac{\sigma_{11}}{E} + \frac{\tilde{\alpha}\sigma_y}{E}\left|\frac{\sigma_{11}}{\sigma_y}\right|^{q-1}. \tag{6}$$

Typical graphs for relation (6) are plotted in Figure 1 for different strain hardening parameters q. If $q = 2$ then relation (3) describes a linear elastic material. For $q > 2$ the material is strain hardening and for $q \to \infty$ the Ramberg/Osgood model is an approximation of the linear elastic, perfect-plastic Hencky model. A mathematical proof of this assertion is given in [27], [2].

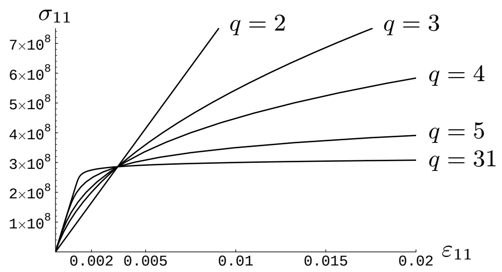

Figure 1. Relation (6) with $E = 197\,\mathrm{GPa}$, $\sigma_y = 286\,\mathrm{MPa}$, $\tilde{\alpha} = 1.378$. Note that $\tilde{\alpha}\sigma_y/E \approx 0.002$. The parameters are taken from [23].

Equations (2)–(5) are the field equations of a physically nonlinear and geometrically linearised elastic material model. However, these equations are applied mainly for the description of aluminium alloys and stainless steel alloys, [23], [26], which show in reality an elasto-plastic behaviour: if a quasi-static cycle of loading and unloading is applied to these materials then in general there remains a small permanent plastic strain after the cycle is finished. This phenomenon cannot be correctly described by the elastic Ramberg/Osgood model which predicts vanishing strains after unloading. Thus the Ramberg/Osgood model can be applied for the description of the metals mentioned above only under the assumption that the applied loading is quasi-static and monotone. Accepting this condition the terms in the constitutive relation (3) can be interpreted as follows:

$$\varepsilon = \underbrace{A\sigma}_{\varepsilon_{\rm el}} + \underbrace{\frac{3\tilde{\alpha}}{2E}\left(\frac{\sigma_e}{\sigma_y}\right)^{q-2}\sigma^D}_{\varepsilon_{\rm pl}} = \varepsilon_{\rm el} + \varepsilon_{\rm pl}. \tag{7}$$

The strain tensor ε is split into an elastic strain $\varepsilon_{\rm el}$, which depends linearly on the stresses, and into a plastic strain $\varepsilon_{\rm pl}$, which depends nonlinearly on the deviatoric part of the stresses. The material behaves nearly linear elastic if the von Mises effective stress σ_e is less than the constant σ_y. If σ_e is larger than σ_y then the plastic strains $\varepsilon_{\rm pl}$ from (7) dominate the strain tensor. This justifies the name "yield stress" for the constant σ_y. One should note that the considered materials have no yield plateau and therefore the yield stress σ_y is not uniquely determinable. Usually σ_y is chosen as the 0.2 % proof stress $\sigma_{0.2}$, see e.g. [23]. Some typical values are listed in Table 1 with $\tilde{\alpha} = 0.002E/\sigma_y$.

	austenitic steel alloys [23]	aluminium alloys [26], [16]
E	180–200 GPa	66–75 GPa
$\sigma_y = \sigma_{0.2}$	300–600 MPa	160–300 MPa
q	5.45–8.9	20–45

Table 1. Typical values of the material parameters.

3. Function spaces and solvability

The special structure of the constitutive law (3) motivates the choice of the function spaces where the weak problem is formulated. The trace of the stress tensor occurs in the linear term of the constitutive law only, whereas the deviatoric part of the stress tensor appears also in the nonlinear term. Therefore, function spaces have to be applied which take into account this structure explicitly. Such spaces were first introduced and investigated by G. Geymonat and P. Suquet [9].

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. For $s > 0$ and $p \in (1, \infty)$ we denote by $W^{s,p}(\Omega)$ the usual Sobolev-Slobodeckij spaces, see e.g. [1]. In [9] the following anisotropic spaces are introduced for $r \in (1, \infty)$, $p \in (1, 2]$ and $q = p' = \frac{p}{p-1} \in [2, \infty)$:

$$L^{r,2}(\Omega) = \big\{\tau\colon \Omega \to \mathbb{R}^{d\times d}_{\mathrm{sym}} : \tau^D \in L^r(\Omega),\ \operatorname{tr}\tau \in L^2(\Omega)\big\}, \tag{8}$$

$$U^{p,2}(\Omega) = \big\{u \in L^p(\Omega) : \varepsilon(u) \in L^{p,2}(\Omega)\big\}. \tag{9}$$

These spaces are reflexive and separable Banach spaces endowed with the following norms:

$$\|\tau\|_{L^{r,2}(\Omega)} = \big\|\tau^D\big\|_{L^r(\Omega)} + \|\operatorname{tr}\tau\|_{L^2(\Omega)},$$
$$\|u\|_{U^{p,2}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\varepsilon(u)\|_{L^{p,2}(\Omega)}.$$

The spaces $U^{p,2}(\Omega)$ are embedded in $W^{1,p}(\Omega)$ for $p \le 2$. Therefore, the traces of functions from $U^{p,2}(\Omega)$ on parts of the boundary are well defined. Moreover, the trace operator

$$\gamma|_{\Gamma_D} : U^{p,2}(\Omega) \to W^{1-1/p,p}(\Gamma_D), \quad u \mapsto u|_{\Gamma_D}, \tag{10}$$

is a surjective mapping [9]. For an open set $\Gamma_D \subset \partial\Omega$ and $p \in (1, 2]$ we denote by

$$V^{p,2}(\Omega) = \{v \in U^{p,2}(\Omega) : v|_{\Gamma_D} = 0\}$$

the space of test functions. In order to define Neumann boundary conditions we introduce the following spaces for $q = p'$

$$\widetilde{W}^{1-1/p,p}(\Gamma_N) = \big\{v \in W^{1-1/p,p}(\partial\Omega) : \operatorname{supp} v \subset \overline{\Gamma_N}\big\}, \tag{11}$$

$$W^{-1/q,q}(\Gamma_N) = \big(\widetilde{W}^{1-1/p,p}(\Gamma_N)\big)'. \tag{12}$$

The dual pairing for elements u of a Banach space X and elements f of its dual X' is denoted by $\langle f, u\rangle = \langle f, u\rangle_X$. Finally, $A : B$ is the inner product for matrices $A, B \in \mathbb{R}^{d\times d}$ which is defined as

$$A : B = A^D : B^D + \frac{1}{d}\operatorname{tr} A \operatorname{tr} B = \sum_{i,j=1}^{d} a_{ij} b_{ij}, \quad |A| = (A : A)^{1/2}.$$

Since the constitutive law (3) describes the strains as a function of the stresses and since this relation cannot be explicitly solved for the stresses, an appropriate framework for the weak formulation is the stress based or dual formulation.

Definition 3.1 (Stress-based complementary energy)**.** Let $\alpha > 0$, $q \in [2,\infty)$ and let $A \in \mathrm{Lin}(\mathbb{R}^{d\times d}, \mathbb{R}^{d\times d})$ be symmetric with $(A\sigma) : \sigma \geq c_A|\sigma|^2$ for every $\sigma \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$. The function

$$W_c \colon \mathbb{R}^{d\times d}_{\mathrm{sym}} \to \mathbb{R}, \quad \sigma \mapsto \frac{1}{2}(A\sigma) : \sigma + \frac{\alpha}{q}\left|\sigma^D\right|^q, \tag{13}$$

is called complementary energy density for Ramberg/Osgood materials. The functional

$$J_c \colon L^{q,2}(\Omega) \to \mathbb{R}, \quad \sigma \mapsto \int_\Omega W_c(\sigma)\,\mathrm{d}x, \tag{14}$$

describes the corresponding stress based complementary energy.

Let us note that the constitutive law (3) can be rewritten as

$$\varepsilon = DW_c(\sigma) = \left(\frac{\partial W_c(\sigma)}{\partial \sigma_{ij}}\right)_{i,j\in\{1,\dots,d\}} = A\sigma + \alpha\left|\sigma^D\right|^{q-2}\sigma^D,$$

where the material constants from relation (3) are now accumulated in the new constant α. Since typical values for q are larger than two, see Table 1, we assume in Definition 3.1 that $q \geq 2$. The functional J_c is well defined for elements of $L^{q,2}(\Omega)$, strictly convex, coercive on $L^{q,2}(\Omega)$ and Fréchet differentiable. The existence of a stress field σ and a displacement field u solving the Ramberg/Osgood equations (2)–(5) in a weak sense can be deduced from a minimization problem for the complementary energy J_c and the corresponding weak Euler-Lagrange equations.

We assume now that

$$\begin{aligned} &q \geq 2,\ p = q' \in (1,2],\ f \in \left(V^{p,2}(\Omega)\right)', \\ &g \in W^{1-1/p,p}(\Gamma_D),\ h \in W^{-1/q,q}(\Gamma_N). \end{aligned} \tag{15}$$

We remark that there exists a function $g_0 \in U^{p,2}(\Omega)$ with $g_0|_{\Gamma_D} = g$ (see [9]).

3.1. Minimisation problem for the complementary energy. The set of admissible stress fields satisfying the equilibrium of forces is defined as

$$\mathcal{M} = \Big\{\tau \in L^{q,2}(\Omega) : \int_\Omega \tau : \varepsilon(v)\,\mathrm{d}x = \langle f, v\rangle_{W^{1,p}(\Omega)} + \langle h, v\rangle_{\widetilde{W}^{1-1/p,p}(\Gamma_N)} \quad \text{for every } v \in V^{p,2}(\Omega)\Big\}.$$

With this definition the minimisation problem reads as follows:

Find $\sigma \in \mathcal{M}$ such that

$$J_c(\sigma) - \int_\Omega \varepsilon(g_0) : \sigma\,\mathrm{d}x \leq J_c(\tau) - \int_\Omega \varepsilon(g_0) : \tau\,\mathrm{d}x, \quad \tau \in \mathcal{M}. \tag{16}$$

Thereby, J_c denotes the complementary energy of Definition 3.1.

3.2. Weak formulation. The corresponding weak Euler-Lagrange equations read as follows:

Find $\sigma_0 \in L^{q,2}(\Omega)$ *and* $u_0 \in U^{p,2}(\Omega)$ *with* $u_0|_{\Gamma_D} = g$ *such that*

$$\int_\Omega DW_c(\sigma_0) : \tau \,\mathrm{d}x = \int_\Omega \varepsilon(u_0) : \tau \,\mathrm{d}x, \tag{17}$$

$$\int_\Omega \sigma_0 : \varepsilon(v) \,\mathrm{d}x = \langle f, v\rangle_{V^{p,2}(\Omega)} + \langle h, v\rangle_{\widetilde{W}^{1-1/p,p}(\Gamma_N)} \tag{18}$$

for every $\tau \in L^{q,2}(\Omega)$ *and* $v \in V^{p,2}(\Omega)$.

Theorem 3.2 (Existence of weak solutions). *Let the assumptions from* (15) *be satisfied. If* $\Gamma_N = \partial\Omega$, *we require in addition that* f *and* h *satisfy the following solvability condition*

$$\langle f, r\rangle_{U^{p,2}(\Omega)} + \langle h, r\rangle_{W^{1-1/p,p}(\partial\Omega)} = 0 \quad \textit{for every } r \in \mathcal{R},$$

where $\mathcal{R}$ *is the set of linearised rigid body displacements,*

$$\mathcal{R} = \{r \colon \Omega \to \mathbb{R}^d : r(x) = a + Bx,\ a \in \mathbb{R}^d,\ B \in \mathbb{R}^{d\times d},\ B + B^T = 0\}.$$

Then the minimisation problem (16) *is uniquely solvable. Moreover, equations* (17)–(18) *are the corresponding Euler-Lagrange equations and have a unique solution* $\sigma_0 \in L^{q,2}(\Omega)$ *and* $u_0 \in U^{p,2}(\Omega)$ *with* $u_0|_{\Gamma_D} = g$ *provided that* $\Gamma_D \neq \emptyset$. *If* $\Gamma_D = \emptyset$, *then* $\sigma_0 \in L^{q,2}(\Omega)$ *is still unique and the displacement field* u_0 *is unique up to elements from* $\mathcal{R}$. *The minimiser of* (16) *coincides with* σ_0.

This theorem can be proved by an adaption of the Ljusternik theorem on the solvability of minimisation problems with constraints (see [31, Theorem 43.D]). Another possibility is to use the ideas of R. Temam [27].

A simple conclusion of the existence of a solution pair $u \in U^{p,2}(\Omega)$ and $\sigma \in L^{q,2}(\Omega)$ for the two-dimensional case describes rough bounds for the exponents for singular terms:

Corollary 3.3. *Let* $d = 2$ *and let* (r, φ) *be the polar coordinates with respect to some corner point* $S \in \partial\Omega$. *Assume that there is a weak solution pair of the form*

$$u(r, \varphi) = r^\beta s_\alpha(\varphi) + u_{\mathrm{reg}}(r, \varphi), \tag{19}$$

$$\sigma(r, \varphi) = r^\gamma \tau_\beta(\varphi) + \sigma_{\mathrm{reg}}(r, \varphi). \tag{20}$$

Then $\beta > -1 + 2/q$, $\gamma > -2/q$.

It is of great interest to get more information about the exponents γ and β, and to study the influence of the material and the geometry on them.

4. Global regularity on Lipschitz domains

The regularity results which are presented in this section are proved with a difference quotient technique. To apply this technique we need a careful description of *admissible domains.* This definition depends on the type of the boundary conditions.

Definition 4.1. Let $\Omega \subset \mathbb{R}^d$ be a bounded domain.

(i) If the type of the boundary conditions does not change, i.e. $\partial\Omega = \Gamma_D$ or $\partial\Omega = \Gamma_N$, then Ω is called admissible, if $\partial\Omega$ is a Lipschitz boundary.

(ii) If the type of the boundary conditions changes, then Ω is called admissible, if $\partial\Omega$ is a Lipschitz boundary and the following condition is satisfied (see Figure 2): There exists a finite number of balls B_j with $\bigcup_{j=1}^{J} B_j \supset \partial\Gamma_D \cap \partial\Gamma_N$. Furthermore, for each of these balls B_j there exists an infinite cone $\mathcal{K}_j$ with the vertex in 0 and pair-wise disjoint non-empty domains $\Omega^j_\infty, \Omega^j_{-\infty} \subset B_j$ such that

$$B_j \backslash \Omega = \overline{\Omega^j_\infty \cup \Omega^j_{-\infty}}, \quad \Gamma_D \cap B_j \subset \partial\Omega^j_\infty, \quad \Gamma_N \cap B_j \subset \partial\Omega^j_{-\infty}, \tag{21}$$

$$\big((\Omega \cap B_j) + \mathcal{K}_j\big) \cap \Omega^j_{-\infty} = \emptyset, \tag{22}$$

$$\big(\Omega^j_\infty + \mathcal{K}_j\big) \cap \big((\Omega \cap B_j) \cup \Omega^j_{-\infty}\big) = \emptyset. \tag{23}$$

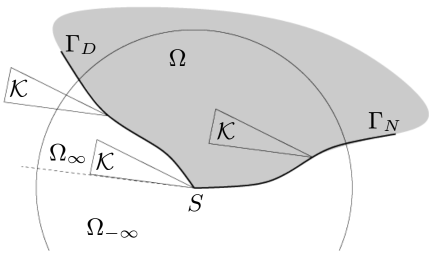

Figure 2. Notation for Definition 4.1.

Remark 4.2. If Ω is a two-dimensional polygonal domain, then the condition for mixed boundary values reduces to a condition for the interior opening angle, namely $\measuredangle(\Gamma_D, \Gamma_N) < \pi$.

Now, we come to the main theorem about the regularity of solutions of mixed boundary value problems in admissible domains.

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be an admissible domain. We assume for the data from (17)–(18) that $f \in L^q(\Omega)$, $g = \gamma|_{\Gamma_D}\hat{g}$, $\hat{g} \in W^{2,p}(\hat{\Omega})$ with $\nabla\hat{g} \in L^\infty(\hat{\Omega})$, where $\hat{\Omega} \supset\supset \Omega$, $h = H\vec{n}$ on Γ_N, $H \in W^{1,q}(\hat{\Omega}, \mathbb{R}^{d\times d}_{\mathrm{sym}}) \cap L^\infty(\hat{\Omega})$.*

Then every solution $u \in U^{p,2}(\Omega)$ and $\sigma \in L^{q,2}(\Omega)$ of (17)–(18) has the following properties.

Locally:

$$u \in W^{2,\tau}_{\mathrm{loc}}(\Omega), \quad \operatorname{div} u,\ \sigma \in W^{2/q-\delta,q}_{\mathrm{loc}}(\Omega) \cap W^{1,2}_{\mathrm{loc}}(\Omega) \tag{24}$$

with $\tau = 2 - \delta$ if $d = 2$ and $\tau = \frac{3q}{2q-1}$ if $d = 3$; $\delta > 0$ is arbitrary.

Globally:

$$u \in W^{3/2-\delta,2dp/(2d-2+p)}(\Omega), \quad \operatorname{div} u,\ \sigma \in W^{1/q-\delta,q}(\Omega) \cap W^{1/2-\delta,2}(\Omega) \tag{25}$$

for all $\delta > 0$. Note that $p \leq \frac{2dp}{2d-2+p} \leq 2$. In the linear case of $p = q = 2$ the well-known regularity results are recovered.

R e m a r k s t o t h e p r o o f. This theorem is proved via the difference quotient technique. The main idea is to insert into the weak formulation test functions of the form

$$v(x) = \varphi^2(x)(u(x + he_i) - u(x)), \tag{26}$$

where φ is a cut-off function, $\{e_1, \ldots, e_d\}$ is a basis of $\mathbb{R}^d$ and $h > 0$. The main goal is to derive uniform estimates of the type

$$h^{-s}\|\nabla u(\cdot + he_i) - \nabla u(\cdot)\|_{L^r(\Omega)} \leq c. \tag{27}$$

Such inequalities follow from the convexity properties of the complementary energy density W_c. From the estimate (27) one obtains the regularity results using embedding theorems for Nikol'skij spaces:

$$\begin{aligned} &\varphi u \in W^{2,r}(\Omega) \quad \text{if } s = 1,\\ &\varphi u \in W^{1+s-\delta,r}(\Omega) \quad \text{if } s \in (0, 1). \end{aligned}$$

Note that the geometric conditions (21)–(23) guarantee the existence of a basis $\{e_1, \ldots, e_d\}$ of $\mathbb{R}^d$ such that v in (26) is an admissible test function. □

4.1. Discussion of the regularity results. It follows immediately from the interior regularity results of Theorem 4.3 that the displacement field u is continuous in the interior of the domain for $d = 2, 3$. Moreover, the results show that the strongest singularities are situated at the boundary.

We discuss now for the two-dimensional case the relation between our regularity results and the HRR-fields. J. W. HUTCHINSON [10], and J. R. RICE and G. F. ROSENGREN [24] have studied the behaviour of stress fields near a crack tip in Ramberg/Osgood materials. Based on the assumption that the displacement and stress fields have an asymptotic structure like in the linear case, they derived a strongly nonlinear eigenvalue problem from which they calculated the dominant terms in the asymptotic expansion. Using the ansatzes

$$u_0(r,\varphi) = r^{\gamma(q-1)+1} v(\varphi), \quad \sigma_0(r,\varphi) = r^{\gamma} \tau(\varphi), \tag{28}$$

they obtained the following eigenvalue problems for the determination of the exponent γ and the functions v and τ:

Find an exponent $\gamma \in \mathbb{R}$ and functions v and τ which do not vanish such that

$$\operatorname{div}_x \sigma_0 = 0 \quad \text{for } r > 0,\ \varphi \in (-\pi, \pi), \tag{29}$$

$$\varepsilon(u_0) - \alpha \left|\sigma_0^D\right|^{q-2} \sigma_0^D = 0 \quad \text{for } r > 0,\ \varphi \in (-\pi, \pi), \tag{30}$$

$$\sigma_{0,\pm} \vec{n}_{\pm} = 0 \quad \text{for } r > 0,\ \varphi \in \{-\pi, \pi\}. \tag{31}$$

Explicit formulae for equations (29)–(31) in polar coordinates are given in [30]. From this eigenvalue problem HUTCHINSON and RICE/ROSENGREN obtained

$$\gamma = -\frac{1}{q}, \quad \gamma(q-1) + 1 = \frac{1}{q} \tag{32}$$

and they calculated τ numerically. Thus, the leading terms in (28) are given by

$$u_0(r,\varphi) = r^{1/q} v(\varphi), \quad \sigma_0 = r^{-1/q} \tau(\varphi).$$

The functions u_0 and σ_0 are called HRR-fields.

We compare this approach with our regularity results. Let

$$u \in W^{3/2-\delta, 4p/(2+p)}(\Omega), \quad \sigma \in W^{1/q-\delta, q}(\Omega) \tag{33}$$

and assume that u and σ admit asymptotic expansions

$$u(r,\varphi) = r^{\tilde{\beta}} \tilde{v}(\varphi) + u_{\text{reg}}, \tag{34}$$

$$\sigma(r,\varphi) = r^{\tilde{\gamma}} \tilde{\tau}(\varphi) + \sigma_{\text{reg}}. \tag{35}$$

From (33) it follows that

$$\tilde{\beta} \geq \frac{1}{q}, \quad \tilde{\gamma} \geq -\frac{1}{q}. \tag{36}$$

This shows a correspondence between the singularities predicted by the HRR approach for cracked domains (see (32)) and the worst possible singularities given in (36). We want to emphasize that it is still an open problem, whether every weak solution admits an asymptotic expansion as in the linear case.

5. Boundary-transmission problems

In this part we consider structures which are composed of different homogeneous materials of Ramberg/Osgood type. Assume that the body Ω is composed of M sub-bodies $\overline{\Omega} = \bigcup_{i=1}^{M} \overline{\Omega_i}$, where each of these sub-bodies is homogeneous and of Ramberg/Osgood type with parameters A_i, E_i, $\tilde{\alpha}_i$, σ_y^i, q_i. The boundary of the domain Ω shall be divided into a part Γ_D, where the displacements are prescribed, and a part Γ_N, where the surface forces are given. Let u_i, σ_i be the restriction of u and σ to Ω_i. The classical boundary transmission problem reads:

Find $u\colon \Omega \to \mathbb{R}^3$ and $\sigma\colon \Omega \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ such that

$$\operatorname{div} \sigma_i + f_i = 0 \quad \text{in } \Omega_i, \tag{37}$$

$$\varepsilon(u_i) - A_i\sigma_i - \frac{3\tilde{\alpha}_i}{2E_i}\left(\frac{\sigma_e^i}{\sigma_y^i}\right)^{q_i-2}\sigma_i^D = 0 \quad \text{in } \Omega_i, \tag{38}$$

$$u_i - u_j = 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega_j, \tag{39}$$

$$\sigma_i\vec{n}_{ij} + \sigma_j\vec{n}_{ji} = 0 \quad \text{on } \partial\Omega_i \cap \partial\Omega_j, \tag{40}$$

$$u_i = g \quad \text{on } \partial\Omega_i \cap \Gamma_D, \tag{41}$$

$$\sigma\vec{n}_i = h \quad \text{on } \partial\Omega_i \cap \Gamma_N. \tag{42}$$

Here, f_i denotes the volume force densities, h the surface force density and g the displacement on Γ_D. Furthermore, $\vec{n}_{ij}$ is the exterior unit normal vector on Ω_i with respect to $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$, $\vec{n}_{ij} = -\vec{n}_{ji}$, $\vec{n}_i$ is the exterior unit normal vector of Ω_i.

In order to give a weak formulation of the boundary transmission problem we have to introduce function spaces which take into account the different nonlinearities on the subdomains. For $1 \leq i \leq M$, $p_i \in (1, \infty)$, $\vec{p} := (p_1, \dots, p_M)$ and $p_{\min} = \min\limits_{1\leq i\leq M} p_i$ we define

$$L^{\vec{p},2}(\Omega) = \big\{\sigma \in L^{p_{\min},2}(\Omega) : \sigma|_{\Omega_i} \in L^{p_i,2}(\Omega_i)\big\}, \tag{43}$$

$$U^{\vec{p},2}(\Omega) = \big\{u \in U^{p_{\min},2}(\Omega) : u|_{\Omega_i} \in U^{p_i,2}(\Omega_i)\big\}. \tag{44}$$

These spaces are endowed with the norms

$$\|\sigma\|_{L^{\vec{p},2}(\Omega)} = \sum_{i=1}^{M} \|\sigma|_{\Omega_i}\|_{L^{p_i,2}(\Omega_i)},$$
$$\|u\|_{U^{\vec{p},2}(\Omega)} = \sum_{i=1}^{M} \|u|_{\Omega_i}\|_{U^{p_i,2}(\Omega_i)}.$$

The trace spaces are introduced for an open subset $\Gamma \subset \partial\Omega$ in the usual way for $p_i \in (1,2]$:

$$W^{1-1/\vec{p},\vec{p}}(\Gamma) = \big\{u \in L^{p_{\min}}(\Gamma) : u = \hat{u}|_\Gamma,\ \hat{u} \in W^{1,\vec{p}}(\Omega)\big\}, \tag{45}$$

with the norm $\|u\|_{W^{1-1/\vec{p},\vec{p}}(\Gamma)} = \inf\{\|\hat{u}\|_{W^{1,\vec{p}}(\Omega)} : \hat{u} \in W^{1,\vec{p}}(\Omega),\ \hat{u}|_\Gamma = u\}$ and

$$\begin{aligned}\widetilde{W}^{1-1/\vec{p},\vec{p}}(\Gamma) = \big\{u \in L^{p_{\min}}(\Gamma) :& \\ &u = \tilde{u}|_\Gamma,\ \tilde{u} \in W^{1-1/\vec{p},\vec{p}}(\partial\Omega),\ \operatorname{supp}\tilde{u} \subset \overline{\Gamma}\big\},\end{aligned} \tag{46}$$

which is endowed with the norm $\|u\|_{\widetilde{W}^{1-1/\vec{p},\vec{p}}(\Gamma)} = \|\tilde{u}\|_{W^{1-1/\vec{p},\vec{p}}(\partial\Omega)}$.

5.1. Weak formulation. Let $\vec{q} \in \mathbb{R}^M$ with $q_i \geq 2$, $\vec{p} = \vec{q}\,'$, $f \in (V^{\vec{p},2}(\Omega))'$, $g \in W^{1-1/\vec{p},\vec{p}}(\Gamma_D)$ and $h \in W^{-1/\vec{q},\vec{q}}(\Gamma_N)$. The weak formulation of the boundary transmission problem is:

Find $\sigma \in L^{\vec{q},2}(\Omega)$ and $u \in U^{\vec{p},2}(\Omega)$ with $u|_{\Gamma_D} = g$ such that the identities

$$\sum_{i=1}^{M} \int_{\Omega_i} A_i\sigma_i : \tau_i + \alpha_i\big|\sigma_i^D\big|^{q_i-2}\sigma_i^D : \tau_i^D \,\mathrm{d}x - \int_{\Omega_i} \tau_i : \varepsilon(u_i)\,\mathrm{d}x = 0, \tag{47}$$
$$\sum_{i=1}^{M} \int_{\Omega_i} \sigma_i : \varepsilon(v_i)\,\mathrm{d}x - \langle f, v\rangle_{V^{\vec{p},2}(\Omega)} - \langle h, v\rangle_{\widetilde{W}^{1-1/\vec{p},\vec{p}}(\Gamma_N)} = 0 \tag{48}$$

hold for every $\tau \in L^{\vec{q},2}(\Omega)$ and $v \in V^{\vec{p},2}(\Omega)$.

Analogously to Theorem 4.3 the existence and uniqueness of a weak solution can be proved.

5.2. Regularity for weak solutions to transmission problems. We start with the classical linear KELLOGG example [11] which illustrates that the regularity of weak solutions can be very low for transmission problems. This example describes the behaviour of weak solutions of the Laplace equation with piecewise constant coefficients near a cross point S of a chess board (see Figure 3).

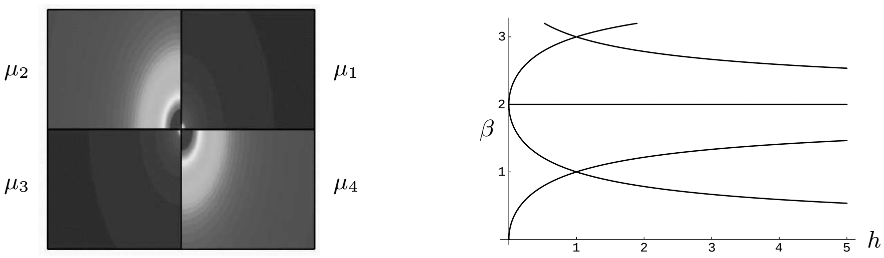

Figure 3. Domain and singular exponents β for Kellogg's example.

The corresponding weak formulation reads:

Find $u \in W^{1,2}(\Omega)$ *such that for every* $v \in W_0^{1,2}(\Omega)$

$$\sum_{i=1}^{4} \int_{\Omega_i} \mu_i \nabla u_i \nabla v_i \,\mathrm{d}x = \int_{\Omega} f v \,\mathrm{d}x. \tag{49}$$

The weak solution of (49) admits an asymptotic expansion near the interior cross point S of the type (1). Denoting by $\beta_{\min}$ the smallest positive exponent in expansion (1), it follows that $u|_{\Omega_i} \in W^{1+\beta_{\min}-\delta,2}(\Omega_i)$ for $1 \leq i \leq 4$, $\delta > 0$. Choosing $\mu_1 = \mu_3 = h$ and $\mu_2 = \mu_4 = 1$ it can be shown that the singular exponents β are solutions of the equation

$$\cos(\beta\pi) = 1 - \frac{8h}{(1+h)^2}.$$

Note that for $h \to 0$ or $h \to \infty$ the exponent $\beta_{\min}$ tends to 0. Therefore, the regularity can be very low.

In order to assure a certain minimum regularity additional assumptions are necessary. In particular, it turns out that a quasi-monotone distribution of the material parameters [6] leads to higher regularity for solutions of the Laplacian and linear isotropic elasticity with piecewise constant coefficients

[22], [17], [12], [20]. In [13] a new condition for nonlinear transmission problems, called quasi-monotone covering condition, is proposed which entails higher regularity for weak solutions.

In this paper we demonstrate how this idea can be applied to transmission problems for Ramberg/Osgood materials. We formulate it for the special case of an internal cross point S.

Definition 5.1 (Quasi-monotonicity)**.** Let be $S \in \Omega$ an interior cross point of the subdomains $\Omega_1, \ldots, \Omega_N$ and let $B(S) \subset\subset \Omega$ be a ball with the centre S. The quasi-monotonicity condition is satisfied on $B(S)$ if there exist numbers $k_1, \ldots, k_N \in \mathbb{R}$ and a cone $\mathcal{K} \subset \mathbb{R}^d$ with the vertex in 0 such that for every $1 \leq i, j \leq N$ the following implication holds:

$$\begin{aligned} &\text{if } \big((\Omega_i \cap B(S)) + \mathcal{K}\big) \cap \big(\Omega_j \cap B(S)\big) \neq \emptyset, \\ &\text{then } W_{c,j}(\tau) + k_j \leq W_{c,i}(\tau) + k_i \text{ for every } \tau \in \mathbb{R}^{d\times d}_{\text{sym}}. \end{aligned}$$

Here, $W_{c,j}(\tau) = \frac{1}{2} A_j \tau : \tau + \frac{\alpha_j}{q_j} \left|\tau^D\right|^{q_j}$ is the complementary energy density.

R e m a r k 5.2. This condition is formulated in [13] also for the case when the cross point S is situated on the boundary.

We give a two-dimensional example in order to illustrate this quasi-monotonicity condition. Let $d = 2$, $S = 0$, $R > 0$ and suppose that the subdomains Ω_i, $1 \leq i \leq N$, are of the following shape: there exist angles $\Phi_0 < \Phi_1 < \cdots < \Phi_N = \Phi_0 + 2\pi$ such that $\Omega_i = \{x \in \mathbb{R}^2 : |x| < R,\ \Phi_{i-1} < \varphi < \Phi_i\}$. The complementary energy densities $W_{c,j}$ satisfy the quasi-monotonicity condition with respect to the ball $B_R(0)$ if there exits a vector $\vec{t} \in \mathbb{R}^2 \setminus \{0\}$, an index $i_0 \in \{2, \ldots, N\}$, and numbers $k_1, \ldots, k_N \in \mathbb{R}$ such that $\vec{t} \in \Omega_1$ and $-\vec{t} \in \Omega_{i_0}$ (see Figure 4), and

$$\begin{aligned} W_{c,1}(\tau) + k_1 &\geq W_{c,2}(\tau) + k_2 \geq \ldots \\ &\geq W_{c,i_0}(\tau) + k_{i_0} \leq \cdots \leq W_{c,N}(\tau) + k_N \leq W_{c,1}(\tau) + k_1 \end{aligned} \tag{50}$$

for every $\tau \in \mathbb{R}^{2\times 2}_{\text{sym}}$. Assuming that $q_i \neq q_j$ for $i \neq j$ and that

$$W_{c,j}(\tau) = \frac{1}{8(\lambda_j + \mu_j)} |\operatorname{tr} \tau|^2 + \frac{1}{2\mu_j} \left|\tau^D\right|^2 + \frac{\alpha_j}{q_j} \left|\tau^D\right|^{q_j},$$

condition (50) reduces to the following inequalities for the material parameters

$$\begin{gathered} \lambda_1 + \mu_1 \leq \cdots \leq \lambda_{i_0} + \mu_{i_0} \geq \cdots \geq \lambda_N + \mu_N \geq \lambda_1 + \mu_1, \\ q_1 > \cdots > q_{i_0} < \cdots < q_N < q_1. \end{gathered}$$

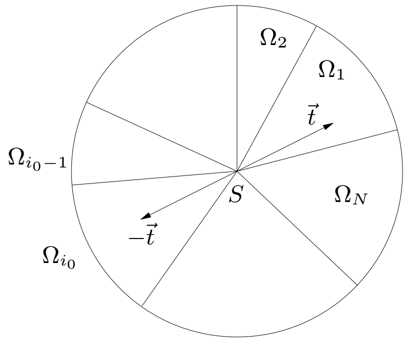

Figure 4. Two-dimensional example for the quasi-monotonicity condition.

Based on the quasi-monotone covering condition we can formulate the main regularity theorem for boundary transmission problems.

Theorem 5.3. *Assume that the densities $W_{c,i}$ and the subdomains Ω_i satisfy the quasi-monotone covering condition of Definition 5.1. Furthermore, let us assume for the loads that $f \in L^{\vec{q}}(\Omega)$, $g = \hat{g}|_{\Gamma_D}$, $\hat{g} \in W^{2,p_{\max}}(\hat{\Omega})$, $p_{\max} = \max_{1\le i\le M} p_i$ with $\nabla\hat{g} \in L^\infty(\hat{\Omega})$, $\hat{\Omega} \supset\supset \Omega$, $h = H\vec{n}$ on Γ_N, $H \in W^{1,p'_{\max}}(\hat{\Omega}, \mathbb{R}^{d\times d}_{\mathrm{sym}}) \cap L^\infty(\hat{\Omega})$.*

Then, for every $\delta > 0$ and i,

$$\begin{aligned} u_i &\in W^{3/2-\delta,2dp_i/(2d-2+p_i)}(\Omega_i), \\ \operatorname{div} u_i,\ \sigma_i &\in W^{1/q_i-\delta,q_i}(\Omega_i) \cap W^{1/2-\delta,2}(\Omega_i). \end{aligned} \tag{51}$$

Remarks to the proof. Again, the main tool is the difference quotient technique. Now, an additional difficulty is that the weak transmission problem is formulated in spaces with variable exponents. In order to use difference quotients as test functions we have to require that translated functions belong to the original spaces, too. This is guaranteed by the quasi-monotone covering condition. □

Example 5.4. Let $\Omega, \Omega_1, \Omega_2 \subset \mathbb{R}^d$, $d = 2,3$, be bounded domains with Lipschitz boundaries, $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$ and $\Omega_1 \subset\subset \Omega$, see Figure 5. For simplicity we assume that the type of the boundary conditions does not change on $\partial\Omega$. Furthermore, let be $2 \ge p_1 > p_2$ and $0 < \lambda_1 + \mu_1 < \lambda_2 + \mu_2$. Then the quasi-monotone covering condition is satisfied and the weak solutions of the corresponding boundary transmission problem (47)–(48) have the regularity (51).

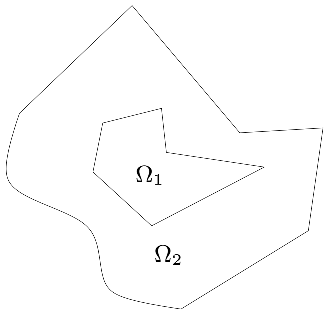

Figure 5. Nested Lipschitz domains.

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