3 The generalized Newtonian model

In the first part of the notes, we introduced the notion of a *Stokes fluid* in (1.5), which has a deviatoric stress tensor of the form

$$S = \alpha \mathrm{Id} + \beta D(v) + \gamma D^2(v),$$

where v is the velocity, $D(v)$ is the symmetric part of the velocity gradient or the deformation tensor, and where α, β, γ are functions of the invariants of D, i.e. the quantities $\mathrm{tr}(D)$, $(\mathrm{tr}(D))^2 - \mathrm{tr}(D^2)$ and $\det(D)$, which are the three coefficients of the characteristic polyomial $p_D(\lambda) = \det(\lambda \mathrm{Id} - D)$ of D, except for the first coefficient, 1. In rheology, the notations I_D,II_D and III_D are used for these invariants. Because they do not depend on the choice of coordinates, they are also called *objective* quantities. By Assumption 1 in 1.3, S was constructed in this way so that its dependence on D is also objective.

If in addition, the incompressibility constraint is taken into account, then $I_D = 0$ and $II_D = -\mathrm{tr}(D^2) = -D : D$.[1] The fluids modelled by the remaining tensor

$$S = \phi_0(II_D, III_D)\mathrm{Id} + \phi_1(II_D, III_D)D(v) + \phi_2(II_D, III_D)D^2(v) \tag{3.1}$$

are called *Reiner-Rivlin fluids.* They are the most general fluids satisfying Assumptions 1-3 in 1.3 and incompressibility. Maybe at this point, we should put the most emphasis on the homogeneity Assumption 3. It says that the stress tensor is not a function of the position we are at in the fluid and that it does not depend on time. These are constitutive assumptions which exclude a particular structure of the fluid and, maybe more importantly, memory effects. It separates the models for Newtonian and generalized Newtonian fluids from models for truly viscoelastic fluids[2]. This assumption implies that the model cannot account for typical normal stress effects like the ones from the second chapter. This can be shown by a calculation in simple shear flow, cf. e.g. [16, pp. 19]. Normal stresses and the corresponding viscoelastic effects can be accounted for e.g. by the Oldroyd-B model, so they will appear in the next part of the notes. In this context, it has to be noted that, although the generalized Newtonian fluid model is widely used in engineering, in application and in mathematics, it has also been criticized by several authors. Therefore, it would be safe to say that it is a “starting-point” for the analysis, cf. [2, p. 380] or [4, pp. 226], which may be more easily accessible then others.

[1] *This was imprecise in the first part of the notes!*

[2] There is also seperate literature on inhomogeneous fluids, but this seems to be independent of the modelling here?

3.1 The generalized Newtonian stress tensor

From (3.1), three arguments lead to the constitutive equations for the stress tensor of generalized Newtonian fluids. The first two are based on experimental data, cf. [16, Section 3.2],

1. If we assume $\phi_2 \neq 0$, in particular in simple flow, this flow behaviour contradicts experiments on non-Newtonian fluids, so it is assumed that real fluids satisfy $\phi_2 = 0$.

2. Likewise, measurements suggest that the dependence of S on III_D is negligible. In simple flow, it holds that $III_D = 0$, so that a second problem is that many rheometers cannot measure III_D.

3. By 1 and 2, (3.1) is reduced to $S = \phi_0(II_D)\mathrm{Id} + \phi_1(II_D)D$. In the third step, we include $\phi_0(II_D)$ in the spherical, pressure part of T, so that
$$S = \phi_1(II_D)D.$$

The last step is simply to introduce a new notation. The invariant II_D is always negative. Instead, we use the quantity $\dot{\gamma} = \sqrt{-4II_D}$, which is called the *shear rate* and we define

$$\eta(\dot{\gamma}^2) := \phi_1(II_D).$$

Assumptions 1-3 on (3.1) now imply that for real fluids, the relation

$$T = -p\mathrm{Id} + \eta(\dot{\gamma})D(v) \tag{3.2}$$

is a good generalization of Newtonian flow, if the fluid is assumed to be homogeneous. Fluids governed by a constitutive law of this kind are called *generalized Newtonian fluids.* The function η is called the (generalized) viscosity of the fluid, as in the definition in (1.6). If η is constant, the fluid is Newtonian. In addition, from the second law of thermodynamics, it follows that $T : D = \eta\mathrm{tr}(D^2) \geq 0$, so that η should satisfy $\eta \geq 0$ in any case, cf. e.g. [16, Section 3.3] and the references therein.

There are very many ways to define η, depending on different numbers of material parameters, in order to model a particular fluid. Some ideas and examples are given in the next section. Note that in general, we see that the viscosity of the fluid will *change* with its rate of shear. Roughly speaking, it may increase with increasing $\dot{\gamma}$, or decrease. Both phenomena appear, whereas the latter is more common. Fluids which increase their viscosity as shear increases are called *shear-thickening* or *dilatant*, if they decrease their viscosity instead, they are called *shear-thinning* or *pseudoplastic*.

An explanation for shear-thinning of polymeric solutions is that the macromolecular particles, the molecules, which have a chain-like structure will align with the fluid velocity as the shear increases, causing the fluid to "flow more easily".

A typical example of a shear-thickening fluid is a solution of corn starch in water. If a lot of stress is applied, the fluid is squeezed out from between the starch and the whole material becomes almost solid. Blood is a fluid which exhibits, among other properties, an interesting shear-thinning behaviour. Whereas it has a high viscosity under low strain rate, it keeps a constant viscosity in a wide higher strain rate region. This is due to clustering and breaking up of clusters under low and higher strain rates, respectively, cf. [10, p. 196].

3.2 Examples

The simplest and first type of generalized Newtonian fluids are the power-law fluids of *Ostwald and de Waele*, where the viscosity is given by

$$\eta(s) = \kappa_0 s^{(d-2)/2}, \tag{3.3}$$

where κ_0 and d are positive constants. In particular, the *power-law exponent* d should satisfy $d \geq 1$. If $d = 2$, the fluid is Newtonian. If $d < 2$, it exhibits shear-thinning behaviour, if $d > 2$, it is shear-thickening. Most models, references and analysis are about/to power-law *type* fluids, cf. also the next chapter. There are some defects to this model: if the fluid is shear-thinning, the viscosity becomes infinite for zero shear rate and if it is shear-thickening, the viscosity will become zero. This does not match experiments with real fluids. The second problem can be overcome if a model of the form

$$\eta(s) = \kappa_1 + \kappa_0 s^{(d-2)/2} \tag{3.4}$$

is considered instead. This was done first by Ladyzhenskaya, e.g. in [11], where fluids of exponent $d \geq \frac{9}{5}$ were considered. Fluids of this type are therefore also called *Ladyzhenskaya fluids.* One further problem is that for power-law fluids of this type, the viscosity is unbounded.

Both issues are resolved by different modifications of (3.3) found in the literature. The *Cross model* uses the relation

$$\frac{\eta(s) - \eta_\infty}{\eta_0 - \eta_\infty} = \frac{1}{1 + (\kappa_0^2 s)^{(2-d)/2}},$$

where η_0, η_∞ are parameters which model a viscosity limit at 0 or infinite shear rate in both shear-thinning and shear-thickening cases (for the latter, the notation η_0, η_∞ should be reversed). The *Ellis model* is a simplification of the Cross model, where $\eta_\infty = 0$,

$$\eta(s) = \frac{\eta_0}{1 + (\kappa_0^2 s)^{(2-d)/2}}.$$

Two more variations of the Cross model are the *Yasuda model*, where even one more modelling parameter a is included,

$$\frac{\eta(s) - \eta_\infty}{\eta_0 - \eta_\infty} = \frac{1}{(1 + \kappa_0^a s)^{(2-d)/a}},$$

and the *Carreau model*, which is the Yasuda model with $a = 2$. This model is often written in the form

$$\eta(s) = \eta_\infty + (\eta_0 - \eta_\infty)(1 + \kappa_0^2 s)^{(d-2)/2}$$

and it is often used and referred to in analysis.

Examples of models for η which are not power-law type are the *Powell-Eyring model*, where

$$\eta(s) = \eta_\infty + (\eta_0 - \eta_\infty)\frac{\sinh^{-1}(\sqrt{s}\lambda)}{\sqrt{s}\lambda},$$

and the *Prandtl-Eyring model*, where $\eta_\infty = 0$ in the Powell-Eyring model.

A different type of models are given for *Bingham fluids.* In general, it is assumed that they behave like a solid until a *yield stress* τ_y is reached. If the yield stress is surpassed, the fluid flows in a Newtonian way. There are modifications of the Bingham model, e.g. the *Herschel-Bulkley model*, which is for shear-thinning yield stress flow. The yield stress may be reached locally in parts of the fluid and therefore a Bingham fluid can be flowing in some parts and at the same time remain solid in others.

4 The generalized Navier-Stokes equations

In the last chapter, we defined the stress tensor of a generalized Newtonian fluid. The corresponding system of equations, the *generalized Navier-Stokes equations* has the form

$$\begin{cases} \rho(\frac{\partial v}{\partial t} + (v \cdot \nabla)v) - \operatorname{div} S(D(v)) + \nabla q &= f, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} v &= 0, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ v|_{\partial\Omega} &= 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \\ v|_{t=0} &= a, \quad \text{on } \Omega, \end{cases} \tag{4.1}$$

cf. (1.8) in the first part. The main difference to the Navier-Stokes equations is that in general, the dependence of S on v will not be linear but given by one of the models for η from the last section. In the next two sections, we look at two different approaches to solving (4.1) in the literature.

The first is about finding strong L^p-solutions to (4.1), like the strong solutions for the Stokes problem, cf. Theorem 1 in Section 1.3.4. It was used by Bothe and Prüss in [2].

The second is more prominent in the mathematical research and it is based on the theory of monotone operators. It yields different results on strong solutions and especially results on global weak solutions. I know much less about this family of methods and hope that we can discuss about it on the seminar day. Some ideas are in Section 4.2.

What can be seen from both methods, in different ways, is that for this generalization of the Navier-Stokes equations, it is so far necessary to work in the L^p-framework and leave Hilbert spaces.

4.1 The result by Bothe and Prüss - the generalized NSEs as a quasi-linear parabolic problem

In the original paper, there is a more general version of the result given in this section, which covers also different types of boundary conditions. Roughly speaking, the idea is to look at (4.1) as a quasi-linear second-order parabolic problem. More precisely, we can calculate $\operatorname{div} S(D(v))$ and define the operator as follows:

$$
\begin{aligned}
(A(v)v)_i := (\operatorname{div} S(D(v)))_i &= \sum_{j=1}^{3} \eta(|D(v)|_2^2)\partial_j d_{ij}^{(v)} + \partial_j(\eta(|D(v)|_2^2)) d_{ij}^{(v)} \\
&= \sum_{j=1}^{3} \big[\eta(|D(v)|_2^2)\partial_j^2 v_i + 2\eta'(|D(v)|_2^2)(\sum_{k,l=1}^{3} d_{kl}^{(v)}\partial_j d_{kl}^{(v)} d_{ij}^{(v)})\big] \\
&= \eta(|D(v)|_2^2)\Delta v_i + 2\eta'(|D(v)|_2^2) \sum_{j,k,l=1}^{3} d_{ik}^{(v)} d_{jl}^{(v)} \partial_k \partial_l v_j,
\end{aligned}
$$

where we use the notation $d_{ij}^{(v)} = (D(v))_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$ and $\dot\gamma = \sqrt{\sum_{i,j}(d_{ij}^{(v)})^2} =: |D(v)|_2$. Clearly, if η were constant, $A(v)$ would just be $\eta\Delta$, regardless of v. The operator is quasi-linear second-order if we consider everything in η and in the $d^{(v)}$'s as "coefficients". Moreover, from this definition it can be seen that some regularity is required of the function η. More precisely, we impose that $\eta \in C^{1,1}(\mathbb{R}_+;\mathbb{R})$, i.e. the function has a Lipschitz-continuous first derivative and the correspoding norm is bounded. Secondly, the positivity and "growth" conditions

$$
\eta(s) > 0 \quad \text{and} \quad \eta(s) + 2s\eta'(s) > 0 \quad \text{for all } s \geq 0 \tag{4.2}
$$

have to be satisfied. We will see below where these conditions come from in the analysis. Note here that both assumptions are physically reasonable. The positivity of the viscosity was already claimed in Section 3.1, the growth condition can be derived if we claim that the norm of the viscous stress $|S|_2 = \eta(\dot\gamma^2)\dot\gamma$ should increase with increasing rate of strain $\dot\gamma^2$. Taking the derivative yields $|S|_2'(\dot\gamma) = \eta(\dot\gamma^2) + 2\dot\gamma^2\eta'(\dot\gamma^2)$.

Under these assumptions, the following is the main theorem on local existence of strong solutions.

Theorem 1. *Assume $p > n+2$, and that Ω is a domain in $\mathbb{R}^n$ with compact boundary of class $C^{2,1}$. Given $a \in W^{2-2/p,p}(\Omega)$, $\operatorname{div} a = 0$ and $a|_{\partial\Omega} = 0$ and $f \in L^p(\mathbb{R}_+; L^p(\Omega))$, there exists a maximal interval $J_{T^*} = (0, T^*)$, $T^* > 0$, such that problem (4.1) admits a unique strong solution*

$$
\begin{aligned}
v &\in X_{p,p}^T := L^p(J_{T^*}; W^{2,p}(\Omega)) \cap W^{1,p}(J_{T^*}; L^p(\Omega)), \\
q &\in L^p(J_{T^*}; \widehat{W}^{1,p}(\Omega)).
\end{aligned}
$$

Moreover, T^* can be characterized as follows: if $T^* < \infty$, then $\lim_{t\to T^*} v(t)$ does not exist in $W^{2-2/p,p}(\Omega)$.

Remark 2. The result holds true regardless of the space dimension (only the condition on p changes). This follows from the general technique used for solving the problem. However, it only gives a criterion, not a result on global solutions in any dimension.

Remark 3. We have to check which types of generalized Newtonian fluids are included in the conditions on η. A first and important observation (from the paper) is the following: The power-law-type functions

$$\eta(s) = \kappa_0(1+s)^{(d-2)/2}$$

satisfy (4.2) for every $d \geq 1$. This model is used in analysis, because it does not have the defects of (3.3) at $\dot{\gamma} \to 0$. It follows that for this model, shear-thinning ($d < 2$) and shear-thickening ($d > 2$) can be covered, for all relevant exponents. Other models from Section 3.2 should be checked, cf. the questions at the end of the notes.

The remainder of the section is for sketching the proof of Theorem 1. In a first step, the quasi-linear problem is linearized by freezing the coefficients of A at a reference solution to the corresponding Stokes problem, which we obtain from Theorem 1 in Section 1.3.4. Let this solution be called v^*. This gives us a linear second-order operator A^* with time- and space-dependent coefficients,

$$\begin{aligned} (A^*v)_i : &= (A(v^*)v)_i = \eta(|D(v^*)|_2^2)\Delta v_i + 2\eta'(|D(v^*)|_2^2)\sum_{j,k,l} d_{ik}^{(v^*)} d_{jl}^{(v^*)} \partial_k\partial_l v_j \\ &=: \sum_{j,k,l} a_{kl}^{ij}\partial_k\partial_l v_j, \end{aligned} \tag{4.3}$$

where

$$a_{kl}^{ij} := \eta(|D(v^*)|_2^2)(\delta_{kl}\delta_{ij}) + 2\eta'(|D(v^*)|_2^2) d_{ik}^{(v^*)} d_{jl}^{(v^*)}.$$

The corresponding linear *generalized Stokes equations* are

$$\begin{cases} \rho\frac{\partial v}{\partial t} - A^*v + \nabla q &= f, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} v &= 0, \quad \text{in } \mathbb{R}_+ \times \Omega, \\ v|_{\partial\Omega} &= 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega, \\ v|_{t=0} &= a, \quad \text{on } \Omega. \end{cases} \tag{4.4}$$

For this system, the authors obtain a maximal regularity result, similar to Theorem 1 in Section 1.3.4. It is by using general theory for maximal regularity of parabolic problems, modified to fit the additional incompressibility constraint and pressure term.

Two non-linearities then remain to be added by a fixed point argument, cf. Subsection 4.1.2.

4.1.1 Ellipticity of A^*

One of the main ingredients for solving (4.4) is the result by Denk, Hieber and Prüss [5] on maximal L^p-regularity for parabolic boundary value problems. We look at (4.4) without the pressure or divergence condition. Following [5], roughly speaking, it has to be shown that

1. A^* is strongly elliptic,

2. the Lopatinskiĭ-Shapiro-condition is satisfied and
3. the coefficients of A^* are bounded in space and time,

to get that the corresponding parabolic initial boundary value problem has maximal regularity.

We check that these conditions are met in our case.

1. A^* is strongly elliptic: By (4.3), the symbol $A^*_\#$ of A^* is given by

$$A^*_\#(x,\xi) = \sum_{k,l} a^{ij}_{kl}(x)\xi_k\xi_l.$$

To show strong ellipticity, we check that for all $x \in \Omega$, $\xi \in \mathbb{R}^n$ and $\mu \in \mathbb{C}^n$ such that $|\xi| = |\mu| = 1$, $\mathrm{Re}(A^*_\#(x,\xi)\mu,\mu) \geq c$ for some positive constant $c > 0$. Indeed, abbreviating $\eta(|D(v^*)(x)|_2^2)$ by $\eta_*(x)$, we get

$$\begin{aligned}(A^*_\#(x,\xi)\mu,\mu) &= \sum_{i,j,k,l} a^{ij}_{kl}(x)\xi_k\xi_l\mu_j\overline{\mu_i} \\ &= \sum_{i,j,k,l} \eta_*(x)\delta_{ij}\delta_{kl}\xi_k\xi_l\mu_j\overline{\mu_i} + 2\eta_*'(x)d^{(v^*)}_{ik}(x)d^{(v^*)}_{jl}(x)\xi_k\xi_l\mu_j\overline{\mu_i} \\ &= \eta_*(x)|\xi|^2|\mu|^2 + 2\eta_*'(x)|(D(v^*)\xi,\eta)|^2.\end{aligned}$$

From (4.2), it follows that $\eta_* > 0$ and we see that this condition is necessary for strong ellicpticity. It would be suffiicient if $\eta_*' \geq 0$. Suppose that $\eta_*' < 0$, then

$$\begin{aligned}\eta_*|\xi|^2|\mu|^2 + 2\eta_*'|(D(v^*)\xi,\eta)|^2 &\geq \eta(|D(v^*)|_2^2)|\xi|^2|\mu|^2 + 2\eta'(|D(v^*)|_2^2)|D(v^*)|_2^2|(\xi,\mu)|^2 \\ &\geq c|(\xi,\mu)|^2\end{aligned}$$

by the Cauchy-Schwarz inequality and by the growth condition in (4.2). If $|(\xi,\mu)|^2 = 0$ or close to 0, the first inequality already yields the claim.

2. The Lopatinskiĭ-Shapiro-condition makes the boundary value problem elliptic by ensuring that somehow the boundary condition matches the operator, cf. e.g the monograph [17]. Here, it refers to Dirichlet boundary conditions for A^*, cf. the short appendix. For the proof for several types of boundary conditions in our case, we refer to [2, p. 386] It is not so difficult.

3. The coefficients a^{ij}_{kl} of A^* depend on η, η' and on $D(v^*)$. In the full quasi-linear problem, they are functions of the the solution v . In [5], boundedness is required. Here, we see that $a^{ij}_{kl} \in C([0,T_*] \times \overline{\Omega})$ because

$$X^{T_*}_{p,p} \hookrightarrow C(J_{T_*}; W^{2-2/p,p}(\Omega)), \tag{4.5}$$

see the short appendix and by Sobolev embedding,

$$W^{2-2/p,p}(\Omega) \hookrightarrow C^1(\overline{\Omega}) \tag{4.6}$$

if $p > n+2$, so that $D(v)$ and $D(v^*)$ are continuous, as well as η and η'. Thus the requirement that $p > n+2$ arises from the dependence of the viscosity on the first-order derivatives of the velocity field.

It is technical and difficult to transform the knowledge on the parabolic problem without pressure and incompressibility to the generalized Stokes problem and we refer to [2, Sections 5-7] for the proof. Here, we can only see that the frozen generalized Stokes operator yields a good parabolic problem. But moreover, the arguments show where the basic assumptions in the main theorem come from and how they connect to the modelling of the fluid. In particular, it shows that the conditions (4.2) on η are exactly the right ones for this kind of analysis, where at the same time, they are physically reasonable.

4.1.2 Fixed-point-argument

It is a standard technique for quasi-linear parabolic problems to use a maximal regularity result on the linearized system and a fixed point argument. For abstract results on this technique, we refer e.g. to [1].

In some of the fixed-point-argument is done explicitly. Solving the linear problem is really the difficult part.

We look at (4.1) and assume $p > n+2, \eta, \Omega$ as in Theorem 1. We fix a time $T_0 > 0$. We can use the embedding $X^T_{p,p} \hookrightarrow C(\overline{J_T}; W^{2-2/p,p}(\Omega)) \hookrightarrow C(\overline{J_T}; C^1(\overline{\Omega}))$ from (4.5) and (4.6) for all $0 < T < T_0$, only the embedding constant blows up as $T \to 0$. Therefore, we want to do the fixed point argument in the space ${}_0X^T_{p,p} = \{u \in X^T_{p,p} : u(0) = 0\}$, where this does not happen. Let the data a and f be as in Theorem 1. For this data, we solve the Stokes problem (1.10) from the first part, Theorem 1 in Section 1.3.4 for viscosity $\mu = 1$ to get a solution $v^* \in X^T_{p,p}$ and $q^* \in Y^T_{p,p}$. For a would-be solution v, q of (4.1), we define $u := v - v^*$ and $p = q - q^*$. By plugging in the Stokes problem and adding some zeroes, this gives a system of equations in u, p, which is equivalent to (4.1),

$$\left\{\begin{array}{rcll} \rho\frac{\partial u}{\partial t} - A^*u + \nabla p &=& F(v^*, u), & \text{in } \mathbb{R}_+ \times \Omega, \\ \operatorname{div} u &=& 0, & \text{in } \mathbb{R}_+ \times \Omega, \\ u|_{\partial\Omega} &=& 0, & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ u|_{t=0} &=& 0, & \text{on } \Omega, \end{array}\right. \tag{4.7}$$

where

$$F(v^*, u) := -\Delta v^* - A^*v^* - (v^*\cdot\nabla)v^* + [A^* - A(v^*+u)](v^*+u) - [(v^*\cdot\nabla)u + (u\cdot\nabla)v^* + (u\cdot\nabla)u].$$

By definition, $F(v^*, u)$ belongs to $L^p(0, T; L^p(\Omega))$. For $R > 0$ let $B^T_R := \{u \in_0 X^T_{p,p} : \|u\|_{X^T_{p,p}} \leq R\}$. Then every fixed point u of the map $\Phi : B^T_R \to_0 X^T_{p,p}$ given by

$$\Phi(\bar{u}) = \mathcal{U}(F(v^*, \bar{u}), 0)$$

and its associated pressure will solve (4.7) and vice versa. Here, $\mathcal{U}(F, 0), \mathcal{P}(F, 0)$ are the solution operators for the generalized Stokes problem (4.4) with right hand side F and initial data 0. By the Banach Fixed Point Theorem, it remains to be shown that Φ is a self-map and that it is contractive, for some $R, T > 0$. We only show the latter in detail, as the arguments are nearly the same for both types of estimates.

Let $\bar{u}_1, \bar{u}_2 \in B_R^T$. By maximal regularity of (4.4), we know that

$$\|\Phi(\bar{u}_1 - \bar{u}_2)\|_{X_{p,p}^T} \leq K\|F(v^*, \bar{u}_1) - F(v^*, \bar{u}_2)\|_{p,p},$$

for some constant $K > 0$ independent of R and T. Furthermore,

$$\begin{aligned} & \|F(v^*, \bar{u}_1) - F(v^*, \bar{u}_2)\|_{p,p} \\ \leq\ & \|[A^* - A(v^* + \bar{u}_1)](\bar{u}_1 - \bar{u}_2)\|_{p,p} \\ & + \|[A(v^* + \bar{u}_1) - A(v^* + \bar{u}_2)](v^* + \bar{u}_2)\|_{p,p} \\ & + \|(v^* \cdot \nabla)(\bar{u}_1 - \bar{u}_2) + ((\bar{u}_1 - \bar{u}_2) \cdot \nabla)v^* + (\bar{u}_1 \cdot \nabla)\bar{u}_1 - (\bar{u}_2 \cdot \nabla)\bar{u}_2)\|_{p,p} \\ =:\ & I + II + III \end{aligned}$$

We use the abbreviation $X^T := C([0,T]; C^1(\overline{\Omega}))$. For the estimates of I, II, III we use the following facts:

- η and η' are Lipschitz by assumption (1),
- $\|\bar{u}_i\|_{X^T} \leq CR$ for a constant C independent of T,
- $K_*^T := \|v^*\|_{X_{p,p}^T} \to 0$ for $T \to 0$ and $\|v^*\|_{X^T} \leq \|v^*\|_{X^{T_0}} \leq C\|v^*\|_{X_{p,p}^{T_0}} \leq C_0$ for some constant C_0 depending on the data a and f .

It follows that

$$I \quad \leq \quad C \sup_{i,j,k,l} \|a_{kl}^{ij}(v^*) - a_{kl}^{ij}(v^* + \bar{u}_1)\|_{\infty,\infty} \|\bar{u}_1 - \bar{u}_2\|_{X_{p,p}^T},$$

where

$$\begin{aligned} & \|a_{kl}^{ij}(v^*) - a_{kl}^{ij}(v^* + \bar{u}_1)\|_{\infty,\infty} \\ \leq\ & \|\eta(|D(v^*)|_2^2) - \eta(|D(v^* + \bar{u}_1)|_2^2)\|_{\infty,\infty} \\ & + \|\eta'(|D(v^*)|_2^2) d_{ik}^{(v^*)} d_{jl}^{(v^*)} - \eta'(|D(v^* + \bar{u}_1)|_2^2) d_{ik}^{(v^*+\bar{u}_1)} d_{jl}^{(v^*+\bar{u}_1)}\|_{\infty,\infty}. \end{aligned}$$

For the first part on the right hand side, we see that

$$\begin{aligned} \|\eta(|D(v^*)|_2^2) - \eta(|D(v^* + \bar{u}_1)|_2^2)\|_{\infty,\infty} \quad & \leq \quad C_\eta \sup_{i,j} \|(d_{ij}^{(v^*)})^2 - (d_{ij}^{(v^*)} + d_{ij}^{(\bar{u}_1)})^2\|_{\infty,\infty} \\ & \leq \quad C_\eta R(R + 2C_0) \end{aligned}$$

and by the same arguments for the second part, it follows that

$$I \leq C_\eta R \|\bar{u}_1 - \bar{u}_2\|_{X_{p,p}^T}.$$

Calculating in a similar way, we get

$$\begin{aligned} II &\leq C_\eta(\|v^*\|_{X^T} + \|\bar{u}_1\|_{X^T} + \|\bar{u}_2\|_{X^T})\|\bar{u}_1 - \bar{u}_2\|_{X^T_{p,p}}\|v^* + \bar{u}_2\|_{X^T_{p,p}} \\ &\leq (C_0 + R)(K^T_* + R)\|\bar{u}_1 - \bar{u}_2\|_{X^T_{p,p}} \end{aligned}$$

and

$$III \leq (K^T_* + R)\|\bar{u}_1 - \bar{u}_2\|_{X^T_{p,p}}.$$

In conclusion, if R and T are chosen to be sufficiently small, then

$$\|\Phi(\bar{u}_1 - \bar{u}_2)\|_{X^T_{p,p}} \leq K \cdot C^T_R \|\bar{u}_1 - \bar{u}_2\|_{X^T_{p,p}}$$

for some $C^T_R < \frac{1}{K}$ and thus Φ is a contraction.

The unique fixed point u and the corresponding pressure p solve (4.7), therefore $v = v^* + u$ and $q = q^* + p$ is a solution of (4.1) in the correct function spaces. If the map

$$t \mapsto \|v(t)\|_{W^{2-2/p,p}(\Omega)}$$

remains bounded, then it yields initial values for problem (4.1), which allow to extend the solution. This argument explains the characterization of the maximal time of existence of solutions in Theorem 1.

4.2 Literature on (4.1) and weak solutions

The classical approach to problem (4.1) goes back to Ladyzhenskaya, [11][1] and it is connected to the development of monotone operator theory for partial differential equations. One result of Ladyzhenskaya is that for $p > \frac{9}{5}$, there is a global weak solution of (4.1) under the assumption that the viscosity satisfies (3.4). Ladyzhenskayas result has been extended and the theory has developed in many directions. I have a very limited overview of the literature, so only some contributions are mentioned here. The references in the references will probably not even give a complete picture.

There is literature on improving the condition on p by various methods, but, the best ist $\frac{6}{5}$ in three space dimensions, using also different notions of weak solutions, cf. [12],[15],[13],[8] and the survey [14], and for more general (inhomogeneous) models, [3] and [9]. On regularity of the solutions, cf. e.g [7] on stationary solutions and e.g. [6] on the existence of strong solutions for $p > \frac{7}{5}$. There are also several results on numerical methods.

In this part of the notes I put some basic ideas about the weak formulation, which hopefully show the connection between p and d and why it is not clear how to go to $d = 1$ for the problem in three space dimensions.

In order to give a weak formulation of (4.1), one can define the solenoidal function spaces that fit the boundary condition, $L^r_\sigma(\Omega)$ and

$$V_r(\Omega) := \{v \in W^{1,r}_0(\Omega) : \operatorname{div} v = 0\}.$$

On the deviatoric stress $S(D(u))$, there are several assumptions,

[1] and earlier works, but I cannot find them...

- *growth:* there is a constant $c > 0$, such that for all $A \in \mathbb{R}^{n\times n}_{sym}$, $|S(z)| \le c(1+|z|)^{p-1}$,
- *p-coercivity:* there is a constant $c_0 > 0$ such that for all $A \in \mathbb{R}^{n\times n}_{sym}$, $S(z) : z \ge c_0|z|^p$,
- *monotonicity:* for all $A \ne B$, $A, B \in \mathbb{R}^{n\times n}_{sym}$, $(S(z) - S(y)) : (z - y) > 0$.

These assumptions in particular include the power-law model and the Carreau model. These are just examples of assumptions from the literature and they may be modified. To get a weak formulation, (4.1) is multiplied by a test function $\varphi \in C^\infty_c(0, T; C^\infty_{c,\sigma}(\Omega))$, and integrated over Ω and $(0, T)$. By using integration by parts, formally, a weak formulation of (4.1) is

$$\int_0^T \int_\Omega u \cdot \varphi_t + \int_0^T \int_\Omega S(D(u)) : D(\varphi) + \int_0^T \int_\Omega u \otimes u : \nabla\varphi = \int_0^T \int_\Omega f \cdot \varphi. \tag{4.8}$$

In order for the expression to be well defined, all parts should be integrable. From the growth condition on S, we get integrability of the second term for $\varphi \in V_p$, so the corresponding operator $A : L^p(0, T; V_p(\Omega)) \to L^{p'}(0, T; V'_p(\Omega))$ given by $(Av, w) = \int_0^T \int_\Omega S(D(v)) : D(w)$ is well-defined (with additional arguments). The growth condition thus defines the function spaces in which to work, and $d = p$. A weak solution of (4.1) is a function $u \in C_w([0, T); L^2_\sigma) \cap L^p(0, T; V_p)$ such that (4.8) and the initial value is satisfied by u. In order for this formulation to work, the space $V_p(\Omega)$ must compactly embed into $L^2_\sigma(\Omega)$, so that a weak solution can be obtained by approximation and the Aubin-Lions lemma. This is the case if $p > \frac{2n}{n+2}$. Monotonicity and coercivity of the operator A follow from the corresponding assumptions on S.

In order for the third term $\int_0^T \int_\Omega u \otimes u : \nabla\varphi$ to be well-defined, also if a solution $u \in V_p$ is inserted for φ, the embedding $V_p(\Omega) \hookrightarrow L^{2p'}_\sigma(\Omega)$ must hold. This is the case if $p \ge \frac{3n}{n+2}$. In three dimensions, the critical exponents are thus $\frac{6}{5}$ and $\frac{9}{5}$ and in order to improve the restrictions they put on the power-law exponent $d = p$, especially for shear-thinning behaviour, special methods have to be applied.

5 Related questions and small appendix

- In Section 3.2, we have seen several examples of generalized Newtonian fluid models. In the fourth chapter, mathematical requirements on η are formulated, based on the methods used for solving the generalized Navier-Stokes problem. How do these requirements fit the examples, created from experimental data and considerations from mechanics?
  Example: Bingham fluids do not fit the requirement $\eta \in C^{1,1}(\mathbb{R}_+)$ in (4.2). As a minimum of regularity, in the modelling, it is assumed that the stress is at least piecewise continuous in space and time, so this should apply to all η in Section 3.2.
- How do the requirements on S or on η in Sections 4.1 and 4.2 compare? Is one of them strictly more general?
- Professor Necasova said that there is a mistake in Ladyzhenskaya's work. Is this somehow well-known?
- In Section 3.2, models for η are just stated, not compared. This is a possible project for the seminar, which would be very much on the modelling side.
- Again, calculations in simple flow on some of the models in Chapter 3 would be interesting.

Appendix

Maximal L^p-regularity and strong solutions Consider the abstract Cauchy problem

$$\begin{cases} u'(t) - Au(t) &= f(t), \quad t \in J_T, \\ u(0) &= 0, \end{cases} \tag{5.1}$$

where $(A, D(A))$ is a closed, densely defined linear operator in some Banach space X with domain $D(A)$ and $f \in L^p(J_T; X)$, $p \in (1, \infty)$ and $T \in \mathbb{R}_+$. The operator A is said to admit *maximal L^p-regularity* on J_T in X, if for every $f \in L^p(J_T; X)$ there exists a unique *strong solution* u of (5.1). A function u is a strong solution of (5.1), if and only if $u \in W^{1,p}(J_T; X) \cap L^p(J_T; D(A))$, u has vanishing trace at time zero and u satisfies (5.1) for almost all $t \in J_T$. We use the short form

$$X_p^T := W^{1,p}(J_T; X) \cap L^p(J_T; D(A))$$

for the space of strong solutions of the abstract Cauchy problem which is also called the space of maximal regularity of the operator A. Let now $Z_p := \{u(0) : u \in X_p^T\}$

be the time trace space of X_p^T. In [1, Section III.4.10] it is shown that if A admits maximal L^p-regularity on an interval J_T, $T > 0$, Z_p can be characterized by

$$Z_p = (X, D(A))_{1-1/p,p}$$

and that X_p^T admits the continuous embedding

$$X_p^T \hookrightarrow C([0,T]; Z_p).$$

For the initial value problem

$$\begin{cases} u'(t) - Au(t) &= f(t), \quad t \in J_T, \\ u(0) &= u_0, \end{cases} \tag{5.2}$$

it can therefore be shown that if A admits maximal L^p-regularity on J_{T_0}, $T_0 > 0$, then for every $u_0 \in Z_p$ and $T \in J_{T_0}$ there exists a unique solution $u \in X_p^T$ of (5.2) such that

$$\|u\|_{X_p^T} \leq C(\|f\|_{L^p(J_T;X)} + \|u_0\|_{Z_p}),$$

where the constant C is independent of T, f, u_0. In this sense, it is often said that the initial value problem (5.2) *has maximal regularity* or the term is used for systems of equations, if they yield a solution operator for *strong solutions* in the appropriate sense, cf. e.g. the problem (4.4).

Lopatinskiĭ-Shapiro Condition Here, this condition has the following form: Show that for every $\xi, \nu \in \mathbb{R}^n$, $|\nu| = 1$ and $(\xi, \nu) = 0$ and $\mathrm{Re}(\lambda) \geq 0$, the ODE

$$\begin{cases} \lambda w(y) + A_{\#}(\xi - \nu D_y)w(y) &= 0, \quad y > 0, \\ ...w(0) &= 0, \end{cases}$$

has only $w = 0$ as a solution in $C_0(\mathbb{R}_+; \mathbb{C}^n)$, cf. [17].

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