Discrete Orthogonal Polynomials and their Difference Equations

Prof. Dr. Wolfram Koepf

Department of Mathematics

University of Kassel

koepf@mathematik.uni-kassel.de

http://www.mathematik.uni-kassel.de/~koepf

Online Demonstrations with Computer Algebra

- I will use the computer algebra system *Maple* to demonstrate and program the algorithms presented.
- Of course, we could also easily use any other system like *Mathematica* or MuPAD.
- We first give a short introduction about the capabilities of *Maple*.

Scalar Products

- Given: a scalar product

$$\langle f, g \rangle := \int_a^b f(x) g(x) d\mu(x)$$

  with non-negative measure μ supported in an interval $[a,b)$.
- Particular cases:
  - absolutely continuous measure $d\mu(x) = \rho(x)dx$,
  - discrete measure $\rho(x)$ supported by $\mathbb{Z}$,
  - discrete measure $\rho(x)$ supported by $q^{\mathbb{Z}}$.

Orthogonal Polynomials

- A family $P_n(x)$ of polynomials

$$P_n(x) = k_n x^n + k_n' x^{n-1} + k_n'' x^{n-1} + \cdots, \quad k_n \neq 0$$

is orthogonal w. r. t. the measure $\mu(x)$ if

$$\langle P_n, P_m \rangle = \begin{cases} 0 & \text{if } m \neq n \\ d_n^2 \neq 0 & \text{if } m = n \end{cases}.$$

Classical Families

- The classical orthogonal polynomials can be defined as the polynomial solutions of the differential equation

$$\sigma(x)P_n''(x) + \tau(x)P_n'(x) + \lambda_n P_n(x) = 0.$$

- Conclusions:
  - $n = 1$ implies $\tau(x) = d\,x + e,\ d \neq 0$
  - $n = 2$ implies $\sigma(x) = a\,x^2 + b\,x + c$
  - coefficient of x^n implies $\lambda_n = -\,n(a(n\text{-}1)+d)$

Classification

- The classical systems can be classified according to the scheme (Bochner 1929)
- $\sigma(x) = 0$ powers x^n
- $\sigma(x) = 1$ Hermite polynomials
- $\sigma(x) = x$ Laguerre polynomials
- $\sigma(x) = x^2$ powers, Bessel polynomials
- $\sigma(x) = x^2 - 1$ Jacobi polynomials

Weight function

- The weight function $\rho(x)$ corresponding to the differential equation satisfies Pearson's differential equation

$$\frac{d}{dx}\left(\sigma(x)\rho(x)\right) = \tau(x)\rho(x)$$

- Hence it is given as

$$\rho(x) = \frac{C}{\sigma(x)} e^{\int \frac{\tau(x)}{\sigma(x)} dx} \ .$$

Classical Discrete Families

- The classical discrete orthogonal polynomials can be defined as the polynomial solutions of the difference equation

$$\sigma(x)\Delta\nabla P_n(x) + \tau(x)\Delta P_n(x) + \lambda_n P_n(x) = 0 .$$

- Conclusions:
  - $n = 1$ implies $\tau(x) = d\,x + e,\ d \neq 0$
  - $n = 2$ implies $\sigma(x) = a\,x^2 + b\,x + c$
  - coefficient of x^n implies $\lambda_n = -\,n(a(n\text{-}1)+d)$

Classification

- The classical discrete systems can be classified according to the scheme (Nikiforov, Suslov, Uvarov 1991)
- $\sigma(x) = 0$ falling factorials
- $\sigma(x) = 1$ translated Charlier pols.
- $\sigma(x) = x$ falling factorials, Charlier, Meixner, Krawtchouk pols.
- $\deg(\sigma(x), x) = 2$ Hahn polynomials

Weight function

- The weight function $\rho(x)$ corresponding to the difference equation satisfies Pearson's difference equation

$$\Delta\bigl(\sigma(x)\rho(x)\bigr)=\tau(x)\rho(x)$$

- Hence it is given as

$$\frac{\rho(x+1)}{\rho(x)}=\frac{\sigma(x)+\tau(x)}{\sigma(x+1)}.$$

Hypergeometric Functions

- The power series

$$ {}_pF_q\left(\begin{matrix} a_1,\cdots,a_p \\ b_1,\cdots,b_q \end{matrix}\middle| z\right) = \sum_{k=0}^{\infty} A_k z^k $$

whose coefficients A_k have rational term ratio

$$ \frac{A_{k+1}z^{k+1}}{A_k z^k} = \frac{(k+a_1)\cdots(k+a_p)}{(k+b_1)\cdots(k+b_q)} \cdot \frac{z}{k+1} $$

is called the generalized hypergeometric function. The summand $A_k z^k$ is called a hypergeometric term.

Coefficients of Hypergeometric Functions

- For the coefficients of the hypergeometric function we get the formula

$$ {}_pF_q\left(\left.\begin{matrix} a_1,\cdots,a_p \\ b_1,\cdots,b_q \end{matrix}\right| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k\cdots(a_p)_k}{(b_1)_k\cdots(b_q)_k}\frac{z^k}{k!}, $$

where $(a)_k = a(a+1)\cdots(a+k-1)$ is called the Pochhammer symbol (or shifted factorial).

Examples of Hypergeometric Functions

$$e^z = {}_0F_0(z)$$

$$\sin z = z \cdot {}_0F_1\left(\begin{matrix} - \\ 3/2 \end{matrix} \middle| -\frac{z^2}{4}\right)$$

Further examples: cos(z), arcsin(z), arctan(z), ln(1+z), erf(z), $L_n^{(\alpha)}(z)$, ..., but for example not tan(z), ...

Classical Discrete Orthogonal Polynomials of Hahn Class as Hypergeometric Functions

- From the difference equation, one can determine a hypergeometric represention.
- As an example, the Hahn polynomials are given by

$$Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\left.\begin{matrix} -n, -x, n+1+\alpha+\beta \\ \alpha+1, -N \end{matrix}\right| 1\right).$$

Notation

- To define *q*-orthogonal polynomials, we need some notation.
- The operator (Hahn 1949)

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}$$

  is called Hahn's *q*-difference operator.
- The *q*-brackets are defined by

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + \cdots + q^{k-1}.$$

Classical q-Families

- The q-orthogonal polynomials of the Hahn class can be defined as the polynomial solutions of the q-difference equation

$$\sigma(x) D_q D_{1/q} P_n(x) + \tau(x) D_q P_n(x) + \lambda_n P_n(x) = 0 .$$

- Conclusions:
  - $n = 1$ implies $\tau(x) = d\,x + e,\ d \neq 0$
  - $n = 2$ implies $\sigma(x) = a\,x^2 + b\,x + c$
  - coefficient of x^n implies $\lambda_n = -a[n]_{1/q}[n-1]_q - d[n]_q$

Classification

- The classical q-systems can be classified according to the scheme
- $\sigma(x) = 0$ powers and q-Pochhammers
- $\sigma(x) = 1$ discrete q-Hermite II pols.
- $\sigma(x) = x$ q-Charlier, q-Laguerre pols.
- $\sigma(x) = x\text{-}b\ q$ q-Meixner polynomials
- $\deg(\sigma(x), x) = 2$ q-Hahn polynomials,

  Big q-Jacobi polynomials

Weight function

- The weight function $\rho(x)$ corresponding to the q-difference equation satisfies the q-Pearson difference equation

$$D_q\big(\sigma(x)\rho(x)\big)=\tau(x)\rho(x)$$

- Hence it is given as

$$\frac{\rho(qx)}{\rho(x)}=\frac{\sigma(x)+(q-1)x\tau(x)}{\sigma(qx)} \quad .$$

Basic Hypergeometric Series

- Instead of considering series whose coefficients A_k have rational term ratio $A_{k+1}/A_k \in \mathbb{Q}(k)$, we can also consider such series whose coefficients A_k have term ratio $A_{k+1}/A_k \in \mathbb{Q}(q^k)$.
- This leads to the q-hypergeometric series

$$ {}_r\phi_s\left(\left.\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}\right| q; x\right) = \sum_{k=0}^{\infty} A_k x^k . $$

Coefficients of the Basic Hypergeometric Series

- Here the coefficients are given by

$$A_k = \frac{(a_1;q)_k \cdots (a_r;q)_k}{(b_1;q)_k \cdots (b_s;q)_k} \frac{x^k}{(q;q)_k} \left((-1)^k q^{\binom{k}{2}} \right)^{1+s-r},$$

where

$$(a;q)_k = \prod_{j=0}^{k-1} \left(1 - aq^j\right)$$

denotes the q-Pochhammer symbol.

q-Orthogonal Polynomials of Hahn Class are Hypergeometric

- All classical orthogonal systems have (several) q-hypergeometric equivalents.
- E.g., the Little and the Big q-Jacobi Polynomials, respectively, are given by

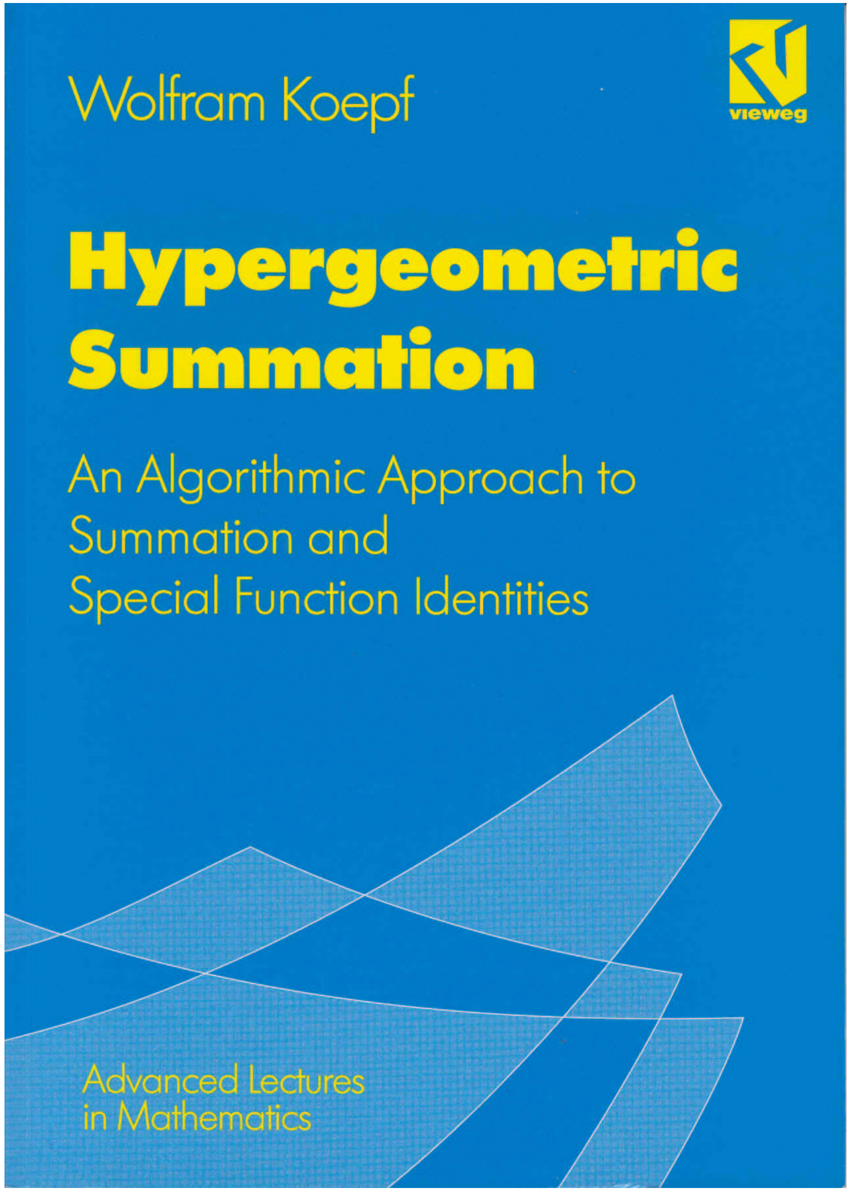
$$p_n(x;a,b\,|\,q) = {}_2\phi_1\left(\left.\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}\right| q; qx\right),$$

$$P_n(x;a,b,c;q) = {}_3\phi_2\left(\left.\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix}\right| q; q\right).$$

Computing Difference Equation from a Recurrence Equation

- From the differential or (q)-difference equation one can determine the three-term recurrence equation for $P_n(x)$ in terms of the coefficients of $\sigma(x)$ and $\tau(x)$.
- Using this information in the opposite direction, one can find the corresponding differential or (q)-difference equation from a given three-term recurrence equation.

The software used here was developed in connection with my book *Hypergeometric Summation, Vieweg, 1998, Braunschweig/ Wiesbaden* and can be downloaded from my home page



http://www.mathematik.uni-kassel.de/~koepf

Example 1

- Given the recurrence equation

$$P_{n+2}(x)-(x-n-1)P_{n+1}(x)+\alpha(n+1)^2 P_n(x)=0$$

one finds that for $\alpha = 1/4$ translated Laguerre polynomials, and for $\alpha < 1/4$, Meixner and Krawtchouk polynomials are solutions.

Example 2

- Given the recurrence equation

$$P_{n+2}(x) - xP_{n+1}(x) + \alpha\, q^n (q^{n+1} - 1) P_n(x) = 0$$

one finds that for every α there are *q*-orthogonal polynomial solutions.

Associated Orthogonal Polynomials

- A monic orthogonal system

  $$P_n(x) = x^n + k_n' x^{n-1} + k_n'' x^{n-1} + \cdots$$

  satisfies a recurrence equation of the form

  $$P_{n+1}(x) = (x - \beta_n) P_n(x) - \gamma_n P_{n-1}(x) .$$

- The polynomials defined by

  $$P_{n+1}^{(r)}(x) = (x - \beta_{n+r}) P_n^{(r)}(x) - \gamma_{n+r} P_{n-1}^{(r)}(x) ,$$

  called the *r*th associated orthogonal polynomials, are orthogonal by Favard’s Theorem.

Representation of the Associated Polynomials

- As examples, we consider the classical discrete polynomials.
- It turns out that the associated polynomials can be represented as linear combinations

$$P_n^{(r)}(x) = \frac{P_{r-1}(x)}{\Gamma_{r-1}} P_{n+r-1}^{(1)}(x) - \frac{P_{r-2}^{(1)}(x)}{\Gamma_{r-1}} P_{n+r}(x),$$

where Γ_n is defined by $\Gamma_n = \prod_{k=1}^{n} \gamma_k$.

The Function of the Second Kind

- Let

$$\sigma(x)\Delta\nabla P_n(x) + \tau(x)\Delta P_n(x) + \lambda_n P_n(x) = 0\,.$$

- This difference equation has a second linearly independent solution given by

$$Q_n(x) = \frac{1}{\rho(x)} \sum_{s=a}^{b-1} \frac{\rho(s)P_n(s)}{s-x}\,.$$

Fourth Order Difference Equation

- The associated polynomials $y(x) = P_n^{(r)}(x)$ satisfy a fourth order recurrence equation of the form

$$R_n^{(r)} y(x) = \sum_{k=0}^{4} J_k(x,n) N^k y(x)$$
$$= \sum_{k=0}^{4} J_k(x,n) y(x+k) = 0\,.$$

with polynomials $J_k(x,n) \in \mathbb{Q}[x,n]$.

Factorization of Difference Operator

- By linear algebra, one can prove that the difference operator a multiple of $R_n^{(r)}$ can be factorized as product of two difference operators of second order

$$X(\sigma,\tau,P_{r-1},\lambda_{r-1})R_n^{(r)} = S_n^{(r)}T_n^{(r)}$$

(joint work with M. Foupouagnigni and

A. Ronveaux, 2002).

Charlier Polynomials

- By computer algebra, in each specific case this factorization can be given explicitly.
- For example, we consider the Charlier polynomials and their associated.
- The monic Charlier polynomials are given by

$$P_n(x) = (-a)^n c_n^{(a)}(x) = (-a)^n {}_2F_0\left(\left. \begin{matrix} -n, -x \\ - \end{matrix} \right| -\frac{1}{a} \right).$$

Second Solution

- A second linearly independent solution of the corresponding difference equation

$$x\Delta\nabla P_n(x) + (a-x)\Delta P_n(x) + nP_n(x) = 0$$

is given by

$$\widetilde{Q}_n(x) = \frac{(-a)^n}{(x+1)(n+1)}\,{}_2F_2\left(\left.\begin{matrix} 1,1 \\ n+2,\, x+2 \end{matrix}\right| a\right).$$

Associated Charlier Polynomials

- The fourth order difference equation of the associated Charlier polynomials is given by

$$
\begin{aligned}
&\Big(a(n+2\zeta)(x+4)N^4 \\
&+(-2ax-4\zeta-2\zeta^3+2n^2-6a+6\zeta^2-3n\zeta^2-n^2\zeta+7n\zeta-2n)N^3 \\
&+(2ax-5an+2\zeta+4\zeta^3-n^2-4\zeta ax-10\zeta a+n^3+4a-6\zeta^2 \\
&\qquad\qquad +6n\zeta^2+4n^2\zeta-4n\zeta-2axn)N^2 \\
&+(2ax+2\zeta-2\zeta^3+4a-3n\zeta^2-n^2\zeta+n\zeta)N \\
&+a(n-2+2\zeta)(x+1)I\Big)P_n^{(r)}(x)=0 \quad \text{with} \quad \zeta=r-x-a-2\,.
\end{aligned}
$$

Factorization

- The factorization yields the second order right factor

$$T_n^{(r)} = \big(P_{r-1}(x+1)P_{r-1}(x)(x+2)^2 aN^2$$
$$+(-(x+1)(n+\zeta+1)(x+2)P_{r-1}(x)^2 - \zeta(n+\zeta+1)(x+2)P_{r-1}(x+1)P_{r-1}(x))N$$
$$+(-a(x+1)(x+2)P_{r-1}(x+1)P_{r-1}(x) - \zeta a(x+2)P_{r-1}(x+1)^2)I\big)$$

where $P_n(x)$ denotes the monic Charlier polynomial.

Solution Basis of Fourth Order Difference Equation

- Using the right factor $T_n^{(r)}$, one can find a solution basis for the fourth order difference equation of the associated polynomials:

$$A_n^{(r)}(x) = \rho(x) P_{r-1}(x) P_{n+r}(x)\,,$$
$$B_n^{(r)}(x) = \rho(x) P_{r-1}(x) Q_{n+r}(x)\,,$$
$$C_n^{(r)}(x) = \rho(x) Q_{r-1}(x) P_{n+r}(x)\,,$$
$$D_n^{(r)}(x) = \rho(x) Q_{r-1}(x) Q_{n+r}(x)\,.$$

Similar Situations

- In a similar manner, the fourth order difference equations and their factorizations of the generalized co-recursive and the generalized co-dilated polynomials can be detected.

Epilogue

- Software development is a time consuming activity!
- Software developers love when their software is used.
- But they need your support.
- Hence my suggestion: If you use a computer algebra package for your research, please cite its use!