# 2 Hypergeometric Identities

In this chapter we deal with hypergeometric identities. These are identities like

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n , \tag{2.1}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \qquad (n \neq 0) , \tag{2.2}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \frac{(2n)!}{n!^2} = \binom{2n}{n} , \tag{2.3}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{(-1)^{n/2}\, n!}{(n/2)!^2} & \text{otherwise} \end{cases} , \tag{2.4}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \dfrac{(-1)^{n/2}\, (3n/2)!}{(n/2)!^3} & \text{otherwise} \end{cases} , \tag{2.5}$$

or

$$\sum_{k=-n}^{n} (-1)^k \binom{n+b}{n+k} \binom{n+c}{c+k} \binom{b+c}{b+k} = \frac{\Gamma(b+c+n+1)}{n!\,\Gamma(b+1)\,\Gamma(c+1)} , \tag{2.6}$$

involving sums of a special type. We will meet the above identities—and many more—in one form or another at several places later on in this book. For the moment, we will not prove any of these identities. However, all of them will be proved by several methods later.

We would like to mention that these kinds of identities can often be interpreted by combinatorial means. Assume a set $S$ with $n$ elements. The left-hand side of (2.1) counts the number of subsets of $S$ with $k$ elements and sums these. The right-hand side counts the total number of subsets of $S$. As soon as we have this combinatorial interpretation and combinatorial proofs for both sides, we have proved (2.1) by combinatorial means. On the other hand, in many cases we have the opposite situation: By combinatorial considerations, a sum of the above type occurs, but we are lacking a (combinatorial) method to evaluate this sum directly. One may ask whether the sum under consideration can be rewritten in simpler form.

We will not deal with combinatorial interpretations of identities in this book. Instead, we will introduce several methods to find simpler form representations for sums of the above type.

What do the above sums have in common? They all are *definite sums* of the type

$$F = \sum_{k=-\infty}^{\infty} a_k \tag{2.7}$$

the sum to be taken over all integers $k$. This is so since (for any $n \in \mathbb{Z}$) all summands vanish outside a finite interval. We say that $a_k$ has *finite support* in this situation. In most of the above cases this is the interval $k = 0, \ldots, n$.

A sum of type (2.7) is called a *hypergeometric series* if the term ratio $a_{k+1}/a_k$ represents a rational function of $k$. In this case we call the summand $a_k$ a *hypergeometric term*.[1]

Note that the summands of the above identities (2.1)–(2.6) do not only represent hypergeometric terms with respect to the summation variable $k$, but form hypergeometric terms with respect to all variables $(n, a, b, c)$ involved.

Without giving a formal definition, we call an equation a *hypergeometric identity* if it represents a hypergeometric series (2.7) by hypergeometric terms like the right-hand sides of (2.1)–(2.6).[2] If the sum is written in terms of products of binomial coefficients, we will frequently also speak of a *binomial sum identity*. Binomial sum identities are hypergeometric ones (at least) if the arguments of the binomial coefficients occurring are *integer-linear* in the summation variable $k$, i.e., they are of the form $\alpha k + \beta$ with $\alpha \in \mathbb{Z}$, $\beta \in \mathbb{K}$, $\mathbb{K}$ denoting any field of characteristic zero, e.g. $\mathbb{K} = \mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$. For simplicity, in the current text, we generally assume $\mathbb{K} = \mathbb{Q}$ or a transcendental extension of $\mathbb{Q}$ with a finite number of variables adjoined. Hence throughout the book $\mathbb{Q}$ is an abbreviation of $\mathbb{Q}(x_1, x_2, \ldots, x_m)$,[3] and it is implicitly understood that the variables $x_1, x_2, \ldots, x_m$ are independent from other variables occurring in the given context.

We will further frequently deal with the case of *rational-linear* input with arguments of the form $\alpha k + \beta$ with $\alpha \in \mathbb{Q}$.

Assume now a hypergeometric series (2.7) is given. In this chapter we begin by considering how to find a representation of $F$ in terms of the *generalized hypergeometric function* ${}_pF_q$ given by

$${}_pF_q\left(\begin{array}{cccc} \alpha_1 & \alpha_2 & \cdots & \alpha_p \\ \beta_1 & \beta_2 & \cdots & \beta_q \end{array} \,\middle|\, x\right) := \sum_{k=0}^{\infty} A_k\, x^k = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdot (\alpha_2)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdot (\beta_2)_k \cdots (\beta_q)_k\, k!} x^k . \tag{2.8}$$

This is the right thing to do since we shall see soon that the term ratio of the coefficient $A_k x^k$ of ${}_pF_q$ is a general rational function in $k$ in factored form.

[1] A hypergeometric term is always the *summand*, never the sum!

[2] The right-hand sides $a_n$ form $m$-fold hypergeometric terms. These are generalizations of hypergeometric terms satisfying a recurrence equation of the type $a_{n+m} = R(n)\, a_n$ for some $m \in \mathbb{N}$ with rational $R(n)$.

[3] $\mathbb{Q}(x_1, x_2, \ldots, x_m)$ denotes the field of rational functions in the variables $x_1, x_2, \ldots, x_m$ over $\mathbb{Q}$, see p. 14.

The numbers $\alpha_k$ are called the *upper* and $\beta_k$ the *lower parameters* of ${}_pF_q$. Note that ${}_pF_q(x)$ is well-defined if no lower parameter is a negative integer or zero and it constitutes a convergent series if $p \le q$, or if $p = q+1$ and $|x| < 1$.

We will, however, deal almost exclusively with the case that ${}_pF_q(x)$ constitutes a polynomial so that convergence is not an issue. This situation occurs if one of the upper parameters is a negative integer. Throughout the present book, the letter $n$ will denote a nonnegative integer and $-n$, $-2n$, or $-n-1$, etc. might denote upper parameters. In such a case, ${}_pF_q(x)$ is a polynomial in $x$ of degree (at most) $n$, $2n$, or $n+1$, respectively.

The coefficients $A_k x^k$ of the generalized hypergeometric function have the rational term ratio

$$\frac{A_{k+1}x^{k+1}}{A_k x^k} = \frac{(k+\alpha_1)\cdot(k+\alpha_2)\cdots(k+\alpha_p)}{(k+\beta_1)\cdot(k+\beta_2)\cdots(k+\beta_q)\cdot(k+1)}\,x \quad (k \in \mathbb{N}) \tag{2.9}$$

by the definition of the shifted factorial, i.e., the recurrence equation

$$(k+\beta_1)\cdot(k+\beta_2)\cdots(k+\beta_q)\cdot(k+1)\,A_{k+1}-(k+\alpha_1)\cdot(k+\alpha_2)\cdots(k+\alpha_p)\,A_k = 0 \tag{2.10}$$

is valid for $A_k$.

Note that the extra factor $(k+1)$ in the denominator of (2.9) which does not occur in the list of lower parameters guarantees that ${}_pF_q(x)$, which is a *power series*, corresponds to a *bilateral sum* (2.7), i.e., for arbitrary $A_0$, the statement $A_{-1} = 0$ can be deduced from (2.10), so that all coefficients $A_k$ with negative $k$ vanish. This fact is also expressed by the $k!$-term in the denominator of the right-hand sum (2.8). The summand therefore vanishes for $k < 0$ according to (1.3). This argument applies whenever none of the upper parameters is a positive integer, whereas in the latter case the lower bound $k = 0$ of ${}_pF_q(x)$ is not the *natural* one, i.e., the summand is not identically zero outside the summation region, and ${}_pF_q(x)$ cannot be considered as a bilateral sum.

The generalized hypergeometric series generalizes the exponential and geometric series: For

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}\ ,$$

we have $A_k := 1/k!$, and therefore $A_{k+1}/A_k = 1/(k+1)$ so that $e^x = {}_0F_0(x)$. For

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k\ ,$$

we have $A_k := 1$ $(k \in \mathbb{N}_0)$, hence $A_{k+1}/A_k = 1$ $(k \in \mathbb{N}_0)$. Note that this term ratio is *not* valid for $k = -1$; however, after multiplication by $(k+1)$, the recurrence equation $(k+1)A_{k+1} - (k+1)A_k = 0$ is valid for all $k \in \mathbb{Z}$, and we have for $|x| < 1$

$$\frac{1}{1-x} = {}_1F_0\left(\left.\begin{array}{c} 1 \\ - \end{array}\right| x\right) .$$

In particular, this is not a sum of type (2.7).

Note that the function ${}_2F_1(x)$ (whose radius of convergence is 1) was introduced by Gauss and is therefore called *Gauss' hypergeometric function*. On the other hand, the series ${}_1F_1(x)$ converges for all $x \in \mathbb{C}$ and is called *Kummer's confluent hypergeometric function*.

If a hypergeometric series (2.7) is given and if $a_0 \neq 0$, then it is easy to represent $F$ in terms of a generalized hypergeometric function if we are able to find polynomials $u_k$ and $v_k$ such that

$$\frac{a_{k+1}}{a_k} = \frac{u_k}{v_k} \tag{2.11}$$

and if we assume a complete factorization of $u_k$ and $v_k$.

We introduce some notation. $\mathbb{K}(k)$ denotes the field of rational functions in the variable $k$ over $\mathbb{K}$, and $\mathbb{K}[k]$ denotes the ring of polynomials in $k$ over $\mathbb{K}$. Similarly, $\mathbb{K}(n,k)$ and $\mathbb{K}[n,k]$ are the field of rational functions and the ring of polynomials in two variables respectively.

**Session 2.1** As we saw above, given $a_k$, it is crucial to find polynomials $u_k, v_k \in \mathbb{Q}[k]$ such that (2.11) is valid. How can we find these with Maple? We saw in Session 1.1 that Maple's `expand` command expands binomial coefficients and factorial and $\Gamma$ function terms. Therefore, we have for example

```
> expand(binomial(n+2,k-1)/binomial(n-1,k+2));
```

$$(n+2)(n+1)\,k\,n\,(k+2)(k+1) \Big/ ((n+3-k)(n+2-k)(n-k+1)$$
$$(n-k-2)(n-k-1)(n-k))$$

However, we must say that this is not a safe procedure. This is shown in the following example

```
> summand:=binomial(n,k)/2^n:
> term:=subs(n=n+1,summand)-summand:
> expr:=expand(subs(k=k+1,term)/term);
```

$$-\frac{1}{2}\,\frac{\text{binomial}(n,k)\,n}{\%1\,(k+1)\,2^n} + \frac{1}{2}\,\frac{\text{binomial}(n,k)}{\%1\,(k+1)\,2^n} + \frac{\text{binomial}(n,k)\,k}{\%1\,(k+1)\,2^n}$$
$$\%1 := \frac{1}{2}\,\frac{\text{binomial}(n,k)\,n}{(n-k+1)\,2^n} + \frac{1}{2}\,\frac{\text{binomial}(n,k)}{(n-k+1)\,2^n} - \frac{\text{binomial}(n,k)}{2^n}$$

```
> normal(expr);
```

$$\frac{(n-k+1)\,(n-1-2\,k)}{(n+1-2\,k)\,(k+1)}$$

You see that in this example, the situation can be resolved by a further application of `normal` or `simplify`. On the other hand, this procedure is not at all efficient in cases like

```
> expand(GAMMA(k+100000)/GAMMA(k+99999));
```

$$k + 99999$$

Issuing this command gives you time to have lunch before you receive the *trivial* result. In the worst case you receive the error message

```
Error, (in expand/GAMMA) object too large
```

depending on the memory situation on your computer! We will now present a better method for the given purpose.

The following algorithm, which is almost trivial but decisive, tells how $u_k$ and $v_k$ can be identified (at least) for input of special type. We will later see that the same algorithm applies for input of more general type.

**Algorithm 2.1** (`simpcomb`)
The following algorithm decides the rationality of terms $a_{k+1}/a_k$:

1. Input: $a_{k+1}/a_k$, where $a_k$ is a ratio of products of rational functions, powers, factorials, $\Gamma$ function terms, binomial coefficients and Pochhammer symbols that are rational-linear in their arguments.

2. (`togamma`)
   Build $a_{k+1}/a_k$, and convert all occurrences of factorials, binomial coefficients, and Pochhammer symbols to $\Gamma$ function terms according to (1.2), (1.5), and (1.10), avoiding negative arguments. The case of binomial coefficients is done by the rules
$$\binom{a}{k} \rightarrow \begin{cases} (-1)^k \frac{\Gamma(k-a)}{\Gamma(k+1)\,\Gamma(-a)} & \text{if } a \in \mathbb{Z},\ a<0 \\ 0 & \text{if } a-k \in \mathbb{Z},\ a-k<0 \\ \frac{\Gamma(a+1)}{\Gamma(k+1)\,\Gamma(a-k+1)} & \text{otherwise} \end{cases} .$$

3. (`simplify_gamma`)
   Rewrite this expression recursively according to the rule (see (1.4))
$$\Gamma(a+j) = a(a+1)\cdots(a+j-1)\cdot\Gamma(a)$$
   whenever the arguments $a$ and $a+j$ of two representing $\Gamma$ function terms have positive integer difference $j$. Reduce the final fraction canceling common $\Gamma$ terms.

4. (`simplify_power`)
   Rewrite the last expression recursively according to the rule
$$b^{a+j} = b^j\, b^a$$
   whenever the arguments $a$ and $a+j$ of two representing power terms have positive integer difference $j$. Reduce the final fraction canceling common power terms.

5. The expression $a_{k+1}/a_k$ is rational if and only if the resulting expression $u_k/v_k$ in step 4 is rational, i.e., $u_k, v_k \in \mathbb{Q}[k]$.

6. Output: $(u_k, v_k)$.

*Proof:* Note that this result follows immediately from the given form of $a_k$ (as a ratio) and therefore of the expression $a_{k+1}/a_k$. The given form guarantees that common $\Gamma$ and power terms in numerator and denominator cancel in steps (3.) and (4.) if $a_{k+1}/a_k$ is rational. $\square$

Note that, for integer-linear input, it is clear that, by the use of the given rewrite rules, all $\Gamma$ and power terms cancel and polynomials $u_k, v_k \in \mathbb{Q}[k]$ are constructed.

**Example 2.1** As an example, the rationality of $a_{k+1}/a_k$ for

$$a_k = \frac{\Gamma(2k)}{4^k\,\Gamma(k)\,\Gamma(k+1/2)}$$

is recognized by the given procedure by the stepwise transformations

$$\begin{aligned}\frac{a_{k+1}}{a_k} &= \frac{\Gamma(2k+2)}{4^{k+1}\,\Gamma(k+1)\,\Gamma(k+3/2)} \Big/ \frac{\Gamma(2k)}{4^k\,\Gamma(k)\,\Gamma(k+1/2)} \\ &= \frac{(2k)(2k+1)\Gamma(2k)4^k\,\Gamma(k)\,\Gamma(k+1/2)}{\Gamma(2k)\,4\cdot 4^k\,k\Gamma(k)(k+1/2)\Gamma(k+1/2)} = \frac{(2k)(2k+1)}{4k(k+1/2)} = 1\ .\end{aligned}$$

From the resulting information, it follows by induction that for $k \in \mathbb{N}$

$$\frac{\Gamma(2k)}{4^k\,\Gamma(k)\,\Gamma(k+1/2)} = a_k = a_1 = \frac{\Gamma(2)}{4\,\Gamma(1)\Gamma(3/2)} = \frac{1}{2\,\Gamma(1/2)} = \frac{1}{2\sqrt{\pi}}\ , \tag{2.12}$$

see (1.14). Note that (2.12) which is called the *duplication formula* of the $\Gamma$ function, is valid for all $k \in \mathbb{C}$, $k$ not the half of a negative integer or zero, a fact which, however, cannot be proved by the present method.

Algorithm 2.1 also applies to

$$a_k = \Gamma(2k) - \alpha\,4^k\,\Gamma(k)\,\Gamma(k+1/2)$$

and the same procedure leads to

$$\frac{a_{k+1}}{a_k} = 2\,k\,\left(2\,k+1\right)$$

(check!) which is true whenever $\alpha \neq \frac{1}{2\sqrt{\pi}}$. If $\alpha = \frac{1}{2\sqrt{\pi}}$, however, by the above computation, $a_k \equiv 0$ and therefore $a_{k+1}/a_k$ is not properly defined.

Note that the occurrence of another variable, $\alpha$, had the side effect that our calculation was not valid for a particular value of $\alpha$. This is a typical situation since we work with rational arithmetic and must make sure that no denominator which might appear in any intermediate calculation is ever equal to zero.

**Example 2.2** Next, we consider the expression ($n \in \mathbb{N}_0$)

$$a_k := \frac{1}{2^{n+1}}\binom{n+1}{k} - \frac{1}{2^n}\binom{n}{k} . \tag{2.13}$$

Note that $a_k$ does not have the form required in the above algorithm since it is not just a ratio but a sum of ratios. On the other hand, it is easily seen that for any expression of the form $a_k = b(n+j,k) \pm b(n,k)$ ($j \in \mathbb{Z}$) for some $b(n,k)$ which satisfies the hypothesis of Algorithm 2.1 *with respect to both $n$ and $k$*, the same algorithm applies. This is the case for $a_k$.

We obtain by the method described

$$\begin{aligned}
\frac{a_{k+1}}{a_k} &= \frac{\frac{1}{2^{n+1}}\binom{n+1}{k+1} - \frac{1}{2^n}\binom{n}{k+1}}{\frac{1}{2^{n+1}}\binom{n+1}{k} - \frac{1}{2^n}\binom{n}{k}} \\
&= \frac{\dfrac{\Gamma(n+2)}{\Gamma(k+2)\Gamma(n-k+1)} - 2\dfrac{\Gamma(n+1)}{\Gamma(k+2)\Gamma(n-k)}}{\dfrac{\Gamma(n+2)}{\Gamma(k+1)\Gamma(n-k+2)} - 2\dfrac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}} \\
&= \frac{\dfrac{(n+1)\Gamma(n+1)}{(k+1)\Gamma(k+1)(n-k)\Gamma(n-k)} - 2\dfrac{\Gamma(n+1)}{(k+1)\Gamma(k+1)\Gamma(n-k)}}{\dfrac{(n+1)\Gamma(n+1)}{\Gamma(k+1)(n-k)(n-k+1)\Gamma(n-k)} - 2\dfrac{\Gamma(n+1)}{\Gamma(k+1)(n-k)\Gamma(n-k)}} \\
&= \frac{\dfrac{n+1}{(k+1)(n-k)} - \dfrac{2}{k+1}}{\dfrac{n+1}{(n-k)(n-k+1)} - \dfrac{2}{n-k}} = \frac{n-k+1}{k+1} \cdot \frac{n+1-2(n-k)}{n+1-2(n-k+1)} \\
&= -\frac{(k-n-1)\,(k-n/2+1/2)}{(k-n/2-1/2)\,(k+1)} .
\end{aligned} \tag{2.14}$$

If we are now interested in

$$F = \sum_{k=-\infty}^{\infty} a_k , \tag{2.15}$$

then, according to (2.9), from the final factored form of (2.14) we can read off the list of upper parameters $(-n-1, -n/2+1/2)$, the lower parameter $(-n/2-1/2)$, and $x = -1$; and by

$$a_0 = \frac{1}{2^{n+1}}\binom{n+1}{0} - \frac{1}{2^n}\binom{n}{0} = -\frac{1}{2^{n+1}} ,$$

we see (by induction, or by the hypergeometric coefficient formula (2.8)) that

$$a_k = \frac{(-n-1)_k\,(-n/2+1/2)_k}{(-n/2-1/2)_k\,k!}\,(-1)^k\,a_0 = -\frac{(-n-1)_k\,(-n/2+1/2)_k}{(-n/2-1/2)_k\,k!}\,(-1)^k\,\frac{1}{2^{n+1}}$$

and therefore, formally,

$$\sum_{k=-\infty}^{\infty}\left(\frac{1}{2^{n+1}}\binom{n+1}{k}-\frac{1}{2^n}\binom{n}{k}\right) = -\frac{1}{2^{n+1}}\,{}_2F_1\left(\left.\begin{array}{c}-n-1,-n/2+1/2\\ -n/2-1/2\end{array}\right|-1\right). \quad (2.16)$$

Note that (2.16) shows in particular that the sum $F$ given by (2.15) for $n \in \mathbb{N}_0$ is a finite sum with summands $k = 0, \ldots, n+1$.

However, for odd $n$, the hypergeometric sum in (2.16) is not well-defined, since in this case the lower parameter is a negative integer. This is equivalent to the fact that we divided by zero in (2.14) so that this deduction is not valid. Therefore, we realize that (2.16) is only valid for even $n \in \mathbb{N}_0$.

According to identity (2.1), we see that $F \equiv 0$ (check!), so that by (2.16) for $n \in \mathbb{N}$

$${}_2F_1\left(\left.\begin{array}{c}-2n-1\,,-n+1\\ -n-1/2\end{array}\right|-1\right) \equiv 0\,.$$

**Example 2.3 (Dixon's Identity)** Identity (2.6) ($n \in \mathbb{N}_0$) is called *Dixon's identity*. We will now give a hypergeometric version. Therefore, for

$$a_k := (-1)^k\binom{n+b}{n+k}\binom{n+c}{c+k}\binom{b+c}{b+k}\,,$$

we calculate by Algorithm 2.1

$$\frac{a_{k+1}}{a_k} = \frac{(k-n)\,(k-b)\,(k-c)}{(k+n+1)\,(k+b+1)\,(k+c+1)} \quad (2.17)$$

(check!), and from

$$a_0 = \binom{n+b}{n}\binom{n+c}{c}\binom{b+c}{b}\,,$$

we are led to the hypergeometric representation

$$\binom{n+b}{n}\binom{n+c}{c}\binom{b+c}{b}\,{}_4F_3\left(\left.\begin{array}{c}-n\,,-b\,,-c\,,1\\ n+1\,,b+1\,,c+1\end{array}\right|1\right)\,,$$

where we had to add the number 1 to the list of upper parameters since the denominator of (2.17) did not contain a factor $(k+1)$.

But, be careful! Did you realize that this hypergeometric function corresponds to the sum of Dixon's term for $k = 0, \ldots, \infty$ rather than for $k = -\infty, \ldots, \infty$? In a later example, we will see that in some instances this might be exactly what we want.

In our case, however, to get rid of this problem, and to deduce a ${}_3F_2$ rather than a ${}_4F_3$ representation, we realize that one of the lower parameters, $n+1$, is an integer. In such a situation we apply a suitable shift. Since the summation is over all $k \in \mathbb{Z}$, a shift of the summation index by an integer does not change the value of the sum. This is the nice thing when working with bilateral sums: Their value is invariant with respect to shifts. Therefore, in our example, we shift the summation index by $-n$, i.e., we consider $b_k = a_{k-n}$ with

$$\sum_{k=-n}^{n} a_k = \sum_{k=-\infty}^{\infty} a_k = \sum_{k=-\infty}^{\infty} b_k = \sum_{k=0}^{2n} b_k$$

and we get from (2.17)

$$\frac{b_{k+1}}{b_k} = \left.\frac{a_{k+1}}{a_k}\right|_{k:=k-n} = \frac{(k-2n)\,(k-n-b)\,(k-n-c)}{(k+1)\,(k-n+b+1)\,(k-n+c+1)}\ .$$

By this procedure, we generated a $(k+1)$-term in the denominator and since

$$b_0 = a_{-n} = (-1)^n \binom{n+c}{c-n} \binom{b+c}{b-n}\ ,$$

we have, finally, the hypergeometric representation

$$F = (-1)^n \binom{n+c}{c-n} \binom{b+c}{b-n}\, {}_3F_2\left(\left.\begin{array}{c} -2n\ ,\, -n-b\ ,\, -n-c \\ -n+b+1\ ,\, -n+c+1 \end{array}\right| 1\right)$$

for the Dixon sum.

Note that we see from this hypergeometric representation and from its discovery that the left-hand side is a sum in the range $k = -n, \ldots, n$. At first glance this might not have been obvious. $\triangle$

Next, we would like to give some more examples that show how one takes care of possible shifts.

**Example 2.4** Let us consider $a_k = k\binom{n}{k}$. Then

$$\frac{a_{k+1}}{a_k} = \frac{n-k}{k}\ .$$

We see that this cannot be the term ratio of a hypergeometric representation since the denominator has a zero root. This corresponds to the fact that $a_0 = 0$, and any hypergeometric representation has $a_0 \neq 0$. By a suitable shift, however, we can overcome this difficulty and, as an important observation, the term ratio given shows us which shift will be successful! Since the denominator root is zero, we shift by one to eliminate it and to construct a $(k+1)$-term. For $b_k := a_{k+1}$, we get

$$\frac{b_{k+1}}{b_k} = \frac{a_{k+2}}{a_{k+1}} = -\frac{k+1-n}{k+1}$$

so that from $b_0 = a_1 = n$ it follows

$$\sum_{k=0}^{n} k\binom{n}{k} = n \cdot {}_1F_0\left(\begin{array}{c} -n+1 \\ - \end{array} \middle| -1\right) .$$

Next, we consider the similar expression $a_k = \frac{1}{k}\binom{n}{k}$. Here we are interested in $\sum\limits_{k=1}^{\infty} a_k$ rather than the bilateral sum. Let's see what can be done nevertheless. We have

$$\frac{a_{k+1}}{a_k} = -\frac{k(k-n)}{(k+1)^2} . \tag{2.18}$$

Since the numerator has a zero root, we must shift by one again. In the given case, there is no chance to keep a $(k+1)$-term in the denominator and therefore we have to increase the number of upper parameters by adding one to them.

The final result is

$$\sum_{k=1}^{n} \frac{1}{k}\binom{n}{k} = n \cdot {}_3F_2\left(\begin{array}{c} -n+1\,,1\,,1 \\ 2\,,2 \end{array} \middle| -1\right) .$$

In this example, the extra factor $(k+1)$ that we put in both numerator and denominator of (2.18) helped us a lot since this step made the sum finite to the left—and that was exactly what we needed.

Both examples given show that for quite similar input, the orders $p$ and $q$ of the corresponding hypergeometric representations can be quite different. $\triangle$

Now we are prepared to state and prove the main result of this chapter. We state it only for bilateral sums and mention that a similar algorithm can be given for sums $k = k_0, \ldots, \infty$.

**Algorithm 2.2 (Conversion of Sums into Hypergeometric Notation)** The following algorithm converts hypergeometric sums into hypergeometric notation:

1. Input: the summand $a_k$, given as ratio of products of rational functions, powers, factorials, $\Gamma$ function terms, binomial coefficients, and Pochhammer symbols that are rational-linear in their arguments, or a sum or difference of such terms like expression (2.13) in Example 2.2.

2. Calculate $a_{k+1}/a_k$ and apply Algorithm 2.1 to it generating $u_k, v_k \in \mathbb{Q}[k]$ such that

$$\frac{a_{k+1}}{a_k} = \frac{u_k}{v_k} .$$

   If Algorithm 2.1 decides that $a_{k+1}/a_k$ is not rational then return: "No hypergeometric representation exists."

3. Factorize $u_k, v_k$ over the rationals.[4] If there are nonlinear factors, then return: "No rational factorization found"; exit. (For factors of degree $\leq 4$, one may use symbolic complex solutions, though.) If the last step was successful, however, then you have a representation

$$u_k = A\,(k+\alpha_1)\,(k+\alpha_2)\cdots(k+\alpha_p) \quad\text{and}\quad v_k = B\,(k+\beta_1)\,(k+\beta_2)\cdots(k+\beta_{q+1})\;.$$

4. If any of the parameters $\beta_1, \ldots, \beta_{q+1}$ is an integer, then calculate the minimal such value[5] $m$ and shift the summation variable by $-m+1$, i.e. shift all upper and lower parameters by $-m+1$. Denote the new upper and lower parameters by $(\alpha_1, \ldots, \alpha_p)$ and $(\beta_1, \ldots, \beta_{q+1})$ again.

5. If none of the shifted lower parameters equals one, then return: "The bilateral sum does not have a hypergeometric representation".

6. Calculate the initial value $b_0 = a_K$, where $K := -m+1$ is the total shift that occurred in step (4.) if applicable, else $K := 0$; set `upper` $:= \{\alpha_1, \ldots, \alpha_p\}$, and `lower` $:= \{\beta_1, \ldots, \beta_q\}$ assuming $\beta_{q+1} = 1$; set $x := A/B$.

7. Output: the hypergeometric function $b_0$·`hypergeom(upper,lower,x)`.

*Proof:* Obviously, in step (2.), Algorithm 2.1 decides whether or not $a_{k+1}/a_k$ is rational. If not, then no hypergeometric representation exists by the definition of a hypergeometric series, whereas in the affirmative case $u_k, v_k \in \mathbb{Q}[k]$ are constructed. Note that this step undoubtedly succeeds if the $\Gamma$-arguments occurring are integer-linear w.r.t. $k$, although this is not a necessary condition.

If the factorization in step (3.) fails, no hypergeometric representation with rational parameters exists. If, on the other hand, a factorization is found, then it obviously defines a hypergeometric representation if none of the corresponding lower parameters is a negative integer. The shift in step (4.)—if applicable—guarantees that all negative integer lower parameters disappear (and at the same time that one of the lower parameters equals 1). This shift corresponds to a shift of the summation variable $k$ and does not change the value of the series.

Finally, if the remaining list of lower parameters does not contain the value 1, then the bilateral sum cannot be represented by a one-sided infinite hypergeometric representation.

If the shift is $K$, then we work with $b_k = a_{k+K}$, and $\sum b_k = \sum a_k$, so that the initial value is given by $b_0 = a_K$. □

A Maple implementation of the algorithm is given in Session 3.1. If the algorithm fails because $a_{k+1}/a_k$ turns out not to be rational, then it may still be possible to find a number $l \in \mathbb{N}$ such that $a_{k+l}/a_k$ is rational; compare Algorithm 8.3. In this case the series under consideration can be written as a sum of $l$ generalized hypergeometric functions.

If other variables are involved then the shifts of steps (4.) and (5.) might depend on the particular values of these variables. As in Example 2.3, the knowledge that any of the variables occurring is an integer might influence this decision. We give some final examples for an application of Algorithm 2.2.

[4]Rational factorization will be considered in more detail on page 64.

[5]If parameters are involved, this might be undecidable, compare e.g. the Dixon case!

**Example 2.5 (Legendre Polynomials)** Let us define the *Legendre polynomials* by the series

$$P_n(x) := \sum_{k=-\infty}^{\infty} \binom{n}{k} \binom{-n-1}{k} \left(\frac{1-x}{2}\right)^k ,$$

which, of course, is a hypergeometric one. For

$$a_k := \binom{n}{k} \binom{-n-1}{k} \left(\frac{1-x}{2}\right)^k ,$$

we get by Algorithm 2.1

$$\frac{a_{k+1}}{a_k} = \frac{(k-n)\,(k+n+1)}{(k+1)^2} \cdot \frac{1-x}{2} ,$$

and therefore we have

$$P_n(x) = {}_2F_1\left(\left.\begin{matrix} -n\,, n+1 \\ 1 \end{matrix}\,\right|\, \frac{1-x}{2}\right) . \tag{2.19}$$

This shows in particular that $P_n(x)$ is a polynomial of degree $n$ with respect to $x$.

Next we consider the family $F = P_{n+1}(x) - P_n(x)$ of consecutive differences of Legendre polynomials. Note that $F$ defines a polynomial of degree $n+1$. Does $F$ constitute a hypergeometric series? By (2.19), it is the *difference* of two hypergeometric functions, but our question is different. Algorithm 2.2 helps us to find the answer.

An application of Algorithm 2.1 gives

$$\frac{a_{k+1}}{a_k} = \frac{(k+n+1)\,(k-n-1)}{k\,(k+1)} \cdot \frac{1-x}{2} .$$

We see that a shift by one is necessary to obtain a hypergeometric representation. For $b_k := a_{k+1}$, we have

$$\frac{b_{k+1}}{b_k} = \frac{(k+n+2)\,(k-n)}{(k+1)\,(k+2)} \cdot \frac{1-x}{2} ,$$

so that with

$$b_0 = a_1 = -(n+1)\,(1-x) ,$$

there follows

$$P_{n+1}(x) - P_n(x) = -(n+1)\,(1-x) \cdot {}_2F_1\left(\left.\begin{matrix} -n\,, n+2 \\ 2 \end{matrix}\,\right|\, \frac{1-x}{2}\right) .$$

**Example 2.6 (Non-Natural Bounds)** In this example, we would like to present a further method which is of value if the upper bound of a hypergeometric sum is not a natural one, i.e., the summand is not identically zero outside the summation region (so that we don't have a bilateral sum). In this case a direct application of Algorithm 2.2 is not possible. We saw how the use of an extra 1 as upper parameter can be used if the left bound is not a natural one for the sum under consideration. But what if the right bound is not natural? Here a change of variable, essentially of the type $k \to -k$, helps (hence reversing the order of summation)!

Let us consider the example

$$F = \sum_{k=0}^{m} a_k := \sum_{k=0}^{m} (-1)^k \binom{n}{k} ,$$

for arbitrary $m$. Here the lower bound is a natural one, but the upper bound is not. We change the summation variable, set $b_k = a_{m-k}$ and get

$$F = \sum_{k=0}^{m} b_k = \sum_{k=0}^{m} (-1)^{m-k} \binom{n}{m-k} ,$$

for which we find

$$\frac{b_{k+1}}{b_k} = \frac{k-m}{k+n-m+1} ,$$

so that by $b_0 = a_m = (-1)^m \binom{n}{m}$, we have

$$F = (-1)^m \binom{n}{m} \, {}_2F_1\left( \left. \begin{array}{c} -m \,, 1 \\ n-m+1 \end{array} \right| 1 \right) .$$

Note that the extra upper parameter 1 made the lower bound a natural one, and the upper bound $m$ was natural from the beginning! We will investigate this example further in later chapters.

**Session 2.2** Maple can discover some hypergeometric identities:

```
> sum(binomial(n,k),k=0..n);
```

$$2^n$$

```
> sum(binomial(n,k)^2,k=0..n);
```

$$\frac{\Gamma(1+2\,n)}{\Gamma(n+1)^2}$$

```
> sum((-1)^k*binomial(n,k)^2,k=0..n);
```

$$\frac{\sqrt{\pi}}{2^{(-n)}\,\Gamma\left(1+\frac{1}{2}\,n\right)\,\Gamma\left(\frac{1}{2}-\frac{1}{2}\,n\right)}$$

This output is equivalent to ours, see Exercise 2.7.

In other cases, Maple may fail:[6]

```
> sum((-1)^k*binomial(n+b,n+k)*binomial(n+c,c+k)*binomial(b+c,b+k),
>     k=-n..n);
```

$$4(-1)^{(-n)}\,\mathrm{binomial}(n+c,\,c-n)\,\mathrm{binomial}(b+c,\,b-n)\,\sqrt{\pi}\,\Gamma(c+1-n)$$
$$\Gamma(b+1-n)\,\Gamma(1+n+c+b)\Big/(2^{(2-2\,n)}\,\Gamma(\frac{1}{2}-n)\,\Gamma(1+c)\,\Gamma(1+b)$$
$$\Gamma(b+c+1))-(-1)^{(n+1)}\,\mathrm{binomial}(n+b,\,2\,n+1)$$
$$\mathrm{binomial}(n+c,\,n+c+1)\,\mathrm{binomial}(b+c,\,n+b+1)$$
$$\mathrm{hypergeom}([1,\,1,\,1+n-c,\,-b+1+n],\,[n+b+2,\,c+2+n,\,2+2\,n],\,1)$$

Loading the package `sumtools`[7] the procedures of Algorithm 2.1 are available. The Maple procedure

```
> ratio:=proc(a,k);
>   simpcomb(subs(k=k+1,a)/a);
> end:
```

therefore, calculates $a_{k+1}/a_k$ and simplifies this expression according to Algorithm 2.1; the Maple procedure `hyperterm(upper,lower,x,k)` generates the hypergeometric term corresponding to the upper parameters `upper`, the lower parameters `lower`, the variable $x$, and the summation variable $k$.

We have for example

```
> ratio((-1)^k*binomial(n+b,n+k)*binomial(n+c,c+k)*binomial(b+c,b+k),k);
```

$$\frac{(b-k)\,(n-k)\,(-c+k)}{(n+k+1)\,(c+k+1)\,(k+1+b)}$$

```
> ratio(hyperterm([-n,n+1],[1],(1-x)/2,k),k);
```

$$\frac{1}{2}\,\frac{(n+k+1)\,(-1+x)\,(n-k)}{(k+1)^2}$$

```
> ratio(subs(n=n+1,hyperterm([-n,n+1],[1],(1-x)/2,k))-
>       hyperterm([-n,n+1],[1],(1-x)/2,k),k);
```

$$\frac{1}{2}\,\frac{(-1+x)\,(n-k+1)\,(n+k+1)}{k\,(k+1)}$$

Since binomial sums come in quite different disguises, it is an important fact that by the notion of the generalized hypergeometric function these sums are classified and hence can be identified. This fact will be stressed in the next chapter.

This point of view has been popularized by Dick Askey and George Andrews.

[6]In any of the Maple Versions V.2, V.3, V.4 you get a different result. The above is the result of Maple V.4 and V.5, whereas Maple V.2 received a hypergeometric term solution.

[7]Part of Maple Version V.4. By a bug of the package, it is wise to load it using `with(sumtools): readlib('sum/simpcomb'):`. Otherwise the `simpcomb` algorithm might not be loaded for all examples. This bug is fixed in Maple Version V.5. If you have an older Maple version, you can use the `hsum` package (see p. 214) which does also include these procedures.

## $q$-Hypergeometric Identities

An important extension of the hypergeometric function is the *$q$-hypergeometric function* (as a general reference for $q$-hypergeometric functions, see [GR90], and for an elementary introduction [Gasper97])

$$ {}_r\phi_s\left(\begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{array}\middle|\, q, x\right) := \sum_{k=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r;\, q)_k}{(b_1, b_2, \ldots, b_s;\, q)_k} \frac{x^k}{(q;\, q)_k} \left((-1)^k\, q^{\binom{k}{2}}\right)^{1+s-r} $$

where $(a_1, a_2, \ldots, a_r;\, q)_k$ is a short form for the product $\prod_{j=1}^{r} (a_j;\, q)_k$, and

$$ (a;\, q)_k := \begin{cases} \prod_{j=0}^{k-1} \left(1 - a\, q^j\right) & \text{if } k > 0 \\ 1 & \text{if } k = 0 \\ \prod_{j=1}^{k} \left(1 - a\, q^{-j}\right)^{-1} & \text{if } k < 0 \\ \prod_{j=0}^{\infty} \left(1 - a\, q^j\right) & \text{if } k = \infty \end{cases} $$

denotes the *$q$-Pochhammer symbol*. The $q$-hypergeometric functions are also called *basic hypergeometric functions* since they come with the base $q$.

An ${}_r\phi_s$ series terminates if one of its numerator parameters is of the form $q^{-n}$ with $n \in \mathbb{N}$. In the non-terminating case the $q$-hypergeometric series converges in its disk of convergence if $|q| < 1$. The additional factor $\left((-1)^k\, q^{\binom{k}{2}}\right)^{1+s-r}$ (which does not occur in the corresponding definition of the generalized hypergeometric function[8]) is due to a *confluence process*. With this factor one gets the simple formula

$$ \lim_{a_r \to \infty} {}_r\phi_s\left(\begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{array}\middle|\, q, \frac{x}{a_r}\right) = {}_{r-1}\phi_s\left(\begin{array}{c} a_1, a_2, \ldots, a_{r-1} \\ b_1, b_2, \ldots, b_s \end{array}\middle|\, q, x\right) . $$

An expression $a_k$ is called a *$q$-hypergeometric term*, if $a_{k+1}/a_k$ is a rational function with respect to $q^k$, a typical example of which is given by the summand of the $q$-hypergeometric series. Using the notion of the $q$-hypergeometric function, series with this property are classified and hence can be identified.

Since for $q \to 1^-$

$$ \lim_{q \to 1^-} \frac{(q^a;\, q)_k}{(q;\, q)_k} = \frac{(a)_k}{k!} \,, \tag{2.20} $$

one has

$$ \lim_{q \to 1^-} {}_r\phi_s\left(\begin{array}{c} q^{a_1}, q^{a_2}, \ldots, q^{a_r} \\ q^{b_1}, q^{b_2}, \ldots, q^{b_s} \end{array}\middle|\, q, (q-1)^{1+s-r}\, x\right) = {}_rF_s\left(\begin{array}{c} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{array}\middle|\, x\right) $$

[8] Note that basic hypergeometric functions and their properties were already considered in [Bailey35] and [Slater66] where the definition of the $q$-hypergeometric function was given *without* this additional factor, though.

which connects the $q$-hypergeometric function with the hypergeometric function.

By

$$[k]_q! := \frac{(q;\, q)_k}{(1-q)^k} \ ,$$

$$\Gamma_q(z) := \frac{(q;\, q)_\infty}{(q^z;\, q)_\infty}\,(1-q)^{1-z} \ ,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;\, q)_n}{(q;\, q)_k \cdot (q;\, q)_{n-k}}$$

and

$$[k]_q := \frac{1-q^k}{1-q}$$

one defines the *$q$-factorial*, the *$q$-Gamma function*, the *$q$-binomial coefficient* and the *$q$-brackets*, and there are $q$-analogues for many hypergeometric identities (see e.g. [GR90]).

We consider an example: Whereas the *binomial theorem* states that

$$\sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = \sum_{k=0}^{\infty} \frac{(-\alpha)_k}{k!}\,(-x)^k = {}_1F_0\left( \begin{array}{c} -\alpha \\ - \end{array} \middle|\, -x \right) = (1+x)^\alpha \ , \qquad (2.21)$$

(a particular case of which is (2.1)), the *$q$-binomial theorem* due to Cauchy, Jacobi and Heine is the identity ($|q| < 1$, $|x| < 1$)

$${}_1\phi_0\left( \begin{array}{c} a \\ - \end{array} \middle|\, q, x \right) = \sum_{k=0}^{\infty} \frac{(a;\, q)_k}{(q;\, q)_k}\, x^k = \frac{(ax;\, q)_\infty}{(x;\, q)_\infty} \ . \qquad (2.22)$$

More details can be found in [Gasper97] and [GR90], and we will continue these considerations in the next chapters.

# Exercises

**2.1** *Show that* (2.12) *remains valid in the limit if $k$ tends to the half of a negative integer or zero.* Hint: *Use Exercise* 1.1.

**2.2** *Prove that the rational term ratio* (2.9) *together with the initial value $A_0 = 1$, implies the hypergeometric coefficient formula* (2.8).

**2.3 (Hypergeometric Differential Equation, see e.g. [Rainville60])** *Show that the generalized hypergeometric function* $F(x) := {}_pF_q\left( \begin{array}{c} \alpha_1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \beta_2, \ldots, \beta_q \end{array} \middle|\, x \right)$ *satisfies the* hypergeometric differential equation

$$\theta(\theta+\beta_1-1)\cdots(\theta+\beta_q-1)F(x) = x(\theta+\alpha_1)(\theta+\alpha_2)\cdots(\theta+\alpha_p)F(x) \qquad (2.23)$$

*where $\theta$ denotes the differential operator $\theta f(x) = x f'(x)$.* Hint: *Substitute the series into* (2.23), *and equate coefficients.*

**2.4 (Hypergeometric Derivative Rule, see e.g. [Rainville60])** *Show that the generalized hypergeometric function* $F_n(x) := {}_pF_q\left(\begin{array}{c}\alpha_1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \beta_2, \ldots, \beta_q\end{array}\middle|\, x\right)$ *satisfies the* derivative rules

$$\theta F_n(x) = n\,\Big(F_{n+1}(x) - F_n(x)\Big)$$

*for any of its numerator parameters* $n := \alpha_k$ $(k = 1, \ldots, p)$ *and*

$$\theta F_n(x) = (n-1)\,\Big(F_{n-1}(x) - F_n(x)\Big)$$

*for any of its denominator parameters* $n := \beta_k$ $(k = 1, \ldots, q)$.

**2.5 (Hypergeometric Recurrence Equation)** *How can Exercises* 2.3 *and* 2.4 *be combined to obtain a recurrence equation with respect to any of the parameters of* ${}_pF_q$? *What is the order of this recurrence equation?*

**2.6** *Use Algorithm* 2.1 *to determine* $a_{k+1}/a_k$ *for* $a_k := b(n+j,k) - b(n,k)$ *for* $j = 1, \ldots, 3$ *when*

(a) $b(n,k) = \binom{n}{k}$, (b) $b(n,k) = \binom{n-k}{k}$,

(c) $b(n,k) = n\binom{n}{k}$, (d) $b(n,k) = (n-k)!$ .

**2.7** *In Session* 2.2, *Maple's result*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = \frac{\sqrt{\pi}\,2^n}{\Gamma\left(1+\frac{1}{2}\,n\right)\,\Gamma\left(\frac{1}{2}-\frac{1}{2}\,n\right)}$$

*was obtained. Show that this result corresponds to* (2.4).

**2.8** *Use Algorithm* 2.2 *to calculate the hypergeometric representations of the sums occurring in* (2.1)–(2.5). *Which hypergeometric terms are determined by the right-hand sides of these identities?*

**2.9** *Use an adaptation of Algorithm* 2.2 *to find hypergeometric representations for* $\sum\limits_{k=A}^{B} a_k$ *with largest possible summation range, if*

(a) $a_k = k(k-1)(k-2)\binom{n}{k}$, (b) $a_k = \binom{n-k}{k}$,

(c) $a_k = \dfrac{1}{k(k-1)(k-2)}\binom{2n}{k}$, (d) $a_k = \binom{n}{k}\binom{2k}{n}$ .

*Which shifts are necessary? Which are the actual ranges* $(A, B)$ *of the hypergeometric representations?*

**2.10** *Show, by a treatment similar to Example 2.2, the identity*

$$ {}_2F_1\left(\left.\begin{array}{c} -n-1\,,-n/2+1/2 \\ -n/2-1/2 \end{array}\right| x\right) = \sum_{k=0}^{n} \frac{(-n-1)_k\,(-n/2+1/2)_k}{(-n/2-1/2)_k} x^k = (1+x)\,(1-x)^n . $$

**2.11** *The following are the standard series representations of the elementary functions. Use Algorithm 2.2 to give their hypergeometric equivalents.*

(a) $\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ ,

(b) $\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k$ ,

(c) $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$ ,

(d) $\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k}$ ,

(e) $(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$ ,

(f) $\arctan(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$ ,

(g) $\arcsin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \binom{-1/2}{k} x^{2k+1}$ .

**2.12** *The Legendre polynomials have the following two different series representations*

$$ P_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k}^2 (x-1)^{n-k}\,(x+1)^k $$

*and*

$$ P_n(x) = \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k} $$

*besides the one presented in Example 2.5. Convert these into hypergeometric notation. Here*

$$ \lfloor x \rfloor := \max\,\{n \in \mathbb{Z} \mid n \le x\} $$

*denotes the floor function.*

*Which identities between hypergeometric functions correspond to the equality of the three given hypergeometric representations for the Legendre polynomials?*

*Note that we are not yet able to prove that the three different series representations for the Legendre polynomials constitute the same family of functions. In Chapter 4, Exercise 4.3, this assertion will be proved.*

**2.13** *Give a hypergeometric representation for the sum of consecutive Legendre polynomials $P_{n+1}(x)+P_n(x)$. Try to give one for $P_{n+2}(x) \pm P_n(x)$, $P_{n+3}(x) \pm P_n(x)$. What happens?*

**2.14 (Apéry Numbers, see [Apéry79])** *Convert the* Apéry numbers

$$A_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 .$$

*into hypergeometric notation.*

◇ **2.15** *Write the Maple function* `hyperterm(upper,lower,x,k)` *that was utilized in Session* 2.2.

**2.16 (Bieberbach Conjecture, see [deBranges85])** *The following sum was an essential tool in the proof of the* Bieberbach conjecture *by de Branges in 1984* [deBranges85] *(see also Example* 7.4*)*

$$\sum_{j=k}^{n} (-1)^{k+j} \binom{2j}{j-k} \binom{n+j+1}{n-j} e^{-jt} .$$

*Convert into hypergeometric notation under the hypothesis that $k$ denotes a positive integer.*

**2.17** *Convert the identity (see* [GKP94], *p.* 171*)*

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k}{\binom{2\,n+2\,b+2\,c+2\,d}{n+b+c+d+k}} \binom{n+b}{n+k} \binom{b+c}{b+k} \binom{c+d}{c+k} \binom{d+n}{d+k} =$$

$$\frac{\Gamma(n+b+c+d+1)\Gamma(n+b+c+1)\Gamma(n+b+d+1)\Gamma(n+c+d+1)\Gamma(b+c+d+1)}{n!\,\Gamma(2n+2b+2c+2d+1)\,\Gamma(n+c+1)\,\Gamma(b+d+1)\,\Gamma(b+1)\,\Gamma(c+1)\,\Gamma(d+1)} .$$

*into hypergeometric notation. What are the natural bounds?*

**2.18** *Maple's* `expand` *procedure expands $\Gamma$ function terms $\Gamma(a+k)$ for integer $k$ in terms of $\Gamma(a)$. This gives an alternative way to decide rationality of expressions involving $\Gamma$ terms.*

*Time the expressions $\Gamma(k+10000)/\Gamma(k+9999)$, and $\Gamma(k+100000)/\Gamma(k+99999)$ using* `expand` *and* `simpcomb`*. Explain!*

**2.19** *Let $a_k$ denote the $k^{th}$ summand of the generalized hypergeometric function ${}_pF_q\left(\left.\begin{smallmatrix}\alpha_1,\ldots,\alpha_p\\ \beta_1,\ldots,\beta_q\end{smallmatrix}\right| x\right)$. Show that the following limit procedure generates the $m^{th}$ partial sum*

$$\sum_{k=0}^{m} a_k = \lim_{\varepsilon\to 0} {}_{p+1}F_{q+1}\left(\left.\begin{matrix}-m,\alpha_1,\ldots,\alpha_p\\ -m+\varepsilon,\beta_1,\ldots,\beta_q\end{matrix}\right| x\right) .$$

**2.20** *Show that Kummer's confluent hypergeometric function is the following limiting case*[9] *of Gauss' hypergeometric function*

$$ {}_1F_1\left(\left.\begin{matrix} a \\ c \end{matrix}\right| x\right) = \lim_{b\to\infty} {}_2F_1\left(\left.\begin{matrix} a, b \\ c \end{matrix}\right| \frac{x}{b}\right) . $$

**2.21** *Prove the following equations for the q-Pochhammer symbol:*

(a) $(a;\, q)_n = \dfrac{(a;\, q)_\infty}{(aq^n;\, q)_\infty}$ ,

(b) $\dfrac{1-aq^{2n}}{1-a} = \dfrac{(q\sqrt{a};\, q)_n\,(-q\sqrt{a};\, q)_n}{(\sqrt{a};\, q)_n\,(-\sqrt{a};\, q)_n}$ ,

(c) $(a;\, q)_n\,(-a;\, q)_n = \left(a^2;\, q^2\right)_n$ ,

(d) $(a;\, q)_n = \left(q^{1-n}/a;\, q\right)_n (-a)^n\, q^{\binom{n}{2}}$ .

**2.22** *Prove* (2.20).

**2.23** *Show that for $k, n \in \mathbb{N}$ the relations*

(a) $[k]_q! = \Gamma_q(k+1)$ ,

(b) $\begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{[n]_q!}{[k]_q!\,[n-k]_q!}$

*are valid.*

**2.24** *Work out the connection between the binomial theorem* (2.21) *and the q-binomial theorem* (2.22).

**2.25** *Prove the q-binomial theorem* (2.22). Hint: *Deduce the functional equation*

$$ f(a,x) = (1-ax)f(aq,x) $$

*for* $f(a,x) := {}_1\phi_0\left(\left.\begin{matrix} a \\ - \end{matrix}\right| q, x\right)$ *by series manipulations, and use induction to show that*

$$ f(a,x) = (ax;\, q)_n\, f(aq^n, x) $$

*which gives*

$$ f(a,x) = (ax;\, q)_\infty\, f(0,x) $$

*for $n \to \infty$ (see* [Gasper97]*).*

**2.26** *Show that*

$$ {}_1\phi_0\left(\left.\begin{matrix} a \\ - \end{matrix}\right| q, x\right) \cdot {}_1\phi_0\left(\left.\begin{matrix} b \\ - \end{matrix}\right| q, ax\right) = {}_1\phi_0\left(\left.\begin{matrix} ab \\ - \end{matrix}\right| q, x\right) . $$

[9] This is a confluence process, again; hence the name of the function.