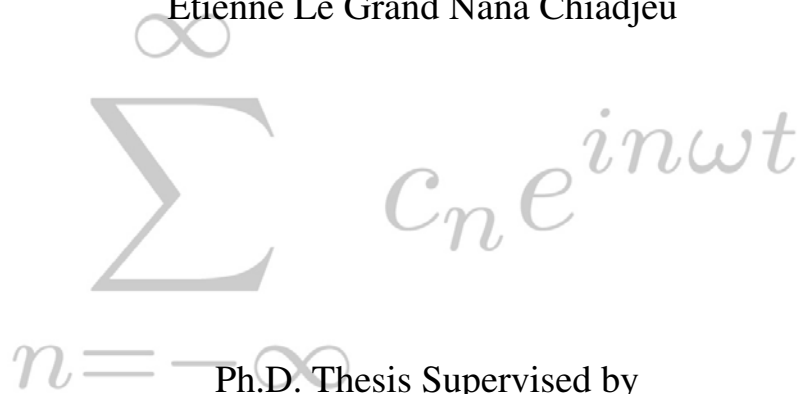# Algorithmic

Computation of

Formal Fourier Series

by

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$$\sum_{n=-\infty}^{\infty} c_n e^{in\omega t}$$

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Department of Mathematics

November 2009

UNIKASSEL
VERSITÄT

Dissertation
zur Erlangung des akademischen Grades
eines Doktors der Naturwissenschaften
(Dr. rer. nat.)

Erstgutachter: Prof. Dr. Wolfram Koepf

Zweitgutachter: Prof. Dr. Werner M. Seiler

Tag der mündlichen Prüfung
04. Februar 2010

Acknowledgment

Contents

Chapter 0

Introduction

0.1 Historical Introduction

The study of trigonometric series has started at the beginning of the nineteenth century. Joseph Fourier made the important observation that every integrable function of a closed interval can be decomposed into the sum of sine and cosine functions. This technique to develop a function into a trigonometric series was published for the first time in 1822 by Joseph Fourier in [Fou22]. The resulting series is nowadays called Fourier series.

Definition 0.1
The real and complex Fourier series of an integrable function $f : [a, b] \to \mathbb{R}$ are the expressions

$$\mathcal{F}(f)(t) := \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\,\omega\,t) + \sum_{n=1}^{\infty} b_n \sin(n\,\omega\,t) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\omega t} \,, \tag{1}$$

where $\omega = \frac{2\pi}{b-a}$ is the circular frequency and the corresponding real Fourier coefficients are given by

$$a_n = \frac{2}{b-a} \int_a^b f(t) \cos(n\omega t) dt \in \mathbb{R} \,, \tag{2}$$

$$b_n = \frac{2}{b-a} \int_a^b f(t) \sin(n\omega t) dt \in \mathbb{R} \,, \tag{3}$$

whereas the complex Fourier coefficients are defined as

$$c_n = \frac{1}{b-a} \int_a^b f(t) e^{-in\omega t} dt \in \mathbb{C} \,. \tag{4}$$

We remark that a finite sum of the form (1) is called a *Fourier polynomial*. Of course, by definition we have the relations $c_0 = \frac{a_0}{2}$ as well as

$$c_n = \frac{1}{2} (a_n - i\, b_n) \quad \text{and} \quad c_{-n} = \frac{1}{2} (a_n + i\, b_n) \quad (n \in \mathbb{Z}_{\geq 1})$$

which—solving for a_n and b_n—gives

$$a_n = c_n + c_{-n} \quad \text{and} \quad b_n = i(c_n - c_{-n}) \quad (n \in \mathbb{Z}_{\geqq 1}) \ .$$

All above formulas are also valid if $f : [a, b] \to \mathbb{C}$ is a complex function. However if the input function is real, then we get moreover $\bar{c}_n = \frac{1}{2}(a_n + ib_n) = c_{-n}$, which—solving for a_n and b_n—gives

$$a_n = c_n + \bar{c}_n \quad \text{and} \quad b_n = i(c_n - \bar{c}_n) \quad (n \in \mathbb{Z}_{\geq 1}) \ .$$

Under the additional assumption that f is continuous in (a, b), it turns out that pointwise $\mathcal{F}(f)(t) = f(t)$ for all $t \in (a, b)$.[1] As general references for elementary properties of Fourier series see e.g. [BSM71],[BSMM98], [Sto99], [CB78] and [Stu74].

Note that we will not study any convergence issue and therefore we mention that the series (1) is only the series which corresponds to $f(t)$. The problem of convergence of Fourier series has been investigated by many authors and one of the first is Lejeune-Dirichlet in [Dir29]. Since we will need in some of our algorithms to compute successive derivatives, all the functions involved in this thesis are assumed to be defined and continuous on an interval $I = [a, b]$ and may be at least N times continuously differentiable on I for suitable $N \geq 0$. Furthermore the considered function f can be periodically continued to $\mathbb{R}$ with period $T = b - a$. Let's denote the resulting periodic function $F : \mathbb{R} \to \mathbb{R}$. By definition, the function F is continuous in $\mathbb{R}$ besides the points $a + kT$ $(k \in \mathbb{Z})$ which are (possible) discontinuities of step size $\Delta := f(a) - f(b)$.
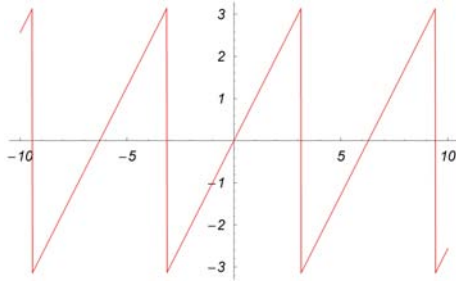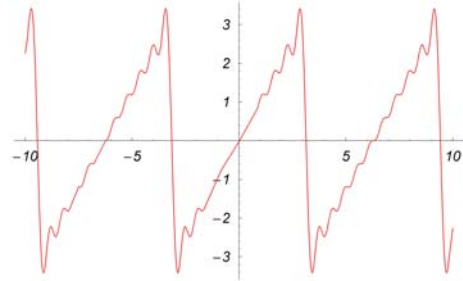
Figure 0.1   Figure 0.2

The sawtooth wave of Figure 0.1 is given by $f : [-\pi, \pi] \to \mathbb{R}, f(t) = t$, and yields the Fourier series

$$\mathcal{F}(f)(t) = \sum_{n=1}^{\infty} 2 \frac{(-1)^{n+1}}{n} \sin(nt)$$

and the partial sum of order 10 of the Fourier series $\sum_{n=1}^{10} 2 \frac{(-1)^{n+1}}{n} \sin(nt)$ is represented by Figure 0.2. So we get for the function $f(t) = t$ the simple formula $a_n = 2 \frac{(-1)^{n+1}}{n}$ and $b_n = 0$.

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[1]One can furthermore prove that at the points of discontinuity one has $\mathcal{F}(f)(a + kT) = \frac{1}{2}(f(a) + f(b))$ $(k \in \mathbb{Z})$.

Since Fourier's time, many different approaches to understand the concept of Fourier series have been discovered, each of which emphasizes different aspects of the topic. Some of the more powerful and elegant approaches are based on mathematical ideas and tools that were not available at the time Fourier completed his original work. Although the original motivation was to solve the heat equation in a metal plate, it later became obvious that the same technique could be applied to a wide array of mathematical and physical problems and has many applications in electrical engineering, vibration analysis, acoustics, optics, signal treatment, image processing, etc .... Despite that increasing demand of those series, the method used until now to compute them via computer algebra systems (CAS) is essentially based on the same principle as Fourier, i.e. using the definitions (2)–(4). Unfortunately this technique is not successful for many functions. Although numeric values of the Fourier coefficients might be available, symbolic values are often not accessible.

Modern CAS like *Maple* or Mathematica can compute such integrals in many cases for a given $n \in \mathbb{Z}$. However if one is interested in the Fourier coefficients for all $n \in \mathbb{Z}$, then n is considered as a given symbolic variable and such integrals can be computed only in few cases.

In general the computation process used by those CAS is as follows: First the corresponding indefinite integral is computed. Secondly, the fundamental theorem of calculus is used to compute the foresaid definite integral. If the first step is successful, then we get in this way a formula for the searched Fourier coefficients. However the success of this first step depends on very complicated algorithms, e.g. the Risch algorithm for elementary integrability [Bro96]. This computation can sometimes be very time consuming although it happens that no elementary anti-derivative for the considered function exists. In those cases there is typically no chance to get the result despite that long computation time. We remark that this case can also happen even if a formula for the searched Fourier coefficients can be found. Thus this method of computing that definite integral is not optimal.

0.2 Summary of the Main Results

0.2.1 Computation of Fourier Series

The computation of the Fourier coefficients of a function f using formulas (2)–(4) is in some cases very complicated, because of the integer parameter n in those formulas. We introduce in this thesis an algorithmic approach to compute those Fourier coefficients, involving differential equations of a particular form, and recurrence equations. This approach extrapolates the computation of the Fourier series for functions whose computation of Fourier coefficients via definitions (2)–(4) is out of reach for current CAS. Consider for example the function given as

$$f(t) = \cos(5t) \ln(2 + \cos(5t)) \ .$$

whose graph is
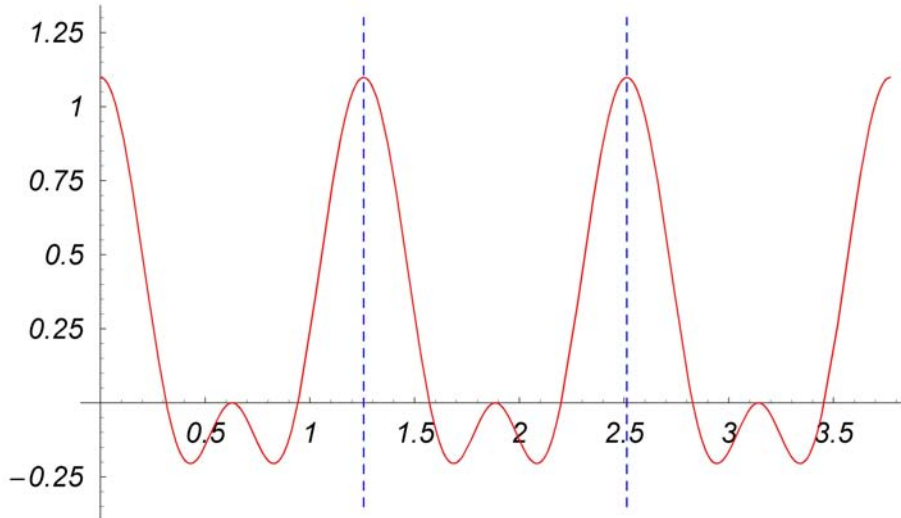
Figure 0.3: Composition of logarithm with trigonometric functions

Using the algorithms presented in this thesis we get that its Fourier series on the interval $I = [0, \frac{2\pi}{5}]$ are

$$f(t) = 2(2-\sqrt{3}) + (2\sqrt{3} - \frac{7}{2} + \ln(2+\sqrt{3}) - \ln(2)) \cos(5t) + \sum_{n=2}^{\infty} \frac{2(-2+\sqrt{3})^n(\sqrt{3}+2n)}{(n+1)(n-1)} \cos(5nt) \ .$$

Note that this algorithmic approach is applicable to a rich family of functions denoted trigonometric holonomic functions, which is defined and characterized in the second chapter.

We would also like to mention that most of the expansions in Fourier series provided in [Sto99] and [BSMM08] are also found with the algorithmic method described in this thesis.

0.2.2   Factorization of Holonomic Recurrence Operators

The search for hypergeometric solutions of *holonomic recurrence equations* is related to the search of first order right factors of *holonomic recurrence operators*, more generally to the factorization of those operators. Marko Petkovšek [Pet92], Mark van Hoeij [Hoe98] and Peter Horn [Hor08] investigated that issue and brought important contributions to the factorization of such operators. We give in this thesis another approach to factorize them by searching for a right factor of a given holonomic recurrence operator, which returns in some cases a lowest order right factor, using this time Fourier coefficients.

0.3   Outline of the Dissertation

In the first chapter we derive in Theorem 1.1 a connection between the complex Fourier coefficients of a function f and those of its first derivative. This connection yields in Theorem 1.8 a more general statement, this time between the complex Fourier coefficients of f and those of its successive

derivatives, from which an explicit formula for the complex Fourier coefficients of polynomials involving successive derivatives is deduced. We give in Theorem 1.7 an explicit formula for the computation of the real Fourier coefficients of powers. Theorem 1.8 will be also used in the third chapter in Theorem 3.9 to convert the differential equations obtained in the second chapter into recurrence equations.

In the second chapter we introduce the set of trigonometric holonomic functions and we give some of its characteristics and properties. This family contains not only some elementary functions, but also many functions whose Fourier coefficients cannot be successfully computed in the classical way. We give some example types of trigonometric holonomic functions. Algorithms 2.1, 2.2 and 2.3 for the computation of the trigonometric holonomic differential equations that those functions satisfy are presented.

In the third chapter we present Algorithm 3.1 (THDEtoRE) and Algorithm 3.8 (ExpTHDEtoRE) for the conversion of trigonometric holonomic differential equations into holonomic recurrence equations for their Fourier coefficients. We focus on the rational trigonometric functions, by giving in Algorithm 3.4 (EfficientPL) a way to detect a recurrence equation of low order satisfied by the complex Fourier coefficients of the considered function. The rest of the chapter is devoted to the computation of the complex Fourier coefficients of the foresaid trigonometric holonomic functions. Algorithms 3.2, 3.3, 3.5 and 3.9 give step by step the way to achieve that purpose.

The fourth chapter is devoted to the factorization of holonomic recurrence operators. We present Algorithm 4.1 to convert a holonomic recurrence equation into a differential equation with side conditions. That algorithm will be involved in Algorithm 4.2, which computes a right factor of a given holonomic recurrence operator.

The use of Parseval's identity in Fourier series is well-known as an efficient tool for the computation of the sum of numeric series. The Fourier series of f being defined as in formulas 1–4, we recall that Parseval's identity is given as

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{T} \int_0^T |f(t)|^2 dt = \|f(t)\|^2 \tag{5}$$

or

$$\|f(t)\|^2 = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) . \tag{6}$$

Generally a finite integral is used to compute a sum. For example if $f(t) = t$ then we get

$$a_0 = \pi, \quad a_n = 0, \quad b_n = \frac{-1}{n} \quad \text{and} \quad \int_0^{2\pi} t^2 dt = \frac{4\pi^2}{3} .$$

Using equation (6) one deduces the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} .$$

In the fifth chapter we emphasize on the opposite way, i.e. we use the computation of infinite sums to deduce the computation of finite integrals. Algorithm 5.1 outlines that technique to compute definite integrals via Parseval's identity using the algorithmic approach of the computation of

Fourier series. Section 5.2 focusses on this approach to deduce the solving of boundary value problems involving initial conditions which are trigonometric holonomic functions and whose Fourier coefficients can not be successfully computed via the classical way.

The appendix is devoted to the computation of the Fourier series of some specific trigonometric holonomic functions. The examples are chosen such that one may remark the difficulties which may be encountered during the computation of Fourier coefficients according to that algorithmic approach. In some instances one gets an explicit solution for some given input, in other cases one gets only a recurrence equation and some initial values. In the worst case only the recurrence equation satisfied by the Fourier coefficients of the given function can be found, whereas the initial values are not accessible in symbolic form. Nevertheless the latter is an important property of the Fourier coefficients considered.

Chapter 1

Some Particular Cases

In this chapter we derive an identity for the Fourier coefficients of a differentiable function $f(t)$ in terms of the Fourier coefficients of its derivative $f'(t)$. This yields an algorithm to compute the Fourier coefficients of $f(t)$ whenever the Fourier coefficients of $f'(t)$ are known, and vice versa. Furthermore this generates an iterative scheme for N times differentiable functions complementing the direct computation of Fourier coefficients via the defining integrals which can be also treated automatically in certain cases, see [Wer98] using *Maple* [Mon03] and in [Den00] using *Mathematica* [Wol99]. As direct consequence of that scheme we deduce an explicit formula for the computation of the complex Fourier coefficients in the case of polynomials. In the third chapter we will use that scheme to present an algorithm for the computation of the complex Fourier coefficients of the set of trigonometric holonomic functions which will be introduced in the second chapter. We would like to mention that [KNC06] is a part of this chapter.

1.1   Notation

Let $f : [a, b] \to \mathbb{R}$ be a continuous function in the interval $I = [a, b]$ which is continuously differentiable in (a, b). Then f' is continuous and has a Fourier series itself, for which we use the following notations

$$\mathcal{F}(f')(t) = \frac{a'_0}{2} + \sum_{n=1}^{\infty} a'_n \cos(n \, \omega \, t) + \sum_{n=1}^{\infty} b'_n \sin(n \, \omega \, t) = \sum_{n=-\infty}^{\infty} c'_n \, e^{in\omega t} \, ,$$

i.e., the Fourier coefficients of the derivative function are denoted by dashes. If $f \in C^N[a, b]$, then we can continue taking derivatives, and for the kth derivative ($k \leqq N$) we use the notation

$$\mathcal{F}(f^{(k)})(t) = \frac{a_0^{(k)}}{2} + \sum_{n=1}^{\infty} a_n^{(k)} \cos(n \, \omega \, t) + \sum_{n=1}^{\infty} b_n^{(k)} \sin(n \, \omega \, t) = \sum_{n=-\infty}^{\infty} c_n^{(k)} \, e^{in\omega t} \, .$$

1.2   An Identity for Fourier Coefficients

Let us recall that the complex Fourier coefficients c_n of a continuous function f on an interval $[a, b]$ are given by the relation

$$c_n = \frac{1}{b-a} \int_a^b f(t) e^{-in\frac{2\pi}{b-a}t} dt = \frac{1}{T} \int_a^{a+T} f(t) e^{-in\omega t} dt \ ,$$

where $b - a = T = \frac{2\pi}{\omega}$. Integrating by parts, using $u(t) = f(t), v'(t) = e^{-in\omega t}$, hence $u'(t) = f'(t)$ and $v(t) = \frac{1}{-in\omega} e^{-in\omega t}$, we get

$$\begin{aligned}
c_n &= \left[\frac{f(t) e^{-in\omega t}}{-in\omega T} \right]_a^{a+T} + \frac{1}{T} \int_a^{a+T} \frac{f'(t) e^{-in\omega t}}{in\omega} dt \\
&= \frac{f(a+T) e^{-in\omega(a+T)} - f(a) e^{-in\omega a}}{-in\omega T} + \frac{1}{in\omega} \left(\frac{1}{T} \int_a^{a+T} f'(t) e^{-in\omega t} dt \right) \\
&= \frac{f(a+T) - f(a)}{-2\pi in} e^{-in\omega a} - \frac{i}{n\omega} c_n' \ .
\end{aligned}$$

Hence we have derived the identity

$$c_n + \frac{i}{n\omega} c_n' = \frac{i}{2\pi n} (f(b) - f(a)) e^{-in\omega a} \ . \tag{1.1}$$

As we shall discuss later, this easy-to-derive relation has interesting applications, and can be used to compute the Fourier coefficients recursively under certain conditions.

Next, we would like to rewrite the above equation in terms of the real Fourier coefficients a_n and b_n. Using the relation $c_n = \frac{1}{2}(a_n - ib_n)$, we get from (1.1)

$$a_n - ib_n + \frac{i}{n\omega}(a_n' - ib_n') = \frac{i}{\pi n}(f(b) - f(a)) e^{-in\omega a} \ ,$$

and separating the real and imaginary parts, we conclude that

$$\begin{aligned}
a_n + \frac{1}{n\omega} b_n' &= \frac{1}{\pi n}(f(b) - f(a)) \sin(n\omega a) \qquad \text{and} \\
-b_n + \frac{1}{n\omega} a_n' &= \frac{1}{\pi n}(f(b) - f(a)) \cos(n\omega a) \ .
\end{aligned} \tag{1.2}$$

Finally, we summarize the above identities in the following

Theorem 1.1 (Fourier coefficients and derivatives)
Let $f : [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and continuously differentiable in (a, b). Then the real and complex Fourier coefficients of $f(t)$ and of $f'(t)$ satisfy the identities

$$c_n + \frac{i}{n\omega} c_n' = \frac{i}{2\pi n} (f(b) - f(a)) e^{-in\omega a} \quad (n \in \mathbb{Z}, n \neq 0)$$

and

$$a_n + \frac{1}{n\omega}b'_n = \frac{1}{\pi n}(f(b) - f(a))\sin(n\omega a) \qquad \text{and}$$

$$-b_n + \frac{1}{n\omega}a'_n = \frac{1}{\pi n}(f(b) - f(a))\cos(n\omega a) \quad (n \in \mathbb{Z}_{\geq 1}),$$

respectively. In particular: If $a = 0$, i.e. if the interval is $I = [0, T]$, then

$$c_n + i\frac{T}{2\pi n}c'_n = \frac{i}{2\pi n}(f(T) - f(0))$$

and

$$a_n + \frac{T}{2\pi n}b'_n = 0 \qquad \text{and}$$

$$-b_n + \frac{T}{2\pi n}a'_n = \frac{1}{\pi n}(f(T) - f(0)).$$

$$(1.3)$$

Furthermore, if the interval is symmetric w.r.t the origin, i.e. if $I = [-\frac{T}{2}, \frac{T}{2}]$, then

$$c_n + i\frac{T}{2\pi n}c'_n = \frac{i}{2\pi n}(f(\tfrac{T}{2}) - f(-\tfrac{T}{2}))(-1)^n$$

and

$$a_n + \frac{T}{2\pi n}b'_n = 0 \qquad \text{and}$$

$$-b_n + \frac{T}{2\pi n}a'_n = \frac{(-1)^n}{\pi n}(f(\tfrac{T}{2}) - f(-\tfrac{T}{2})).$$

$$(1.4)$$

Using these identities, one can easily compute the Fourier coefficients of $f'(t)$, if those of $f(t)$ are known, and vice versa.

1.3   Iterative Computation of Fourier Coefficients

Theorem 1.1 can be used to compute the Fourier coefficients iteratively. We give some examples for this approach.

Example 1.2 (Fourier coefficients of powers)
Let $f(t) = t^m$ for some $m \in \mathbb{Z}_{\geq 1}$. First, we consider the case $I = [0, T]$. For $m = 1$ and $m = 2$, the corresponding periodic functions F are represented in Figure 1.1. Note that the periodic linear function is called a *sawtooth function*.

Figure 1.1: The linear and square functions for $T = 1$

Whereas in these cases, it is easy to compute the Fourier coefficients directly from the defining formulas, namely for $f(t) = t$:

$$a_n = \frac{2}{T} \int_0^T t \cos\left(\frac{2\pi nt}{T}\right) dt = 0 \ ,$$

$$b_n = \frac{2}{T} \int_0^T t \sin\left(\frac{2\pi nt}{T}\right) dt = -\frac{T}{\pi n} \ ,$$

and for $f(t) = t^2$:

$$a_n = \frac{2}{T} \int_0^T t^2 \cos\left(\frac{2\pi nt}{T}\right) dt = \frac{T^2}{\pi^2 n^2} \ ,$$

$$b_n = \frac{2}{T} \int_0^T t^2 \sin\left(\frac{2\pi nt}{T}\right) dt = -\frac{T^2}{\pi n} \ ,$$

we would like to use Theorem 1.1 instead. Let $f(t) = t$. Then the derivative $f'(t) = 1$ obviously has Fourier coefficients $a'_0 = 2$ and $a'_n = 0, b'_n = 0$ $(n \in \mathbb{Z}_{\geq 1})$. Therefore we get from (1.3)

$$a_n + \frac{T}{2\pi n} b'_n = 0 \ ,$$

hence $a_n = 0$ and

$$-b_n + \frac{T}{2\pi n} a'_n = \frac{1}{\pi n}(f(T) - f(0)) = \frac{T}{\pi n} \ ,$$

hence $b_n = -\frac{T}{\pi n}$.

In the next step, we set $f(t) = t^2$. Hence $f'(t) = 2t$ with $a'_n = 0$ and $b'_n = -\frac{2T}{\pi n}$ by our previous computation. Therefore, using (1.3), we get

$$a_n + \frac{T}{2\pi n} b'_n = a_n - \frac{T}{2\pi n}\frac{2T}{\pi n} = a_n - \frac{T^2}{\pi^2 n^2} = 0 \ ,$$

hence $a_n = \frac{T^2}{\pi^2 n^2}$ and

$$-b_n + \frac{T}{2\pi n} a'_n = -b_n = \frac{1}{\pi n}(f(T) - f(0)) = \frac{T^2}{\pi n} \ ,$$

hence $b_n = -\frac{T^2}{\pi n}$.

Obviously, this strategy can be used iteratively (or recursively) to compute the Fourier coefficients of every power $f(t) = t^m$, and by linearity, of every polynomial. This algorithm is considered in more generality in the next section.

Next, we consider the symmetric case $I = [-\frac{T}{2}, \frac{T}{2}]$. The corresponding functions for $m = 1$ and $m = 2$ are drawn in Figure 1.2.
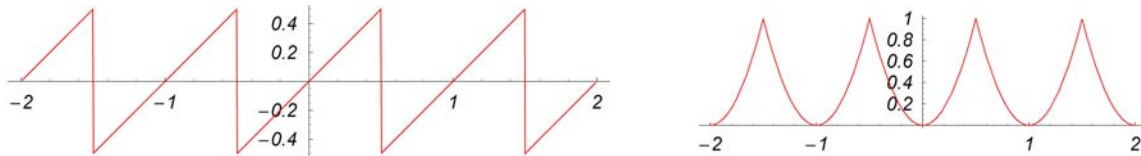
Figure 1.2: The symmetric linear and square functions ($f(t) = t$ and $f(t) = 4t^2$) for $T = 1$

Again one can compute the Fourier coefficients easily from their definition. For $f(t) = t$:

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} t \cos\left(\frac{2\pi nt}{T}\right) dt = 0 \;,$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} t \sin\left(\frac{2\pi nt}{T}\right) dt = -\frac{(-1)^n \, T}{\pi n} \;,$$

and for $f(t) = t^2$:

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} t^2 \cos\left(\frac{2\pi nt}{T}\right) dt = \frac{(-1)^n \, T^2}{\pi^2 n^2} \;,$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} t^2 \sin\left(\frac{2\pi nt}{T}\right) dt = 0 \;.$$

However, these results are contained in Formulas (1.4) and can be also computed iteratively.

Example 1.3 (Fourier coefficients including special functions)

Let $f(t) = \arctan t$. Since $f(t)$ is odd, it is clear that $a_n = 0$. However the coefficients b_n cannot be easily computed by the defining formula:

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} \arctan t \sin\left(\frac{2\pi nt}{T}\right) dt \;.$$

Note that *Maple* fails to compute b_n even if T is given explicitly.

However, we can compute b_n in terms of special functions using (1.4). Notice that for $f(t) = \arctan t$, we have $f'(t) = \frac{1}{1+t^2}$, a rational function. For the even function $f'(t)$ we get $b'_n = 0$, and for a'_n we derive using *Maple*

```
>   anprime:=2/T*int(1/(1+t^2)*cos(2*Pi*n*t/T),t=-T/2..T/2);
```

$$anprime := 2\Bigl(-2\,\mathrm{Si}(\frac{n\,\pi\,(T+2\,I)}{T})\sinh(\frac{n\,\pi}{T})\cosh(\frac{n\,\pi}{T}) + \mathrm{Ci}(-\frac{n\,\pi\,(T+2\,I)}{T})\cosh(\frac{n\,\pi}{T})^2\,I$$

$$-\frac{1}{2}\,I\,\mathrm{Ci}(-\frac{n\,\pi\,(T+2\,I)}{T}) - 2\,\mathrm{Si}(\frac{n\,\pi\,(T-2\,I)}{T})\sinh(\frac{n\,\pi}{T})\cosh(\frac{n\,\pi}{T})$$

$$-\mathrm{Ci}(-\frac{n\,\pi\,(T-2\,I)}{T})\cosh(\frac{n\,\pi}{T})^2\,I + \frac{1}{2}\,I\,\mathrm{Ci}(-\frac{n\,\pi\,(T-2\,I)}{T})$$

$$-\mathrm{Ci}(\frac{n\,\pi\,(T-2\,I)}{T})\cosh(\frac{n\,\pi}{T})^2\,I + \frac{1}{2}\,I\,\mathrm{Ci}(\frac{n\,\pi\,(T-2\,I)}{T})$$

$$+\mathrm{Ci}(\frac{n\,\pi\,(T+2\,I)}{T})\cosh(\frac{n\,\pi}{T})^2\,I - \frac{1}{2}\,I\,\mathrm{Ci}(\frac{n\,\pi\,(T+2\,I)}{T})\Bigr)/T$$

where

$$\mathrm{Ci}(t) = -\int_t^\infty \frac{\cos x}{x}\,dx = \gamma + \ln t + \int_0^t \frac{\cos x - 1}{x}\,dx$$

(γ denoting the Euler-Mascheroni constant) and

$$\text{Si}(t) = \int_0^t \frac{\sin x}{x} dx$$

denote the integral cosine and sine functions, respectively.

Since for $f(t) = \arctan t$ we have $f(-T/2) = \arctan(-T/2) = -\arctan(T/2) = -f(T/2)$ it is now clear that we get for b_n by (1.4)

```
>   f:=arctan(t);
```

$$f := \arctan(t)$$

```
>   bn:=T/(2*Pi*n)*anprime-(-1)^n/(Pi*n)*2*subs(t=T/2,f);
```

$$bn := (-2\,\text{Si}(\frac{n\,\pi\,(T+2\,I)}{T})\sinh(\frac{n\,\pi}{T})\cosh(\frac{n\,\pi}{T}) + \text{Ci}(-\frac{n\,\pi\,(T+2\,I)}{T})\cosh(\frac{n\,\pi}{T})^2\,I$$

$$-\frac{1}{2}\,I\,\text{Ci}(-\frac{n\,\pi\,(T+2\,I)}{T}) - 2\,\text{Si}(\frac{n\,\pi\,(T-2\,I)}{T})\sinh(\frac{n\,\pi}{T})\cosh(\frac{n\,\pi}{T})$$

$$-\text{Ci}(-\frac{n\,\pi\,(T-2\,I)}{T})\cosh(\frac{n\,\pi}{T})^2\,I + \frac{1}{2}\,I\,\text{Ci}(-\frac{n\,\pi\,(T-2\,I)}{T})$$

$$-\text{Ci}(\frac{n\,\pi\,(T-2\,I)}{T})\cosh(\frac{n\,\pi}{T})^2\,I + \frac{1}{2}\,I\,\text{Ci}(\frac{n\,\pi\,(T-2\,I)}{T})$$

$$+\text{Ci}(\frac{n\,\pi\,(T+2\,I)}{T})\cosh(\frac{n\,\pi}{T})^2\,I - \frac{1}{2}\,I\,\text{Ci}(\frac{n\,\pi\,(T+2\,I)}{T}))/(n\,\pi) - \frac{2\,(-1)^n\,\arctan(\frac{T}{2})}{n\,\pi}$$

It turns out that for this particular example, the direct computation with *Maple* is not successful, but *Mathematica* computes it and gives the above result directly from the defining integral. However, our theorem gives an algorithmic approach which will lead to further examples that are out of reach for *Maple* or *Mathematica*. Example 1.11 is an illustration of this.

Let us finish this section with the remark that for rational functions the algorithm described can be used to lower the degree of powers of the denominator polynomial recursively.

1.4   Iterated Derivatives

In this section, we assume that $f \in C^N[0, T]$ for some $N \in \mathbb{Z}_{\geq 1}$. Then by Theorem 1.1 we can write down successive identities for successive derivatives of f. Using (1.1), we get for the complex Fourier coefficients and for $n \in \mathbb{Z}, n \neq 0$

$$c_n + i\frac{T}{2\pi n}c_n' = \frac{i}{2\pi n}(f(T) - f(0))$$
$$c_n' + i\frac{T}{2\pi n}c_n'' = \frac{i}{2\pi n}(f'(T) - f'(0))$$
$$c_n'' + i\frac{T}{2\pi n}c_n^{(3)} = \frac{i}{2\pi n}(f''(T) - f''(0))$$
$$\vdots$$
$$c_n^{(N-1)} + i\frac{T}{2\pi n}c_n^{(N)} = \frac{i}{2\pi n}(f^{(N-1)}(T) - f^{(N-1)}(0)) .$$

In order to manipulate the previous relations easier, let us rewrite them in the following way:

$$c_n + \tau\, c_n' = \alpha_0$$
$$c_n' + \tau\, c_n'' = \alpha_1$$
$$c_n'' + \tau\, c_n^{(3)} = \alpha_2$$
$$\vdots$$
$$c_n^{(N-1)} + \tau\, c_n^{(N)} = \alpha_{N-1}$$

with the abbreviations $\tau = i\frac{T}{2\pi n}$ and $\alpha_k = \frac{i}{2\pi n}(f^{(k)}(T) - f^{(k)}(0))$.

Then multiplying the kth equation by $(-1)^k \tau^k$ and summing up obviously yields a telescoping sum with the result

$$c_n = \alpha_0 - \tau\alpha_1 + \tau^2\alpha_2 - \tau^3\alpha_3 + \cdots + (-1)^{N-1}\tau^{N-1}\alpha_{N-1} + (-1)^N\tau^N c_n^{(N)}$$

which finally leads to the following theorem

Theorem 1.4 (Fourier coefficients and iterated derivatives)
For $f \in C^N[0,T]$ the following identity for the complex Fourier coefficients is valid ($n \in \mathbb{Z}, n \neq 0$):

$$c_n - (-1)^N \left(\frac{Ti}{2\pi n}\right)^N c_n^{(N)} = \sum_{k=0}^{N-1} (-1)^k \left(\frac{Ti}{2\pi n}\right)^k \frac{i}{2\pi n}(f^{(k)}(T) - f^{(k)}(0)) . \qquad (1.5)$$

As a consequence, since a polynomial $f(t)$ of degree N satisfies $f^{(N)}(t) = \text{constant}$, and therefore $c_n^{(N)} = 0$ for $n \in \mathbb{Z}, n \neq 0$, Theorem 1.4 implies the following corollary

Corollary 1.5 (Fourier coefficients of polynomials)
Let $f : [0,T] \to \mathbb{R}$ be a polynomial of degree N. Then the complex Fourier coefficients of f can be written in the form ($n \in \mathbb{Z}, n \neq 0$)

$$c_n = \sum_{k=0}^{N-1} (-1)^k \left(\frac{Ti}{2\pi n}\right)^k \frac{i}{2\pi n} (f^{(k)}(T) - f^{(k)}(0)) \,.$$

Similarly, we can treat the interval $I = [-\frac{T}{2}, \frac{T}{2}]$. In this case, $\alpha_k = i\frac{(-1)^n}{2\pi n}(f^{(k)}(T) - f^{(k)}(0))$, hence we have

Theorem 1.6 (Fourier coefficients and iterated derivatives in symmetric intervals)
For
$f \in C^N[-\frac{T}{2}, \frac{T}{2}]$ the following identity for the complex Fourier coefficients is valid:

$$c_n - (-1)^N \left(\frac{Ti}{2\pi n}\right)^N c_n^{(N)} = \sum_{k=0}^{N-1} (-1)^{k+n} \left(\frac{Ti}{2\pi n}\right)^k \frac{i}{2\pi n} \left(f^{(k)}\left(\frac{T}{2}\right) - f^{(k)}\left(-\frac{T}{2}\right)\right) \,. \quad (1.6)$$

In particular: Let $f : [-\frac{T}{2}, \frac{T}{2}] \to \mathbb{R}$ be a polynomial of degree N. Then the complex Fourier coefficients of f can be written in the form ($n \in \mathbb{Z}, n \neq 0$)

$$c_n = \sum_{k=0}^{N-1} (-1)^{k+n} \left(\frac{Ti}{2\pi n}\right)^k \frac{i}{2\pi n} \left(f^{(k)}\left(\frac{T}{2}\right) - f^{(k)}\left(-\frac{T}{2}\right)\right) \,. \quad (1.7)$$

Note that the computation of the Fourier coefficients of t^m, e.g., using (1.7) is much more efficient than the computation using the definition. But for polynomials, we can do even more.

1.5   Fourier Coefficients of Polynomials

Although the algorithm of the previous section can be easily used to compute the Fourier coefficients of every polynomial efficiently, in the current section we would like to mention that the Fourier coefficients of polynomials can be even written down explicitly. By the linearity of the Fourier coefficients, it is enough to know them for powers $f(t) = t^m$ ($m \in \mathbb{Z}_{\geqq 1}$). In this case, we have

Theorem 1.7 (Fourier coefficients of powers)
Let $f(t) = t^m$ ($m \in \mathbb{Z}_{\geqq 1}$). The Fourier coefficients in the interval $I = [-\frac{T}{2}, \frac{T}{2}]$ are given as ($n \in \mathbb{Z}_{\geqq 1}$)

$$a_n = \begin{cases} \left(\frac{T}{2\pi n}\right)^m (-1)^n \sum\limits_{k=0}^{\frac{m}{2}-1} \frac{2\,m!}{(2k+1)!} (n\pi)^{2k} (-1)^{\frac{m}{2}-1+k} & \text{if } m \text{ is even} \\ \qquad\qquad\qquad 0 & \text{otherwise} \end{cases} \quad (1.8)$$

and

$$b_n = \begin{cases} \left(\frac{T}{2\pi n}\right)^m (-1)^{n+1} \sum_{k=0}^{\frac{m-1}{2}} \frac{2m!}{(2k+1)!} (n\pi)^{2k} (-1)^{\frac{m-1}{2}+k} & \text{if } m \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \qquad (1.9)$$

Proof: The formulas are obvious consequences of Theorem 1.6.

On the other hand, these formulas can also be obtained more directly from the definition. We would like to find

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} t^m \cos\left(n\frac{2\pi}{T}t\right) dt .$$

To do so it is enough to check the antiderivative

$$\int t^m \cos\left(n\omega t\right) dt = \frac{1}{(n\omega)^{m+1}} \times$$

$$\left[\frac{1+(-1)^m}{2} \left(\sum_{k=0}^{\frac{m}{2}} (n\omega t)^{2k} \frac{m!}{(2k)!} (-1)^{\frac{m}{2}+k} \sin(n\omega t) + \sum_{k=0}^{\frac{m}{2}-1} (-1)^{\frac{m+2}{2}+k} (n\omega t)^{2k+1} \frac{m!}{(2k+1)!} \cos(n\omega t) \right) + \right.$$

$$\left. + \frac{1-(-1)^m}{2} \sum_{k=0}^{\frac{m-1}{2}} (-1)^{\frac{m-1}{2}+k} \left((n\omega t)^{2k} \frac{m!}{(2k)!} \cos(n\omega t) + (n\omega t)^{2k+1} \frac{m!}{(2k+1)!} \sin(n\omega t) \right) \right]$$

of the integrand $t^m \cos \omega t$ by differentiation. Formula (1.8) then follows by the fundamental theorem of calculus. In a similar fashion the Fourier coefficient b_n can be treated. $\square$

We would like to mention that similar formulas can be obtained if the interval is given as $I = [0, T]$.

1.6 Efficiency Considerations

Whereas it seems obvious that the formulas (1.8)–(1.9) should yield the fastest computation for the Fourier coefficients of $f(t) = t^m$ whereas (1.7) should be weaker, and the direct computation using the definition

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} t^m \exp\left(-int\frac{2\pi}{T}\right) dt \qquad (1.10)$$

should lead to the longest computation times, reality is a little more complicated. Undoubtedly, the computation via the definition is the weakest method. Whether (1.7) or (1.8)–(1.9) are faster, depends on implementation details, however, and the above assumption is true only for small values of m. If (1.7) is programmed directly as sum, then (1.8)–(1.9) are faster since (1.7) needs the repeated computation of high order derivatives. On the other hand, if (1.7) is programmed by computing the high order derivatives iteratively, then this is faster than (1.8)–(1.9) for large m.

The reason is that the calculation of large factorials $m!$ and $(2k+1)!$ and their gcd computation in (1.8)–(1.9) is avoided. Detailed timings can be seen in Tables 1.1–1.3.[1] The fastest codes for large m in *Maple* and *Mathematica*, respectively, based on (1.7), are given in the appendix.

$m = 100$	(1.10)	(1.7) with Σ	(1.7) iteratively	(1.8)–(1.9)
Maple	4.88	0.015	0.00	0.00
Mathematica	3.64	0.016	0.015	0.00

Table 1.1: Timings for the computation of c_n in $[-\frac{T}{2}, \frac{T}{2}]$ for $f(t) = t^{100}$ with different algorithms

$m = 1.000$	(1.10)	(1.7) with Σ	(1.7) iteratively	(1.8)–(1.9)
Maple	$\diamond$	0.859	0.203	0.031
Mathematica	31.80	1.13	0.25	0.06

Table 1.2: Timings for the computation of c_n for $f(t) = t^{1.000}$

$m = 10.000$	(1.10)	(1.7) with Σ	(1.7) iteratively	(1.8)–(1.9)
Maple	$\diamond$	143.90	33.13	92.09
Mathematica	157.17	117.02	11.13	39.03

Table 1.3: Timings for the computation of c_n for $f(t) = t^{10.000}$

We note that the relations (1.5) and (1.6) can be summarized in the following theorem:

Theorem 1.8 (Fourier coefficients and iterated derivatives on the interval [a,b])
For $f \in C^N[a,b]$ and $\omega = \frac{2\pi}{b-a}$, the following identity for the complex Fourier coefficients is valid:

$$c_n - \left(\frac{-i}{n\omega}\right)^N c_n^{(N)} = \sum_{j=0}^{N-1} (-1)^j (b-a)^j \left(\frac{i}{2n\pi}\right)^{j+1} (f^{(j)}(b) - f^{(j)}(a)) e^{-in\omega a} . \qquad (1.11)$$

Remark 1.9
The complex Fourier coefficients of a function $f \in C^N[a,b]$ and those of its successive derivatives satisfy the matrix representation

$$\mathbf{c_n^{(P)}} = M_P \mathbf{c_n} + \mathbf{B_P}, \qquad P \le N \qquad (1.12)$$

---

[1] All timings are in seconds and were done with *Maple* 12.0 / *Mathematica* 6.0.3.0 with a PC AMD Athlon(tm) 64 Processor 3200+, 2.21 GHz CPU and 1.00 GB RAM. The iteration is most efficient with *Maple* using a `for` loop, and with *Mathematica* generating a list `tab` and using `Apply[Plus,tab]`. $\diamond$ indicates that the computation was not successful within one hour.

where

$$
M_P = \begin{pmatrix}
in\omega & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & (in\omega)^2 & 0 & \cdot & \cdot & \cdot & \cdot \\
\cdot & 0 & (in\omega)^3 & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 0 & \cdot & 0 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 0 & \cdot & 0 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & 0 & (in\omega)^P
\end{pmatrix} \tag{1.13}
$$

with $\omega = \frac{2\pi}{b-a}$.

Proof: From the relation (1.11) we deduce that

$$
c_n^{(N)} = (in\omega)^N c_n - (in\omega)^N \sum_{j=0}^{N-1} (-1)^j (b-a)^j \left(\frac{i}{2n\pi}\right)^{j+1} (f^{(j)}(b) - f^{(j)}(a)) e^{-in\omega a} \tag{1.14}
$$

Writing (1.14) successively for N ranging from 1 to P we get:

$$
c_n' = (in\omega) c_n - (in\omega) \frac{i}{2n\pi} (f(b) - f(a)) e^{-in\omega a}
$$

$$
c_n'' = (in\omega)^2 c_n + (in\omega)^2 \left(\frac{i}{2n\pi} (f(b) - f(a)) - (b-a)(\frac{i}{2n\pi})^2 (f'(b) - f'(a)) \right) e^{-in\omega a}
$$

$$
\vdots
$$

$$
c_n^{(P)} = (in\omega)^P c_n - (in\omega)^P \sum_{j=0}^{P-1} (-1)^j (b-a)^j (\frac{i}{2n\pi})^{j+1} (f^{(j)}(b) - f^{(j)}(a)) e^{-in\omega a}
$$

which may be brought into the form

$$
\begin{pmatrix}
c_n' \\
c_n'' \\
c_n^{(3)} \\
\cdot \\
\cdot \\
\cdot \\
c_n^{(P)}
\end{pmatrix}
=
\begin{pmatrix}
in\omega & 0 & \cdot & \cdot & \cdot & \cdot & 0 \\
0 & (in\omega)^2 & 0 & \cdot & \cdot & \cdot & \cdot \\
\cdot & 0 & (in\omega)^3 & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 0 & \cdot & 0 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 0 & \cdot & 0 & \cdot \\
\cdot & \cdot & \cdot & \cdot & 0 & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & 0 & (in\omega)^P
\end{pmatrix}
\begin{pmatrix}
c_n \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
c_n
\end{pmatrix}
+
$$

$$
+
\begin{pmatrix}
-(in\omega)\frac{i}{2n\pi}(f(b) - f(a))e^{-in\omega a} \\
(in\omega)^2 c_n + (in\omega)^2 \left(\frac{i}{2n\pi}(f(b) - f(a)) - (b-a)(\frac{i}{2n\pi})^2(f'(b) - f'(a)) \right) e^{-in\omega a} \\
\cdot \\
\cdot \\
\cdot \\
(in\omega)^P c_n - (in\omega)^P \sum_{j=0}^{P-1} (-1)^j (b-a)^j (\frac{i}{2n\pi})^{j+1} (f^{(j)}(b) - f^{(j)}(a)) e^{-in\omega a}
\end{pmatrix}
$$

which finally leads to the result. $\square$

We can immediately see that (1.11) describes for a given function a relation between c_n and $c_n^{(N)}$. From this relation we may also deduce by iteration a more general relation between $c_n^{(p)}$ and $c_n^{(q)}$ for $p, q \leq N$.

Remark 1.10

For many functions, the computations of their Fourier coefficients on an interval $[a, b]$ via definitions (2)–(4) using current CAS is not successful. However one can compute those of one of their successive derivatives. In those cases Theorem 1.8 outlines the computation of the Fourier coefficients of the foresaid functions. We summarize that process in the following algorithm:

Algorithm 1.1: Computation of the Fourier coefficients c_n of a function from those of one of its successive derivatives or anti-derivatives $c_n^{(N)}$. ($c_n\mathrm{From}c_n^{(N)}$)

---

    input : A function $f \in C^N[a, b]$ such that the computation of its Fourier coefficients on an interval $[a, b]$ also via definitions (2)–(4) using current CAS is not successful, and however one can compute those of one of its successive derivatives or anti-derivatives.

    output: The complex Fourier coefficients of f on the interval $[a, b]$ or 'the complex Fourier coefficients of f cannot be computed in a reasonable time using this algorithm'.

  1 **begin**
  2 $m \leftarrow 1$.
  3   **while** $m \leq M$ **do**
  4   Compute the complex Fourier coefficients of $f^{(m)}$.
  5   **if** *the computation is successful* **then**
  6   use relation (1.11) to achieve the computation of the Fourier coefficients of f.
  7   **return** *the complex Fourier coefficients of f on the interval $[a, b]$.*
  8   **end**
  9 $m \leftarrow m + 1$.
 10   **end**
 11   The complex Fourier coefficients of f cannot be computed in a reasonable time using this algorithm.
 12 **end**

Note that the goal of the appearing M in the algorithm is to stop it after a reasonable run.

Example 1.11

Consider the function defined by

$$f(t) = \arctan(2 + \cos(t)e^{it}) \ .$$

The complex Fourier coefficients of f and those of its successive derivatives cannot be successfully computed using formulas (2)–(4). However we remark that the first derivative f' of f belongs to the set of trigonometric holonomic functions which will be defined in the next chapter. Using Algorithm 3.9 which will be presented in the fourth chapter, we can compute the Fourier coefficients of f' and finally using Algorithm 1.1, we deduce those of f and we get:

$$c_n = \begin{cases} \frac{-i}{n}\left(\frac{1+(-1)^n}{2}\right)\left(\left(\frac{1}{29}\sqrt{-145+58i}\right)^n - \left(\frac{1}{29}\sqrt{-145-58i}\right)^n\right) & \forall n \geq 1 \\ \frac{1}{2}(\pi - \arctan(\frac{20}{21})) & \text{if} \quad n = 0 \\ 0 & \text{otherwise} \ . \end{cases}$$

Note that Algorithm 1.1 may also be used to give a simpler form of the Fourier coefficients in certain cases where the computation is successful.

Chapter 2

Trigonometric Holonomic Functions

For a moment, let's have a break with the computation of Fourier coefficients. We introduce in this chapter the set of trigonometric holonomic functions TH for which we will present in the next chapter algorithms for the computation of their Fourier coefficients. We will present some particular subsets of TH. Then we will give some algebraic properties of TH, focussing on the aspects concerning the aims of the dissertation.

2.1   Notations and Recall

Let $\mathbb{K}$ denote the field $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ or any subfield or transcendental extension, and $\mathbb{K}^\star = \mathbb{K} \setminus \{0\}$. For simplification purposes we write $\mathbb{K}[\cos(t), \sin(t)]$ for the set of trigonometric polynomials instead of $\mathbb{K}[x, y]/\langle x^2 + y^2 - 1 \rangle$ and analogously in similar cases. We will also understand $\cos(2t)$, e. g., by $\cos(2t) = \cos(t)^2 - \sin(t)^2$ as a member of $\mathbb{Q}[\cos(t), \sin(t)]$. It is well-known that every trigonometric polynomial $p = \sum_{i=0}^{I} \sum_{j=0}^{J} a_{ij} \cos^i(t) \sin^j(t) \in \mathbb{K}[\cos(t), \sin(t)]$ can be written as a Fourier polynomial in the form $\sum_{k=0}^{K} (a_k \cos(kt) + b_k \sin(kt))$ and vice versa via the following addition theorems

$$\cos(a + b) = \cos(a) \cos(b) - \sin(a) \sin(b) \tag{2.1}$$

$$\cos(a - b) = \cos(a) \cos(b) + \sin(a) \sin(b) \tag{2.2}$$

$$\sin(a + b) = \sin(a) \cos(b) + \cos(a) \sin(b) \tag{2.3}$$

$$\sin(a - b) = \sin(a) \cos(b) - \cos(a) \sin(b) \tag{2.4}$$

and the substitution rules

$$\cos(a) \cos(b) = \frac{1}{2} \cos(a - b) + \frac{1}{2} \cos(a + b) \tag{2.5}$$

$$\sin(a) \cos(b) = \frac{1}{2} \sin(a - b) + \frac{1}{2} \sin(a + b) \tag{2.6}$$

$$\sin(a)\sin(b) = \frac{1}{2}\cos(a-b) - \frac{1}{2}\sin(a+b) \ . \tag{2.7}$$

For more details about this conversion see ([Koe06], Chapter 9).

We recall also that the sum $f = f_1 + f_2 + \cdots + f_n$ of a finite sequence of periodic functions $\{f_1, f_2, \ldots, f_n\}$ is periodic if and only if their periods $T_1, T_2, \ldots, T_n$, respectively are commensurable.

The commensurability of $T_1, T_2, \ldots, T_n$ means that there exist n integers $N_1, \ldots, N_n$ such that

$$N_1 T_1 = N_2 T_2 = \cdots = N_n T_N \ . \tag{2.8}$$

It follows from (2.8) that a period of f is $T = N_1 T_1 = N_2 T_2 = \cdots = N_n T_N$.

Example 2.1

   a) Consider the three functions $g(t) = \sin(2\pi t)$, $h(t) = \cos(\frac{\pi}{5}t)$ and $r(t) = \sin(\frac{7\pi}{10}t)$ of periods $T_1 = 1, T_2 = 10$ and $T_3 = \frac{20}{7}$, respectively. Here $20T_1 = 2T_2 = 7T_3$ hence T_1, T_2 and T_3 are commensurable. Therefore it can be conclued that the sum $f(t) = \sin(2\pi t) + \cos(\frac{\pi}{5}t) + \sin(\frac{7\pi}{10}t)$ of the functions $g(t), h(t)$ and $r(t)$ is periodic of period T, where $T = 20T_1 = 2T_2 = 7T_3 = 20$

   b) Let g and h be the functions defined by: $g(t) = \cos(t), h(t) = \sin(\sqrt{3}t)$ of period $T_1 = 2\pi$ and $T_2 = \frac{2\pi\sqrt{3}}{3}$, respectively. Since $\frac{T_1}{T_2} = \sqrt{3}$ is not a rational number, then T_1 and T_2 are not commensurable and therefore the sum $f(t) = \cos(t) + \sin(\sqrt{3}t)$ of the functions $g(t)$ and $h(t)$ is not periodic.

2.2   Definitions

Definition 2.2 (Trigonometric holonomic functions)
Let $\omega \neq 0$ be a given real number: We call ω-trigonometric holonomic functions $\mathrm{TH}(\omega)$ the set of functions satisfying differential equations of the form

$$\sum_{p=0}^{P}\sum_{l=0}^{L}\left(\alpha_{pl}\cos(l\omega t) + \beta_{pl}\sin(l\omega t)\right)f^{(p)}(t) = 0 \tag{2.9}$$

for appropriate integers $P \geq 1$ and $L \geq 0$, where α_{pl} and β_{pl} are constants.
A function f is said to be trigonometric holonomic if there exist $\omega \in \mathbb{R}^\star$ such that $f \in \mathrm{TH}(\omega)$. We denote by TH the set of those functions, i.e. $\bigcup \mathrm{TH}(\omega) = \mathrm{TH}$.
Differential equations of the form (2.9) are called trigonometric holonomic differential equations.

Example 2.3
The following differential equation

$$DE1 : (\sin(2\sqrt{7}t) + 2)F(t) + 5F'(t) + (3 + \cos(4\sqrt{7}t))F''(t) = 0$$

is a trigonometric holonomic differential equation, since its non-constant coefficients $\sin(2\sqrt{7}t)+2$ and $3 + \cos(4\sqrt{7}t)$ are periodic and of commensurable periods. But

$$DE2 : (\sin(8t)) + \cos(16t))F(t) + (\cos(\sqrt{3}t) + 3)F'(t) + 5F'''(t) = 0$$

is not a trigonometric holonomic differential equation, since the periods of its non-constant coefficients $\sin(8t) + \cos(16t)$ and $\cos(\sqrt{3}t) + 3$ are not commensurable.

Remark 2.4
If for a given couple of integers (P_0, L_0), a function f satisfies a differential equation of the form (2.9), then for all $L \geq L_0$ and for all $P \geq P_0$, f satisfies other differential equations of the form (2.9).

Proof: Deriving equation (2.9) with respect to t, a new differential equation of the same form is obtained, but of higher order. hence for all $P \geq P_0$ f satisfies a differential equation of the form (2.9). The proof for $L \geq L_0$ is completed in Section 3.5.2.   □

Definition 2.5 (Type)
A function $f \in \mathrm{TH}(\omega)$ is said to be of type $L \in \mathbb{N}$ if L is the smallest positive integer for which it satisfies a differential equation of the form (2.9).

Definition 2.6 (Degree)
Let f be a function of $\mathrm{TH}(\omega)$. The smallest integer P for which f satisfies a differential equation of the form (2.9) is called the degree of f in $\mathrm{TH}(\omega)$.

Example 2.7
W.l.o.g., set $\omega = 1$ and let f be the function defined by

$$f(t) = \frac{\sin(t)}{\cos(t) + 2} .$$

f satisfies infinitely many trigonometric holonomic differential equations. Two of them are given by

DE1: $-2\ (1 + 2\ \cos(t))\ F(t) + (4\ \sin(t) + \sin(2t))\ F'(t) = 0$ and

$DE2 : +4F(t) + (2 - 2\ \cos(t) - 4\ \sin(t))\ F'(t)$
$\qquad + (-3\ \sin(t) + 4 + 2\ \cos(t))\ F''(t) + (2 + \cos(t))\ F'''(t) = 0.$

1. In the above differential equations, the type of f can be read off from DE2. We get $L_0 = 1$.

2. The smallest integer P for which f satisfies a differential equation of the form (2.9) is also 1, and appears in DE1. Hence f is of degree $P_0 = 1$ in $\mathrm{TH}(1)$.

Theorem 2.8
If a function $f \in \mathrm{TH}(\omega)$, then $\forall n \in \mathbb{N}^\star$, $f \in \mathrm{TH}(\frac{\omega}{n})$. I.e. $\mathrm{TH}(n\omega) \subseteq \mathrm{TH}(\omega)$.

Proof: The proof is obvious.   □

2.3　Example Types of Trigonometric Holonomic Functions

2.3.1　Polynomials

Theorem 2.9

For all $\omega \in \mathbb{R}^\star$, the set $\mathbb{K}[t]$ of all polynomials is a subset of $\mathrm{TH}(\omega)$ and each polynomial f of degree N is of type 0 and of degree $N + 1$ in $\mathrm{TH}(\omega)$.

Proof:　Let f be a polynomial of degree N. Then its derivative of order N is a constant and therefore $f^{(N+1)}(t) = 0$. Hence for $P = N + 1$ and $L = 0$, f satisfies a differential equation of the form (2.9). One may not expect to have a trigonometric holonomic differential equation of lower order, otherwise all its coefficients α_{pl} and β_{pl} will vanish. Since if we assume that $f(t) = t^N$ satisfies a trigonometric holonomic differential equation of order $P \leq N$, then substituting its successive derivatives $f^{(p)}(t) = \frac{N!}{(N-p)!} t^{N-p}$ in (2.9), we obtain an equation analogous to (2.14)—in the proof that $\frac{1}{t} \notin \mathrm{TH}$—which leads to vanishing coefficients. Hence $\mathbb{K}[t] \subset \mathrm{TH}(\omega)$ and f is of type 0 and of degree $N + 1$ in $\mathrm{TH}(\omega)$.　　□

2.3.2　*Exp-like* Functions

Exp-like functions are those which satisfy differential equations with constant coefficients, see [Koe92], i.e. differential equations of the form

$$a_n F^{(n)}(t) + a_{n-1} F^{(n-1)}(t) + \cdots + a_2 F''(t) + a_1 F'(t) + a_0 F(t) = 0, \qquad (2.10)$$

$$a_n \neq 0, \quad a_i \in \mathbb{C} \quad \forall i \in \{0, \ldots, n\} \ .$$

Since such differential equations are obtained with $L = 0$, we deduce that *exp-like* functions are trigonometric holonomic functions $\forall \omega \in \mathbb{R}^\star$ and of type 0. Searching for solutions of (2.10) of the form $f(t) = e^{\lambda t}, \lambda \in \mathbb{C}$ leads to the characteristic equation

$$a_n \lambda^n + a_{n-1} \lambda^{(n-1)}(t) + \cdots + a_2 \lambda^2 + a_1 \lambda + a_0 = 0 \ . \qquad (2.11)$$

The solutions of (2.10) depend on those of (2.11), whether its roots are all distinct and real, or its roots are repeated, or some of the roots are complex. Hence the general solution of (2.10) may be written as linear combination of expressions of the form

$$e^{\lambda t}, \quad e^{pt} \sin(qt), \quad e^{pt} \cos(qt), \quad t^j e^{\lambda t}, \quad t^j e^{pt} \sin(qt), \quad t^j e^{pt} \cos(qt), \quad j \in \{0, \ldots, m-1\} \ ,$$

see [SG95] where $m \geq 2$ refers to the multiplicity of the roots of (2.11), with λ, p and $q \in \mathbb{R}$. Since the hyperbolic cosine and the hyperbolic sine functions may be written in the form

$$\cosh(t) = \frac{e^t + e^{-t}}{2} \quad \text{and} \quad \sinh(t) = \frac{e^t - e^{-t}}{2} \ ,$$

we deduce, w.l.o.g. the following theorem

Theorem 2.10
$\mathbb{K}[t, e^{-\alpha t}, e^{\beta t}, \cos(\theta t), \sin(\psi t), \cosh(\delta t), \sinh(\mu t)]$ is a subset of TH , α, β, θ, ψ, δ and $\mu \in \mathbb{R}$.
Each of these functions is of type 0.

Example 2.11
Consider the function

$$f(t) = t\cos^2(t)\sinh(3t)(e^t)^5 ,$$

f is solution of the differential equation

$$DE1 : 75759616F(t) - 206110720F'(t) + 258281472F''(t) - 197550080F'''(t)$$

$$+102421504F^{(4)}(t) - 37716480F^{(5)}(t) + 10052416F^{(6)}(t) - 1940160F^{(7)}(t)$$

$$+267504F^{(8)}(t) - 25600F^{(9)}(t) + 1612F^{(10)}(t) - 60F^{(11)}(t) + F^{(12)}(t) = 0 .$$

Note that f satisfies also a lower order differential equation, namely

$$DE2 : -32(-692\cos(2t) + 359\sin(4t) + 808\cos(4t) - 3690 + 4522\sin(2t))F(t)$$

$$(10544\cos(4t) + 19056\sin(4t) - 36832\cos(2t) + 61536\sin(2t) - 45840)F'(t)$$

$$+(5696\cos(2t) - 52\cos(4t) - 4408\sin(4t) + 5748 - 8048\sin(2t))F''(t) + (1284 + 16\sin(2t)$$

$$+300\cos(4t) + 8\sin(4t) + 1584\cos(2t))F'''(t) + (-119\cos(4t) + 32\sin(4t) - 357$$

$$-476\cos(2t) + 64\sin(2t))F^{(4)}(t) + (32\cos(2t) + 8\cos(4t) + 24)F^{(5)}(t) = 0 .$$

But as we will see in the next chapter, for the purpose of the computation of Fourier coefficients we will prefer DE1 to DE2, because the type of the differential equation is more important than the order. Hence a trigonometric holonomic differential equation obtained for the smallest possible L is preferable.

Remark 2.12
Note that $\mathbb{K}[t, e^{-\alpha t}, e^{\beta t}, \cos(\theta t), \sin(\psi t), \cosh(\delta t), \sinh(\mu t)] \subseteq \mathrm{TH}(\omega)$ for all $\omega \in \mathbb{R}^\star$, α, β, θ, ψ, δ and $\mu \in \mathbb{R}$.

2.3.3 Trigonometric Polynomials

Although trigonometric polynomials are *exp-like* functions, we would like to emphasize on them in this section because we would like to characterize them according to their type and degree. One can directly deduce a differential equation of the form (2.9) they satisfy.

Theorem 2.13
For all $\omega \in \mathbb{R}^\star$, $\mathbb{K}[\cos(\omega t), \sin(\omega t)]$ is a subset of $\mathrm{TH}(\omega)$. Each function f of $\mathbb{K}[\cos(\omega t), \sin(\omega t)]$ is of type 0 and of degree 1 in $\mathrm{TH}(\omega)$ and f satisfies the differential equation

$$f(t)F'(t) - f'(t)F(t) = 0 . \tag{2.12}$$

Proof: Consider $f \in \mathbb{K}[\cos(\omega t), \sin(\omega t)]$ defined by $f(t) = \sum\limits_{k=0}^{N}(a_k \cos(k\omega t) + b_k \sin(k\omega t))$. Then f is *exp-like*, hence f is of type 0. Since $f'(t) \in \mathbb{K}[\cos(\omega t), \sin(\omega t)]$ and $f'f - ff' = 0$, we deduce that f is solution of the first order differential equation

$$f(t)F'(t) - f'(t)F(t) = 0$$

from which it appears that f is of degree 1 in $\mathrm{TH}(\omega)$.   □

Example 2.14

Consider the function $f(t) = \cos(2t)$, then (2.12) reads as

$$2\sin(2t)F(t) + \cos(2t)F'(t) = 0 \ .$$

Note that as *exp-like* function f satisfies also the differential equation

$$4F(t) + F''(t) = 0 \ .$$

2.3.4   Rational Trigonometric Functions

Theorem 2.15

For all $\omega \in \mathbb{R}^\star$ the set of rational trigonometric functions $\mathbb{K}(\cos(\omega t), \sin(\omega t))$ is a subset of $\mathrm{TH}(\omega)$ and each function of $\mathbb{K}(\cos(\omega t), \sin(\omega t))$ is of degree 1 in $\mathrm{TH}(\omega)$.

Proof: Consider the function f defined by

$$f(t) = \frac{g(t)}{h(t)}, \qquad \text{with} \quad h(t) \neq 0$$

where $g(t)$ and $h(t) \in \mathbb{K}[\cos(\omega t), \sin(\omega t)]$. Then we get:

$$f'(t) = \frac{g'(t)h(t) - g(t)h'(t)}{h^2(t)} \iff f'(t)h^2(t) = g'(t)h(t) - g(t)h'(t) \ .$$

Multiplying the left and the right hand side of the last equation by $f(t) = \frac{g(t)}{h(t)}$, we deduce that the function f is solution of the first order differential equation

$$(g(t)h(t))F'(t) - (g'(t)h(t) - g(t)h'(t))F(t) = 0 \tag{2.13}$$

which is of degree 1 in $\mathrm{TH}(\omega)$.   □

Note that in contrast to the cases of $\mathbb{K}[t]$ and $\mathbb{K}[\cos(\omega t), \sin(\omega t)]$ where the type is a priori known for all functions, in the case of $\mathbb{K}(\cos(\omega t), \sin(\omega t))$ the type depends on each particular function. In this case an efficient algorithm for the determination of the type for a given rational trigonometric function will be presented in Section 3.5.2.

Example 2.16
Let f be the function defined by

$$f(t) = \frac{\sin(t)}{2 + \cos(2t)} \ .$$

Applying relation (2.13) we get that f is solution of the first order differential equation

$$(3\sin(t) + \sin(3t))F'(t) + (-7\cos(t) + \cos(3t))F(t) = 0 \ .$$

We may immediately see that the degree of f is 1 in $\mathrm{TH}(1)$. Although the type of f cannot be read off from the previous differential equation, one may compute it via Algorithm (3.4), which will be presented in the next chapter, and gets two and the corresponding differential equation

$$(2 - 3\cos(2t))F(t) - 4\sin(2t)F'(t) + (2 + \cos(2t))F''(t) = 0 \ .$$

Theorem 2.17
The set of functions of the form $g(t) \cdot h(t)$ where $g(t)$ is *exp-like* and $h(t) \in \mathbb{K}(\cos(\omega t), \sin(\omega t))$ is a subset of $\mathrm{TH}(\omega)$.

Proof: Since $g(t) \cdot h(t)$ may be read as the product of two ω-trigonometric holonomic functions, according to Theorem 2.37 which will be given in Section 2.6 we deduce that $g(t) \cdot h(t) \in \mathrm{TH}(\omega)$.   □

Example 2.18
W.l.o.g. set $\omega = 1$ and consider the function

$$f(t) = \frac{t\cosh(t)e^t}{2 + \cos(t)} \ .$$

f satisfies the differential equation

$$-(4\sin(t) + 3\cos(t))F(t) + (12\cos(t) - 4\sin(t))F'(t) + (8 - 2\cos(t) + 12\sin(t))F''(t)$$

$$+(-8 - 4\cos(t) - 4\sin(t))F'''(t) + (2 + \cos(t))F^{(4)}(t) = 0$$

from which we deduce that f is of type $1 \in \mathrm{TH}(1)$.

2.3.5   Function which is not Trigonometric Holonomic

In this section we give an example of a function which is not trigonometric holonomic. W.l.o.g. we may assume that $\omega = 1$. We will show that the rational function

$$f(t) = \frac{1}{t}$$

is not a trigonometric holonomic function.
Proof: We use a proof by contradiction. Let us assume that f is a trigonometric holonomic function. Then there exist integers $P \geq 1$ and $L \geq 0$ and coefficients α_{pl} and β_{pl} for which f satisfies a differential

equation of the form (2.9). At least one of α_{pl} and β_{pl} is non-vanishing. The successive derivatives of f are given by

$$f^{(p)}(t) = \frac{(-1)^p p!}{t^{p+1}} \ .$$

The substitution of those derivatives in (2.9) leads to

$$\sum_{p=0}^{P} \sum_{l=0}^{L} \left(\alpha_{pl} \cos(lt) + \beta_{pl} \sin(lt) \right) \frac{(-1)^p p!}{t^{p+1}} = 0 \iff$$

$$\frac{1}{t} \sum_{l=0}^{L} (\alpha_{0l} \cos(lt) + \beta_{0l} \sin(lt)) - \frac{1}{t^2} \sum_{l=0}^{L} (\alpha_{1l} \cos(lt) + \beta_{1l} \sin(lt)) + \cdots$$

$$+ \frac{(-1)^P P!}{t^{P+1}} \sum_{l=0}^{L} (\alpha_{Pl} \cos(lt) + \beta_{Pl} \sin(lt)) = 0 \ .$$

Multiplying the previous equation by t^{P+1} we get

$$t^P \sum_{l=0}^{L} (\alpha_{0l} \cos(lt) + \beta_{0l} \sin(lt)) - t^{P-1} \sum_{l=0}^{L} (\alpha_{1l} \cos(lt) + \beta_{1l} \sin(lt)) + \cdots \qquad (2.14)$$

$$+ (-1)^P P! \sum_{l=0}^{L} (\alpha_{Pl} \cos(lt) + \beta_{Pl} \sin(lt)) = 0 \ .$$

Collecting the previous equation with respect to the expressions $\cos(lt), \sin(lt)$, $l \in \{0, \dots, L\}$, we get:

$$\left(\alpha_{00} t^P - \alpha_{10} t^{P-1} + \cdots + (-1)^P P! \alpha_{P0} \right) + \left(\alpha_{01} t^P - \alpha_{11} t^{P-1} + \cdots + (-1)^P P! \alpha_{P1} \right) \cos(t) +$$

$$\left(\beta_{01} t^P - \beta_{11} t^{P-1} + \cdots + (-1)^P P! \beta_{P1} \right) \sin(t) + \cdots + \left(\alpha_{0L} t^P - \alpha_{1L} t^{P-1} + \cdots + (-1)^P P! \alpha_{PL} \right) \cos(Lt) +$$

$$\left(\beta_{0L} t^P - \beta_{1L} t^{P-1} + \cdots + (-1)^P P! \beta_{PL} \right) \sin(Lt) = 0 \ .$$

The previous equation is satisfied for every t if and only if all the polynomial coefficients vanish, i.e.

$$\alpha_{00} = \alpha_{10} = \cdots = \alpha_{PL} = \beta_{00} = \beta_{10} = \cdots = \beta_{PL} = 0 \ ,$$

which is in contradiction with our initial assumption. $\qquad\qquad\square$

Up to now we have not seen in a more general setting how to determine a differential equation satisfied by an ω-trigonometric holonomic function. We present in the next section an algorithm for the determination of a differential equation of the form (2.9).

2.4   Differential Equation for Trigonometric Holonomic Functions

We give in this section an algorithm to determine for each function of the set $\mathrm{TH}(\omega)$ a differential equation of the form (2.9). We assume that f is defined on the interval $[a, b]$, connected with ω in the way $\omega = \frac{2\pi}{b-a}$.

Algorithm 2.1: Determination of a trigonometric holonomic differential equation for a trigonometric holonomic function (THDE)

    input : A real number $\omega = \frac{2\pi}{b-a}$, a function $f \in \mathrm{TH}(\omega)$ such that $f \in C^{(N)}[a, b]$ for N large enough and an initial positive integer value vi.

    output: A differential equation satisfied by f in the form (2.9).

1 **begin**
2      $L_{max} \leftarrow 0$, $P_{max} \leftarrow 0$.
3   **repeat**
4          $L_{max} \leftarrow L_{max} + vi$, $P_{max} \leftarrow P_{max} + vi$.
5   **for** $L = 0$ **to** L_{max} **do**
6   **for** $P = P_{max} - vi$ **to** P_{max} **do**
7   search for coefficients α_{pl} and β_{pl} such that the equation
8                  $\sum_{p=0}^{P} \sum_{l=0}^{L} (\alpha_{pl} \cos(l\omega t) + \beta_{pl} \sin(l\omega t)) f^{(p)}(t) = 0$ is valid.
9   **if** *the search is successful* **then**
10   **return** *the differential equation of f in the form (2.9)*.
11   **end**
12   **end**
13   **end**
14   **until** *the search is successful*;
15 **end**

Remark 2.19

Note that if the input function is not trigonometric holonomic, then Algorithm 2.1 does not terminate.

Example 2.20

Consider the function defined on the interval $[0, \frac{2\pi}{5}]$ by

$$f(t) = \cos(5t) \ln(2 + \cos(5t)) \ .$$
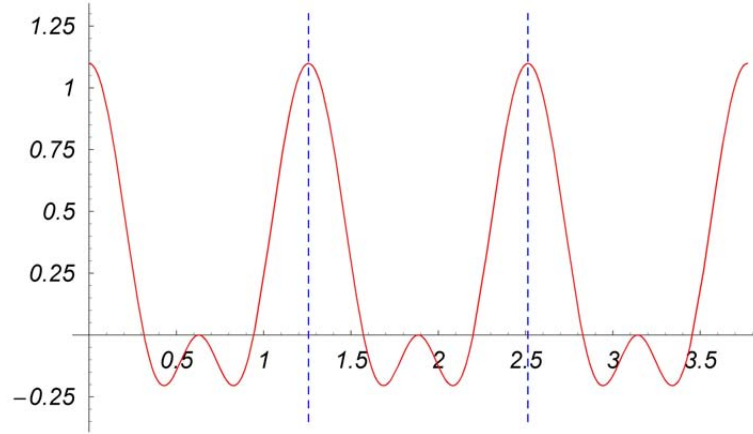
Figure 2.1: Composition of logarithm with trigonometric functions

f is periodic of period $T = \frac{2\pi}{5}$. Setting $\omega = \frac{2\pi}{T} = 5$ and applying the previous algorithm, we get that for $L = 1$ and $P = 7$, f satisfies the trigonometric holonomic differential equation

$$DE := (-500000 + 843750 \cos(5t))F'(t) + 28125 \sin(5t)F''(t) + (54375 \cos(5t) - 45000)F^{(3)}(t) +$$

$$+ 4625 \sin(5t)F^{(4)}(t) + (825 \cos(5t) - 1200)F^{(5)}(t) + 120 \sin(t)F^{(6)}(t) + (-4 \cos(5t) - 8)F^{(7)}(t) = 0 .$$

Remark 2.21

The search for a differential equation for functions of $\mathrm{TH}(\omega)$ using Algorithm 2.1 may be time-consuming in some cases because we don't know in advance for which choices of P and L a differential equation of the form (2.9) exists. In Section 3.5.2, for the case of the functions of $\mathbb{R}(\cos(\omega t), \sin(\omega t))$, we will present an algorithm for the determination of the best possible choices of P and L, for which we can get directly a differential equation of the form (2.9). Assuming that the best choice (P, L) is known, then Algorithm 2.1 will be simplified in the following form:

Algorithm 2.2: Determination of a trigonometric holonomic differential equation for a trigonometric holonomic function knowing the efficient P and L(THDEPL)

---

    input : A real number $\omega = \frac{2\pi}{b-a}$, a function $f \in \mathrm{TH}(\omega)$ such that $f \in C^{(N)}[a, b]$ for N large enough and the numbers P and L.

    output: A differential equation of the form (2.9) satisfied by f.

1 begin

2 | Look for coefficients α_{pl} and β_{pl} such that the equation

3 | $\displaystyle\sum_{p=0}^{P} \sum_{l=0}^{L} (\alpha_{pl} \cos(l\omega t) + \beta_{pl} \sin(l\omega t)) f^{(p)}(t) = 0$ is valid.

4 end

2.5   Functions Satisfying Differential Equations with Coefficients in $\mathbb{K}[e^{i\omega t}, e^{-i\omega t}]$

For a given $\omega \in \mathbb{R}^{\star}$ one can convert a trigonometric holonomic differential equation (as we have defined until now) into a differential equation with coefficients in $\mathbb{K}[e^{-i\omega t}, e^{i\omega t}]$ and conversely. In this section we will emphasize on differential equations whose coefficients are *linear* polynomials of either $e^{-i\omega t}$ or $e^{i\omega t}$ exclusively. As we will see in the next chapter, the particular importance of that type of differential equations is that they lead to first order holonomic recurrence equations for the Fourier coefficients of the considered function on an interval of length T, where ω is chosen according the equality $\omega = \frac{2\pi}{T}$. We remark that such recurrence equations (of first order) do not result once the coefficients of the considered differential equation are not of the foresaid form.

Theorem 2.22
For a given $\omega \in \mathbb{R}^{\star}$, functions satisfying a differential equation of the form

$$\sum_{p=0}^{P} \sum_{l=0}^{L} \left(\gamma_{pl} e^{-il\omega t} + \delta_{pl} e^{il\omega t} \right) f^{(p)}(t) = 0 \qquad (2.15)$$

for appropriate integers $P \geq 1, L \geq 0$, where γ_{pl} and $\delta_{pl} \in \mathbb{C}$ are ω-trigonometric holonomic functions.

Proof: The proof is obvious.   □

2.5.1   Example of Functions Satisfying Differential Equations whose Coefficients are Linear Polynomials of either $e^{-i\omega t}$ or $e^{i\omega t}$

Definition 2.23
The set of functions satisfying trigonometric holonomic differential equations leading to first order recurrence equations for their complex Fourier coefficients are called *simple trigonometric holonomic functions* (sTH). The coefficients of such trigonometric holonomic differential equations are all linear polynomials of either $e^{i\omega t}$ or $e^{-i\omega t}$, but not of both.

Example 2.24
Set $\omega = 1$ and consider the function defined by

$$f(t) = e^{ie^{it}} .$$

f satisfies the differential equation

$$DE : e^{it} F(t) + F'(t) = 0 ,$$

from which it follows that f is of degree 1 in $\mathrm{TH}(1)$. Since both coefficients of DE are linear polynomials of e^{it}, we deduce that $f \in \mathrm{sTH}$. As we will see in the next chapter, choosing an

interval of length $T = \frac{2\pi}{\omega} = 2\pi$, e.g. $I = [0, 2\pi]$, DE leads to a first order recurrence equation satisfied by the complex Fourier coefficients of f on I.

Note that the search for a differential equation of the form (2.9) satisfied by f gives, however

$$DE2 : -i(i + i\sin(t) + \cos(t))F(t) + (-i + \cos(t) - i\sin(t))F'(t) = 0 ,$$

which converts into a second order recurrence equation for the complex Fourier coefficients of f according to Algorithm 3.1 (THDEtoRE).

We remark also that if we aim now to convert $DE2$ into a differential equation of the form (2.15) via the substitution of $\cos(t)$ and $\sin(t)$ in terms of e^{-it} and e^{it}, we get

$$DE3 : (-i + e^{-it})F'(t) - i(i + e^{it})F(t) = 0 .$$

The coefficients of $DE3$ are *not* linear polynomials of either e^{-it} or e^{it}, but simultaneously of both of them. Hence $DE3 \notin \mathrm{sTH}$ and will lead to a second order recurrence equation this time according to Algorithm 3.8. Due to the non uniqueness of the factorization in $\mathbb{K}[\cos(t), \sin(t)]/\langle\cos(t)^2 + \sin(t)^2 - 1\rangle$, one cannot expect to get a differential equation of the exact form as DE (where the coefficients are linear polynomials of either e^{it}, or e^{-it}) by the conversion of $DE2$ into a differential equation of the form (2.15).

Example 2.25

Set $\omega = 5$ and consider the function defined by

$$f(t) = \frac{e^{-4it}}{2 + e^{5it}} \in \mathrm{TH}(5) .$$

f is solution of the first degree differential equation

$$DE : (8e^{-5it} + 9)F(t) - i(2e^{-5it} + 1)F'(t) = 0 .$$

Since the coefficients of DE are linear polynomials of e^{-5it}, then $f \in \mathrm{sTH}$.

2.5.2   Some More Example Types of Trigonometric Holonomic Functions

Remark 2.26

Since $e^{it} = \cos(t) + i\sin t$ and $e^{-it} = \cos(t) - i\sin t$, we can deduce without proof many theorems and properties which have been shown previously for the functions with arguments $\cos(t)$ and $\sin(t)$.

From Theorem 2.10 we may deduce the following theorem.

Theorem 2.27

The set $\mathbb{K}[t, e^{-\alpha t}, e^{\beta t}, e^{-i\gamma t}, e^{i\theta t}]$ where $\alpha, \beta, \gamma, \theta \in \mathbb{R}$ is a subset of $\mathrm{TH}(\omega)$ for all $\omega \in \mathbb{R}^\star$. Each of its functions is of type 0 in $\mathrm{TH}(\omega)$.

Example 2.28

Consider the trigonometric holonomic function defined by

$$f(t) = t\sin(t)e^{3it} + 3e^{it}\cos(t)$$

f is solution of the following complex differential equation

$$-64iF'(t) + 96F''(t) + 52iF'''(t) - 12F^{(4)}(t) - iF^{(5)}(t) = 0 \ .$$

From Theorem 2.15 we may also deduce the following one:

Theorem 2.29

The set $\mathbb{K}(e^{-i\omega t}, e^{i\omega t})$ is a subset of $\mathrm{TH}(\omega)$ and each function of $\mathbb{K}(e^{-i\omega t}, e^{i\omega t})$ is of degree 1 in $\mathrm{TH}(\omega)$.

As we will see in the next chapter, such a trigonometric holonomic differential equation whose coefficients have only the argument $e^{i\lambda t}$ for a fixed $\lambda \in \mathbb{R}$, leads to a first order holonomic recurrence equation for the corresponding Fourier coefficients.

Theorem 2.30

$\mathbb{K}(\cos(\omega t), \sin(\omega t), e^{-i\omega t}, e^{i\omega t})$ is a subset of $\mathrm{TH}(\omega)$.

We deduce from Theorem 2.17 that

Theorem 2.31

Functions of the form $g(t) \cdot h(t)$ where g(t) is *exp-like* and $h(t) \in \mathbb{K}[e^{-i\omega t}, e^{i\omega t}]$ is a subset of $\mathrm{TH}(\omega)$.

Example 2.32

Set $\omega = 1$ and consider the function

$$f(t) = \frac{te^{i\sqrt{3}t}\cos(\frac{3}{2}t)}{2 + e^{it}} \ .$$

f may be read as $te^{i\sqrt{3}t}\cos(\frac{3}{2}t) \cdot \frac{1}{2+e^{it}}$ where $te^{i\sqrt{3}t}\cos(\frac{3}{2}t)$ is *exp-like* and $\frac{1}{2+e^{it}} \in \mathbb{K}(e^{-it}, e^{it})$. Hence f satisfies the trigonometric holonomic differential equation

$$DE: -(-18 - 241e^{it} + 112\sqrt{3}e^{it})F(t) + 16i(6\sqrt{3} + 15\sqrt{3}e^{it} - 31e^{it})F'(t)$$

$$+(-312e^{it} + 192\sqrt{3}e^{it} - 432)F''(t) - 64i(2\sqrt{3} + e^{it}\sqrt{3} - e^{it})F'''(t) + (16e^{it} + 32)F^{(4)}(t) = 0 \ .$$

Note that the coefficients of DE are linear polynomials of e^{it}, i.e. $f \in \mathrm{sTH}$.

Remark 2.33

We cannot give a complete list of families of functions satisfying a differential equation either of the form (2.9) or (2.15). In the following example we list some functions whose form has not been mentioned previously.

Example 2.34

1. Consider the function

$$f(t) = \arctan(2 + e^{it}) \in \mathrm{TH}(1) .$$

   f is solution of the trigonometric holonomic differential equation

$$DE : (-e^{it} + 5e^{-it})F'(t) + i(4 + 5e^{-it} + e^{it})F''(t) .$$

   We deduce that f of degree 2 in $\mathrm{TH}(1)$.

2. Consider now the function

$$f(t) = \frac{\sqrt{e^{it} + 3}}{e^{it} + 2} \in \mathrm{TH}(1) .$$

   f satisfies the trigonometric holonomic differential equation

$$i(4 + e^{it})F(t) + (10 + 12e^{-it} + 2e^{it})F'(t) = 0 .$$

   f is of type 1 and of degree 1 in $\mathrm{TH}(1)$.

3. Set $\omega = \sqrt{7}$ and consider the function

$$f(t) = \cos(\sqrt{7}t)\ln(2 + \sin(\sqrt{7}t)) .$$

   f satisfies the trigonometric holonomic differential equation

$$DE : 98\sqrt{7}F(t)(4 + 3\sin(\sqrt{7}t)) + 343\cos(\sqrt{7}t)F'(t) + 7\sqrt{7}(10 + 3\sin(\sqrt{7}t))F''(t)$$

$$+ 21\cos(\sqrt{7}t)F'''(t) + \sqrt{7}(2 + \sin(\sqrt{7}t))F^{(4)}(t) = 0$$

   from which we deduce that $f \in \mathrm{TH}(\sqrt{7})$.

Analogously to Algorithm 2.1, we present in Algorithm 2.3 the determination of a differential equation with coefficients in $\mathbb{K}[e^{i\omega t}, e^{-i\omega t}]$ satisfied by a trigonometric holonomic function. In contrast to that algorithm, it may return in some cases trigonometric holonomic differential equations whose coefficients are linear polynomials of either $e^{i\omega t}$ or $e^{-i\omega t}$, which will lead to first order holonomic recurrence equations for the complex Fourier coefficients of the considered function on an interval on length $T = \frac{2\pi}{\omega}$.

Algorithm 2.3: Determination of a trigonometric holonomic differential equation for a trigonometric holonomic function satisfying a differential equation with coefficients in $\mathbb{K}[e^{-i\omega t}, e^{i\omega t}]$ (ExpTHDE).

> **input** : A real number $\omega = \frac{2\pi}{b-a}$, a function $f \in \mathrm{TH}(\omega)$ such that f is continuous in the interval $[a, b]$ and an initial positive integer value vi.
>
> **output**: A differential equation satisfies by f in the form (2.15).

1 begin
2 $L_{max} \leftarrow 0, P_{max} \leftarrow 0$.
3   **repeat**
4 $L_{max} \leftarrow L_{max} + v_i, P_{max} \leftarrow P_{max} + v_i$.
5   **for** $L = 0$ **to** L_{max} **do**
6   **for** $P = P_{max} - v_i$ **to** P_{max} **do**
7   Search for coefficients γ_{pl} and δ_{pl} such that either the equation
8

$$\sum_{p=0}^{P} \sum_{l=0}^{L} (\gamma_{pl}(\cos(l\omega t) - i\sin(l\omega t)) + \delta_{pl}(\cos(l\omega t) + i\sin(l\omega t))) f^{(p)}(t) = 0 \text{ or}$$

$$\sum_{p=0}^{P} \sum_{l=0}^{L} \left(\gamma_{pl} e^{-il\omega t} + \delta_{pl} e^{il\omega t}\right) f^{(p)}(t) = 0 \text{ is valid.}$$

9   **if** *the search is successful* **then**
10   **return** *the differential equation of f in the form (2.9).*
11   **end**
12   **end**
13   **end**
14   **until** *the search is successful*;
15 end

Remark 2.35

Note that if the input function does not satisfy a differential equation of the form (2.9), the previous algorithm would not terminate.

2.6 Algebraic Properties of Trigonometric Holonomic Functions

Theorem 2.36

If f is a ω-trigonometric holonomic function, then its derivative and anti-derivative are also trigonometric holonomic functions.

Proof: **Anti-derivative**

If $f \in \mathrm{TH}(\omega)$, then it satisfies a differential equation of the form

$$\sum_{p=0}^{P} \sum_{l=0}^{L} (\alpha_{pl} \cos(l\omega t) + \beta_{pl} \sin(l\omega t)) f^{(p)}(t) = 0 .$$

Let F be an anti-derivative of f. Then we get:

$$\sum_{p=0}^{P}\sum_{l=0}^{L}\big(\alpha_{pl}\cos(l\omega t)+\beta_{pl}\sin(l\omega t)\big)f^{(p)}(t)=0 \Longleftrightarrow \sum_{p=0}^{P}\sum_{l=0}^{L}\big(\alpha_{pl}\cos(l\omega t)+\beta_{pl}\sin(l\omega t)\big)(F')^{(p)}(t)=0$$

$$\Longleftrightarrow \sum_{p=0}^{P}\sum_{l=0}^{L}\big(\alpha_{pl}\cos(l\omega t)+\beta_{pl}\sin(l\omega t)\big)(F)^{(p+1)}(t)=0$$

$$\Longleftrightarrow \sum_{p=1}^{P+1}\sum_{l=0}^{L}\big(\alpha_{pl}\cos(l\omega t)+\beta_{pl}\sin(l\omega t)\big)(F)^{(p)}(t)=0$$

which means that the anti-derivative F of f is also a ω-trigonometric holonomic function.

Derivative

Since $f \in \mathrm{TH}(\omega)$, then f satisfies a differential equation of the form

$$\sum_{p=0}^{P}\sum_{l=0}^{L}\left(\alpha_{pl}\cos(l\omega t)+\beta_{pl}\sin(l\omega t)\right)f^{(p)}(t)=0 \Longleftrightarrow$$

$$\sum_{l=0}^{L}\left(\alpha_{0l}\cos(l\omega t)+\beta_{0l}\sin(l\omega t)\right)f(t)+\sum_{p=1}^{P}\sum_{l=0}^{L}\left(\alpha_{pl}\cos(l\omega t)+\beta_{pl}\sin(l\omega t)\right)f^{(p)}(t)=0 . \qquad (2.16)$$

Deriving the previous (2.16) we get

$$\sum_{l=0}^{L}\left(-l\alpha_{0l}\sin(l\omega t)+l\beta_{0l}\cos(l\omega t)\right)f(t)+\sum_{l=0}^{L}\left(\alpha_{0l}\cos(l\omega t)+\beta_{0l}\sin(l\omega t)\right)f'(t)+$$

$$(2.17)$$

$$\sum_{p=1}^{P}\sum_{l=0}^{L}\left(-l\alpha_{pl}\sin(l\omega t)+l\beta_{pl}\cos(l\omega t)\right)f^{(p)}(t)+\sum_{p=1}^{P+1}\sum_{l=0}^{L}\left(\alpha_{pl}\cos(l\omega t)+\beta_{pl}\sin(l\omega t)\right)f^{(p)}(t).$$

Computing the linear combination

$$(2.17)\cdot\sum_{l=0}^{L}\left(\alpha_{0l}\cos(l\omega t)+\beta_{0l}\sin(l\omega t)\right)-(2.16)\cdot\sum_{l=0}^{L}\left(-l\alpha_{0l}\sin(l\omega t)+l\beta_{0l}\cos(l\omega t)\right) ,$$

the expressions connected with $f(t)$ vanish and it remains a differential equation for the derivative f' of f, which may be brought into the form (2.9). $\qquad\qquad\square$

Theorem 2.37

1. $\mathrm{TH}(\omega)$ is closed under addition and multiplication.

2. If $f(t)$ and $g(t)$ are two functions of degree P and Q in $\mathrm{TH}(\omega)$, respectively, then $f(t)+g(t)$ is of degree $\leq P+Q$ in $\mathrm{TH}(\omega)$ and $f(t)\cdot g(t)$ is of degree $\leq P\cdot Q$ in $\mathrm{TH}(\omega)$.

Proof:

Addition: We will show in the proof that the sum of two functions of $\mathrm{TH}(\omega)$ is a function of $\mathrm{TH}(\omega)$, compare [Koe06]. W.l.o.g. we set $\omega = 1$. Let $f(t)$ be a function of the set $\mathrm{TH}(1)$ and let us consider the vector space $V(f) = \langle f(t), f'(t), \ldots \rangle$ over the field of rational trigonometric functions $\mathbb{K}(\cos(t), \sin(t))$, generated by the successive derivatives of $f(t)$. The family $\{f(t), f'(t), \ldots, f^{(P-1)}\}$ is a basis of $V(f)$ of dimension P, since f satisfies no differential equation of the form (2.9) of order $P-1$ and because each higher derivative of f can be written over the field $\mathbb{K}(\cos(t), \sin(t))$ as a linear combination of the functions $f(t), f'(t), \ldots, f^{(P-1)}$. In the same way we construct for $g(t)$, a function of degree Q in TH, the vector space $V(g) = \langle g(t), g'(t), \ldots \rangle$ of dimension Q, generated by the successive derivatives of the function $g(t)$. $\{g(t), g'(t), \ldots, g^{(Q-1)}\}$ is a basis of $V(g)$.

We deduce that $V(f) + V(g)$ is a vector space of dimension $\leq P + Q$ and that the functions $h = f + g$, $h' = f' + g', \ldots, h^{(k)} = f^{(k)} + g^{(k)}, \ldots$ are vectors of $V(f) + V(g)$. I.e. h satisfies a differential equation of order $\leq P + Q$ with coefficients in $\mathbb{K}(\cos(t), \sin(t))$.

Multiplication: The proof consists in showing that the product of two functions of $\mathrm{TH}(\omega)$ is a function of $\mathrm{TH}(\omega)$. W.l.o.g. we set $\omega = 1$. f and g being two functions of $\mathrm{TH}(1)$ of degree P and Q respectively, we construct analogously to the case of the addition, the vector spaces $V(f)$ and $V(g)$. Let us put $h = fg$. Using Leibniz's product rule, we compute the successive derivatives of h and we get: $h' = f'g + fg', \ldots, h^{(K)} = \sum_{j=0}^{K} \binom{K}{j} f^{(j)} g^{(K-j)}$. Due to the substitution rules (2.19), the previous successive derivatives of h can be written using only the successive derivatives of f of maximal order $P-1$ and of g of maximal order $Q-1$, respectively. The set generated by the product of elements of $V(f)$ by those of $V(g)$ is also a vector space, of dimension $\leq PQ$. I.e. the successive derivatives $h^{(K)} = (fg)^{(K)} = \sum_{j=0}^{K} \binom{K}{j} f^{(j)} g^{(K-j)}$ are elements of that vector space, K running from zero to PQ. Hence there exist $A_j \in \mathbb{K}(\cos(t), \sin(t))$ such that $\sum_{j=0}^{K} A_j h^{(j)} = 0$. We deduce that the product $h = fg$ satisfies a differential equation of type (2.9) of degree $\leq PQ$. $\qquad \square$

Remark 2.38

The previous proof does not show directly how to compute the differential equations of the sum or of the product of two functions, starting from the trigonometric holonomic differential equations they satisfy, respectively. In the two following sections, we will present algorithms to compute those differential equations.

2.6.1 Algorithm for the Sum

The following algorithm computes the differential equation of the sum of two functions of $\mathrm{TH}(\omega)$, from the given trigonometric holonomic differential equations they satisfy.

Algorithm 2.4: Determination of the trigonometric holonomic differential equation satisfied by the sum of the solutions of two trigonometric holonomic differential equations (SumTHDE).

> **input** : Two trigonometric holonomic differential equations DE1 and DE2 of degree P and Q respectively, satisfied by the functions f and g of $\mathrm{TH}(\omega)$
>
> **output**: The trigonometric holonomic differential equation satisfied by the sum $h = f + g$.

1 **begin**

2 $\quad$ Use DE1 and DE2 to deduce the following equations

$$f^{(P)} = \sum_{i=0}^{P-1} a_i f^{(i)} \quad \text{and} \quad g^{(Q)} = \sum_{j=0}^{Q-1} b_j g^{(j)} \tag{2.18}$$

$\quad$ where a_i and b_j are rational functions of $K(\cos(\omega t), \sin(\omega t))$.

3 $\quad$ $N \leftarrow \max\{P, Q\}$.

4 $\quad$ Use (2.18) and recursive substitutions to rewrite the higher derivatives of f and those of g as linear combinations of their respective successive derivatives and obtain trigonometric holonomic differential equations of the form

$$f^{(N)} = \sum_{i=0}^{P-1} a_i^N f^{(i)} \quad (N \geq P) \quad \text{and} \quad g^{(N)} = \sum_{j=0}^{Q-1} b_j^N g^{(j)} \quad (N \geq Q) \tag{2.19}$$

5 $\quad$ $K \leftarrow P + Q + 1$.

6 $\quad$ **for** $I = N$ **to** K **do**

7 $\quad\quad$ Put $h = f + g$ and write the linear combination

$$\sum_{k=0}^{I} A_k h^{(k)} = 0 \quad \text{i.e.} \quad \sum_{k=0}^{I} A_k (f^{(k)} + g^{(k)}) = 0.$$

8 $\quad\quad$ Collect with respect to the successive derivatives of $f(t)$ and $g(t)$ in order to get a system of $P + Q$ linear equations (S), where the unknowns are the $A_k's$.

9 $\quad\quad$ Solve (S) .

10 $\quad\quad$ **if** *(S) has non-vanishing solutions* **then**

11 $\quad\quad\quad$ **return** *the differential equation*

$$\sum_{k=0}^{I} A_k H^{(k)} = 0 \tag{2.20}$$

$\quad\quad\quad$ *satisfied by the sum $h = f + g$.*

12 $\quad\quad$ **end**

13 $\quad$ **end**

14 **end**

Remark 2.39

The previous algorithm terminates: The system (S) has exactly $P + Q$ equations independently of K. In the worst case for $K = P + Q + 1$, the number of the variables K is greater than the number equations $P + Q$. Hence the system (S) is under-determinate, i.e. we will get a non-trivial solution.

Example 2.40

Consider the following trigonometric holonomic differential equations where $\omega = 1$:

$$DE1 : (\cos(t) + 2)F'(t) + \sin(t)F(t) = 0$$

$$DE2 : \sin(t)G(t) - 2\cos(t)G'(t) + (-3 - \sin(t))G''(t) = 0$$

from which we deduce that

$$F'(t) = \frac{-\sin(t)F(t)}{\cos(t) + 2} \quad \text{and}$$

$$G''(t) = \frac{\sin(t)G(t) - 2\cos(t)G'(t)}{3 + \sin(t)} .$$

Using both previous relations and the substitution rules (2.19) with $K = 3$, we get:

$$F'(t) = \frac{-\sin(t)F(t)}{\cos(t) + 2}, \quad F''(t) = \frac{-\cos(t)F(t)}{\cos(t) + 2}, \quad F'''(t) = \frac{\sin(t)F(t)}{\cos(t) + 2} ,$$

$$G''(t) = \frac{\sin(t)G(t) - 2\cos(t)G'(t)}{3 + \sin(t)} ,$$

$$G'''(t) = \frac{(-\sin(2t) + 3\cos(t))G(t) + (3\cos(t)^2 + 3 + 9\sin(t))G'(t))}{\sin(t) + 3} .$$

We set $h = f + g$ and form now the equations

$$\sum_{k=0}^{3} A_k h^{(k)} = 0 \quad i.e. \quad \sum_{k=0}^{3} A_k(f^{(k)} + g^{(k)}) = 0$$

where the A_k are unknowns. Collecting with respect to the successive derivatives of $f(t)$ and $g(t)$, we get:

$$\left(A_0 - \frac{A_1 \sin(t)}{\cos(t) + 2} - \frac{A_2 \cos(t)}{\cos(t) + 2} + \frac{A_3 \sin(t)}{\cos(t) + 2}\right) F(t) + \left(\frac{A_2 \sin(t)}{\sin(t) + 3} + A_0 - \frac{3A_3 \cos(t)\sin(t)}{(\sin(t) + 3)^2} + \right.$$

$$\left. \frac{A_3 \cos(t)}{(\sin(t) + 3)}\right) G(t) + \left(\frac{3A_3 \sin(t)}{\sin(t) + 3} + A_1 - \frac{2A_2 \cos(t)}{\sin(t) + 3} + \frac{6A_3 \cos(t)^2}{(\sin(t) + 3)^2}\right) G'(t) = 0 .$$

Equating coefficients leads to a system of 3 linearly independent homogeneous equations with 4 unknowns. Solving the system and substituting its solution in (2.20), we obtain the differential equation satisfied by the sum of the functions $h = f + g$:

$$2(-\sin(3t)+3\sin(t)+3)H(t)+(14\cos(t)+2\cos(3t)+10-2\cos(2t)+3\sin(2t))H'(t)$$

$$+(-4\sin(2t)-2\sin(3t)+18\cos(2t)-12-28\sin(t))H''(t)$$

$$+(-20\cos(t)-12\sin(t)+2\cos(3t)-15\sin(2t)+2\cos(2t))H'''(t)=0$$

which corresponds to $P=3$ and $L=3$.

Due to the non-uniqueness of the factorization in $\mathbb{K}(\cos t, \sin t)$, the *content* of that differential equation (greatest common divisor of its coefficients) is not well defined. Hence one cannot expect the uniqueness of such a solution, since one cannot divide the coefficients by their content in order to get the simplest form. We note that the previous differential equation for the sum of the functions f and g was found without knowing the functions explicitly. One may verify that the functions

$$f(t)=\cos(t)+2 \quad \text{and} \quad g(t)=\frac{t}{\sin(t)+3}$$

are solutions of DE1 and DE2, respectively. Knowing in advance that

$$h=f+g=\cos(t)+2+\frac{t}{\sin(t)+3}\in \text{TH}(1)$$

and computing a differential equation of the form (2.9) satisfied by h directly via Algorithm 2.1, we get the following differential equation

$$-6\cos(t)H'(t)+(12-11\sin(t))H''(t)+(15-10\sin(t))H^{(4)}(t)+6\cos(t)H^{(5)}(t)+(3+\sin(t))H^{(6)}(t)=0$$

which corresponds this time to $P=6$ and $L=1$.

Example 2.41

Let us now consider the following trigonometric holonomic differential equations:

$$DE1=(4+2\cos(t))F'(t)+\sin(t)F(t) \quad \text{and} \quad DE2=(-2-\sin(t))G'(t)+\cos(t)G(t)$$

satisfied by two functions f and g of $\text{TH}(1)$, respectively. Applying Algorithm 2.4 to $DE1$ and $DE2$ we obtain that the differential equation satisfied by the sum $f+g$ is given by

$$DE=-(\cos(6t)-430+63\cos(2t)-18\cos(4t))F(t)+(-4\cos(5t)+12\cos(3t)-32\cos(4t)$$

$$-776\cos(t)+128\cos(2t)+36\sin(4t)+3\sin(6t)+111\sin(2t)-96-120\sin(3t)$$

$$+24\sin(5t)+1776\sin(t))F'(t)+(-72\sin(3t)+8\sin(5t)-72\cos(3t)+1584\cos(t)+24\cos(5t)$$

$$+60\cos(4t)+944\sin(t)+126\cos(2t)+580-64\sin(2t)+2\cos(6t)+32\sin(4t))F''(t)$$

corresponding to $P=2$ and $L=6$.

2.6.2 Algorithm for the Product

Although for the determination of the trigonometric holonomic differential equation of the product of two functions of $\text{TH}(\omega)$ we may use a method similar as in Algorithm 2.4, we will use in this case an approach involving elimination.

Algorithm 2.5: Determination of the trigonometric holonomic differential equation satisfied by the product of the solutions of two trigonometric holonomic differential equations (ProductTHDE).

---

input : Two trigonometric holonomic differential equations $DE1$ and $DE2$ satisfied by the functions f and g of degree P and Q in $\text{TH}(\omega)$, respectively.

output: The trigonometric holonomic differential equation satisfied by the product $h = fg$.

---

1 **begin**

2   Use $DE1$ and $DE2$ to deduce the following equations

$$f^{(P)} = \sum_{i=0}^{P-1} a_i f^{(i)} \quad \text{and} \quad g^{(Q)} = \sum_{j=0}^{Q-1} b_j g^{(j)}$$

    where a_i and b_j are rational functions of $K(\cos(\omega t), \sin(\omega t))$.

3     $N \leftarrow \max\{P, Q\}$.

4   Build the substitution rules

$$f^{(N)} = \sum_{i=0}^{P-1} a_i^N f^{(i)} \quad (K \geq P) \quad \text{and} \quad g^{(N)} = \sum_{j=0}^{Q-1} b_j^N g^{(j)} \quad (K \geq Q) \qquad (2.21)$$

    Put $h(t) = f(t)g(t)$ and use Leibniz's product rule to compute the successive derivatives $(fg)^{(K)}$ and we get

$$
\begin{aligned}
h &= fg \\
h' &= f'g + g'f \\
&\;\;\vdots \\
h^{(K)} &= \sum_{j=0}^{K} \binom{K}{j} f^{(j)} g^{(K-j)}
\end{aligned}
$$

    which constitute a system of equations with the PQ variables $f^{(j)}g^{(l)}$ $(j = 0, \ldots, P-1, l = 0, \ldots, Q-1)$.

5     $K \leftarrow PQ + 1$.

6   **for** $k = N$ **to** K **do**

7   Build the system (S):

8         $\{h - fg = 0, h' - (f'g + g'f) = 0, h'' - (f''g + 2f'g' + fg'') =$

        $0, \ldots, h^{(k)} - \sum_{j=0}^{k} \binom{k}{j} f^{(j)} g^{(k-j)} = 0\}$ and eliminate the previous variables in (S) in order to get a linear combination of the successive derivatives of h.

9   **if** *the elimination process is successful* **then**

10   **return** *the differential equation satisfied by the product h.*

11   **end**

12   **end**

13 **end**

Remark 2.42

In the worst case $K = PQ$, we will have solutions, because in this case we will have $PQ + 1$ equations in which we want to eliminate PQ variables. Hence the algorithm terminates.

Example 2.43

Consider the following trigonometric holonomic differential equations

$$DE1 : (3 + \cos(2t))F(t) - 2\sin(2t)F'(t) + (1 - \cos(2t))F''(t) = 0$$

$$DE2 : \sin(t)G(t) + (-2 - \cos(t))G(t)' = 0$$

from which we deduce that

$$F''(t) = \frac{-(3 + \cos(2t))F(t) + 2\sin(2t)F'(t)}{(1 - \cos(2t))} \quad \text{and}$$

$$G'(t) = \frac{\sin(t)G(t)}{(2 + \cos(t))} .$$

For $K = 2$ we get

$$G''(t) = \frac{(\cos(t)^2 + 2\cos(t) + 2\sin(t)^2)G(t)}{(\cos(t) + 2)^2} .$$

Let us put $h = fg$ and form the system

$$I = \Big\{ H(t) - F(t)G(t) = 0, H'(t) - F'(t)G(t) + \frac{F(t)\sin(t)G(t)}{\cos(t) + 2} = 0,$$

$$H''(t) - \frac{\big(2\sin(2t)F'(t) - 3F(t) - \cos(2t)F(t)\big)G(t)}{-1 + \cos(2t)}$$

$$- \frac{2\sin(t)F'(t)G(t)}{\cos(t) + 2} - \frac{(\cos(t)^2 + 2\cos(t) + 2\sin(t)^2)F(t)G(t)}{(\cos(t) + 2)^2} = 0 \Big\}$$

which is composed on the one hand of the 2 products $F^{(j)}G^{(l)}$ ($j \in \{0, 1\}, l = 0$), considered as variables and on the other hand of $H^{(k)}$ ($k \in \{0, 1, 2\}$), taking into consideration the substitution rules (2.19). The last step is to use linear elimination, to get rid of the previous variables which leads to the following differential equation

$$-4(\cos(2t) + 3 + 2\cos(t))H(t) + (8\sin(2t) + 8\sin(t))H'(t)$$

$$(-4 + \cos(3t) - \cos(t) + 4\cos(2t))H''(t) = 0$$

corresponding to $P = 2$ and $L = 3$.

One may verify that the product of the following functions

$$f = t\sin(t) \quad \text{and} \quad g = \frac{1}{\cos(t) + 2}$$

is solution of the previous differential equation, although it has been found without any knowledge of them. Knowing in advance that

$$h = fg = \frac{t\sin(t)}{\cos(t) + 2} \in \mathrm{TH}(1) \,,$$

the computation of a trigonometric holonomic differential equation of the form (2.9) satisfied by h using Algorithm 2.1 gives

$$2F(t) + (4 - 4\cos(t))H''(t) - 4\sin(t)H'''(t) + (\cos(t) + 2)H^{(4)}(t) = 0$$

which corresponds to $P = 4$ and $L = 1$.

Corollary 2.44
If f is an ω-trigonometric holonomic function, then $\forall n \in \mathbb{N}$, f^n is also an ω-trigonometric holonomic function.

Remark 2.45
1. The algorithms considered cannot detect a differential equation of smaller order than the given ones, even if such a differential equation is valid. A trivial example of this type is: $g = -f$, and $h = f + g = 0$.

2. From Theorem 2.37 we may deduce that $(\mathrm{TH}, +, \cdot)$ is a commutative unitary ring.

Chapter 3

Fourier Coefficients of Trigonometric Holonomic Functions

In the previous chapter we defined the set of trigonometric holonomic functions, and we gave some of their properties. In this chapter we present a general algorithm for the computation of the complex Fourier coefficients of trigonometric holonomic functions. Then for the particular case of rational trigonometric functions, we investigate conditions under which one may get those coefficients most efficient and we present two algorithms to compute them.

3.1 Hypergeometric Terms and Closed Forms

In the rest of the dissertation we will deal with recurrence equations (for Fourier coefficients), having interest in their solutions. The type of solution in which we are interested is the "closed form". We will make this notion more precise in a moment. Mark van Hoeij presented in [Hoe98] an algorithm to solve recurrence equations in closed form when such solutions exist. That algorithm is a reviewed and improved version of Petkovšek's algorithm [Pet92], see also [CvH06]. Nevertheless in the cases where a closed form solution does not exist, we may return that solution in another form, if possible, rather than not to give any output.

Definition 3.1 (Hypergeometric term)
An expression a_n is called hypergeometric term if the ratio $\frac{a_{n+1}}{a_n}$ represents a rational function in n.

Example 3.2
Consider the function

$$a_n = \frac{\binom{n}{k-1}(n+k)!}{n\binom{n+1}{k}} \; .$$

We get

$$\frac{a_{n+1}}{a_n} = \frac{(n+k+1)n}{n+2} \in \mathbb{K}(n) \quad \text{with} \quad \mathbb{K} = \mathbb{Q}(k) \; .$$

Definition 3.3 (Closed Form)
A function f_n is said to be of closed form if it is equal to a linear combination of a fixed number of hypergeometric terms, see also [PWZ96].

Example 3.4
Consider the sum

$$s_n = \sum_{k=0}^{\left[\frac{n}{3}\right]} \binom{n-2k}{k} \left(\frac{-4}{27}\right)^k ,$$

see Example 9.8 in [Koe98]. Using Zeilberger's algorithm ([Zei90],[Zei91b],[Zei91a]) it can be proved that s_n satisfies the following recurrence equation

$$RE : 9(n+2)s_{n+2} - 3(n+4)s_{n+1} - 2(n+3)s_n = 0 .$$

Solving it via Petkovšek's algorithm [Pet92], we get the following solution $\frac{1}{9}\left(\frac{-1}{3}\right)^n + \frac{2(3n+4)}{9}\left(\frac{2}{3}\right)^n$, which is obviously a linear combination of hypergeometric terms. Thus

$$s_n = \sum_{k=0}^{\left[\frac{n}{3}\right]} \binom{n-2k}{k} \left(\frac{-4}{27}\right)^k = \frac{1}{9}\left(\frac{-1}{3}\right)^n + \frac{2(3n+4)}{9}\left(\frac{2}{3}\right)^n$$

is in closed form, i.e. RE has closed form solutions. Each solution of RE is a linear combination of the two hypergeometric terms $\left(\frac{-1}{3}\right)^n$ and $(3n+4)\left(\frac{2}{3}\right)^n$.

Remark 3.5
Since the complex Fourier coefficients c_n of a given function f are searched for all $n \in \mathbb{Z}$, in the successful cases it will be returned in the following form:

$$c_n = \begin{cases} \text{Closed form I} & \forall n > N \\ \text{Some initial values} & M \leq n \leq N \\ \text{Closed form II} & \forall n < M . \end{cases}$$

where N and M are some integers related to the order of the recurrence equation satisfied by c_n and also to the zeros of the leading and trailing coefficients of that recurrence equation, see [CvH06].

3.2   Holonomic Recurrence Equation for Trigonometric Holonomic Functions

Definition 3.6 (Holonomic recurrence equation)
A recurrence equation c_n is holonomic when it is homogeneous and linear, and has polynomial coefficients $\in \mathbb{K}[n]$, see [Koe06].

Remark 3.7

Since an inhomogeneous recurrence equation with polynomial coefficients whose right-hand side is a hypergeometric term can be converted into a homogeneous recurrence equation with polynomial coefficients, we include in the setting of this thesis these recurrence equations in the class of holonomic recurrence equations.

Example 3.8

Consider the inhomogeneous recurrence equation

$$RE1 : nc_{n-1} + c_n + (n+1)c_{n+1} = n2^n \ .$$

Substituting n by $n+1$ in $RE1$, we get

$$RE2 : (n+1)c_n + c_{n+1} + (n+2)c_{n+2} = (n+1)2^{n+1} \ .$$

$$\frac{RE2}{RE1} \iff \frac{(n+1)c_n + c_{n+1} + (n+2)c_{n+2}}{nc_{n-1} + c_n + (n+1)c_{n+1}} = \frac{(n+1)2^{n+1}}{n2^n} = \frac{2(n+1)}{n} \ .$$

Clearing denominators leads to the homogeneous recurrence equation

$$RE3 : -2n(n+1)c_{n-1} + (n+1)(n-2)c_n + (-3n - 2n^2 - 2)c_{n+1} + n(n+2)c_{n+2} = 0 \ .$$

3.2.1 Conversion of a Trigonometric Holonomic Differential Equation Into a Recurrence Equation

We recall that the set of the trigonometric holonomic functions consists of those functions satisfying relation (2.9), namely

$$\sum_{p=0}^{P} \sum_{l=0}^{L} \left(\alpha_{pl} \cos(l\omega t) + \beta_{pl} \sin(l\omega t) \right) f^{(p)}(t) = 0$$

for appropriate integers $P \geq 1, L \geq 0$ and $\omega \in \mathbb{R}^\star$, where α_{pl} and $\beta_{pl} \in \mathbb{K}$. This may be written in the form

$$\sum_{p=0}^{P} \alpha_{p0} f^{(p)}(t) + \sum_{p=0}^{P} \sum_{l=1}^{L} \left(\alpha_{pl} \cos(l\omega t) + \beta_{pl} \sin(l\omega t) \right) f^{(p)}(t) = 0 \ . \tag{3.1}$$

Theorem 3.9 (DE into RE)

Let $f : [a, b] \to \mathbb{R}$ satisfy a differential equation of the form (3.1). Then the complex Fourier coefficients $c_n^{(p)}$ of the derivatives $f^{(p)}(t)$ satisfy the recurrence equation

$$\sum_{p=0}^{P} 2\alpha_{p0} c_n^{(p)} + \sum_{p=0}^{P} \sum_{l=1}^{L} \left(c_{n+l}^{(p)}(\alpha_{pl} + i\beta_{pl}) + c_{n-l}^{(p)}(\alpha_{pl} - i\beta_{pl}) \right) = 0 \ . \tag{3.2}$$

After replacing $c_{n\pm l}^{(p)}$ for $p > 0$ according to equation (1.14) in terms of $c_{n\pm l}$ this yields a holonomic recurrence equation for the complex Fourier coefficients c_n of $f(t)$, given as

$$\sum_{p=0}^{P} 2\alpha_{p0}(-1)^p \left(\frac{2n\pi}{i(T)}\right)^p \left(c_n - \sum_{j=0}^{p-1}(-1)^j(T)^j(\frac{i}{2n\pi})^{j+1}(f^{(j)}(b) - f^{(j)}(a))e^{-in\omega a}\right) +$$

$$\sum_{p=0}^{P}\sum_{l=1}^{L}\left[(\alpha_{pl} + i\beta_{pl})\left(\frac{2(n+l)\pi i}{T}\right)^p\left(c_{n+l} - \sum_{j=0}^{p-1}(-1)^j(T)^j(\frac{i}{2(n+l)\pi})^{j+1}(f^{(j)}(b) - f^{(j)}(a))e^{-i(n+l)\omega a}\right) +\right.$$

$$(\alpha_{pl} - i\beta_{pl})\left(\frac{2(n-l)\pi i}{T}\right)^p\left(c_{n-l} - \sum_{j=0}^{p-1}(-1)^j(T)^j(\frac{i}{2(n-l)\pi})^{j+1}(f^{(j)}(b) - f^{(j)}(a))e^{-i(n-l)\omega a}\right)\right] = 0 \;.$$

$$(3.3)$$

Proof: Starting from the differential equation (3.1), we will construct two relations which are linear combinations of the real Fourier coefficients of the successive derivatives of the function f. Then we will combine both relations to get a linear combination between the complex Fourier coefficients of the successive derivatives of f, which will be converted into a recurrence equation for the complex Fourier coefficients of f itself.

Multiplying (3.1) by $\frac{2}{T} \cdot \cos(n\omega t)$ and integrating over the interval $[a, b]$ we get:

$$\frac{2}{T}\int_a^b \left(\sum_{p=0}^{P}\alpha_{p0}f^{(p)}(t) + \sum_{p=0}^{P}\sum_{l=1}^{L}(\alpha_{pl}\cos(l\omega t) + \beta_{pl}\sin(l\omega t))f^{(p)}(t)\right)\cos(n\omega t)dt = 0$$

$$\Longleftrightarrow \frac{2}{T}\int_a^b\left[\sum_{p=0}^{P}\alpha_{p0}f^{(p)}(t)\cos(n\omega t) + \sum_{p=0}^{P}\sum_{l=1}^{L}(\alpha_{pl}\cos(l\omega t)\cos(n\omega t) + \right.$$

$$\beta_{pl}\sin(l\omega t)\cos(n\omega t))f^{(p)}(t)\bigg]dt = 0 \;.$$

Using the trigonometric addition theorems (2.1)–(2.4) the previous equation becomes:

$$\frac{2}{T}\int_a^b\left[\sum_{p=0}^{P}\alpha_{p0}f^{(p)}(t)\cos(n\omega t) + \sum_{p=0}^{P}\sum_{l=1}^{L}\left(\frac{1}{2}\alpha_{pl}\left(\cos((n+l)\omega t) + \cos((n-l)\omega t)\right) + \right.\right.$$

$$\frac{1}{2}\beta_{pl}\left(\sin((n+l)\omega t) - \sin((n-l)\omega t)\right)\bigg)f^{(p)}(t)\bigg]dt = 0$$

which leads to the relation

$$\sum_{p=0}^{P}\alpha_{p0}\frac{2}{T}\int_a^b f^{(p)}(t)\cos(n\omega t)dt+$$

$$\sum_{p=0}^{P}\sum_{l=1}^{L}\left[\frac{1}{2}\alpha_{pl}\left(\frac{2}{T}\int_a^b f^{(p)}(t)\cos((n+l)\omega t)dt + \frac{2}{T}\int_a^b f^{(p)}(t)\cos((n-l)\omega t)dt\right) + \right.$$

$$\frac{1}{2}\beta_{pl}\left(\frac{2}{T}\int_a^b f^{(p)}(t)\sin((n+l)\omega t)dt - \frac{2}{T}\int_a^b f^{(p)}(t)\sin((n-l)\omega t)dt\right)\right] = 0$$

and finally we obtain:

$$\sum_{p=0}^P \alpha_{p0} a_n^{(p)}(f) + \sum_{p=0}^P \sum_{l=1}^L \left[\frac{1}{2}\alpha_{pl}\left(a_{n+l}^{(p)}(f) + a_{n-l}^{(p)}(f)\right) + \frac{1}{2}\beta_{pl}\left(b_{n+l}^{(p)}(f) - b_{n-l}^{(p)}(f)\right)\right] = 0. \quad (3.4)$$

Hence we get an identity between the sine Fourier coefficients $b_{n+l}^{(p)}(f)$ and the cosine Fourier coefficients $a_{n+l}^{(p)}(f)$ of the successive derivatives of a function f.

Starting from the same relation (3.1) and using the same process as previously, but multiplying this time by $\frac{2}{b-a}\cdot\sin(n\omega t)$ instead of $\frac{2}{b-a}\cdot\cos(n\omega t)$ we get:

$$\frac{2}{b-a}\int_a^b\left(\sum_{p=0}^P \alpha_{p0}f^{(p)}(t) + \sum_{p=0}^P\sum_{l=1}^L (\alpha_{pl}\cos(l\omega t) + \beta_{pl}\sin(l\omega t))\,f^{(p)}(t)\right)\sin(n\omega t)dt = 0$$

which leads to the following second relation which is again a linear combination of the sine Fourier coefficients $b_{n+l}^{(p)}(f)$ and the cosine Fourier coefficients $a_{n+l}^{(p)}(f)$ of the successive derivatives of the function f

$$\sum_{p=0}^P \alpha_{p0} b_n^{(p)}(f) + \sum_{p=0}^P \sum_{l=1}^L \left[\frac{1}{2}\alpha_{pl}\left(b_{n+l}^{(p)}(f) + b_{n-l}^{(p)}(f)\right) + \frac{1}{2}\beta_{pl}\left(-a_{n+l}^{(p)}(f) + a_{n-l}^{(p)}(f)\right)\right] = 0. \quad (3.5)$$

Aiming to find a recurrence equation for the complex Fourier coefficients

$$c_n(f) = \frac{1}{2}(a_n(f) - ib_n(f))$$

of the function f, we put together the relations (3.4)and (3.5) in the way

$$(3.4) - i(3.5)$$

which leads after some simplifications to (3.2) and this finishes the first part of the proof.

Let us convert (3.2) into a recurrence equation for the complex Fourier coefficients of f. From (1.14) we deduce that

$$c_{n-l}^{(p)} = \left(\frac{2(n-l)\pi i}{T}\right)^p\left(c_{n-l} - \sum_{j=0}^{p-1}(-1)^j(T)^j\;(\frac{i}{2(n-l)\pi})^{j+1}(f^{(j)}(b) - f^{(j)}(a))e^{-i(n-l)\omega a}\right) \quad (3.6)$$

and that

$$c_{n+l}^{(p)} = \left(\frac{2(n+l)\pi i}{T}\right)^p\left(c_{n+l} - \sum_{j=0}^{p-1}(-1)^j(T)^j\;(\frac{i}{2(n+l)\pi})^{j+1}(f^{(j)}(b) - f^{(j)}(a))e^{-i(n+l)\omega a}\right) \quad (3.7)$$

The substitution of (3.6) and (3.7) in (3.2) leads to (3.3) which is a holonomic recurrence equation satisfied by the complex Fourier coefficients of f. $\qquad\square$

Example 3.10

Consider the function defined by

$$f(t) = \frac{t}{2 + \sin(t)} \in \mathrm{TH}(1) \ .$$

f satisfies the trigonometric holonomic differential equation

$$\sin(t)F(t) - 2\cos(t)F'(t) + (-2 - \sin(t))F''(t) = 0 \ . \tag{3.8}$$

Retrieving the coefficients α_{pl} and β_{pl} for which DE is of the form (2.9) and substituting them in (3.2), we get the recurrence equation

$$- ic_{n-1} + ic_{n+1} - 2c'_{n-1} - 2c'_{n+1} - 4c''_n + ic''_{n-1} - ic''_{n+1} = 0 \tag{3.9}$$

which is a linear combination of the complex Fourier coefficients of the successive derivatives of f. Using (3.6) and (3.7), we obtain:

$$
\begin{aligned}
c'_{n-1} &= i(n-1)c_{n-1} + \frac{1}{2}, \quad c'_{n+1} = i(n-1)c_{n+1} + \frac{1}{2} \\
c''_n &= -n^2 c_n + \frac{2ni - 1}{4}, \quad c''_{n-1} = -(n-1)^2 c_{n-1} + \frac{2(n-1)i - 1}{4} \\
c''_{n+1} &= -(n+1)^2 c_{n-1} + \frac{2(n+1)i - 1}{4} \ .
\end{aligned}
$$

Substituting the previous values in (3.9), we deduce that the complex Fourier coefficients of the function f satisfy the inhomogeneous second order recurrence equation

$$-in^2 c_{n-1} + 4n^2 c_n + in^2 c_{n+1} = 2in \ .$$

This is what (3.3) gives in one step.

Remark 3.11

In practice, to convert a differential equation of the form (2.9) into a recurrence equation, we substitute the coefficients α_{pl} and β_{pl} into (3.3) to get the recurrence equation satisfied by the Fourier coefficients of the considered function directly. We summarize this process in the following algorithm.

Algorithm 3.1: Conversion of a trigonometric holonomic differential equation into a recurrence equation (THDEtoRE)

---

input : An interval $[a, b]$ with $\omega = \frac{2\pi}{b-a}$, a linear differential equation DE with coefficients in $\mathbb{K}[\cos(\omega t), \sin(\omega t)]$ and a function f solution of DE.

output: A recurrence equation for the complex Fourier coefficients of the function f on an interval of period $T = \frac{2\pi}{\omega}$.

1 **begin**
2 $\quad$ Retrieve from DE the coefficients α_{pl} and β_{pl} for which DE is of the form (2.9). Substitute those coefficients in (3.3) to get the recurrence equation satisfied by the complex Fourier coefficients of f.
3 $\quad$ Return the recurrence equation satisfied by the complex Fourier coefficients of f.
4 **end**

Example 3.12

Consider the differential equation of Example 3.10 (page 50).

$$DE := \sin(t)F(t) - 2\cos(t)F'(t) + (-2 - \sin(t))F''(t) = 0 \,,$$

where $\omega = 1$, one of whose solutions is the function defined on the interval $[0, 2\pi]$ by

$$f(t) = \frac{t}{2 + \sin(t)} \ .$$

The coefficients α_{pl} and β_{pl} for which DE is of the form (2.9) are

$$\alpha_{00} = 0, \alpha_{01} = 0, \alpha_{10} = 0, \alpha_{11} = -2, \alpha_{20} = -2, \alpha_{21} = 0$$

and

$$\beta_{00} = 0, \beta_{01} = 1, \beta_{10} = 0, \beta_{11} = 0, \beta_{20} = 0, \beta_{21} = -1 \ .$$

Substituting the previous coefficients and f in (3.3) we get the following recurrence equation

$$-in^2 c_{n-1} + 4n^2 c_n + in^2 c_{n+1} = 2in$$

again. Unfortunately the previous recurrence equation does not have any solution in closed form which is proved by the Petkovšek-Van Hoeij algorithm.

Remark 3.13

Depending on the input function and the considered interval, Theorem 3.9 does not always return a holonomic recurrence equation, namely if the right-hand side is not a hypergeometric term. In the setting of this thesis we will restrict ourselves to the case of holonomic recurrence equations. Such recurrence equations are always obtained if the input function is periodic and the length of the considered interval is one period.

3.3   Fourier Coefficients of *Exp-Like* Functions

We devote this section to the computation of the Fourier coefficients of *exp-like* functions. We have seen in Section 2.3.2 that those functions satisfy trigonometric holonomic differential equations with constant coefficients, corresponding to $L = 0$. The particularity here is that such differential equations lead to recurrence equations of order zero. These recurrence equations are obtained by substituting L by zero in the equation (3.3). Of course the Fourier coefficients of those functions can also be directly computed via definitions (2)–(4), but we present here an algorithmic approach of that computation.

Example 3.14
Consider the function defined on the interval $[0, 2l]$ by

$$f(t) = t^3 .$$

f satisfies the trigonometric holonomic differential equation with constant coefficients.

$$DE : F^{(4)}(t) = 0 .$$

Using Algorithm 3.1 DE is converted to the following recurrence of order zero

$$RE : 2n^4\pi^4 c_n - 4i(2\pi^2 n^2 - 3in\pi - 3)n\pi l^3 = 0 .$$

from wich we deduce that the algorithmic computation of the Fourier coefficients of f are

$$c_n = \begin{cases} 2l^3 & \text{if } n = 0 \\ \frac{4il^3\pi^2 n^2 + 6l^3\pi n - 6il^3}{(\pi n)^3} & \text{otherwise} . \end{cases}$$

Example 3.15
Consider now the function defined on the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ by:

$$f(t) = t^2 \cos(t)(e^{it})^5$$

f satisfies the trigonometric holonomic differential equation

$$DE : 13824F(t) + 17280iF'(t) - 8928F''(t) - 2440iF'''(t) + 372F^{(4)}(t) + 30iF^{(5)}(t) - F^{(6)}(t) = 0 .$$

Converted into a recurrence equation, DE gives

$$RE : 4(n-2)^3(n-3)^3 c_n + 30(-1)^{n+1} + 37(-1)^n n + 15(-1)^{n+1}n^2 + 2(-1)^n n^3 = 0$$

which is an inhomogeneous holonomic recurrence equation of order zero, from which we deduce that the complex Fourier coefficients of f on the interval $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ are given as:

$$c_n = \begin{cases} \frac{-1}{4} + \frac{\pi^2}{24} & \text{if } n = 2 \vee n = 3. \\ \frac{-2(-1)^n n + 5(-1)^n}{4n^4 - 40n^3 + 148n^2 - 240n + 144} & \text{otherwise} \end{cases}$$

3.4 General Algorithm for the Computation of the Fourier Coefficients of Trigonometric Holonomic Functions

Although we will emphasize further on the rational trigonometric functions, we give here an algorithm for the computation in the general case of the complex Fourier coefficients of trigonometric holonomic functions.

Algorithm 3.2: Computation of the Fourier coefficients of a trigonometric holonomic function (FouCoeffTH)

---

    input : An ω-trigonometric holonomic function and an interval $I = [a, b]$ such that $\omega = \frac{2\pi}{b-a}$.

    output: Either the complex Fourier coefficients of f on I, or RE and enough initial values.

1 **begin**
2   Apply Algorithm 2.1 to f on I to get a trigonometric holonomic differential equation DE satisfied by f.
3   Apply Algorithm 3.1 to DE to convert it into a recurrence equation RE.
4   **if** RE *is not holonomic* **then**
5   **return** RE *and enough initial values.*
6   **end**
7   Solve RE with appropriate initial values.
8   **if** *solving the previous RE is successful* **then**
9   **return** *the complex Fourier coefficients of f on I.*
10   **else**
11   **return** RE *and enough initial values.*
12   **end**
13 **end**

Remark 3.16
The order N of the recurrence equation (3.3) depends only on L, namely $N = 2L$. Hence in the search for a differential equation of the form (2.9), we should keep L as low as possible. In the next section, we will emphasize on the case of rational trigonometric functions. We will optimize P and L so that the obtained recurrence equation is of smallest possible order.[1]

---

[1]Because of the non-uniqueness of the factorization in $\mathbb{K}(\cos t, \sin t)$, the method and therefore the best choice of L and P depends on the representation of $f(t)$.

3.5 Rational Trigonometric Functions

3.5.1 First Algorithm for Rational Trigonometric Functions

In this section we present an algorithm for the computation of the complex Fourier coefficients in the particular case of rational trigonometric functions. This algorithm may get directly a recurrence equation satisfied by the Fourier coefficients of the input function, without involving differential equations.

W.l.o.g. we set $\omega = 1$ and consider the function

$$f(t) \in \mathbb{K}(\cos(t), \sin(t)) \ .$$

We write $Z = e^{it}$. Then $f(t)$ can be rewritten in the form

$$F(Z) = f(t) = \frac{\sum_{k=0}^{M} p_k Z^k}{\sum_{k=0}^{N} q_k Z^k} = \frac{p(Z)}{q(Z)} \tag{3.10}$$

assuming $\gcd(p, q) = 1$, via the transformations $\cos(t) = \frac{1}{2}(Z + \frac{1}{Z})$ and $\sin(t) = \frac{1}{2i}(Z - \frac{1}{Z})$. Since $Z = e^{it}$, (3.10) may be written in the form

$$F(Z) = f(t) = \frac{\sum_{k=0}^{M} p_k e^{ikt}}{\sum_{k=0}^{N} q_k e^{ikt}} \ .$$

Clearing denominator leads to

$$f(t) \sum_{k=0}^{N} q_k e^{ikt} = \sum_{k=0}^{M} p_k e^{ikt} \ .$$

Multiplying by $\frac{1}{2\pi} e^{-int}$ we get

$$\sum_{k=0}^{N} q_k \frac{1}{2\pi} e^{-i(n-k)t} f(t) = \sum_{k=0}^{M} p_k \frac{1}{2\pi} e^{-i(n-k)t} \tag{3.11}$$

Aiming to get a holonomic recurrence equation for $c_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int} dt$, let us integrate (3.11) over the interval $[0, 2\pi]$ and we have

$$\sum_{k=0}^{N} q_k \frac{1}{2\pi} \int_0^{2\pi} e^{-i(n-k)t} f(t) dt = \sum_{k=0}^{M} \frac{1}{2\pi} \int_0^{2\pi} p_k e^{-i(n-k)t} dt$$

which leads to the recurrence equation

$$\sum_{k=0}^{N} q_k c_{n-k} = 0 \quad \text{for} \quad n \leq 0 \quad \text{or} \quad n \geq M \tag{3.12}$$

and

$$\sum_{k=0}^{N} q_k c_{n-k} = \sum_{k=0}^{M} p_k \quad \text{for} \quad 0 \leq n \leq M \ . \tag{3.13}$$

We summarize the previous computations in the following algorithm.

Algorithm 3.3: First algorithm for the computation of the Fourier coefficients of rational trigonometric holonomic functions (1FouCoeffRatTH)

---

    input : A function $f \in \mathbb{K}(\cos(t), \sin(t))$
    output: Either the complex Fourier coefficients of f on I, or RE and enough initial values.

1 **begin**
2   Substitute $\cos(t) = \frac{1}{2}(Z + \frac{1}{Z})$ and $\sin(t) = \frac{1}{2i}(Z - \frac{1}{Z})$ in f, bring f into rational normal form (3.10), read off M, N , and the coefficients p_k, q_k.
3   Deduce the following recurrence equation RE:

$$\sum_{k=0}^{N} q_k c_{n-k} = 0 \quad \text{for} \quad n \leq 0 \quad \text{or} \quad n \geq M \tag{3.14}$$

     and
$$\sum_{k=0}^{N} q_k c_{n-k} = \sum_{k=0}^{M} p_k \quad \text{for} \quad 0 \leq n \leq M \tag{3.15}$$

     Solve RE with N initial values in the positiv direction and with N initial values in the negativ direction.
4   **if** *Solving the previous RE is successful* **then**
5   **return** *the complex Fourier coefficients of f on I in the form*

$$c_n = \begin{cases} \text{\textit{Closed form I}} & \forall n > M \\ \text{\textit{Some initial values}} & 0 \leq n \leq M \\ \text{\textit{Closed form II}} & \forall n < 0 \ . \end{cases}$$

6   **else**
7   **return** *RE and enough initial values.*
8   **end**
9 **end**

We may deduce from the previous computation the following

Theorem 3.17

The complex Fourier coefficients of rational trigonometric functions satisfy a linear recurrence equation with constant coefficients, except for a finite number of initial values $0 \leq n \leq M$. In particular these Fourier coefficients are of closed form for all $n \leq 0$ and also for all $n \geq M$.

Proof: Algorithm 3.3 produces recurrence equations with constant coefficients. Such recurrence equations can be always solved in closed form using enough initial values, see e.g. [Sta00] in the chapter on rational generating functions.   □

Remark 3.18

Recurrence equations (3.14) and (3.15) can be brought into a single holonomic recurrence equation for all $n \in \mathbb{Z}$ by multiplying (3.14) with $n(n-1) \cdots (n-M)$ and we get

$$\sum_{k=0}^{N} \prod_{i=0}^{M} (n-i) q_k c_{n-k} = 0 \ .$$

Example 3.19

1. Consider the function defined on the interval $[0, 2\pi]$ by

$$f(t) = \frac{1}{2 + \cos(t)} \ .$$

   Algorithm 3.3 produces the recurrence equation

$$RE := c_{n-2} + 4 c_{n-1} + c_n = 0 \ .$$

   Solving RE we get the closed form

$$c_n = \begin{cases} \frac{\sqrt{3}}{3} \left(\sqrt{3} - 2 \right)^n & \text{if } n \geq 0 \\ \frac{\sqrt{3}}{3} \left(-\sqrt{3} - 2 \right)^n & \text{if } n \leq 0 \ . \end{cases}$$

   which are the complex Fourier coefficients of f on the interval $[0, 2\pi]$.

2. Consider now the function
$$g(t) = \frac{1}{(2 + \cos(t))^{20}} \ .$$

   Using the previous Algorithm 3.3, we get the following recurrence equation of order 40

$$c_{n-40} + 80 c_{n-39} + 3060 c_{n-38} + 74480 c_{n-37} + 1295230 c_{n-36} + 17130096 c_{n-35}$$

$$+ 179072340 c_{n-34} + 1518229200 c_{n-33} + 10629547245 c_{n-32} + 62257759040 c_{n-31}$$

$$+ 307948690960 c_{n-30} + 1295267106240 c_{n-29} + 4655673046120 c_{n-28}$$

$$+ 14349560273600 c_{n-27} + 14349560273600 c_{n-27} + 38010466639440 c_{n-26}$$

$$+86645955115584c_{n-25} + 170074452183570c_{n-24} + 287490594872160c_{n-23}$$

$$+418426332826200c_{n-22} + 524194439193120c_{n-21} + 565107853947444c_{n-20}$$

$$+524194439193120c_{n-19} + 418426332826200c_{n-18} + 287490594872160c_{n-17}$$

$$+170074452183570c_{n-16} + 86645955115584c_{n-15} + 38010466639440c_{n-14}+$$

$$14349560273600c_{n-13} + 4655673046120c_{n-12} + 1295267106240c_{n-11}$$

$$+307948690960c_{n-10} + 62257759040c_{n-9} + 10629547245c_{n-8} + 1518229200c_{n-7}$$

$$+179072340c_{n-6} + 17130096c_{n-5} + 1295230c_{n-4} + 74480c_{n-3}$$

$$+3060c_{n-2} + 80c_{n-1} + c_n = 0 \ .$$

A high order recurrence equation has a direct consequence on the time consumption of the computation of initial values. In the next section we will present an algorithm which will search for the best values of P and L for which we may get the recurrence equation of lowest order for the particular case of rational trigonometric functions. For the previous example instead of a recurrence equation of order 40 this algorithm will return the following second order recurrence equation

$$i(n+19)c_{n-1} + 4inc_n + i(-19+n)c_{n+1} = 0 \ ,$$

so that only two initial values are needed.

3.5.2 Efficient Computation of P and L

W.l.o.g. we set $\omega = 1$. We deal in this section with functions of $\mathbb{K}(\cos(t), \sin(t))$, which are elements of the ring $\mathbb{K}[\cos(t), \sin(t)]/\langle \cos(t)^2 + \sin(t)^2 - 1\rangle$ which is not a unique factorization domain. Jamie Mulholland and Michael Monagan presented in [MM01] algorithms for simplifying ratios of trigonometric polynomials, and algorithms for dividing, factoring and computing common divisors of trigonometric polynomials. The output of the algorithm we will present in this section depends on the form in which the input rational trigonometric function is given in $\mathbb{K}[\cos(t), \sin(t)]$, which in some case is not in the simplest form according to our purpose. Thus it may happen in some cases that the output is not the optimal one. We assume in this section that a function $f(t)$ of $\mathbb{K}(\cos t, \sin t)$ is given in the following form

$$f(t) = \frac{\displaystyle\prod_{i=0}^{I} [A_i(\cos(t), \sin(t))]^{\alpha_i}}{\displaystyle\prod_{j=0}^{J} [B_j(\cos(t), \sin(t))]^{\beta_j}}, \quad B_j(\cos(t), \sin(t)) \neq 0, \quad (j = 0, \dots, J).$$

where $A_i(\cos(t), \sin(t))$ and $B_j(\cos(t), \sin(t))$ are trigonometric polynomial functions of degrees n_i and m_j respectively.

It is easy to prove that the pth derivative of the function f may be given by the relation

$$f^{(p)}(t) = \left(\frac{\prod\limits_{i=0}^{I} [A_i(\cos(t), \sin(t))]^{\alpha_i}}{\prod\limits_{j=0}^{J} [B_j(\cos(t), \sin(t))]^{\beta_j}} \right)^{(p)} = \frac{C_p(\cos(t), \sin(t))}{\prod\limits_{j=0}^{J} [B_j(\cos(t), \sin(t))]^{\beta_j+p}}$$

where $C_p(\cos(t), \sin(t)) \in \mathbb{K}[\cos(t), \sin(t)]$ of degree $p \sum\limits_{j=0}^{J} m_j + \sum\limits_{i=0}^{I} \alpha_i n_i$.

For simplification reasons, we omit the arguments so that we have

$$f^{(p)}(t) = \left(\frac{\prod\limits_{i=0}^{I} A_i^{\alpha_i}}{\prod\limits_{j=0}^{J} B_j^{\beta_j}} \right)^{(p)} = \frac{C_p}{\prod\limits_{j=0}^{J} B_j^{\beta_j+p}} \quad \text{with} \quad \text{degree}(C_p) = p \sum\limits_{j=0}^{J} m_j + \sum\limits_{i=0}^{I} \alpha_i n_i \ .$$

Since f is solution of the differential equation (2.9) then

$$\sum_{p=0}^{P} \sum_{l=0}^{L} (\alpha_{pl} \cos(lt) + \beta_{pl} \sin(lt)) f^{(p)}(t) = 0$$

$$\Longleftrightarrow \sum_{p=0}^{P} \sum_{l=0}^{L} (\alpha_{pl} \cos(lt) + \beta_{pl} \sin(lt)) \left(\frac{\prod\limits_{i=0}^{I} A_i^{\alpha_i}}{\prod\limits_{j=0}^{J} B_j^{\beta_j}} \right)^{(p)} = 0$$

$$\Longleftrightarrow \sum_{p=0}^{P} \sum_{l=0}^{L} (\alpha_{pl} \cos(lt) + \beta_{pl} \sin(lt)) \frac{C_p}{\prod\limits_{j=0}^{J} B_j^{\beta_j+p}} = 0$$

$$\Longleftrightarrow \frac{\sum\limits_{l=0}^{L} (\alpha_{0l} \cos(lt) + \beta_{0l} \sin(lt)) C_0 \prod\limits_{j=0}^{J} B_j^{P} + \sum\limits_{l=0}^{L} (\alpha_{1l} \cos(lt) + \beta_{pl} \sin(lt)) C_1 \prod\limits_{j=0}^{J} B_j^{P-1} + \cdots}{\prod\limits_{j=0}^{J} B_j^{\beta_j+P}}$$

$$\frac{\cdots + \sum\limits_{l=0}^{L} (\alpha_{kl} \cos(lt) + \beta_{kl} \sin(lt)) C_k \prod\limits_{j=0}^{J} B_j^{P-k} + \cdots + \sum\limits_{l=0}^{L} (\alpha_{Pl} \cos(lt) + \beta_{Pl} \sin(lt)) C_P}{\prod\limits_{j=0}^{J} B_j^{\beta_j+P}} = 0$$

$$\Longleftrightarrow$$

$$\sum_{l=0}^{L} (\alpha_{0l} \cos(lt) + \beta_{0l} \sin(lt)) C_0 \prod_{j=0}^{J} B_j^{P} + \sum_{l=0}^{L} (\alpha_{1l} \cos(lt) + \beta_{pl} \sin(lt)) C_1 \prod_{j=0}^{J} B_j^{P-1} + \cdots$$

$$\cdots + \sum_{l=0}^{L} \left(\alpha_{kl} \cos(lt) + \beta_{kl} \sin(lt) \right) C_k \prod_{j=0}^{J} B_j^{P-k} + \cdots + \sum_{l=0}^{L} \left(\alpha_{Pl} \cos(lt) + \beta_{Pl} \sin(lt) \right) C_P \ . \tag{3.16}$$

Let us investigate an arbitrary term of the relation (3.16), namely the expression

$$\sum_{l=0}^{L} \left(\alpha_{kl} \cos(lt) + \beta_{kl} \sin(lt) \right) C_k \prod_{j=0}^{J} B_j^{P-k} \ .$$

We get:

$$\sum_{l=0}^{L} \left(\alpha_{kl} \cos(lt) + \beta_{kl} \sin(lt) \right) C_k \prod_{j=0}^{J} B_j^{P-k} =$$

$$\alpha_{k0} C_k \prod_{j=0}^{J} B_j^{P-k} + \cdots + \left(\alpha_{kL} \cos(Lt) + \beta_{kL} \sin(Lt) \right) C_k \prod_{j=0}^{J} B_j^{P-k}$$

Since C_k is of degree $\sum_{i=0}^{I} \alpha_i n_i + k \sum_{j=0}^{J} m_j$ and $\prod_{j=0}^{J} B_j^{P-k}$ are of degree $(P-k) \sum_{j=0}^{J} m_j$, then the expression $\cos(Lt) C_k \prod_{j=0}^{J} B_j^{P-k}$ and $\sin(Lt) C_k \prod_{j=0}^{J} B_j^{P-k}$ can be rewritten in terms of Fourier polynomial functions of degree

$$L + \sum_{i=0}^{I} \alpha_i n_i + k \sum_{j=0}^{J} m_j + (P-k) \sum_{j=0}^{J} m_j = L + \sum_{i=0}^{I} \alpha_i n_i + k \sum_{j=0}^{J} m_j + P \sum_{j=0}^{J} m_j - k \sum_{j=0}^{J} m_j =$$

$$= L + \sum_{i=0}^{I} \alpha_i n_i + P \sum_{j=0}^{J} m_j \ .$$

Collecting (3.16) with respect to $\cos(kt)$, k running from 0 to L, we obtain a maximum of $L + \sum_{i=0}^{I} \alpha_i n_i + P \sum_{j=0}^{J} m_j + 1$ expressions, and doing the same thing with respect to $\sin(kt)$, k running from 0 to L, we obtain also a maximum of $L + \sum_{i=0}^{I} \alpha_i n_i + P \sum_{j=0}^{J} m_j$ expressions. We get that (3.16) gives us a system of maximum

$$L + \sum_{i=0}^{I} \alpha_i n_i + P \sum_{j=0}^{J} m_j + L + \sum_{i=0}^{I} \alpha_i n_i + P \sum_{j=0}^{J} m_j + 1 = 2L + 2 \sum_{i=0}^{I} \alpha_i n_i + 2P \sum_{j=0}^{J} m_j + 1$$

equations, while the number of variables remains the same, namely $2(P+1)(L+1)$. To be sure that the obtained system of equations will get solutions, we should look for conditions on L and P

such that the number of variables is always greater than the one of equations, assuring by the same way the non-triviality of the solutions. We get

$$2(L+1)(P+1) > 2L + 2\sum_{i=0}^{I} \alpha_i n_i + 2P\sum_{j=0}^{J} m_j + 1$$

$$\Longleftrightarrow 2(LP + L + P + 1) > 2L + 2\sum_{i=0}^{I} \alpha_i n_i + 2P\sum_{j=0}^{J} m_j + 1$$

$$\Longleftrightarrow 2LP > 2\sum_{i=0}^{I} \alpha_i n_i + 2P\sum_{j=0}^{J} m_j - 2P - 1$$

and finally

$$L > \frac{2\sum_{i=0}^{I} \alpha_i n_i + 2P\sum_{j=0}^{J} m_j - 2P - 1}{2P} \ . \tag{3.17}$$

Choosing L and P such that (3.17) is satisfied assures the existence of non-vanishing solutions of the system of equations obtained from (3.16). But our aim is to optimize that choice such that L is as small as possible. For simplification reasons, let us put $N = 2\sum_{i=0}^{I} \alpha_i n_i$ and $M = \sum_{j=0}^{J} m_j$, then (3.16) becomes

$$L > \frac{2P(M-1) + N - 1}{2P} \ .$$

- **Efficient Choice of P and L**

Since we want to keep L as small as possible while P can be as large as necessary, let us now consider the function h defined by:

$$h(P) = \frac{2P(M-1) + N - 1}{2P} \ . \tag{3.18}$$

We have: $\lim_{P \to \infty} h(P) = M - 1 = \sum_{j=0}^{J} m_j - 1$. Since $L > h(P)$ there is no guarantee for (3.16) to have solutions if $L \le \sum_{j=0}^{J} m_j - 1$. Our optimal choice of L such that we can be sure to get a solution for P is $L = M = \sum_{j=0}^{J} m_j$.

- **Optimal Choice of P**

We now want to look for the smallest value of P, for which choosing $L = M$, (3.16) should get a solution. This is the case when the number of variables is larger than the number of equations in (3.16), i.e.

$$2(L+1)(P+1) > 2L + 2\sum_{i=0}^{I} \alpha_i n_i + 2P \sum_{j=0}^{J} m_j + 1$$

$$\iff 2(M+1)(P+1) > 2M + N + 2MP + 1$$

$$\iff P > \frac{N-1}{2} \iff P \geq \frac{N-1}{2} + 1 = \frac{N+1}{2}$$

$$P \geq \frac{2\sum_{i=0}^{I} \alpha_i n_i + 1}{2} .$$

Since P is an entire number, the smallest P corresponding to the optimal $L = M$ is the entire part of $\frac{2\sum_{i=0}^{I} \alpha_i n_i + 1}{2}$, i.e. $P = E\left[\frac{1}{2} + \sum_{i=0}^{I} \alpha_i n_i\right] = 1 + \sum_{i=0}^{I} \alpha_i n_i$. We can summarize the previous result in the following theorem.

Theorem 3.20
Let f be the rational trigonometric function given in the form:

$$f(t) = \frac{\prod_{i=0}^{I} [A_i(\cos(t), \sin(t))]^{\alpha_i}}{\prod_{j=0}^{J} [B_j(\cos(t), \sin(t))]^{\beta_j}}, \quad B_j(\cos(t), \sin(t)) \neq 0, \quad (j = 0, \ldots, J).$$

where $A_i(\cos(t), \sin(t))$ and $B_j(\cos(t), \sin(t))$ are trigonometric polynomial functions of degree n_i and m_j respectively. The optimal choice of P and L for which the complex Fourier coefficients of f satisfy a recurrence equation of lowest order (depending on the given representation of $f(t)$) is $P = 1 + \sum_{i=0}^{I} \alpha_i n_i$ and $L = \sum_{j=0}^{J} m_j$, i.e. there is no guarantee to obtain a differential equation of the form (2.9) if $L < \sum_{j=0}^{J} m_j$, and even when $L = \sum_{j=0}^{J} m_j$ there is also no guarantee if $P < 1 + \sum_{i=0}^{I} \alpha_i n_i$.

Let us summarize the previous result in the following algorithm:

Algorithm 3.4: Determination of the efficient P and L (EfficientPL)

---

input : A rational trigonometric function in the following form:

$$f(t) = \frac{\prod_{i=0}^{I} [A_i(\cos(\omega t), \sin(\omega t))]^{\alpha_i}}{\prod_{j=0}^{J} [B_j(\cos(\omega t), \sin(\omega t))]^{\beta_j}}, \quad B_j(\cos(\omega t), \sin(\omega t)) \neq 0, \quad (j = 0, \ldots, J).$$

where $A_i(\cos(\omega t), \sin(\omega t))$ and $B_j(\cos(\omega t), \sin(\omega t))$ are trigonometric polynomial functions.

output: The pair (P, L).

1 **begin**

2 | Determine I, the degree n_i of the numerators $A_i(\cos(\omega t), \sin(\omega t))$, $\quad (i = 0, \ldots, I)$ and their multiplicity α_i.

3 | Determine J, the degree m_j of the denominators $B_j(\cos(\omega t), \sin(\omega t))$, $\quad (j = 0, \ldots, J)$

4 | Apply Algorithm 3.1 to DE to convert it into a recurrence equation RE.

5 | Deduce the numbers P and L as in the following: $L = \sum_{j=0}^{J} m_j$, $P = 1 + \sum_{i=0}^{I} \alpha_i n_i$.

6 | Return the pair (P, L).

7 **end**

Example 3.21

1. Consider the function from Example 3.19 (page 56) defined by:

$$f(t) = \frac{1}{(2 + \cos(t))^{20}} \ .$$

We have: $I = 0, n_0 = 0, \alpha_0 = 0, J = 0, m_0 = 1$.

So we deduce that the best choice for L is $L = \sum_{j=0}^{J} m_j = 1$ and the best choice of P for this L is

$P = 1 + \sum_{i=0}^{0} \alpha_i n_i = 1$. For those choices we obtain the recurrence equation

$$i(n + 19)c_{n-1} + 4inc_n + i(-19 + n)c_{n+1} = 0 \ .$$

2. Consider now the function defined by

$$g(t) = \frac{\cos(t)(1 + \sin^2(t))^2}{(2 + \cos^2(t))^3 (3 + \sin^2(t))^2} \ .$$

In this case we have: $I = 1, n_0 = 1, \alpha_0 = 1, n_1 = 2, \alpha_1 = 2$ $J = 1, m_0 = 2, m_1 = 2$. So we deduce that $P = 1 + \sum_{i=0}^{1} \alpha_i n_i = 1 + 1 + 2 \cdot 2 = 6$ and $L = \sum_{j=0}^{1} m_j = 2 + 2 = 4$.

The corresponding recurrence equation of order 8 is

$$-(2n^4 + 8n^3 - 40n^2 - 96n + 27)(n+1)^2 c_{n-4} + (-232n^4 - 6580n^2 - 1116n + 8n^6$$

$$+72n^5 - 3312n^3 + 2052)c_{n-2} + (-11388n^4 - 39042 + 276n^6 + 59262n^2)c_n$$

$$+(2052 + 3312n^3 - 72n^5 - 232n^4 + 8n^6 + 1116n - 6580n^2)c_{n+2}$$

$$-(2n^4 - 8n^3 - 40n^2 + 96n + 27)(n-1)^2 c_{n+4} = 0 .$$

3.5.3   Second Algorithm for the Rational Trigonometric Functions

The computation of the complex Fourier coefficients in the case of rational trigonometric functions may be summarized in the following algorithm

---

Algorithm 3.5: Second algorithm for the computation of the Fourier coefficients of a rational trigonometric holonomic function (2FouCoeffRatTH)

---

input : A rational trigonometric function in the following form:

$$f(t) = \frac{\prod_{i=0}^{I} [A_i(\cos(t), \sin(t))]^{\alpha_i}}{\prod_{j=0}^{J} [B_j(\cos(t), \sin(t))]^{\beta_j}}, \quad B_j(\cos(t), \sin(t)) \neq 0, \quad (j = 0, \dots, J).$$

output: Either the complex Fourier coefficients of f on I, or RE and enough initial values.

1 **begin**
2   Apply Algorithm 3.4 (EfficientPL) to determine the efficient P and L.
3   Apply Algorithm 2.2 (THDEPL) to f, on the interval $[0, 2\pi]$, for P and L obtained previously, to get a trigonometric holonomic differential equation DE satisfied by f.
4   Apply Algorithm 3.1 (THDEtoRE) to DE, and f to convert DE into a recurrence equation RE satisfied by the complex Fourier coefficients of f.
5   **if** *RE is not holonomic* **then**
6   **return** *RE and enough initial values.*
7   **end**
8   Solve RE with appropriate initial values.
9   **if** *solving the previous RE is successful* **then**
10   **return** *the complex Fourier coefficients of f on I.*
11   **else**
12   **return** *RE and enough initial values.*
13   **end**
14 **end**

Example 3.22

Let us aim to compute the complex Fourier coefficients of the function

$$f(t) = \frac{1}{(2 + \cos(t))^{20}}$$

on the interval $[0, 2\pi]$. We get from Algorithm 3.4 $L = 1$ and $P = 1$ and Algorithm 2.2 gives the differential equation

$$DE : (2 + \cos(t))F'(t) - 20\sin(t)F(t) = 0 .$$

Algorithm 3.1 converts DE into the following recurrence equation RE

$$RE : i(n + 19)c_{n-1} + 4inc_n + i(-19 + n)c_{n+1} = 0 . \tag{3.19}$$

Solving RE we get $\forall n \geq 0$

$$c_n = \frac{(-2 + \sqrt{3})^n}{2154906460212336230400} \big(2598373260585253340700n + 207784608n^{15} + 1179246535908242448n^7$$

$$+ 67868187250174724208n^3 + 3n^{19} + 68400n^{17} + 956086266762532871940n^2\sqrt{3}$$

$$+ 4189258320n^{14}\sqrt{3} + 2587230n^{16}\sqrt{3} + 4686690914390935200n^6\sqrt{3} + 115724365700595819470\sqrt{3}n^4$$

$$+ 2507083241260n^{12}\sqrt{3} + 380n^{18}\sqrt{3} + 658724628579160n^{10}\sqrt{3} + 81271048798518540n^8\sqrt{3}$$

$$+ 198221547000n^{13} + 77548709950608n^{11} + 45298720378942521900n^5$$

$$+ 113911880303046800975 0\sqrt{3} + 13910290044027000n^9 \big) .$$

and $\forall n \leq 0$

$$c_n = \frac{(-2 + \sqrt{3})^{-n}}{2154906460212336230400} \big(-2598373260585253340700n - 207784608n^{15} - 1179246535908242448n^7$$

$$- 67868187250174724208n^3 - 3n^{19} + 68400n^{17} + 956086266762532871940n^2\sqrt{3}$$

$$+ 4189258320n^{14}\sqrt{3} + 2587230n^{16}\sqrt{3} + 4686690914390935200n^6\sqrt{3} + 115724365700595819470\sqrt{3}n^4$$

$$+ 2507083241260n^{12}\sqrt{3} + 380n^{18}\sqrt{3} + 658724628579160n^{10}\sqrt{3} + 81271048798518540n^8\sqrt{3}$$

$$- 198221547000n^{13} - 77548709950608n^{11} - 45298720378942521900n^5$$

$$+ 113911880303046800975 0\sqrt{3} - 13910290044027000n^9 \big) .$$

Note that the recurrence equation (3.19) has its coefficients in $\mathbb{Q}$. But the solution contains $\sqrt{3}$, since the Petkovšek-Van-Hoeij algorithm needed an algebraic field extension to compute this solution.

Conclusion 3.23

For the computation of the complex Fourier coefficients of rational trigonometric functions, if the degree of the denominator of the input function is 1, then it is preferable to use Algorithm 3.3, because we get a constant coefficient recurrence equation of second order. However for rational trigonometric functions whose denominator has degree ≥ 2, Algorithm 3.5 is preferable, because we may obtain a lower order recurrence equation which needs consequently fewer initial values.

We summarize the computation of the complex Fourier coefficients of rational trigonometric holonomic function f in the following algorithm:

Algorithm 3.6: Summarizing algorithm for the computation of the Fourier coefficients of rational trigonometric functions (FouCoeffRatTH)

---

input : A given rational trigonometric holonomic function $f : I = [a, b] \to \mathbb{C}$ where $\omega = \frac{2\pi}{b-a}$.

output: Either the complex Fourier coefficients of f on I, or RE and enough initial values.

1 **begin**
2   **if** *f has a denominator of degree 1* **then**
3   Apply Algorithm 3.3 to f on I.
4   **else**
5   Apply Algorithm 3.5 to f on I .
6   **end**
7 **end**

Remark 3.24

For a given ω one can search if a given function is in $\mathrm{TH}(\omega)$. Conversely one can also be interested in knowing if for a given function f, ratio of trigonometric polynomials, there exist an ω (or how can ω be chosen) such that $f \in \mathrm{TH}(\omega)$. The following algorithm gives a rational trigonometric membership test, and how ω can be chosen in such cases.

Algorithm 3.7: Membership test for rational trigonometric function (RatMembTest)

---

input : A function f, ratio of trigonometric polynomials.

output: Either ω for which f is in $\mathbb{K}(\cos(\omega t), \sin(\omega t))$ or f is not a rational trigonometric function.

1 **begin**
2      $Num \leftarrow$ numerator of f.
3      $Den \leftarrow$ denominator of f.
4   **if** *the periods of Num and Den, respective are commensurable* **then**
5   return ω, the circular frequency associated to the period of the function $g(t) = (Num, Den)$.
6   **else**
7   return: f is not a rational trigonometric function.
8   **end**
9 **end**

Of course ω is not unique, Theorem 2.8 implies that $n\omega$ for each integer n is also a good choice.

Example 3.25

Consider the function defined by

$$f(t) = \frac{\sin(\frac{abt}{2}) + \cos(\frac{abt}{3})}{a^2 + 2 + \cos(\frac{2abt}{3})} .$$

$Num = \sin(\frac{abt}{2}) + \cos(\frac{abt}{3})$ is of period $T_1 = \frac{12\pi}{ab}$, i.e. $Num \in \mathbb{K}[\cos(\frac{ab}{6}), \sin(\frac{ab}{6})]$. Similarly we deduce that $Den = a^2 + 2 + \cos(\frac{2abt}{3})$ is of period $T_2 = \frac{3\pi}{ab}$, i.e. $Den \in \mathbb{K}[\cos(\frac{2ab}{3}), \sin(\frac{2ab}{3})]$. Since $T_1 = 3T_2$, then Num and Den are commensurable, i.e. f is a rational trigonometric function. (Num, Den) is of period $\frac{12\pi}{ab}$, hence $\omega = \frac{ab}{6}$. Thus $f \in \mathbb{K}(\cos(\frac{ab}{6}), \sin(\frac{ab}{6}))$, i.e. $f \in \text{TH}(\frac{ab}{6})$. Applying Algorithm 3.4 to f, we get $P = 4$ and $L = 4$ as the efficient values for which we get the lowest order recurrence equation satisfied by the complex Fourier coefficients of f. The trigonometric holonomic differential equation satisfied by f for $\omega = \frac{ab}{6}$, $P = 4$ and $L = 4$ is

$$DE : b^4 a^4 (6 + 7\cos(\frac{2ab}{3}t) + 3a^2)F(t) + 76a^3 b^3 \sin(\frac{2ab}{3}t)F'(t) + 3b^2 a^2 (26 - 83\cos(\frac{2ab}{3}t)$$

$$+13a^2)F''(t) - 288ab\sin(\frac{2ab}{3}t)F'''(t) + (216 + 108\cos(\frac{2ab}{3}t) + 108a^2)F^{(4)}$$

and the complex Fourier coefficients of f satisfies the following holonomic recurrence equation of order 8

$$b^4 a^4 (n-2)(n-3)(n+3)(n+2)c_{n-4} + 2b^4 a^4 (n-2)(n-3)(n+3)(n+2)(a^2+2)c_n$$

$$+b^4 a^4 (n-2)(n-3)(n+3)(n+2)c_{n+4} = 0 .$$

Remark 3.26

Until now the smallest order of the recurrence equations obtained using the algorithms presented is 2 and there is no possibility to get a lower order. However this is possible in the case of some functions satisfying a differential equation of a particular form. In the next section we will present an algorithm for the computation of the Fourier coefficients in those cases.

3.6 Fourier Coefficients of Functions Satisfying Differential Equations with Coefficients in $\mathbb{K}[e^{i\omega t}, e^{-i\omega t}]$

This section is devoted to the computation of the Fourier coefficients of the trigonometric holonomic functions satisfying differential equations with coefficients in $\mathbb{K}[e^{-i\omega t}, e^{i\omega t}]$. As we announced in Section 2.5, the particularity in such differential equations is that they may lead in some cases to first order recurrence equations, wich cannot be get neither via Algorithm 3.2 nor Algorithm 3.6.

3.6.1 Conversion of Differential Equations with Coefficients in $\mathbb{K}[e^{i\omega t}, e^{-i\omega t}]$ into Holonomic Recurrence Equations

Theorem 3.27 (Holonomic Complex Re)

Let $f : [a, b] \to \mathbb{C}$ satisfy a differential equation of the form (2.15) with $\omega = \frac{2\pi}{b-a}$. Then its complex Fourier coefficients are given by

$$\sum_{p=0}^{P} \sum_{l=1}^{L} \left[\delta_{pl} \left(\frac{2(n-l)\pi i}{T} \right)^p \left(c_{n-l} - \sum_{j=0}^{p-1} (-1)^j (T)^j \left(\frac{i}{2(n-l)\pi} \right)^{j+1} (f^{(j)}(b) - f^{(j)}(a)) e^{-i(n-l)\omega a} \right) + \right.$$

$$\left. \gamma_{pl} \left(\frac{2(n+l)\pi i}{T} \right)^p \left(c_{n+l} - \sum_{j=0}^{p-1} (-1)^j (T)^j \left(\frac{i}{2(n+l)\pi} \right)^{j+1} (f^{(j)}(b) - f^{(j)}(a)) e^{-i(n+l)\omega \alpha} \right) \right] = 0 \, .$$

$$(3.20)$$

Proof: Starting from the differential equation (2.15), namely

$$\sum_{p=0}^{P} \sum_{l=0}^{L} \left(\gamma_{pl} e^{-il\omega t} + \delta_{pl} e^{il\omega t} \right) f^{(p)}(t) = 0$$

and putting $T = b - a$ and $\omega = \frac{2\pi}{b-a}$, multiplying (2.15) by $\frac{1}{T} \cdot e^{-in\omega t}$ and integrating over the interval $[a, b]$, one gets

$$\frac{1}{T} \int_a^b \left(\gamma_{pl} e^{-il\omega t} + \delta_{pl} e^{il\omega t} \right) f^{(p)}(t) e^{-in\omega t} dt = 0$$

$$\iff \quad \sum_{p=0}^{P} \sum_{l=0}^{L} \left(\delta_{pl} \frac{1}{T} \int_a^b f^{(p)}(t) e^{-i\omega(n-l)t} dt + \gamma_{pl} \frac{1}{T} \int_a^b f^{(p)}(t) e^{-i\omega(n+l)t} dt \right) = 0$$

which leads to the relation

$$\sum_{p=0}^{P} \sum_{l=0}^{L} \left(\delta_{pl} c_{n-l}^{(p)} + \gamma_{pl} c_{n+l}^{(p)} \right) = 0 \tag{3.21}$$

where $c_{n-l}^{(p)}$ and $c_{n+l}^{(p)}$ denote the shifted complex Fourier coefficients of the pth derivative of f, given by (3.6) and (3.7). Substituting them in (3.21), we obtain that the complex Fourier coefficients of the function f satisfy the recurrence equation (3.20). $\qquad\square$

Remark 3.28

In practice for the conversion of a differential equation of the form (2.15) into a recurrence equation, we will retrieve from the DE the coefficients γ_{pl} and δ_{pl} for which f DE is of the form (2.15) and substitute them in (3.20). That process may be summarized in the following algorithm:

Algorithm 3.8: Conversion of a trigonometric holonomic differential equation with coefficients in $\mathbb{K}[e^{-i\omega t}, e^{i\omega t}]$ into a holonomic recurrence equation (Exp-THDEtoRE)

---

input : A number $\omega \in \mathbb{R}^\star$, a trigonometric holonomic differential equation DE of the form (2.15) and a function f solution of DE.

output: A recurrence equation for the complex Fourier coefficients of the function f on an interval of period $T = \frac{2\pi}{\omega}$.

1 **begin**
2   Retrieve from DE the coefficients γ_{pl} and δ_{pl} for which DE is of the form (2.15).
3   Substitute those coefficients in (3.20) to get the recurrence equation satisfied by the complex Fourier coefficients of f.
4   Return the recurrence equation satisfied by the complex Fourier coefficients of f.
5 **end**

Example 3.29

Consider the function $f : [0, 2\pi] \to \mathbb{C}$ defined by

$$f(t) = e^{ie^{it}} .$$

f satisfies the differential equation

$$f'(t) + e^{it} f(t) = 0 . \tag{3.22}$$

The retrieval of the coefficients γ_{pl} and δ_{pl} from the previous differential equation gives

$$\gamma_{00} = 0, \quad \gamma_{01} = 0, \quad \gamma_{10} = 0, \quad \gamma_{11} = 0$$

and

$$\delta_{00} = 0, \quad \delta_{01} = 1, \quad \delta_{10} = 1, \quad \delta_{11} = 0 .$$

Substituting the previous γ_{pl} and δ_{pl} in (3.20) we obtain the following recurrence equation

$$c_{n-1} + inc_n = 0 .$$

Solving with one initial value we get that the Fourier coefficients of f are given as

$$c_n = \begin{cases} \frac{i^n}{n!} & \forall n \geq 0 \\ 0 & \text{otherwise} . \end{cases}$$

Note that if we use Algorithm 3.2 to compute the Fourier coefficients of the previous function, of course we get the same result, but this time via a second order recurrence equation.

Let us compute it via Algorithm 3.2. f satisfies the differential equation

$$DE : -i(i + i\sin(t) + \cos(t))F(t) + (-i + \cos(t) - i\sin(t))F'(t) = 0$$

which, converted into a recurrence equation via Algorithm 3.1, gives the second order holonomic recurrence equation

$$RE : -2ic_{n-1} + (2n + 2)c_n + 2i(n + 1)c_{n+1} = 0 .$$

Solving RE with two initial values leads again to

$$c_n = \begin{cases} \frac{i^n}{n!} & \forall n \geq 0 \\ 0 & \text{otherwise}^2 . \end{cases}$$

Note that in this case an application of the Petkovšek-Van-Hoeij algorithm is necessary.

3.6.2 Algorithm for the Computation of the Fourier Coefficients of Functions Satisfying a Differential Equation with Coefficients in $\mathbb{K}[e^{i\omega t}, e^{-i\omega t}]$

Algorithm 3.9: Computation of the complex Fourier coefficients of functions satisfying a differential equation with coefficienrs in $\mathbb{K}[e^{-i\omega t}, e^{i\omega t}]$ (Exp-FouCoeffTH)

---

    input : A function $f \in C^{(N)}[a, b] \to \mathbb{C}$ for N large enough satisfying a differential equation with coefficients in $\mathbb{K}[e^{i\omega t}, e^{-i\omega t}]$.
    output: Either the complex Fourier coefficients of f on I, or RE and enough initial values.

1 **begin**
2   Apply Algorithm 2.3 to f on the interval I to determine a differential equation DE of the form (2.15) satisfied by f.
3   Apply Algorithm 3.8 to DE and f to convert DE into a recurrence equation RE satisfied by the complex Fourier coefficients of f.
4   **if** *RE is not holonomic* **then**
5   **return** *RE and enough initial values.*
6   **end**
7   Solve RE and deduce the complex Fourier coefficients of f.
8   **if** *the solution of the previous RE is successful* **then**
9   **return** *the complex Fourier coefficients of f on I.*
10   **else**
11   **return** *RE and enough initial values.*
12   **end**
13 **end**

Remark 3.30
One may easily remark that a differential equation of the form (2.15) may be converted into a differential equation of the form (2.9) and conversely, using Euler's formula $e^{it} = \cos(t) + i\sin(t)$

and Moivre's formula $(\cos(t) + i\sin(t))^n = \cos(nt) + i\sin(nt)$. Doing so, the conversion of the obtained differential equation into the corresponding recurrence equation for the Fourier coefficients of f may be done using appropriate conversion algorithms which may be either Algorithm 3.8 or Algorithm 3.1.

3.6.3   Fourier Coefficients of Simple Trigonometric Holonomic Functions

Example 3.31

Let f be the function defined on the interval $[0, \frac{2\pi\sqrt{7}}{7}]$ by:

$$f(t) = e^{i\sqrt{7}t}\sqrt{2 + e^{i\sqrt{7}t}} \ .$$

f satisfies the trigonometric holonomic differential equation

$$-2i\sqrt{7}(4 + 3e^{i\sqrt{7}t})F(t) + (8 + 4e^{i\sqrt{7}t})F'(t) = 0 \ ,$$

where $\omega = \sqrt{7}$ and its complex Fourier coefficients satisfy the first order holonomic recurrence equation

$$RE := 2i\sqrt{7}(-5 + 2n)c_{n-1} + 8i\sqrt{7}(n - 1)c_n = 0 \ .$$

Solving RE we get the closed form

$$c_n = \begin{cases} \frac{(-1)^n 2^{-n}\sqrt{2}(n - \frac{5}{2})!}{\sqrt{\pi}(n-1)!} & \forall n \geq 1 \\ 0 & \text{otherwise} \ . \end{cases}$$

Example 3.32

Consider the following function defined on the interval $[0, 2\pi]$ by:

$$f(t) = \frac{e^{-4it}}{2 + e^{it}} \ .$$

f is solution of the trigonometric holonomic differential equation

$$DE := (8 + 5e^{it})F(t) - i(2 + e^{it})F'(t) = 0 \ .$$

The conversion of DE into a holonomic recurrence equation gives

$$(8 + 2n)c_n + (n + 4)c_{n-1} = 0$$

which is of first order and whose solution in closed form is

$$c_n = \begin{cases} \frac{1}{32}(\frac{-1}{2})^n & \forall n \geq -4 \\ 0 & \text{otherwise} \ . \end{cases}$$

Remark 3.33
Note that for some trigonometric holonomic functions in general, we may combine Algorithm 1.1 of the first chapter and Algorithm 3.2 or Algorithm 3.9 to get a closed form of the Fourier coefficients. For example if for a given trigonometric holonomic function we may not get a closed form of its Fourier coefficients only because of the lack of the computation of enough initial values, or the difficulty to solve the obtained recurrence equation, if we may overcome this gap with one of its successive derivatives, then we use Algorithm 1.1 to achieve the Fourier coefficients of the foresaid function. Example 1.11 ist also an illustration of this method.

Example 3.34
Consider the function defined by

$$f(t) = \arctan(ie^{-it} + (1-i)e^{it}) \, .$$

f is a trigonometric holonomic function, satisfying the differential equation

$$-i(-2 + 3i + 3ie^{-2it} - 6e^{2it}))F'(t) + (4ie^{-2it} + 8e^{2it})F''(t) + i(ie^{-2it} - 2e^{2it} + 2 - 3i)F'''(t) = 0$$

and whose Fourier coefficients are solution of the fourth order recurrence equation

$$-2(n-1)(n-2)(n+1)c_{n-2} + (2-3i)(n-1)(n+1)nc_n + i(n-1)(n+2)(n+1)c_{n+2} = 0 \, .$$

Unfortunately *Maple* cannot compute any of its initial values. We remark that the derivative $g = f'$ of f has an easier form than f, namely

$$g = \frac{e^{-it} + (i+1)e^{it}}{1 + (ie^{-it} + (1-i)e^{it})^2} \, .$$

Applying Algorithm 3.9 to g we get:

$$c'_n = \begin{cases} \frac{-1+(-1)^n}{2}\left(\frac{1}{2} - \frac{1}{2}\sqrt{5+4i}\right)^n & \forall n \geq 0 \\ \frac{1-(-1)^n}{2}\left(\frac{1}{2} + \frac{1}{2}\sqrt{5+4i}\right)^n & \forall n \leq 0 \, . \end{cases}$$

Combining now Algorithm 1.1 and Algorithm 3.9 we deduce that the Fourier coefficients of f on the interval $[0, 2\pi]$ are given by

$$c_n = \begin{cases} \frac{i(1-(-1)^n)}{2n}\left(\frac{1}{2} - \frac{1}{2}\sqrt{5+4i}\right)^n & \forall n \geq 1 \\ 0 & \text{if} \quad n = 0 \\ \frac{i(-1+(-1)^n)}{2n}\left(\frac{1}{2} + \frac{1}{2}\sqrt{5+4i}\right)^n & \forall n \leq 1 \, . \end{cases}$$

3.7 Recapitulation of the Algorithms for the Computation of Fourier Coefficients

In this section we put together all the algorithms we designed to compute the complex Fourier coefficients for trigonometric holonomic functions.

Algorithm 3.10: Fourier coefficients in the general case

---

input : A function $f \in C^{(N)}[a,b]$ for N large enough, which may be w.l.o.g. $I = [0, T]$.

output: Either the complex Fourier coefficients of f on I, or the recurrence equation
satisfied by the complex Fourier coefficients of f and enough initial values, or
"This algorithm is not appropriate for the computation of the Fourier
coefficients of f".

1 **begin**
2   Compute the complex Fourier coefficients of f with existing method.
3   **if** *the computation is successful* **then**
4   **return** *The complex Fourier coefficients of f on the interval I.*
5   **end**
6   Compute the complex Fourier coefficients of one of the successive derivatives of
    anti-derivatives of f on I.
7   **if** *the computation is successful* **then**
8   Apply Algorithm 1.1 to f on the interval I.
9   **end**
10   **if** *f is a trigonometric holonomic function* **then**
11   **if** *f is a rational trigonometric function* **then**
12   apply Algorithm 3.6 to f on I.
13   **else**
14   **if** *f is a simple trigonometric holonomic function* **then**
15   apply Algorithm 3.9 to f on I.
16   **else**
17   Apply Algorithm 3.2 to f on I.
18   **end**
19   **end**
20   **if** *a closed form of the Fourier coefficients of f may not be get* **then**
21   Apply a combination of Algorithm 1.1 and Algorithm 3.2 or Algorithm 3.9
            to f on I.
22   **end**
23   **end**
24   **return** *this algorithm is not appropriate for the computation of the complex Fourier
coefficients of f.*
25 **end**

Chapter 4

Factorization of Holonomic Recurrence Operators

This chapter deals with the factorization of holonomic recurrence operators. An algorithm for computing a first order right factor of such operators was first given by Petkovšek in [Pet92]. Its application is limited to the cases in which the product of the leading and trailing coefficients of the considered operators do not have too many factors, because the algorithm computes more combinations than necessary. Mark Van Hoeij addressed those problems in [Hoe98] by introducing the concept of *finite singularities*. Peter Horn in [Hor08] presented an adapted version of the Petkovšek algorithm and computes fewer combinations than in [Pet92], particularly for the cases involving *q shift operators*. We present in the second section a different method, involving Fourier series, to compute a right factor of holonomic recurrence operators, which in some cases returns the smallest order right factor. In the first section we give some resources to achieve that goal.

4.1 Conversion of a Holonomic Recurrence Equation Into a Trigonometric Holonomic Differential Equation

Chapter three described the conversion of trigonometric holonomic differential equations into recurrence equations for the Fourier coefficients, which may be homogeneous or not. In this section we do the reverse of that conversion, focussing on homogeneous recurrence equations. To do so we look if for a given homogeneous holonomic recurrence equation RE one may find coefficients α_{pl} and β_{pl} (resp. γ_{pl} and δ_{pl}) such that RE is the conversion of a differential equation of the form (2.9) (resp. (2.15)) with some initial values. In this case the searched trigonometric holonomic differential equation will be the one satisfied by a function defined on an interval $[a, b]$ such that $F^{(i)}(a) = F^{(i)}(b)$, $(i = 0, \ldots, P - 1)$, see Example 4.2.

Theorem 4.1
For a given real number $\omega = \frac{2\pi}{b-a}$ with $a < b$, each holonomic recurrence equation can be converted

into a differential equation with side conditions either of the form

$$
\begin{cases}
\sum_{p=0}^{P} \sum_{l=0}^{L} \left(\alpha_{pl} \cos(l\omega t) + \beta_{pl} \sin(l\omega t) \right) f^{(p)}(t) = 0 \\
f^{(j)}(a) = f^{(j)}(b) \qquad (j = 0, \ldots, P-1)
\end{cases}
\tag{4.1}
$$

for appropriate integers $P \geq 1$ and $L \geq 0$, where α_{pl} and β_{pl} are constants, or of the form

$$
\begin{cases}
\sum_{p=0}^{P} \sum_{l=0}^{L} \left(\gamma_{pl} e^{-il\omega t} + \delta_{pl} e^{il\omega t} \right) f^{(p)}(t) = 0 \\
f^{(j)}(a) = f^{(j)}(b) \qquad (j = 0, \ldots, P-1)
\end{cases}
\tag{4.2}
$$

for appropriate integers $P \geq 1, L \geq 0$, where γ_{pl} and δ_{pl} are constant.

Proof: We prove only the case of relation (4.2) since the case of relation (4.1) can be shown analogously. Let $RE = \sum_{i=0}^{I} a_i c_{n+i} = 0$ with $a_i \in \mathbb{K}[n]$ denote the holonomic recurrence equation to be converted. According to Theorem 3.27, subtracting RE from the relation (3.20) gives

$$
\sum_{p=0}^{P} \sum_{l=1}^{L} \left[\delta_{pl} \left(\frac{2(n-l)\pi i}{T} \right)^p \left(c_{n-l} - \sum_{j=0}^{p-1} (-1)^j (T)^j \left(\frac{i}{2(n-l)\pi} \right)^{j+1} (f^{(j)}(b) - f^{(j)}(a)) e^{-i(n-l)\omega a} \right) + \right.
$$

$$
\left. \gamma_{pl} \left(\frac{2(n+l)\pi i}{T} \right)^p \left(c_{n+l} - \sum_{j=0}^{p-1} (-1)^j (T)^j \left(\frac{i}{2(n+l)\pi} \right)^{j+1} (f^{(j)}(b) - f^{(j)}(a)) e^{-i(n+l)\omega \alpha} \right) \right]
$$

$$
- \sum_{i=0}^{I} a_i c_{n+i} = 0 .
$$

First case: I is even.
Choosing $L = \frac{I}{2}$ and applying an appropriate shift if necessary to the relation (3.20), we can collect the previous equation in a maximum of $L + 1 + L + 1 = 2L + 2$ terms with respect to c_{n+j}, $(j = -L, \ldots, L)$ or $(j = 0, \ldots, 2L)$. Setting P_m as the maximum of the degree of the coefficients a_i of RE, we can collect each coefficient of c_{n+j} of the previous equation in a maximum of $P_m + 1$ terms, this time with respect to n. Hence we can build from that equation a system of maximum $(2L + 2)(P_m + 1)$ equations where the unknowns are the γ_{pl}s and the δ_{pl}s. On the other hand, the number of unknowns is $2(L + 1)(P + 1)$. We get surely a non-vanishing solution when the number of unknowns is greater than the number of equations, i.e. if

$$
2(L + 1)(P + 1) > (2L + 2)(P_m + 1) \iff P > P_m .
$$

Thus taking $L = \frac{I}{2}$ and $P > P_m$, the problem is solved. We do similarly for the case where I is odd, taking $L = I$ and setting the γ_{pl}s to zero. $\qquad \square$

We summarize this search in the following algorithm:

Algorithm 4.1: Conversion of a holonomic recurrence equation into differential equation with side conditions (ReverseTHDEtoRE).

> **input** : A holonomic recurrence equation RE and an interval $[a, b]$.
> **output**: A differential equation with side conditions of the form (4.1) (resp. (4.2)).

1 begin

2 $\quad \omega \leftarrow \frac{2\pi}{b-a}$.

3 $\quad$ Search for coefficients α_{pl} and β_{pl} (resp. γ_{pl} and δ_{pl}) such that an equation of the form (3.3) $- RE = 0$ (resp. (3.20) $- RE = 0$) is valid.

4 $\quad$ Return a differential equation with side conditions of the form (4.1) (resp. (4.2)).

5 end

Example 4.2

Consider the holonomic recurrence equation

$$RE := (2n - 3)c_{n-1} + 8nc_n + (3 + 2n)c_{n+1} = 0$$

1. On the interval $[0, 2\pi]$ RE is the conversion into a recurrence equation of the differential equation with side conditions

$$DE : \begin{cases} \sin(t)F(t) + (4 + 2\cos(t))F'(t) = 0 \\ F(0) = F(2\pi) \, . \end{cases}$$

   one of whose solutions is $f(t) = \sqrt{2 + \cos(t)}$

2. Taking now in consideration the symbolic interval $[a, b]$ the previous algorithm returns

$$DE1 : \begin{cases} \sin(\frac{2\pi}{b-a}t)F(t) + \frac{b-a}{\pi}(2 + \cos(\frac{2\pi}{b-a}t))F'(t) = 0 \\ F(a) = F(b) \, . \end{cases}$$

   One of the solutions of $DE1$ is $f(t) = \sqrt{2 + \cos(\frac{2\pi}{b-a}t)}$.

Consider now the recurrence equation

$$RE2 : 10(n-1)(n+1)c_n + (n-1)(n+1)c_{n+2} + (n-1)(n+1)c_{n-2} = 0 \, .$$

On the symbolic interval $[a, b]$ Algorithm 4.1 returns the differential equation with side conditions

$$DE2 : \begin{cases} (-5 + 3\cos(\frac{4\pi}{b-a}t))F(t) + \frac{2(b-a)}{\pi}\sin(\frac{4\pi}{b-a}t)F'(t) - \frac{(b-a)^2}{4\pi^2}(5 + \cos(\frac{4\pi}{b-a}t))F''(t) = 0 \\ F^{(j)}(a) = F^{(j)}(b) \qquad (j = 0, 1) \, . \end{cases}$$

4.2   Holonomic Recurrence Operators

Let $\mathbb{K}$ be a field of characteristic zero. We denote by $\mathbb{K}^{\mathbb{N}}$ the set of all sequences $(a_n)_{n=0}^{\infty}$ whose terms belong to $\mathbb{K}$.

Definition 4.3
The function $N : \mathbb{K}^{\mathbb{N}} \longrightarrow \mathbb{K}^{\mathbb{N}}$ which acts on each a_n in the following way: $N(a_n) = a_{n+1}$, is defined as the *shift operator*.

Note that N is linear, and that the set of all linear operators with addition defined pointwise and with the functional composition as multiplication is a (non-commutative) ring. N satisfies the commutation relation $Nn = (n+1)N$.

Definition 4.4
Operators of the form

$$L = \sum_{k=0}^{r} a_k N^k \ ,$$

where $a_k \in \mathbb{K}^{\mathbb{N}}$ and N is the shift operator are called *recurrence operators* on $\mathbb{K}^{\mathbb{N}}$, see [PWZ96]. If $a_r \neq 0$ and $a_0 \neq 0$, then the *order* of L is r. Equations of the form

$$L(u_n) = 0, \quad \text{i.e.} \quad a_r u_{n+r} + a_{r-1} u_{n+r-1} + \cdots + a_0 u_n = 0$$

are called *recurrence equations*. A recurrence equation is *holonomic* if $a_k \in \mathbb{K}[n]$ $(0 \leq k \leq r)$.

4.2.1   A New Factorization Method

The idea behind the method described in this study is to consider the holonomic recurrence equation RE corresponding to a given holonomic recurrence operator as the one satisfied by the Fourier coefficients of a trigonometric holonomic function on an interval $[a, b]$, which may be w.l.o.g. the interval $[0, 2\pi]$. To do so, we compute the differential equation with side conditions corresponding to RE either of the form (4.1) or of the form (4.2). Then we solve it to get possible choices of the corresponding trigonometric holonomic functions. Applying Algorithm 2.1 or 2.3, a new trigonometric holonomic differential equation satisfied by f is found, which may lead to a recurrence equation of lower order. In the last step, the corresponding holonomic recurrence operator is deduced from the above obtained lower order recurrence equation. This holonomic recurrence operator is taken as a right factor of the input holonomic recurrence operator. Algorithm 4.2 below provides an overview of this.

Algorithm 4.2: Search for a right factor of a holonomic recurrence operator.

input : A holonomic recurrence operator L.
output: A right factor of L or "a right factor of L cannot be found using this algorithm, since DE cannot be solved".

1 **begin**
2   Convert L into a recurrence equation RE.
3   Use Algorithm 4.1 to convert RE into a differential equation with side conditions DE.
4   Solve that differential equation with side conditions.
5   **if** *DE cannot be solved* **then**
6   **return** *a right factor of the given operator cannot be found using this algorithm, since DE cannot be solved*
7   **else**
8   Set f the solution DE.
9   Apply Algorithm 2.1 or Algorithm 2.3 to f to search for a new holonomic differential equation DE satisfied by f which leads this time to the lowest order holonomic recurrence equation that can be obtained via Algorithm 3.1 or Algorithm 3.8.
10   Convert this holonomic recurrence equation into a holonomic recurrence operator.
11   Return the right factor of the given holonomic recurrence operator.
12   **end**
13 **end**

4.2.2   Some Examples of Factorization of Holonomic Recurrence Operators

Example 4.5

Consider the following holonomic recurrence operator, as given in [PWZ96].

$$L = (n - 1)N^2 - (n^2 + 3n - 2)N + 2n(n + 1) \ .$$

The corresponding holonomic recurrence equation with respect to c_n is

$$(n - 1)c_{n+2} - (n^2 + 3n - 2)c_{n+1} + 2n(n + 1)c_n = 0 \ .$$

Applying Algorithm 4.1 returns the differential equation with side conditions

$$DE : \begin{cases} (-4e^{-it} + 3e^{-2it})F(t) + i(-e^{-it} + e^{-2it} + 2)F'(t) - (e^{-it} - 2)F''(t) = 0 \\ f^{(j)}(0) = f^{(j)}(2\pi) \qquad (j = 0, 1) \ . \end{cases}$$

Solving this DE we get one of the solutions

$$f(t) = e^{-it - e^{-it}} \in \text{sTH} \ .$$

Applying Algorithm 2.3 to f with $\omega = \frac{2\pi}{2\pi - 0} = 1$ leads to

$$DE : (-1 + e^{it})F(t) - ie^{it}F'(t) = 0 .$$

Now converting DE, considering the interval $[0, 2\pi]$, into a holonomic recurrence equation we get

$$RE : c_{n+1} - (n+1)c_n = 0 .$$

The above RE shows that a first order right factor of the holonomic recurrence operator L is given by $N - n - 1$, which exactly corresponds to one of the right factors found in [PWZ96].

Example 4.6

Consider the following holonomic recurrence operator L of order 6

$$L = N^6 - 5N^4 + (14 - n)N^3 + (-n^2 - n + 2)N^2 + (n^2 + 11n - 14)N + (24 - 12n) .$$

Its lowest order right factor, which is of order 3, was found by Peter Horn in [Hor08]. The following shows the computation of the same factor with the method of this study. The conversion of L into a recurrence equation with respect to $c(n)$ is

$$RE : c_{n+6} - 5c_{n+4} + (14 - n)c_{n+3} + (-n^2 - n + 2)c_{n+2}$$

$$+(n^2 + 11n - 14)c_{n+1} + (24 - 12n)c_n = 0 .$$

The differential equation with side conditions corresponding to RE is

$$DE : \begin{cases} -(-24 + 24e^{-it} - 17e^{-3it} + 5e^{-4it} - e^{-6it})F(t) \\ -i(3e^{-2it} - 12 + 9e^{-it} - e^{-3it})F'(t) + (-e^{-it} + e^{-2it})F''(t) = 0 \\ f^{(j)}(0) = f^{(j)}(2\pi) \qquad (j = 0, 1) . \end{cases}$$

One of the solutions of this DE is

$$f = e^{12e^{it} - \frac{1}{2}e^{-2it}} .$$

Applying Algorithm 2.3 to f, we get that f satisfies the trigonometric holonomic differential equation

$$(1 + 12e^{3it})F(t) + ie^{2it}F'(t) = 0$$

which leads to the third order holonomic recurrence equation

$$RE : c_{n+3} + (-n - 1)c_{n+1} + 12c_n = 0 .$$

Converting the above RE in terms of operators leads to

$$N^3 + (-n - 1)N + 12$$

which corresponds exactly to the one found in [Hor08].

Remark 4.7

The method presented in this chapter is time efficient. For the particular case described in Example 4.6 the computation time of the right factor is less than 5 seconds and needs much less memory capacity in comparison to the computation method described in [Hor08], which needs more than 21 hours and utilizes 32 GB memory capacity to get the same result. The timing in [Hor08] is high because the algorithm first needs to compute a left factor of a recurrence operator of order 20.

Additionally an analogous method via power series can be used to get the previous right factors, see [Koe09]. This method does not require to solve a differential equation, which in some instances turns out to be complicated. Unfortunately this method does not always find a right factor even when one exists. For the following example, a right factor cannot be found using the method involving power series which shows that this method is quite rigid whereas our approach is more flexible.

Example 4.8

Consider the following holonomic recurrence operator

$$L = (-5 - n)N^4 + (-20 - 4n)N^3 - 4N^2 + (4n + 4)N + (n + 1) .$$

The corresponding holonomic recurrence equation is

$$RE := (n + 1)c_n + (4n + 4)c_{n+1} - 4c_{n+2} + (-20 - 4n)c_{n+3} + (-5 - n)c_{n+4}$$

which converted into a differential equation with side conditions returns

$$DE : \begin{cases} -(-1 + 4e^{-2it} + 8e^{-3it} + e^{-4it})F(t) + i(-1 - 4e^{-it} + 4e^{-3it} + e^{-4it}F'(t) = 0 . \\ F(0) = F(2\pi) . \end{cases}$$

One of the solutions of the above DE is

$$f(t) = \frac{e^{it} - e^{-it}}{e^{2it} + 4e^{it} + 1}$$

whose Fourier coefficients on the interval $[0, 2\pi]$ satisfy the second order holonomic recurrence equation

$$RE1 = (n + 1)(n + 3)c_{n+2} + 4(n + 1)(n + 3)c_{n+1} + (n + 1)(n + 3)c_n$$

from which we deduce the second order right factor

$$L1 = (n + 1)(n + 3)N^2 + 4(n + 1)(n + 3)N + (n + 3)(n + 1) .$$

Chapter 5

Some Applications of Fourier Series

The classical way of computing the exact value of definite integrals is via the search for an anti-derivative. However in some cases where integer parameters are involved, one may get those exact values by solving a recurrence equation satisfied by those integrals. In the first section of this chapter we combine Parseval's identity and the algorithms developed in chapter 3 for the computation of the Fourier coefficients, to present an algorithm for the computation of some definite integrals of trigonometric holonomic functions on an interval $[a, b]$. Its direct consequence is the computation of some integrals which are out of reach of existing computer algebra systems. Section 5.2 shows some examples using those algorithms of how boundary value problems may be solved when the initial values satisfy certain conditions.

5.1   Applications of Fourier Series in Integral Computation

5.1.1   Recall of Parseval's Identity

Let f be a continuous function on the interval $[0, T]$. As also mentioned in the introduction of this thesis, the Fourier coefficients of f are given as

$$a_n = \frac{2}{T} \int_0^T f(t) \cos(n\frac{2\pi}{T}t)dt, \quad b_n = \frac{2}{T} \int_0^T f(t) \sin(n\frac{2\pi}{T}t)dt, \quad \forall n \geq 0$$

and

$$c_n = \frac{1}{T} \int_0^T f(t)e^{-in\frac{2\pi}{T}t}dt \quad \forall n \in \mathbb{Z} \,.$$

Then Parseval's identity tells that

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{T} \int_0^T |f(t)|^2 dt = \|f(t)\|^2 \tag{5.1}$$

or

$$\|f(t)\|^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \,. \tag{5.2}$$

We would like to mention that the previous identity is also valid for the interval $[a, a + T]$, $a \in \mathbb{R}$.

5.1.2   Algorithm for Computing Integrals via Fourier Series

Let g be the function defined by

$$g(t) = f(t)^2$$

where f is continuous on $I = [0, T]$.

The following algorithm deals with the question of how to compute the definite integral

$$\int_0^T g(t)dt$$

via a combination of the algorithms described in the third chapter and the Parseval's identity.

Algorithm 5.1: Computation of a definite integral using Parseval's identity.

    input : A function $g(t) = f(t)^2$ and an interval $I = [0, T]$, such that the Fourier
              coefficients of f can be explicitly computed via Algorithm 3.2 or Algorithm 3.9
              on I.

    output: The exact value of the definite integral

$$\int_0^T g(t)dt.$$

1 **begin**

2   Apply Algorithm 3.2 or Algorithm 3.9 to f on I to get the explicit expression of the
         Fourier coefficients of f on I.

3   Use Parseval's identity (5.1) or (5.2) to deduce via summation[1] the searched integral,
         namely either as

$$\int_0^T g(t)dt = T \sum_{k=-\infty}^{\infty} |c_n|^2, \quad \text{or} \quad \int_0^T g(t)dt = T \left(\frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right).$$

4   (IntCompPars)

5 **end**

---

[1] Summation algorithms can be found e.g. in [Koe98] and [PWZ96].

Remark 5.1

Algorithm 5.1 is a way of computing integrals without involving the notion of anti-derivative. Moreover this algorithm is capable of computing, via some special functions, integrals which are out of reach of current computer algebra systems.

5.1.3 Some Examples of the Computation of Integrals via Fourier Series

Integral Computation in Simple Cases

This section describes the computation of some definite integrals. Although these integrals may be computed using existing computer algebra systems, we compute them using Algorithm 5.1.

Example 5.2

Let us to compute the integral

$$\int_0^{\frac{2\pi}{3}} \left(\frac{\cos(3t)}{(2 + \sin(3t))^2} \right)^2 dt = \int_0^{\frac{2\pi}{3}} f(t)^2 dt \quad \text{with} \quad f(t) = \frac{\cos(3t)}{(2 + \sin(3t))^2} \ .$$
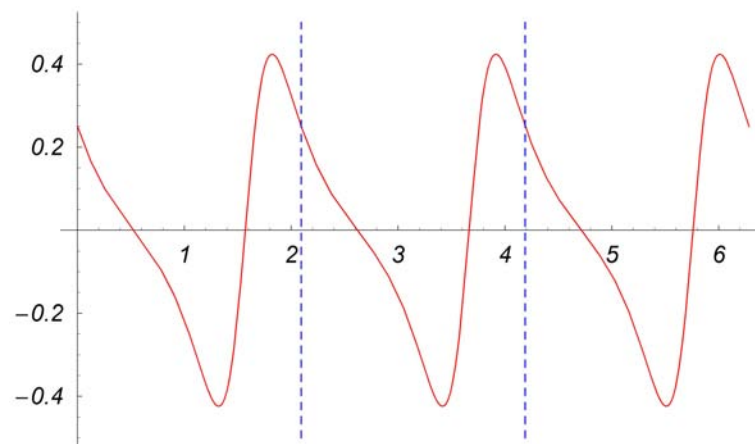
Figure 5.1: Graph of f on $I = [0, 2\pi]$

Applying Algorithm 3.2 to f on the interval $[0, \frac{2\pi}{3}]$, we get that the complex Fourier coefficients of f are given by

$$c_n = \begin{cases} -\frac{1}{3}i(2i - i\sqrt{3})^n n\sqrt{3} & n \geq 0 \\ -\frac{1}{3}i\sqrt{3}(2i + i\sqrt{3})^n n & n \leq 0. \end{cases}$$

Using Parseval's identity (5.1) we deduce that

$$
\begin{aligned}
\int_0^{\frac{2\pi}{3}} f(t)^2 dt &= \frac{2\pi}{3} \sum_{n=-\infty}^{\infty} |c_n|^2 \\
&= \frac{2\pi}{3} \left(\sum_{n=-\infty}^{-1} \left| -\frac{in\sqrt{3}}{3}(2i+i\sqrt{3})^n \right|^2 + \sum_{n=0}^{\infty} \left| -\frac{in}{3}(2i-i\sqrt{3})^n \right|^2 \right) \\
&= \frac{2\pi\sqrt{3}}{81}
\end{aligned}
$$

Hence

$$
\int_0^{\frac{2\pi}{3}} \left(\frac{\cos(t)}{(2+\sin(t))^2} \right)^2 dt = \frac{2\pi\sqrt{3}}{81} \ .
$$

We remark that the previous integral may be computed by existing CAS, since the anti-derivative of the considered function can be computed.

Example 5.3

Consider the function defined by

$$
f(t) = \frac{e^{i\sqrt{2}t}}{2+e^{i\sqrt{2}t}} \ .
$$

Our purpose is to compute

$$
\int_0^{\sqrt{2}\pi} g(t)dt = \int_0^{\sqrt{2}\pi} \left| \frac{e^{i\sqrt{2}t}}{2+e^{i\sqrt{2}t}} \right|^2 dt = \int_0^{\sqrt{2}\pi} |f(t)|^2 \, dt
$$

The computation of the complex Fourier coefficients of f via Algorithm 3.2 on the interval $[0, \sqrt{2}\pi]$ gives

$$
c_n = \begin{cases} \frac{-(-1)^n}{2^n} & n \geq 1 \\[2mm] 0 & \text{otherwise} \end{cases}
$$

Parseval's identity (5.1) applied to f leads to the following integration

$$
\begin{aligned}
\int_0^{\sqrt{2}\pi} g(t) = \int_0^{\sqrt{2}\pi} |f(t)|^2 dt &= \sqrt{2}\pi \sum_{n=-\infty}^{\infty} |c_n|^2 \\
&= \sqrt{2}\pi \left(\sum_{n=1}^{\infty} \left| \frac{-(-1)^n}{2^n} \right|^2 \right) \\
&= \frac{\sqrt{2}\pi}{3}
\end{aligned}
$$

Hence, we deduce that

$$
\int_0^{\sqrt{2}\pi} \left| \frac{e^{i\sqrt{2}t}}{2+e^{i\sqrt{2}t}} \right|^2 dt = \frac{\sqrt{2}\pi}{3} \ .
$$

Integral Computation in Non-Trivial Cases

In this section we deal with definite integrals which cannot be computed by current computer algebra systems. By applying Algorithm 5.1 we compute these integrals explicitly, via some special functions.

Example 5.4
Our aim in this example is to compute the integral

$$A = \int_0^{2\pi} \left(\sin(t) \ln(2 + \cos(t)) \right)^2 dt$$

We get

$$A = \int_0^{2\pi} f(t)^2 dt \quad \text{with} \quad f(t) = \sin(t) \ln(2 + \cos(t)) \ .$$
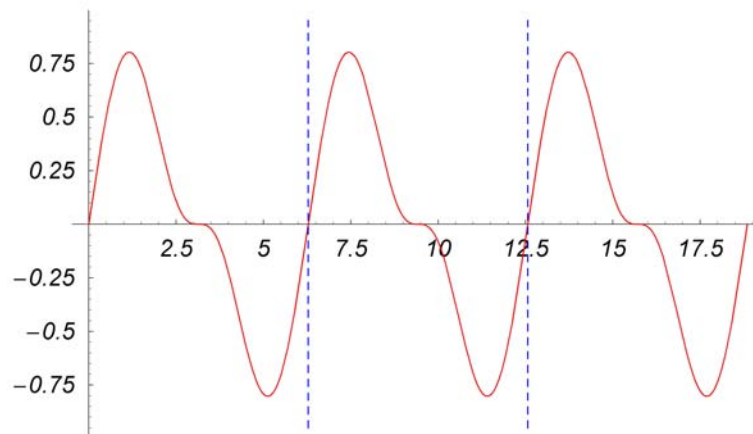
Figure 5.2: Graph of f on $I = [0, 6\pi]$

The Fourier coefficients of f via Algorithm 3.2 are given by:

$$c_n = \begin{cases} \frac{i\sqrt{3}(-2+\sqrt{3})^n(2\sqrt{3}+3n)}{3(n^2-1)} & \forall n \geq 2 \\ \frac{i}{4}(-7 + 4\sqrt{3} + \ln(7 - 4\sqrt{3}) + 2\ln(2)) & \text{if } n = 1 \\ 0 & \text{if } n = 0 \\ \frac{-i}{4}(-7 + 4\sqrt{3} + \ln(7 - 4\sqrt{3}) + 2\ln(2)) & \text{if } n = -1 \\ \frac{i\sqrt{3}(-2+\sqrt{3})^{-n}(2\sqrt{3}-3n)}{3(n^2-1)} & \forall n \leq -2 \ . \end{cases}$$

Since f is odd, the real and the complex Fourier coefficients of f are connected in the following way: $a_n = 0$ and $b_n = 2ic_n$ for $n \geq 0$. Parseval's identity (5.2) leads to

$$\int_0^{2\pi} f(t)^2 dt = \pi \left(\sum_{n=1}^{\infty} b_n^2 \right)$$

and we get:

$$\int_0^{2\pi} \left(\sin(t) \ln(2 + \cos(t)) \right)^2 =$$

$$\frac{\pi}{16} \left| 4\sqrt{3} - 2\ln(2 + \sqrt{3}) - 7 + 2\ln(2) \right|^2 + \pi \sum_{n=2}^{\infty} \left| \frac{-2\sqrt{3}(-2 + \sqrt{3})^n (2\sqrt{3} + 3n)}{3(n^2 - 1)} \right|^2$$

$$= \frac{-\pi}{4(7 - \sqrt{3})} (1961 - 1132\sqrt{3} + (-384 + 224\sqrt{3})\ln(4\sqrt{3} - 6) + (-56 + 32 \text{ polylog}(2, 7 - 4\sqrt{3})))$$

$$\frac{\pi}{16} \left| 4\sqrt{3} - 2\ln(2 + \sqrt{3}) - 7 + 2\ln(2) \right|^2 .$$

Hence

$$A = \frac{-\pi}{4(7 - \sqrt{3})} (1961 - 1132\sqrt{3} + (-384 + 224\sqrt{3})\ln(4\sqrt{3} - 6) + (-56 + 32 \text{ polylog}(2, 7 - 4\sqrt{3})))$$

$$\frac{\pi}{16} \left| 4\sqrt{3} - 2\ln(2 + \sqrt{3}) - 7 + 2\ln(2) \right|^2 ,$$

where the polylogarithm (also known as de Jonquire's function) is a special function denoted $Li_s(z)$ or $\text{polylog}(s, z)$ and is defined by

$$\text{polylog}(s, z) = Li_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s} .$$

Example 5.5

Consider the complex function

$$f(t) = \ln(2 + e^{it}) .$$

Our goal is to compute

$$A = \int_0^{2\pi} |\ln(2 + e^{it})|^2 dt = \int_0^{2\pi} |f(t)|^2 dt ,$$

The Fourier coefficients of f are given by

$$c_n = \begin{cases} \frac{(-1)^{n+1}}{n2^n} & \forall n \geq 1 \\ \ln(2) \quad \text{if} & \text{if } n = 0 \\ 0 & \text{otherwise} . \end{cases}$$

According to Parseval's identity (5.1), we get

$$
\begin{aligned}
A = \int_0^{2\pi} |\ln(2 + e^{it})|^2 dt = \int_0^{2\pi} |f(t)|^2 dt &= 2\pi \sum_{n=-\infty}^{\infty} |c_n|^2 \\
&= 2\pi \left(|c_0|^2 + \sum_{n=1}^{\infty} |c_n|^2 \right) \\
&= 2\pi \left(\ln(2)^2 + \sum_{n=2}^{\infty} \left(\frac{(-1)^n}{(n-1)2^{n-1}} \right)^2 \right) \\
&= 2\pi \left(\ln(2)^2 + \operatorname{polylog}(2, \frac{1}{4}) \right) .
\end{aligned}
$$

Hence

$$
A = 2\pi \left((\ln(2))^2 + \operatorname{polylog}(2, \frac{1}{4}) \right) .
$$

5.2   Application in the Solution of Boundary Value Problems

5.2.1   Solving the Heat Equation

Joseph Fourier has developed in [Fou22] a method to solve the heat equation. This method has been extended in a more general setting to solve boundary value problems, see [GD04], [CB78], [Spi74]. One can obtain symbolic solutions to these problems only when the initial values may be developed in Fourier series.

Consider the rod to be the segment $[0, 2\pi]$ of the real axis x, and denote by $u(x, t)$ the temperature of an element of abscise x at the time t. Then u satisfies the partial differential equation

$$
\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} . \tag{5.3}
$$

where a is a non-zero constant and we impose the following initial condition

$$
u(x, 0) = f(x) = \sin(2x) \ln(2 + \cos(2x)) \tag{5.4}
$$

and the following boundary conditions

$$
u(0, t) = u(\pi, t) = 0 . \tag{5.5}
$$

A family of non-trivial solutions of the system of equation constituted by (5.3)–(5.5) are given by

$$
u_n(x, t) = \sin(nx) e^{-a^2 n^2 t} \quad \text{and therefore} \quad u_n(x, t) = \sum_{n=0}^{\infty} b_n \sin(nx) e^{-a^2 n^2 t} .
$$

The last step is to determine b_n and this gives a representation of the solution. For $t = 0$, we have:

$$u(x, 0) = \sum_{n=0}^{\infty} b_n \sin(nx) = f(x) \ .$$

Therefore b_n can be considered as the sine Fourier coefficients of the function f, i.e.

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx = \frac{2}{\pi} \int_0^{\pi} \sin(2t) \ln(2 + \cos(2t)) \sin(nx) dx \ .$$

Since the initial function f is a holonomic trigonometric function, applying Algorithm 3.2 to f we get the following complex Fourier coefficients:

$$c_n = \begin{cases} \frac{-i\sqrt{3}(-2+\sqrt{3})^n(2\sqrt{3}+3n)}{3(n^2-1)} & \forall n \geq 2 \\ \frac{i}{4}(-7 + 4\sqrt{3} + \ln(7 - 4\sqrt{3}) + 2\ln(2)) & \text{if } n = 1 \\ 0 & \text{if } n = 0 \\ \frac{-i}{4}(-7 + 4\sqrt{3} + \ln(7 - 4\sqrt{3}) + 2\ln(2)) & \text{if } n = -1 \\ \frac{i\sqrt{3}(-2+\sqrt{3})^{-n}(2\sqrt{3}-3n)}{3(n^2-1)} & \forall n \leq -2 \ . \end{cases}$$

Since f is odd, only the sine Fourier coefficients are needed here. Therefore we get

$$a_n = 0, \quad b_n = 2ic_n \Longleftrightarrow$$

$$b_1 = \frac{-1}{2}(-7 + 4\sqrt{3} + \ln(7 - 4\sqrt{3}) + 2\ln(2)), \quad b_n = \frac{2\sqrt{3}(-2 + \sqrt{3})^n(2\sqrt{3} + 3n)}{3(n^2 - 1)} \quad (n \geq 2).$$

We can now deduce the solution of the system (5.3)-(5.4)-(5.5) and we get:

$$u_n(x, t) = \left(\frac{-1}{2}(-7 + 4\sqrt{3} + \ln(7 - 4\sqrt{3}) + 2\ln(2)) + \frac{2\sqrt{3}(-2 + \sqrt{3})^n(2\sqrt{3} + 3n)}{3(n^2 - 1)}\right) \sin(nx) e^{-a^2 n^2 t} \ .$$

5.2.2   Solving Ordinary Differential Equations in Terms of Fourier Series

Analogous to holonomic differential equations whose solutions may be searched in terms of power series, we give in this section a generalization for solving a trigonometric holonomic differential equation, this time in terms of Fourier series. A direct consequence of that algorithmic generalization is the solution of differential equations of higher order, which cannot be solved explicitly via current CAS. Example 5.8 is an illustration of this.

Let us aim to solve the differential equation

$$\begin{cases} \sum_{p=0}^{P} \sum_{l=0}^{L} (\alpha_{pl} \cos(l\omega t) + \beta_{pl} \sin(l\omega t)) f^{(p)}(t) = 0 \\ f^{(i)}(a) = a_i \quad (i = 0, \dots, P - 1) \\ f^{(i)}(b) = b_i \quad (i = 0, \dots, P - 1) \end{cases} \tag{5.6}$$

or

$$\begin{cases} \sum_{p=0}^{P} \sum_{l=0}^{L} \left(\gamma_{pl} e^{-il\omega t} + \delta_{pl} e^{il\omega t} \right) f^{(p)}(t) = 0 \\ f^{(i)}(a) = a_i \qquad (i = 0, \dots, P-1) \\ f^{(i)}(b) = b_i \qquad (i = 0, \dots, P-1) \,. \end{cases} \tag{5.7}$$

We may use Algorithm 3.1(resp. Algorithm 3.8) to convert (5.6) (resp. (5.7)) into a recurrence equation. In the last step we solve the obtained recurrence equation to deduce the solution of the given differential equation in terms of a Fourier series. That process can be summarized in the following algorithm:

Algorithm 5.2: Solving a trigonometric holonomic differential equation in terms of a Fourier Series (SolTHDEFou)

    input : A differential equation DE of the form (5.6) or (5.7).
    output: The solution of the considered differential equation in term of Fourier series or a
              recurrence equation satisfied by the Fourier coefficients of the solution of the
              considered DE

1 **begin**
2   Apply Algorithm 3.1 or Algorithm 3.8 to the considered differential equation to
       convert it into a recurrence equation.
3   Solve the obtained recurrence equation RE and deduce the solution of the
       considered differential equation in terms of a Fourier series.
4 **end**

Example 5.6

Let us search for a solution in terms of a Fourier series of the differential equation

$$DE : \begin{cases} (-44\cos(4t) - 28)F'(t) - 12F''(t))\sin(4t) + (\cos(4t) - 7)(F'''(t) + 48F(t)\sin(4t) = 0 \\ f(0) = f(\pi) \\ f'(0) = f'(\pi) \end{cases}$$

The conversion of the previous DE using Algorithm 3.8 leads to the following recurrence equation

$$RE : -16i(2n+1)n(2n-1)c_{n+1} + 224i(2n+1)n(2n-1)c_n - 16i(2n+1)n(2n-1)c_{n-1} \,.$$

Solving the previous RE considering the initial values $c_0 = a$ and $c_1 = b$, we get

$$c_n = \frac{1}{24}(7a + 4a\sqrt{3} - b)\sqrt{3}(7 - 4\sqrt{3})^n + \frac{1}{24}(-7a + 4a\sqrt{3} + b)\sqrt{3}(7 + 4\sqrt{3})^n$$

and we deduce that the solution of the previous trigonometric holonomic differential equation in terms of a Fourier series is given as

$$f(t) = \sum_{n=0}^{\infty} \left(\frac{1}{12}(7a + 4a\sqrt{3} - b)\sqrt{3}(7 - 4\sqrt{3})^n + \frac{1}{12}(-7a + 4a\sqrt{3} + b)\sqrt{3}(7 + 4\sqrt{3})^n \right) \cos(2nt)$$

Note that solving the previous differential equation with existing methods leads to

$$f(t) = \frac{C_1(1 + \cos(2t)) + C_2 \sin(2t)}{\cos(4t) - 7}, \quad C_1, C_2 \in \mathbb{K} \,.$$

Example 5.7

Consider the differential equation with side conditions

$$\begin{cases} e^{it} F(t) + F^{(99)}(t) = 0 \\ F^{(j)}(0) = F^{(j)}(2\pi), \quad (j = 0, \ldots, 98) \,. \end{cases} \tag{5.8}$$

Note that an explicit solution of (5.8) may not be found. Algorithm 3.8 converts the previous differential equation into the following recurrence equation

$$c_{n-1} - in^{99} c_n = 0$$

which can be solved using the initial value $c_0 = a$, to get the closed form

$$c_n = \frac{a(-i)^n}{n!^{99}} \,.$$

We deduce that the solution of (5.8) in terms of a Fourier series is given as

$$f(t) = \sum_{n=0}^{\infty} \left(\frac{a(-i)^n}{n!^{99}} \right) e^{int} \,.$$

Chapter 6

General Conclusion and Perspectives

We presented in this thesis algorithms for the computation of the Fourier series for a family of functions satisfying a differential equation of a particular type. These algorithms can be applied to many functions whose Fourier coefficients cannot be computed via the classical way. In some cases symbolic expressions of those series could not be obtained because the resulting recurrence equations could not be solved. However these cases nevertheless provide important informations on the Fourier coefficients of the considered functions. From these algorithms we deduced an algorithm for the computation of definite integrals, without involving the knowledge of the anti-derivatives. This other possibility of computing definite integrals via Fourier series enables the computation of many definite integrals whose computations were out of reach of current CAS. Another consequence of that algorithm for Fourier series is that it offers the possibility to get an explicit solution of certain boundary value problems. An algorithm for the factorization of holonomic recurrence operators via Fourier series is deduced.

We restricted ourselves in this thesis on univariate functions. Multivariate Fourier series (see e.g. [KK71], [DM72], [Spi74]) have also many applications, for example in solving partial differential equations. In particular one notable application of Fourier series on the square is in image compression. One may be interested to know if similar algorithms can be found for multivariate Fourier series, just as well as the issue to know if connections exist between the computation of multiple integrals and multivariate Fourier series.

Appendix A

A.1 Fourier Series of Some Trigonometric Holonomic Functions

Example A.1

Consider the function defined on the interval $I = [0, 2\pi]$ by:

$$f(t) = \left(\frac{\cos t}{2 + \cos^2 t} \right)^3 \ .$$

Then $\omega = \frac{2\pi}{2\pi - 0} = 1$ and f satisfies the differential equation
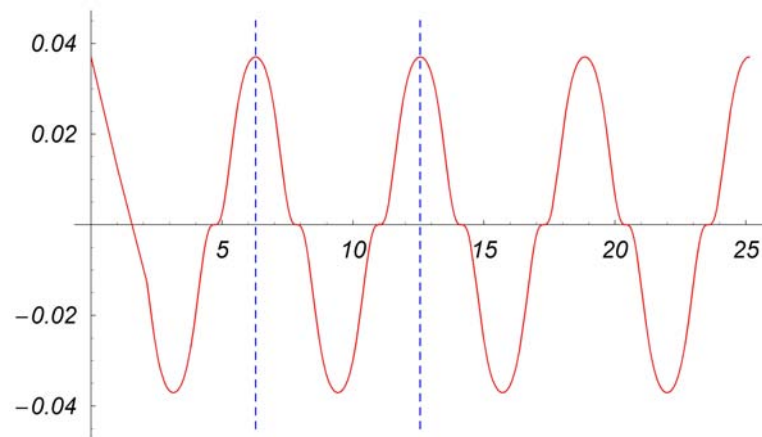
Figure A.1: Case of a rational trigonometric function

$$DE : (11 \, \cos t + \cos(3 \, t)) \, F'(t) + 3 \, (7 \, \sin(t) - \sin(3t)) \, F(t) = 0 \ .$$

Therefore, its complex Fourier coefficients on I are solution of the recurrence equation

$$RE : n \, c_{n+3} + (32 + 11 \, n) \, c_{n+1} + (-32 + 11 \, n) \, c_{n-1} + n \, c_{n-3} = 0 \ .$$

Using 6 initial values, for $n \geqq 0$ and 6 initial values, for $n \leq 0$, we get finally that those coefficients are given by[1]

$$
c_n = \begin{cases}
\frac{-1}{144}(\sqrt{3} - \sqrt{2})^n (2\sqrt{3}n^2 - 4\sqrt{3} - 3n\sqrt{2})\left(\frac{1-(-1)^n}{2}\right)(-1)^{\frac{n-1}{2}} & \forall n \geq 1 \\
0 & \text{if} \quad n = 0 \\
\frac{-1}{144}(\sqrt{3} - \sqrt{2})^{-n}(2\sqrt{3}n^2 - 4\sqrt{3} + 3n\sqrt{2})\left(\frac{1-(-1)^{-n}}{2}\right)(-1)^{\frac{-n-1}{2}} & \forall n \leq -1 \ .
\end{cases}
$$

Since f is even, $a_n = 2c_n, b_n = 0$ $(n \geqq 0)$. We deduce that the Fourier series of f on I can be written as

$$
f(t) = \left(\frac{\cos t}{2 + \cos^2 t}\right)^3
$$

$$
= \sum_{n=1}^{\infty} \frac{-1}{72}(\sqrt{3} - \sqrt{2})^n (2\sqrt{3}\, n^2 - 4\sqrt{3} - 3\, n\, \sqrt{2})\left(\frac{1 - (-1)^n}{2}\right)(-1)^{\frac{n-1}{2}} \cos(nt) \ .
$$

Example A.2

Consider now the function defined on $I = [0, \frac{2\pi}{5}]$ by

$$
f(t) = \cos(5t)\ln(2 + \cos(5t)) \ .
$$

$\omega = \frac{2\pi}{\frac{2\pi}{5} - 0} = 5$. f satisfies the following differential equation DE
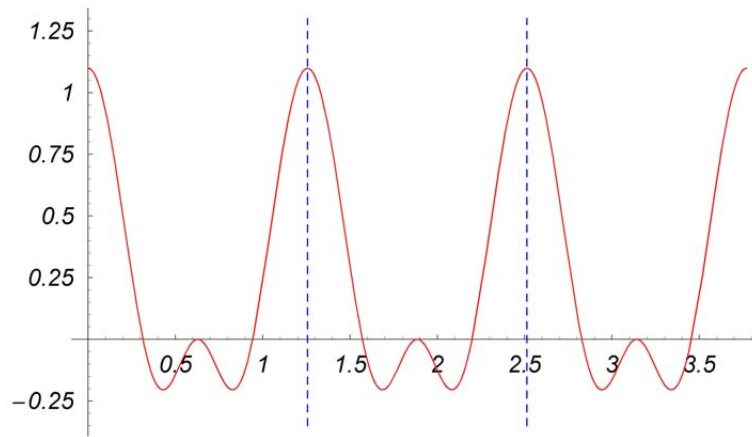
Figure A.2: Composition of logarithm with trigonometric functions

$$
DE := (-500000 + 843750\cos(5t))F'(t) + 28125\sin(5t)F''(t)
$$

---

[1] which is a linear combination of hypergeometric terms.

$$+(54375\cos(5t)-45000)F^{(3)}(t)+4625\sin(5t)F^{(4)}(t)+(825\cos(5t)-1200)F^{(5)}(t)$$
$$+120\sin(t)F^{(6)}(t)+(-4\cos(5t)-8)F^{(7)}(t)=0 .$$

The conversion of DE into a recurrence equation satisfied by the complex Fourier coefficients c_n of f gives

$$RE := i(n-1)(n+2)(n+1)(n-2)^2(2n+1)^2 c_{n-1}+16i(n-2)(n+2)n(n+1)^2(n-1)^2 c_n$$
$$+i(n-1)(n-2)(n+1)(n+2)^2(2n-1)^2 c_{n+1}=0 .$$

Solving RE using 2 initial values, we get:

$$c_n = \begin{cases} \frac{(-2+\sqrt{3})^n(\sqrt{3}+2n)}{(n+1)(n-1)} & \forall n \geq 2 \\ \sqrt{3}-\frac{7}{4}+\frac{1}{2}\ln(2+\sqrt{3})-\frac{1}{2}\ln(2) & \text{if} \quad n=1 \\ 2-\sqrt{3} & \text{if} \quad n=0 \\ \sqrt{3}-\frac{7}{4}+\frac{1}{2}\ln(2+\sqrt{3})-\frac{1}{2}\ln(2) & \text{if} \quad n=-1 \\ \frac{(-2+\sqrt{3})^{-n}(\sqrt{3}-2n)}{(-n+1)(-n-1)} & \forall n \leq -2 . \end{cases}$$

Since f is even, $b_n=0$ and $a_n=2c_n$. Hence the Fourier series of f on I is given as

$$f(t)=2(2-\sqrt{3})+(2\sqrt{3}-\frac{7}{2}+\ln(2+\sqrt{3})-\ln(2))\cos(5t)+\sum_{n=2}^{\infty}\frac{2(-2+\sqrt{3})^n(\sqrt{3}+2n)}{(n+1)(n-1)}\cos(5nt) .$$

Example A.3
We investigate in this example the case of a trigonometric holonomic function satisfying a differential equation with coefficients in $\mathbb{K}[e^{-i\omega t}, e^{i\omega t}]$. Consider the complex function defined on the interval $[0, 2\pi]$ by

$$f(t) = \ln(4 + e^{-it} + e^{it}) \in \text{TH}(1)$$

satisfying the differential equation

$$DE : 4F'(t) + 2i(e^{it} - e^{-it})F''(t) + (e^{it} + 4 + e^{-it})F'''(t) .$$

The conversion of DE into a recurrence equation for the Fourier coefficients of f using Algorithm 3.8 gives:

$$RE : -i(n+1)(n-1)^2 c_{n-1} - 4i(n+1)n(n-1)c_n - i(n-1)(n+1)^2 c_{n+1} = 0 .$$

Solving RE with enough initial values leads to the solution

$$c_n = \begin{cases} \frac{-(-2+\sqrt{3})^n}{n} & \forall n > 0 \\ \ln(1+\frac{\sqrt{3}}{2}) & \text{if} \quad n=0 \\ \frac{(-2-\sqrt{3})^n}{n} & \forall n < 0 \end{cases}$$

and we deduce that the Fourier series of f is given by

$$f(t) = \sum_{n=1}^{\infty}\left(\frac{-(-2+\sqrt{3})^n}{n}\right)e^{-int} + \ln(1+\frac{\sqrt{3}}{2}) + \sum_{n=1}^{\infty}\left(\frac{-(-2+\sqrt{3})^n}{n}\right)e^{int} .$$

A.2   Some Particular Cases

A.2.1   First Case

In the following examples, for the given function we obtain a recurrence equation with appropriate initial values, but its solution is rather complicated and involves special functions.

Example A.4

Consider the function defined on $[0, 2\pi]$ by

$$f(t) = \frac{t^2}{2 + \cos(t)} \ .$$
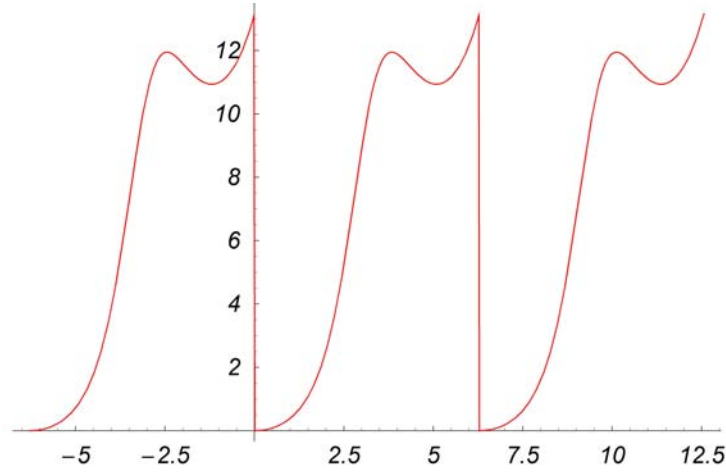
Figure A.3: Case of product of polynomial and rational trigonometric function

With $\omega = 1$ f satisfies the following differential equation:

$$DE := \sin(t)F(t) - 3\cos(t)F'(t) - 3\sin(t)F''(t) + (2 + \cos(t))F'''(t) \ ,$$

and its corresponding inhomogeneous recurrence equation of second order is

$$RE : -in^3 c_{n-1} - 4in^3 c_n - in^3 c_{n+1} = 4n(-i + \pi n) \ .$$

The initial values are given by

$$
\begin{aligned}
c_0 &= \frac{4(\pi^2 + 3\operatorname{polylog}(2, -2 + \sqrt{3})}{3}\sqrt{3}\pi \ , \\
c_1 &= \frac{1}{9}\big((12 - 8\sqrt{3})\pi^2 + 6i\pi((3 + 2\sqrt{3})\ln(3 - \sqrt{3}) \\
&\quad + (-3 + 2\sqrt{3})\ln(\frac{3 + \sqrt{3}}{6})) - 24\sqrt{3}\operatorname{polylog}(2, -2 + \sqrt{3})\big) \ .
\end{aligned}
$$

The previous recurrence equation does not have any closed form solution, see [Hoe98] and [Pet92]. However *Maple* finds a huge representation of the solution, constituted of a linear combination of special functions such as hypergeometric2F1, LerchPhi, and polylog.

Example A.5

Consider now the function

$$f(t) = \frac{t^2 e^{it}}{2 + e^{it}} .$$

f satisfies with $\omega = 1$ the following differential equation:

$$DE := -2iF(t) + 6F'(t) + 6iF''(t) + (-2 - e^{it}F'''(t) = 0$$

which leads to the first order inhomogeneous recurrence equation

$$RE := i(n-1)^3 c_{n-1} + 2i(n-1)^3 c_n = -2(n-1)(-\pi + n\pi - i)$$

and initial values

$$
\begin{aligned}
c_0 &= -2i\pi \ln(3) + 2i\pi \ln(2) - 2\operatorname{polylog}(2, -\frac{1}{2}) , \\
c_1 &= \frac{2}{3}\pi^2 + i\pi \ln(3) - i\pi \ln(2) + \operatorname{polylog}(2, -\frac{1}{2}) , \\
c_2 &= \frac{-1}{3}\pi^2 - \frac{1}{2}i\pi \ln(3) + \frac{1}{2}i\pi \ln(2) - \frac{1}{2}\operatorname{polylog}(2, -\frac{1}{2}) + i\pi + 1 .
\end{aligned}
$$

As in the previous example, a closed form solution does not exist.

A.2.2   Second case

In this case we deal with functions whose Fourier coefficients cannot be brought in an explicit form, because the obtained recurrence equation and the appropriate initial values cannot be solved.

Example A.6

Consider the function defined on $I = [0, 2\pi]$ by

$$f(t) = e^{-2\cos(t)} .$$

f satisfies the first order differential equation

$$DE := -2\sin(t)F(t) + F'(t) = 0 .$$

for $\omega = 1$. The conversion of DE into a recurrence equation for the complex Fourier coefficients of f gives

$$-c_{n-1} - nc_n + c_{n+1} = 0 .$$

Therefore the Fourier coefficients are given by

$$
\begin{cases}
-c_{n-1} - nc_n + c_{n+1} = 0 \\
c_0 = \operatorname{BesselI}(0, 2), \quad c_1 = -\operatorname{BesselI}(1, 2) .
\end{cases}
$$

An explicit solution is not accessible.
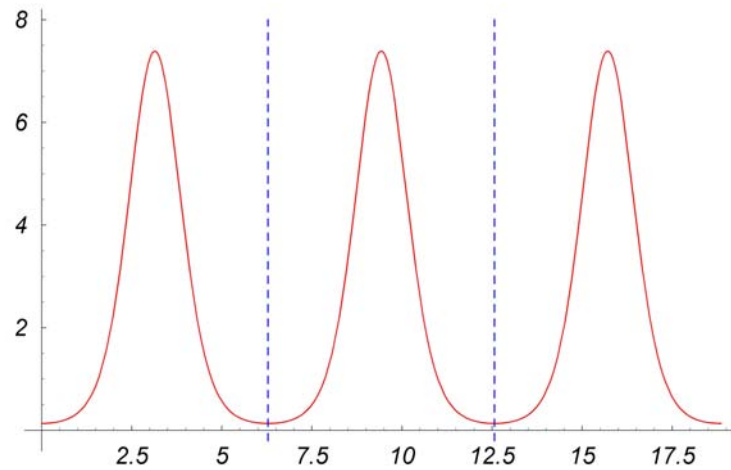
Figure A.4: Composition of exponential with trigonometric functions

A.2.3   Third case

Although an explicit form of the Fourier coefficients in Example A.6 was not accessible, we could at least compute sufficiently many initial values. In the following examples for the considered functions, we get a recurrence equation for their complex Fourier coefficients, but because of lack of sufficiently many symbolic initial values, the Fourier coefficients c_n can not be computed.

Example A.7
Consider the function defined on the interval $[0, 2\pi]$ by

$$f(t) = \cos(t)\sqrt{2 + \cos(t)} \ .$$

A differential equation of the form (2.9) with $\omega = 1$ satisfied by f is

$$DE : (-62\cos(t) - 52)F'(t) + 48\sin(t)F''(t) + (-64 - 32\cos(t))F'''(t) + 123\sin(t)F(t) = 0 \ .$$

The corresponding recurrence for the complex Fourier coefficients of f is

$$RE : i(2n-5)(16n^2 + 16n + 9)\pi c_{n-1} + 8i\pi n(16n^2 - 13)c_n$$

$$+i(16n^2 - 16n + 9)(2n + 5)\pi c_{n+1} = 0 \ .$$

$$c_0 = \frac{2\sqrt{3}}{3\pi}\left(-\operatorname{EllipticK}(\frac{1}{3}\sqrt{6}) + 2\operatorname{EllipticE}(\frac{1}{3}\sqrt{6})\right) \ .$$

The symbolic computation of c_1 is not successful. The same remark occurs with analogous functions such as $\ln(2 + \cos(t))\sqrt{2 + \cos(t)}$.
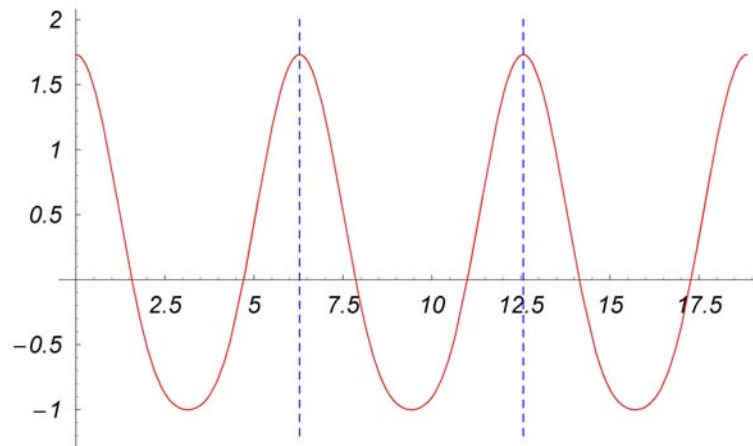
Figure A.5: Composition of square root with trigonometric functions

Example A.8

Consider the function

$$f(t) = \ln(2 + e^{it})\sqrt{2 + e^{it}} \ .$$

f satisfies the differential equation

$$DE := -2ie^{it}F(t) + (8e^{it} + 32 + 32e^{-it})F'(t) + 8i(e^{it} + 4 + 4e^{-it})F''(t) = 0 \ .$$

We deduce from DE that the complex Fourier coefficients of f satisfies the recurrence relation

$$-2i(-3 + 2n)^2 c_{n-1} - 32in(n - 1)c_n - 32in(n + 1)c_{n+1} = 0 \ .$$

However there is not possibility to compute the initial values of c_n symbolically.

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List of Algorithms

Index

Eidesstattliche Erklärung

Hiermit versichere ich, dass ich die vorliegende Dissertation selbständig und ohne unerlaubte Hilfe angefertigt und andere als die in der Dissertation angegebenen Hilfsmittel nicht benutzt habe. Alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten oder unveröffentlichten Schriften entnommen sind, habe ich als solche kenntlich gemacht. Kein Teil dieser Arbeit ist in einem anderen Promotions- oder Habilitationsverfahren verwendet worden.

Kassel, den 27. November 2009