Moments
of
Classical Orthogonal Polynomials

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Abstract

The aim of this work is to find simple formulas for the moments μ_n for all families of classical orthogonal polynomials listed in the book by Koekoek, Lesky and Swarttouw [30]. The generating functions or exponential generating functions for those moments are given.

To my dear parents

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Contents

Chapter 1

Introduction

The xyz-axes of three-dimensional space are pairwise orthogonal with each other. This is very convenient since for that reason many formulas are extremely simple. Every point of three-dimensional space is written as linear combination of such orthogonal coordinates. In a similar fashion, many functions can be written as linear combinations of orthogonal polynomials which play the role of the coordinates. For this reason orthogonal polynomials play a very prominent role in applications.

Monic polynomial families orthogonal with respect to the measure $d\alpha(x)$

$$\int_a^b P_n(x)P_m(x)d\alpha(x) = k_n\,\delta_{n,m},\ k_n \neq 0,\ n \geq 0,$$

are given explicitly in terms of the moments $\mu_n = \displaystyle\int_a^b x^n\,d\alpha(x),\ n \geq 0$, by [49]

$$P_n(x) = \frac{1}{d_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{n-1} & \mu_{n+1} & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix},$$

where

$$d_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} \neq 0,\ n \geq 0.$$

The previous representation shows that the moments characterize fully the orthogonal family $(P_n)_n$.

Also, the moments are involved in the representation of the Stieltjes series

$$S(x) = \sum_{n=0}^{\infty} \frac{\mu_n}{x^{n+1}},$$

which is useful for the characterization of families of orthogonal polynomials via the Riccati equation and also for the determination of the measure $d\alpha(x)$ by means of the Stieltjes inverse formula [48]

$$\alpha(t) - \alpha(s) = -\frac{1}{\pi} \lim_{y \to 0^+} \int_s^t \operatorname{Im}(S(x + i\,y)\,dx.$$

For the moments $(\mu_n)_{n\in\mathbb{N}}$ of an orthogonal family, the generating function is defined by

$$G_0(z) = \sum_{n=0}^{\infty} \mu_n z^n,$$

while the exponential generating function is defined by

$$G_1(z) = \sum_{n=0}^{\infty} \mu_n \frac{z^n}{n!},$$

and the q-exponential generating functions are defined by

$$G_2(z) = \sum_{n=0}^{\infty} \mu_n \frac{z^n}{(q;q)_n},$$
$$G_3(z) = \sum_{n=0}^{\infty} \mu_n \frac{z^n}{[n]_q!}.$$

Generating functions, exponential generating functions and q-exponential generating functions contain the information of *all* moments of the orthogonal polynomial family at the same time.

Despite the important role that the moments play in various topics of orthogonal polynomials and applications to other domains such as statistics and probability theory, no exhaustive repository of moments for the well-known classical orthogonal polynomials can be found in the literature. The book by Koekoek, Lesky and Swarttouw [30] which is one of the best and most famous documents containing almost all kinds of formulas and relations for various classical orthogonal polynomials does not provide information about the moments. It becomes therefore imperative to investigate this topic in order to complete such missing important information.

Classical orthogonal polynomials of a continuous, discrete and q-discrete variable are known to be orthogonal with respect to a weight function ρ satisfying respectively the Pearson, the discrete Pearson and the q-discrete Pearson equation

$$(\sigma(x)\rho(x))' = \tau(x)\rho(x), \tag{1.1}$$
$$\Delta\left(\sigma(x)\rho(x)\right) = \tau(x)\rho(x), \tag{1.2}$$
$$D_q\left(\sigma(x)\rho(x)\right) = \tau(x)\rho(x), \tag{1.3}$$

where $\sigma(x) = ax^2 + bx + c$ is a non-zero polynomial of degree at most two, $\tau(x) = dx + e$ is a first degree polynomial, D_q is the Hahn operator $D_q f(x) = \frac{f(qx)-f(x)}{(q-1)x}$, $q \neq 1$, and Δ is the forward difference operator $\Delta f(x) = f(x+1) - f(x)$.

In addition, classical orthogonal polynomials of a continuous, discrete and q-discrete variable satisfy the following second-order hypergeometric differential, difference or q-difference equations, respectively,

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda_n y(x) = 0, \tag{1.4}$$
$$\sigma(x)\Delta\nabla y(x) + \tau(x)\Delta y(x) + \lambda_n y(x) = 0, \tag{1.5}$$
$$\sigma(x)D_q D_{\frac{1}{q}} y(x) + \tau(x)D_q y(x) + \lambda_{n,q} y(x) = 0, \tag{1.6}$$

where λ_n and $\lambda_{n,q}$ are constants given by

$$\lambda_n = -n((n-1)a + d), \quad \lambda_{n,q} = -a[n]_{1/q}[n-1]_q - d[n]_q,$$

with $[n]_q = \frac{1-q^n}{1-q}$, and ∇ is the backward difference operator

$$\nabla f(x) = f(x) - f(x-1).$$

The corresponding moments of these three classical families (called here "very classical orthogonal polynomials") satisfy a second-order recurrence relation of the form

$$\mu_{n+1} = a(n)\mu_n + b(n)\mu_{n-1},$$

where $a(n)$ and $b(n)$ are rational functions of n or q^n.

There are other classes of classical orthogonal polynomials whose variable $x(s)$ is a quadratic or q-quadratic lattice of the form

$$x(s) = c_1\, q^{-s} + c_2\, q^s + c_3, \tag{1.7}$$

or

$$x(s) = c_4\, s^2 + c_5\, s + c_6 \tag{1.8}$$

These polynomials are known to satisfy a second-order divided-difference equation [8, 17]

$$\phi(x(s))\, \mathbb{D}_x^2 P_n(x(s)) + \psi(x(s))\, \mathbb{S}_x \mathbb{D}_x P_n(x(s)) + \lambda_n\, P_n(x(s)) = 0, \tag{1.9}$$

where λ_n is a constant term, ϕ and ψ are polynomials of degree at most two and of degree one, respectively, and the divided-difference operators $\mathbb{D}_x$ and $\mathbb{S}_x$ are defined by [17]

$$\mathbb{D}_x\, f(x(s)) = \frac{f(x(s+\frac{1}{2})) - f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})}, \quad \mathbb{S}_x\, f(x(s)) = \frac{f(x(s+\frac{1}{2})) + f(x(s-\frac{1}{2}))}{2}. \tag{1.10}$$

Combining all the previous orthogonal families leads to the families of the so-called *Askey-Wilson scheme*, defined explicitly in [30].

The work is presented in five chapters.

Chapter 1 is the introduction.

In *Chapter 2*, we give many definitions and recall known and useful results concerning special functions and orthogonal polynomials. Some useful difference operators are introduced and some of their properties are proved.

In *Chapter 3*, using some classical well known formulas, we compute canonical moments of some orthogonal polynomials, next, interesting generating functions for some of these moments are provided. It is seen for example that the function

$$\sqrt{\pi} e^{z^2/4} = \sum_{n=0}^{\infty} \mu_n \frac{z^n}{n!}$$

generates the Hermite moments that are

$$\mu_n = \frac{1+(-1)^n}{2} \Gamma(\frac{n+1}{2}),$$

or the function

$$\frac{\Gamma(\alpha+1)}{(1-z)^{\alpha+1}} = \sum_{n=0}^{\infty} \mu_n \frac{z^n}{n!}$$

generates the canonical Laguerre moments that are

$$\mu_n = \Gamma(n+\alpha+1).$$

It is not always easy to get those canonical moments by direct computations.

In *Chapter 4*, we provide results for the inversion problem for all the polynomials in the Askey scheme. These inversion formulas will enable in Chapter 5 to get explicit representations of generalized moments.

In *Chapter 5*, we compute explicitly canonical moments for all the fifty one polynomials listed in [30]. The fundamental idea here is to use Theorem 50, which gives a link between the inversion coefficients and the generalized moments combined with the obvious links (see pages 13 and 13) between canonical and generalized moments to get the canonical moments.

In order to get those links, we have proved Taylor formulas with respect to particular bases, for example:

$$f(x) = \sum_{k=0}^{n} \frac{(\mathsf{D}_\varepsilon^k f)(0)}{k!} \xi_k(x,\varepsilon), \quad \text{see page 52,}$$

$$f(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \mathcal{D}^k f\left(i\left(a+\frac{k}{2}\right)\right) \eta_k(a,x), \quad \text{see page 53,}$$

where

$$\mathsf{D}_\varepsilon f(x) = \frac{\Delta f(u(x))}{\Delta u(x)}, \quad u(x) = -x(x+\varepsilon),$$

and

$$\mathcal{D} f(x) = f\left(x+\frac{i}{2}\right) - \left(x-\frac{i}{2}\right), \quad \text{with} \quad i^2 = -1.$$

Combining these results we get for example the following explicit formulas for the canonical moments:

- canonical Wilson moments (see page 76)

$$\mu_n = 2\pi \frac{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)}{\Gamma(a+b+c+d)} \times \sum_{k=0}^{n}\sum_{l=0}^{k} \frac{(-k)_l}{k!l!} \frac{(a+b)_k(a+c)_k(a+d)_k}{(a+b+c+d)_k} \frac{(-2a-2k+2l)}{(-2a-2k+l)_{k+1}} (a+k-l)^{2n}.$$

- canonical Racah moments (see page 76)

$$\mu_n = \mu_0 \sum_{k=0}^{n} \frac{\mathsf{D}_\varepsilon^k [x(x+\varepsilon)]^n_{|x=0}}{k!} \frac{(\alpha+1)_k(\beta+\delta+1)_k(\gamma+1)_k}{(\alpha+\beta+2)_k},$$

  where

$$\mu_0 = \begin{cases} \dfrac{(-\beta)_N(\gamma+\delta+2)_N}{(-\beta+\gamma+1)_N(\delta+1)_N} & \text{if} \quad \alpha+1=-N \\ \dfrac{(-\alpha+\delta)_N(\gamma+\delta+2)_N}{(-\alpha+\gamma+\delta+1)_N(\delta+1)_N} & \text{if} \quad \beta+\delta+1=-N \\ \dfrac{(\alpha+\beta+2)_N(-\delta)_N}{(\alpha-\delta+1)_N(\beta+1)_N} & \text{if} \quad \gamma+1=-N. \end{cases}$$

The contribution of this work can be seen at three levels:

- the work is a good database for the inversion formula of *all* the orthogonal families listed in [30]; the inversion formulas
  - for the quadratic case (the Wilson polynomials, the Continuous Dual Hahn polynomials, the Racah polynomials, the Continuous Hahn polynomials, the Dual Hahn polynomials and the Meixner Pollaczek polynomials),
  - for the q-quadratic case (the Continuous q-Hahn polynomials, the Dual q-Hahn polynomials, the Al-Salam-Chihara polynomials, the q-Meixner-Pollaczek polynomials, the Continuous q-Jacobi polynomials, the continuous q-Ultraspherical polynomials, the Continuous q-Legendre polynomials, the Dual q-Krawtchouk polynomials, the Continuous big q-Hermite polynomials and the Continuous q-Laguerre polynomials)

  are new;
- the work is a good database for *all the moments* of all the orthogonal families listed in [30]; as far as we know, all the generalized moments given in Chapter 5 are new. Concerning the canonical moments

- **for the classical continuous orthogonal polynomials**, two new representations for the Jacobi canonical moments are given;
- **for the classical discrete orthogonal polynomials**, the representations of the canonical Hahn moments and the canonical Krawtchouk moments we have given are new;
- **for the classical q-discrete orthogonal polynomials**, the representations of the canonical Big q-Jacobi moments, the canonical q-Hahn moments, the canonical Big q-Laguerre moments, the canonical q-Meixner moments, the canonical Quantum q-Krawtchouk moments, the canonical q-Krawtchouk moments and the canonical Affine q-Krawtchouk moments we have given are new.
- **for the classical quadratic orthogonal polynomials**, the representations of the canonical Wilson moments, the canonical Racah moments, the canonical Continuous Dual Hahn moments, the canonical Continuous Hahn moments, the canonical Dual Hahn moments, the canonical Meixner-Pollaczek moments are new.
- **for the classical q-quadratic orthogonal polynomials**, as far as we know, we have encountered only the canonical Askey-Wilson moments in the literature, the rest seems to be new;

• important generating functions for those moments are provided.

Chapter 2

Definitions and Miscellaneous Relations

2.1 Special functions

2.1.1 Gamma and Beta functions

Definition 1. [30, P. 3] *The Gamma function is defined by*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad z \in \mathbb{C}, \, \mathrm{Re}(z) > 0. \tag{2.1}$$

Note that for a complex number z such that $\mathrm{Re}(z) > 0$,

$$\Gamma(z+1) = z\Gamma(z) \tag{2.2}$$

and particularly, for a nonnegative integer n, the following relation is valid

$$\Gamma(n+1) = n!.$$

Note that formula (2.2) is used to extend progressively the validity of the Gamma function to any complex number which is not a negative integer by writing

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \; \cdots .$$

Definition 2. [30, P. 3] *The Beta function is defined by*

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt \quad z, w \in \mathbb{C}, \, \mathrm{Re}(z) > 0, \, \mathrm{Re}(w) > 0.$$

The connection between the Beta function and the Gamma function is given by the relation

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \mathrm{Re}(x) > 0, \quad \mathrm{Re}(y) > 0.$$

2.1.2 Hypergeometric functions

Definition 3. [30, P. 4] *The Pochhammer symbol or shifted factorial is defined by*

$$(a)_0 := 1 \quad and \quad (a)_n = a(a+1)(a+2)\cdots(a+n-1), \quad a \neq 0 \quad n = 1, 2, 3, \ldots .$$

The following notation (falling factorial) will also be used:

$$a^{\underline{0}} := 1 \quad \text{and} \quad a^{\underline{n}} = a(a-1)(a-2)\cdots(a-n+1), \quad n = 1, 2, 3, \ldots .$$

It should be noted that the Pochhammer symbol and the falling factorial are linked as follows:

$$(-a)_n = (-1)^n a^{\underline{n}}.$$

Definition 4. [30, P. 5] *The hypergeometric series ${}_rF_s$ is defined by*

$$ {}_rF_s\left(\begin{matrix} a_1,\cdots,a_r \\ b_1,\cdots,b_s \end{matrix}\middle| z\right) := \sum_{n=0}^{\infty} \frac{(a_1,\cdots,a_r)_n}{(b_1,\cdots,b_s)_n}\frac{z^n}{n!}, $$

where

$$ (a_1,\ldots,a_r)_n = (a_1)_n \cdots (a_r)_n. $$

An example of a summation formula for the hypergeometric series is given by the binomial theorem ([30, P. 7])

$$ {}_1F_0\left(\begin{matrix} a \\ - \end{matrix}\middle| -z\right) = \sum_{n=0}^{\infty}\binom{a}{n} z^n = (1+z)^a, \quad |z|<1, \tag{2.3} $$

where

$$ \binom{a}{n} = \frac{(-1)^n}{n!}(-a)_n. $$

2.1.3 Basic hypergeometric functions

An important extension of the hypergeometric function is the q-hypergeometric function (general references for q-hypergeometric functions are [19], [3] or [50], [46]).

Definition 5. [30, P. 11] *The q-variant of the shifted factorial is defined by*

$$ \begin{aligned} &(a;q)_0 = 1, \\ &(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}),\ \ n=1,2,\ldots\,. \end{aligned} $$

When $n=\infty$, we set

$$ (a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n),\ \ |q|<1. $$

Definition 6. [30, P. 15] *The q-hypergeometric function denoted by ${}_r\phi_s$ is defined by*

$$ {}_r\phi_s\left(\begin{matrix} a_1,a_2,\cdots,a_r \\ b_1,b_2,\cdots,b_s \end{matrix}\middle| q;z\right) = \sum_{n=0}^{\infty} \frac{(a_1,\cdots,a_r;q)_n}{(b_1,\cdots,b_s;q)_n}\left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r}\frac{z^n}{(q,q)_n}, $$

where

$$ (a_1,a_2,\cdots,a_m;q)_n = (a_1;q)_n(a_2,q)_n\cdots(a_m;q)_n. $$

We will also use the following common notations

$$ [a]_q = \frac{1-q^a}{1-q}, \quad a\in\mathbb{C}, \quad q\neq 1, \tag{2.4} $$

$$ \begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}, \quad 0\le m\le n. \tag{2.5} $$

called the q-bracket and the q-binomial coefficient, respectively.
A q-analogue of the binomial theorem (2.3) is called the q-binomial theorem [30, P. 16]:

$$ \sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}, \quad |x|<1,\ 0<|q|<1. \tag{2.6} $$

Some consequences of the q-binomial theorem are the Euler formulas:

$$\sum_{n=0}^{\infty} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}, \quad |x| < 1, \quad |q| < 1, \tag{2.7}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} x^n}{(q;q)_n} = (x;q)_\infty, \quad |q| < 1. \tag{2.8}$$

The Ramanujan summation formula [3, P. 502] is also valid for $|q| < 1$ and $|ba^{-1}| < |x| < 1$,

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} x^n = \frac{(ax;q)_\infty (q/ax;q)_\infty (q;q)_\infty (b/a;q)_\infty}{(x;q)_\infty (b/ax;q)_\infty (b;q)_\infty (q/a;q)_\infty}. \tag{2.9}$$

Another important formula is the Jacobi triple product identity [3, P. 497]

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} x^k = (x;q)_\infty (q/x;q)_\infty (q;q)_\infty, \ \ |q| < 1, \ \ x \in \mathbb{C} - \{0\}. \tag{2.10}$$

In order to deal with some families of orthogonal polynomials and other basic hypergeometric functions, the following notation (see [28])

$$(x \ominus y)_q^n = (x-y)(x-qy)\cdots(x-q^{n-1}y), \tag{2.11}$$

which is the so-called q-power basis, will be used.

2.1.4 q-Exponential functions

For the exponential function, we have two different natural q-extensions, denoted by $e_q(z)$ and $\mathrm{E}_q(z)$ which can be defined by [30, P. 22]

$$e_q(z) := {}_1\phi_0 \left(\begin{array}{c|c} 0 & \\ & q, q \\ - & \end{array} \right) = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n}, \quad 0 < |q| < 1, \quad |z| < 1, \tag{2.12}$$

and

$$\mathrm{E}_q(z) := {}_0\phi_0 \left(\begin{array}{c|c} - & \\ & q, -z \\ - & \end{array} \right) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}}{(q;q)_n} z^n, \quad 0 < |q| < 1. \tag{2.13}$$

Note that by Euler's formulas (2.7) and (2.8), we have

$$e_q(x) = \frac{1}{(z;q)_\infty}, \quad \text{and} \quad E_q(x) = (-z;q)_\infty.$$

These q-analogues of the exponential function are therefore related by

$$e_q(z)\mathrm{E}_q(-z) = 1.$$

2.2 Difference operators

2.2.1 The operators Δ and ∇

Definition 7. *Let f be a function of the variable x. The forward and the backward operators Δ and ∇ are, respectively, defined by:*

$$\Delta f(x) = f(x+1) - f(x), \qquad \nabla f(x) = f(x) - f(x-1).$$

For $m \in \mathbb{N}^\star = \{1, 2, 3, \dots\}$, one sets

$$\Delta^{m+1} f(x) = \Delta(\Delta^m f(x)).$$

It should be noted that Δ and ∇ transform a polynomial of degree n ($n \geq 1$) in x into a polynomial of degree $n-1$ in x and a polynomial of degree 0 into the zero polynomial.

The operator Δ fulfils the following properties

Proposition 8. *Let f and g be two functions in the variable x, a and b be two complex numbers. The following properties are valid.*

1. $\Delta(af(x)+bg(x)) = a\Delta f(x) + b\Delta g(x)$ *(linearity);*
2. $\Delta[f(x)g(x)] = f(x+1)\Delta g(x) + g(x)\Delta f(x) = f(x)\Delta g(x) + g(x+1)\Delta f(x)$, *(product rule);*
3. $\Delta\Big[\dfrac{f(x)}{g(x)}\Big] = \dfrac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+1)}$ *(quotient rule).*

Note that these operators play an essential role for orthogonal polynomials of a discrete variable.

2.2.2 The operator D_q

Definition 9. *Let f be a function of the variable x. The q-difference operator D_q is defined as:*

$$D_q f(x) := \frac{f(x) - f(qx)}{(1-q)x} \text{ if } x \neq 0,$$

and $D_q f(0) = f'(0)$ provided that f is differentiable at $x = 0$.
If m is a nonnegative integer, we have

$$D_q^{m+1} f = D_q\left(D_q^m f\right); \qquad D_q^0 f = f.$$

The operator D_q fulfils the following properties

Proposition 10. *Let f and g be two functions in x, a and b be two complex numbers. The q-difference operator D_q fulfil the following rules.*

1. $D_q(af(x)+bg(x)) = aD_q f(x) + bD_q g(x)$ *(linearity);*
2. $D_q(f(x)g(x)) = f(qx)D_q g(x) + g(x)D_q f(x) = g(qx)D_q f(x) + g(x)D_q f(x)$ *(product rule);*
3. $D_q\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x)D_q f(x) - f(x)D_q g(x)}{g(x)g(qx)} = \dfrac{g(qx)D_q f(x) - f(qx)D_q g(x)}{g(x)g(qx)}$ *(quotient rule).*

One should note that the operator D_q plays an important role for the polynomials of a q-discrete variable.

2.2.3 The operators $\mathcal{D}$ and $\mathcal{S}$

Definition 11. *Let f be a function of the variable x. The difference operator $\mathcal{D}$ and its companion operator $\mathcal{S}$ are defined as follows:*

$$\mathcal{D}f(x) = f\left(x+\frac{i}{2}\right) - f\left(x-\frac{i}{2}\right) \qquad \mathcal{S}f(x) = \frac{f(x+\frac{i}{2}) + f(x-\frac{i}{2})}{2},$$

where $i^2 = -1$.

The operator $\mathcal{D}$ transforms a polynomial of degree n ($n \geq 1$) in x into a polynomial of degree $n-1$ in x, and a polynomial of degree 0 into the zero polynomial. The operator $\mathcal{S}$ transforms a polynomial of degree n in x into a polynomial of degree n in x.
Note that the operators $\mathcal{D}$ and $\mathcal{S}$ play an important role for the Continuous Hahn and the Meixner-Pollaczek polynomials.

2.2.4 The operators D and S

Definition 12. *Let f be a function of the variable x. The difference operator* **D** *and its companion operator* **S** *are defined as follows:*

$$\mathbf{D}f(x^2) = \frac{f((x+\frac{i}{2})^2) - f((x-\frac{i}{2})^2)}{2ix} \qquad \mathbf{S}f(x^2) = \frac{f((x+\frac{i}{2})^2) + f((x-\frac{i}{2})^2)}{2},$$

where $i^2 = -1$.

The operator **D** transforms a polynomial of degree n ($n \geq 1$) in x^2 into a polynomial of degree $n-1$ in x^2, and a polynomial of degree 0 into the zero polynomial. The operator **S** transforms a polynomial of degree n in x^2 into a polynomial of degree n in x^2.
Note that the operators **D** and **S** play an important role for the Wilson polynomials and the Continuous Dual Hahn polynomials.

2.2.5 The operator D_ε.

Definition 13. *Let ε be a complex number, u be the polynomial of the variable x defined by $u(x) = -x(x+\varepsilon)$. Let f be a function of the variable x. We define the difference operator D_ε as follows:*

$$\mathsf{D}_\varepsilon f(u(x)) = \frac{\Delta f(u(x))}{\Delta u(x)} = \frac{f(u(x)) - f(u(x+1))}{2x+1+\varepsilon}.$$

The operator D_ε transforms a polynomial of degree n ($n \geq 1$) in $-x(x+\varepsilon)$ into a polynomial of degree $n-1$ in $-x(x+\varepsilon)$ and a polynomial of degree 0 into the zero polynomial.
Note that the operators D_ε plays an important role for the Racah and the Dual Hahn polynomials.

2.2.6 The operators $\mathbb{D}_x$ and $\mathbb{S}_x$

Definition 14. *Let f be a function of the variable $x(s)$. The difference operator $\mathbb{D}_x$ and its companion operator $\mathbb{S}_x$ are defined as follows:*

$$\mathbb{D}_x f(x(s)) = \frac{f(x(s+\frac{1}{2})) - f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})}, \quad \mathbb{S}_x f(x(s)) = \frac{f(x(s+\frac{1}{2})) + f(x(s-\frac{1}{2}))}{2},$$

where $x(s)$ is a lattice defined by (1.7) or (1.8).

The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ play an important role for the polynomials of quadratic and q-quadratic lattices.

2.3 q-integration

In this section, we recall the definition of the concept of the q-integration with the assumption $0 < q < 1$ and give some properties. More details can be found in [27], [19], [28] and [43].

2.3.1 The q-integration in the interval $(0; a)$, $a > 0$

Let f be a real function defined in the interval $(0; a)$ and $\mathcal{P}_q((0; a))$ the q-partition of the interval $(0; a)$ defined by

$$\mathcal{P}_q((0;a)) = \{\cdots < aq^{n+1} < aq^n < \cdots < aq < a\}.$$

For any integer N, consider the Riemann type sum

$$A_N(f) = \sum_{n=0}^{N} (aq^n - aq^{n+1}) f(aq^n) = a(1-q) \sum_{n=0}^{N} q^n f(aq^n).$$

If the limit of $A_N(f)$ when $N \to \infty$ is finite, then f is said to be q-integrable and the q-integral of f in the interval $(0;a)$, denoted $\int_0^a f(s)d_qs$, is given by

$$\int_0^a f(s)d_qs = \lim_{N\to\infty} A_N(f) = a(1-q)\sum_{n=0}^{\infty} q^n f(aq^n). \tag{2.14}$$

2.3.2 The q-integration in the interval $(a;0)$, $a<0$

Let f be a real function defined in the interval $(a;0)$ and $\mathcal{P}_q((a;0))$ the q-partition of the interval $(0;a)$ defined by

$$\mathcal{P}_q((a;0)) = \{a < aq < \cdots < aq^n < aq^{n+1} < \ldots\} = \{aq^n, n \in \mathbb{N}\}.$$

For any integer N, consider the Riemann type sum

$$A_N(f) = \sum_{n=0}^{N} (aq^{n+1} - aq^n) f(aq^n) = -a(1-q)\sum_{n=0}^{N} q^n f(aq^n).$$

If the limit of $A_N(f)$ when $N \to \infty$ is finite, then f is said to be q-integrable and the q-integral of f in the interval $(a;0)$, denoted $\int_a^0 f(s)d_qs$, is given by

$$\int_a^0 f(s)d_qs = \lim_{N\to\infty} A_N(f) = -a(1-q)\sum_{n=0}^{\infty} q^n f(aq^n). \tag{2.15}$$

2.3.3 The q-integration in the interval $(a;\infty)$, $a>0$

Let f be a real function defined in the interval $(a;\infty)$ and $\mathcal{P}_q((a;\infty))$ the q-partition of the interval $(a;\infty)$ defined by

$$\mathcal{P}_q((a;\infty)) = \{a < aq^{-1} < \cdots < aq^{-n-1} < \ldots\} = \{aq^{-n},\ n \in \mathbb{N}\}.$$

For any integer N, consider the Riemann type sum

$$A_N(f) = \sum_{n=0}^{N} (aq^{-n-1} - aq^{-n}) f(aq^{-n-1}) = a(q^{-1}-1)\sum_{n=0}^{N} q^{-n} f(aq^{-n-1}).$$

If the limit of $A_N(f)$ when $N \to \infty$ is finite, then f is said to be q-integrable and the q-integral of f in the interval $(a;\infty)$, denoted $\int_a^\infty f(s)d_qs$, is given by

$$\int_a^\infty f(s)d_qs = \lim_{N\to\infty} A_N(f) = a(q^{-1}-1)\sum_{n=0}^{\infty} q^{-n} f(aq^{-n-1}). \tag{2.16}$$

2.3.4 The q-integration in the interval $(-\infty;a)$, $a<0$

Let f be a real function defined in the interval $(-\infty;a)$ and $\mathcal{P}_q((-\infty;a))$ the q-partition of the interval $(-\infty;a)$ defined by

$$\mathcal{P}_q((-\infty;a)) = \{a > aq^{-1} > \cdots > aq^{-n-1} > \ldots\} = \{aq^{-n},\ n \in \mathbb{N}\}.$$

For any integer N, consider the Riemann type sum

$$A_N(f) = \sum_{n=0}^{N} (aq^{-n} - aq^{-n-1}) f(aq^{-n-1}) = -a(q^{-1}-1)\sum_{n=0}^{N} q^{-n} f(aq^{-n-1}).$$

If the limit of $A_N(f)$ when $N \to \infty$ is finite, then f is said to be q-integrable and the q-integral of f in the interval $(-\infty; a)$, denoted $\int_{-\infty}^{a} f(s)d_qs$, is given by

$$\int_{-\infty}^{a} f(s)d_qs = \lim_{N\to\infty} A_N(f) = -a(q^{-1}-1)\sum_{n=0}^{\infty} q^{-n} f(aq^{-n-1}). \tag{2.17}$$

Remark 15. *The q-integration is extended to the whole real line by using relations (2.14)-(2.17) and the following rules*

$$\begin{aligned}
\int_a^b f(s)d_qs &= \int_a^0 f(s)d_qs + \int_0^b f(s)d_qs \quad \forall a,b \in \mathbb{R}\\
\int_a^\infty f(s)d_qs &= \int_a^b f(s)d_qs + \int_b^\infty f(s)d_qs \quad \forall a,b \in \mathbb{R},\ a<0,\ b>0\\
\int_{-\infty}^b f(s)d_qs &= \int_{-\infty}^a f(s)d_qs + \int_a^b f(s)d_qs \quad \forall a,b \in \mathbb{R},\ a<0,\ b>0\\
\int_{-\infty}^\infty f(s)d_qs &= \int_{-\infty}^a f(s)d_qs + \int_a^b f(s)d_qs + \int_b^\infty f(s)d_qs \quad \forall a,b \in \mathbb{R}.
\end{aligned}$$

Like the usual integration, the q-integration enjoys several important properties. We give some of them in the following proposition.

Proposition 16. [28]

1. *If f is a real function continuous at 0, then we have*

$$\int_0^x D_q f(s)d_qs = f(x) - f(0).$$

2. *For any function f q-integrable in $(0; x)$, we have*

$$D_q \int_0^x f(s)d_qs = f(x),$$

assuming that the operator D_q acts on the variable x.

3. *If f is a real function continuous in the interval $(0; a)$, then f is q-integrable on $(0; a)$ and obeys*

$$\lim_{q\to 1}\int_0^a f(s)d_qs = \int_0^a f(s)ds.$$

4. *If f and g are two real functions, q-integrable in the interval $(0; a)$, then we have*

$$\int_0^a f(s)D_qg(s)d_qs = fg\Big|_0^a - \int_0^a D_qf(s)g(qs)d_qs = f(s/q)g(s)\Big|_0^a - \frac{1}{q}\int_0^a g(s)D_{\frac{1}{q}}f(s)d_qs,$$

with $fg\Big|_0^a = f(a)g(a) - f(0)g(0)$.

2.4 Orthogonal polynomials

Let $\mathcal{P}$ be the linear space of polynomials with complex coefficients. A polynomial sequence $\{P_n\}_{n\geq 0}$ in $\mathcal{P}$ is called a polynomial set if and only if $\deg P_n = n$ for all nonnegative integers n.
Let α denote a nondecreasing function with a finite or an infinite number of points of increase in the interval $(a; b)$. The latter interval may be infinite. We assume that the numbers μ_n defined by

$$\mu_n = \int_a^b x^n d\alpha(x) \tag{2.18}$$

exist for $n = 0, 1, 2, \ldots$. These numbers are called *canonical moments of the measure* $d\alpha(x)$. The integral (2.18) can be considered as a Riemann-Stieltjes integral (with nondecreasing $\alpha(x)$) or equivalently as measure integral with measure $d\alpha(x)$. In the continuous case, $d\alpha(x) = \alpha'(x)\,dx$. In the discrete case, the measure $d\alpha(x)$ is a weighted sum of Dirac measures (point measures) ϵ_x at the points of increase x_k of $\alpha(x)$,

$$d\alpha(x) = \sum_{k=0}^{N} \alpha_k \epsilon_{x_k}$$

where α_k denotes the increment of $\alpha(x)$ at x_k, $N \in \mathbb{N}$ or $N = \infty$. In this case, the integral can be computed as the sum

$$\int_a^b x^n d\alpha(x) = \sum_{k=0}^{N} \alpha_k\, x_k^n\ .$$

Note that the Dirac measure ϵ_x at the point y is defined by

$$\epsilon_x(y) = \begin{cases} 1 \text{ if } & y = x \\ 0 \text{ if } & y \neq x. \end{cases}$$

Definition 17. [3, P. 244, Def. 5.2.1] *We say that a polynomial set* $\{p_n(x)\}_0^\infty$ *is orthogonal with respect to the measure* $d\alpha(x)$ *if* $\forall n, m \in \mathbb{N}$

$$\int_a^b p_n(x)p_m(x)d\alpha(x) = h_n\delta_{nm}, \quad h_n \neq 0. \tag{2.19}$$

Definition 18. *Let* $\theta_n(x)$ *be a polynomial set. The numbers*

$$\mu_n(\theta_k(x)) = \int_a^b \theta_n(x)d\alpha(x), \quad n = 0, 1, 2, \ldots \tag{2.20}$$

are the moments with respect to $\theta_n(x)$ *of the family* $\{p_n(x)\}_0^\infty$*, they are called generalized moments.*

Note that it is possible to obtain the canonical moments from the generalized moments if one can find explicit representations for $C_m(n)$ and $D_m(n)$ in the expansions

$$x^n = \sum_{m=0}^{n} C_m(n)\theta_m(x), \tag{2.21}$$

and

$$\theta_n(x) = \sum_{m=0}^{n} D_m(n)x^n. \tag{2.22}$$

In these cases, we have the obvious relations

$$\mu_n = \sum_{m=0}^{n} C_m(n)\mu_m(\theta_k(x)), \tag{2.23}$$

and

$$\mu_n(\theta_k(x)) = \sum_{m=0}^{n} D_m(n)\mu_m. \tag{2.24}$$

2.4.1 Classical continuous orthogonal polynomials

A polynomial set

$$y(x) = p_n(x) = k_n x^n + \ldots \quad (n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\},\ k_n \neq 0) \tag{2.25}$$

is a family of classical continuous orthogonal polynomials if it is the solution of a differential equation of the type

$$\sigma(x)y''(x) + \tau(x)y'(x) + \lambda_n y(x) = 0, \tag{2.26}$$

where $\sigma(x) = ax^2 + bx + c$ is a polynomial of at most second order and $\tau(x) = dx + e$ is a polynomial of first order. Here, the measure $d\alpha(x)$ takes the form

$$d\alpha(x) = \rho(x)dx,$$

where ρ is the non-negative solution on (a, b) of the Pearson equation

$$\frac{d}{dx}(\sigma(x)\rho(x)) = \tau(x)\rho(x).$$

The function $\rho(x)$ is called weight function. Up to a linear change of variable, these polynomials can be classified as (see e.g. [30], [33]):

(a) The Jacobi polynomials [30, P. 216]

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{matrix} \middle| \frac{1-x}{2}\right).$$

Special cases are:

(a-1) The Gegenbauer / Ultraspherical polynomials [30, P. 222]
They are Jacobi polynomials for $\alpha = \beta = \lambda - \frac{1}{2}$.

$$\begin{aligned} C_n^{(\lambda)} &= \frac{(2\lambda)_n}{\left(\lambda+\frac{1}{2}\right)_n} P_n^{\left(\lambda-\frac{1}{2},\lambda-\frac{1}{2}\right)}(x) \\ &= \frac{(2\lambda)_n}{n!} {}_2F_1\left(\begin{matrix} -n, n+2\lambda \\ \lambda+\frac{1}{2} \end{matrix} \middle| \frac{1-x}{2}\right), \quad \lambda \neq 0. \end{aligned}$$

(a-2) The Chebyshev polynomials [30, P. 225]
The Chebyshev polynomials of the first kind can be obtained from the Jacobi polynomials by taking $\alpha = \beta = -\frac{1}{2}$:

$$T_n(x) = \frac{P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x)}{P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(1)} = {}_2F_1\left(\begin{matrix} -n, n \\ \frac{1}{2} \end{matrix} \middle| \frac{1-x}{2}\right),$$

and the Chebyshev polynomials of the second kind can be obtained from the Jacobi polynomials by taking $\alpha = \beta = \frac{1}{2}$:

$$U_n(x) = (n+1)\frac{P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(x)}{P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(1)} = (n+1){}_2F_1\left(\begin{matrix} -n, n+2 \\ \frac{3}{2} \end{matrix} \middle| \frac{1-x}{2}\right).$$

(a-3) The Legendre polynomials
They are Jacobi polynomials with $\alpha = \beta = 0$:

$$P_n(x) = P_n^{(0,0)}(x) = {}_2F_1\left(\begin{matrix} -n, n+1 \\ 1 \end{matrix} \middle| \frac{1-x}{2}\right).$$

(b) The Laguerre polynomials [30, P. 241]

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix}\,\middle|\, x\right).$$

(c) The Hermite polynomials [30, P. 250]

$$H_n(x) = (2x)^n {}_2F_0\left(\begin{matrix} -\frac{n}{2}, -\frac{n-1}{2} \\ - \end{matrix}\,\middle|\, -\frac{1}{x^2}\right).$$

(d) The Bessel polynomials [30, P. 244]

$$B_n^{(\alpha)}(x) = {}_2F_0\left(\begin{matrix} -n, n+\alpha+1 \\ - \end{matrix}\,\middle|\, -\frac{x}{2}\right).$$

Usually, Bessel polynomials fulfil an orthogonality relation on a unit circle. However, it should be mentioned that they also fulfil a real orthogonality. In this case, the family obtained is finite. In this work, we consider the real orthogonality provided by Lesky and Masjed-Jamei [39, 40, 41, 42].

2.4.2 Classical discrete orthogonal polynomials

A polynomial set $p_n(x)$, given by (5.27), is a family of discrete classical orthogonal polynomials (also known as the Hahn class) if it is the solution of a difference equation of the type

$$\sigma(x)\Delta\nabla y(x) + \tau(x)\Delta y(x) + \lambda_n y(x) = 0, \tag{2.27}$$

Here the measure $d\alpha(x)$ takes the form

$$d\alpha(x) = \sum_{k=0}^{N} \rho(k)\epsilon_k, \quad N \in \mathbb{N} \quad \text{or} \quad N = \infty.$$

where ρ is the non-negative solution of the Pearson type equation

$$\Delta(\sigma(x)\rho(x)) = \tau(x)\rho(x).$$

The function $\rho(x)$ is again called weight function.

These polynomials can be classified as (see e.g. [30], [33]):

(a) The Hahn polynomials [30, P. 204]

$$Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\begin{matrix} -n, n+\alpha+\beta+1, -x \\ \alpha+1, -N \end{matrix}\,\middle|\, 1\right).$$

(b) The Krawtchouk polynomials [30, P. 237]

$$K_n(x;p,N) = {}_2F_1\left(\begin{matrix} -n, -x \\ -N \end{matrix}\,\middle|\, \frac{1}{p}\right).$$

(c) The Meixner polynomials [30, P. 234]

$$M_n(x;\beta,c) = {}_2F_1\left(\begin{matrix} -n, -x \\ \beta \end{matrix}\,\middle|\, 1-\frac{1}{c}\right).$$

(d) The Charlier polynomials [30, P. 247]

$$C_n(x;a) = {}_2F_0\left(\begin{matrix} -n, -x \\ - \end{matrix}\,\middle|\, -\frac{1}{a}\right).$$

2.4.3 Classical q-discrete orthogonal polynomials

A polynomial set $p_n(x)$ given by (5.27), is a family of classical q-discrete orthogonal polynomials (also known as the polynomials of the q-Hahn tableau) if it is the solution of a q-difference equation of the type

$$\sigma(x)D_qD_{q^{-1}}y(x) + \tau(x)D_qy(x) + \lambda_n y(x) = 0. \tag{2.28}$$

Here the polynomials $\sigma(x)$ and $\tau(x)$ are known to satisfy a Pearson type equation

$$D_q(\sigma(x)\rho(x)) = \tau(x)\rho(x),$$

where the function $\rho(x)$ is the q-discrete weight function associated to the family. Here, once more, the measure $d\alpha(x)$ takes the form

$$d\alpha(x) = \sum_{k\in\mathbb{Z}}\left(\rho(q^k)\epsilon_{q^k} + \rho(-q^k)\epsilon_{-q^k}\right).$$

These polynomials can be classified as (see e.g. [18], [30]):

(a) The Big q-Jacobi polynomials [30, P. 438]

$$p_n(x;a,b,c;q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix}\,\middle|\, q;q\right)$$

A special case when $a = b = 1$ are the Big q-Legendre polynomials

$$P_n(x;c;q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{n+1}, x \\ q, cq \end{matrix}\,\middle|\, q;q\right).$$

(b) The q-Hahn polynomials [30, P. 445]

$$Q_n(q^{-x};\alpha,\beta,N;q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, \alpha\beta q^{n+1}, q^{-x} \\ \alpha q, q^{-N} \end{matrix}\,\middle|\, q;q\right)$$

(c) The Big q-Laguerre polynomials [30, P. 478]

$$P_n(x,a,b;q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, 0, x \\ aq, bq \end{matrix}\,\middle|\, q;q\right) = \frac{1}{(b^{-1}q^{-n};q)_n}{}_2\phi_1\left(\begin{matrix} q^{-n}, aqx^{-1} \\ aq \end{matrix}\,\middle|\, q;\frac{x}{b}\right).$$

(d) The Little q-Jacobi polynomials [30, P. 482]

$$p_n(x;a,b|q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}\,\middle|\, q;qx\right).$$

A special case when $a = b = 1$ are the little q-Legendre polynomials given by

$$p_n(x;q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{n+1} \\ q \end{matrix}\,\middle|\, q;qx\right).$$

(e) The q-Meixner polynomials [30, P. 488]

$$M_n(q^{-x};b,c;q) = {}_2\phi_1\left(\begin{matrix} q^{-n},q^{-x} \\ bq \end{matrix}\middle| q;-\frac{q^{n+1}}{c}\right).$$

(f) The Quantum q-Krawtchouk polynomials [30, P. 493]

$$K_n^{qtm}(q^{-x};p,N;q) = {}_2\phi_1\left(\begin{matrix} q^{-n},q^{-x} \\ q^{-N} \end{matrix}\middle| q;pq^{n+1}\right).$$

(h) The q-Krawtchouk polynomials [30, P. 496]

$$\begin{aligned} K_n(q^{-x};p,N;q) &= {}_3\phi_2\left(\begin{matrix} q^{-n},q^{-x},-pq^n \\ q^{-N},0 \end{matrix}\middle| q;q\right) \\ &= \frac{(q^{x-N};q)_n}{(q^{-N};q)_n q^{nx}}{}_2\phi_1\left(\begin{matrix} q^{-n},q^{-x} \\ q^{N-x-n+1} \end{matrix}\middle| q;-pq^{n+N+1}\right), \quad n=0,1,2,\dots,N. \end{aligned}$$

(g) The Affine q-Krawtchouk polynomials [30, P. 501]

$$\begin{aligned} K_n^{Aff}(q^{-x};p,N;q) &= {}_3\phi_2\left(\begin{matrix} q^{-n},0,q^{-x} \\ pq,q^{-N} \end{matrix}\middle| q;q\right) \\ &= \frac{(-pq)^n q^{\binom{n}{2}}}{(pq;q)_n}{}_2\phi_1\left(\begin{matrix} q^{-n},q^{x-N} \\ q^{-N} \end{matrix}\middle| q;\frac{q^{-x}}{p}\right), \quad n=0,1,2,\dots,N. \end{aligned}$$

(i) The Little q-Laguerre polynomials [30, P. 518]

$$p_n(x,a|q) = {}_2\phi_1\left(\begin{matrix} q^{-n},0 \\ aq \end{matrix}\middle| q;qx\right) = \frac{1}{(a^{-1}q^{-n};q)_n}{}_2\phi_0\left(\begin{matrix} q^{-n},x^{-1} \\ 0 \end{matrix}\middle| q;\frac{x}{a}\right).$$

(j) The q-Laguerre polynomials [30, P. 522]

$$L_n^{(\alpha)}(x) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}{}_1\phi_1\left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix}\middle| q;-q^{n+\alpha+1}x\right) = \frac{1}{(q;q)_n}{}_2\phi_1\left(\begin{matrix} q^{-n},-x \\ 0 \end{matrix}\middle| q;q^{n+\alpha+1}\right).$$

(k) The Alternative q-Charlier (also called q-Bessel) polynomials [30, P. 526]

$$\begin{aligned} K_n(x;a;q) &= {}_2\phi_1\left(\begin{matrix} q^{-n},-aq^{-n} \\ 0 \end{matrix}\middle| q;qx\right) \\ &= (q^{-n+1}x;q)_n {}_1\phi_1\left(\begin{matrix} q^{-n} \\ q^{-n+1}x \end{matrix}\middle| q;-aq^{n+1}x\right) \\ &= (-aqx)^n {}_2\phi_1\left(\begin{matrix} q^{-n},x^{-1} \\ 0 \end{matrix}\middle| q;-\frac{q^{-n+1}}{a}\right) \end{aligned}$$

(l) The q-Charlier polynomials [30, P. 530]

$$C_n(q^{-x};a;q) = {}_2\phi_1\left(\begin{matrix} q^{-n}, q^{-x} \\ 0 \end{matrix}\middle| q; -\frac{q^{n+1}}{a}\right) = (-a^{-1}q;q)_n {}_1\phi_1\left(\begin{matrix} q^{-n} \\ -a^{-1}q \end{matrix}\middle| q; -\frac{q^{n+1-x}}{a}\right).$$

(m) The Al Salam-Carlitz I polynomials [30, P. 534]

$$U_n^{(a)}(x;q) = (-a)^n q^{\binom{n}{2}} {}_2\phi_1\left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix}\middle| q; \frac{qx}{a}\right).$$

(n) The Al Salam-Carlitz II polynomials [30, P. 537]

$$V_n^{(a)}(x;q) = (-a)^n q^{-\binom{n}{2}} {}_2\phi_0\left(\begin{matrix} q^{-n}, x \\ 0 \end{matrix}\middle| q; \frac{q^n}{a}\right).$$

(o) The Stieltjes-Wigert polynomials [30, P. 544]

$$S_n(x;q) = \frac{1}{(q;q)_n} {}_1\phi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix}\middle| q; -q^{n+1}x\right).$$

(p) The Discrete q-Hermite I polynomials [30, P. 547]

$$h_n(x;q) = q^{\binom{n}{2}}\ {}_2\phi_1\left(\begin{matrix} q^{-n};\ x^{-1} \\ 0 \end{matrix}\middle| q; -qx\right) = x^n\ {}_2\phi_0\left(\begin{matrix} q^{-n};\ q^{-n+1} \\ - \end{matrix}\middle| q^2; \frac{q^{2n-1}}{x^2}\right).$$

(q) The Discrete q-Hermite II polynomials [30, P. 550]

$$\tilde{h}_n(x;q) = i^{-n} q^{-\binom{n}{2}}\ {}_2\phi_0\left(\begin{matrix} q^{-n};\ ix \\ - \end{matrix}\middle| q; -q^n\right) = x^n\ {}_2\phi_1\left(\begin{matrix} q^{-n};\ q^{-n+1} \\ 0 \end{matrix}\middle| q^2; -\frac{q^2}{x^2}\right).$$

2.4.4 Classical orthogonal polynomials on a quadratic lattice

A family $p_n(x)$ of polynomials of degree n, given by (5.27), is a family of classical quadratic orthogonal polynomials (also known as orthogonal polynomials on non-uniform lattices) if it is the solution of a divided difference equation of the type ([36])

$$\phi(x^2)\mathbf{D}^2 y(x^2) + \psi(x^2)\mathbf{SD}y(x^2) + \lambda_n y(x^2) = 0. \tag{2.29}$$

These polynomials can be classified as:

(a) The Wilson polynomials [30, P. 185]

$$\frac{W_n(x^2;a,b,c,d)}{(a+b)_n(a+c)_n(a+d)_n} = {}_4F_3\left(\begin{matrix} -n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix}\middle| 1\right).$$

(b) The Racah polynomials [30, P. 190]

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta) = {}_4F_3\left(\begin{matrix} -n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1 \\ \alpha+1, \beta+\delta+1, \gamma+1 \end{matrix}\middle| 1\right),\quad n = 0,1,2,\dots,N, \tag{2.30}$$

where

$$\lambda(x) = x(x+\gamma+\delta+1)$$

and

$$\alpha+1=-N, \quad \text{or} \quad \beta+\delta+1=-N \quad \text{or} \quad \gamma+1=-N$$

with N a non-negative integer.

(c) The Continuous Dual Hahn polynomials [30, P. 196]

$$\frac{S_n(x^2;a,b,c)}{(a+b)_n(a+c)_n} = {}_3F_2\left(\left.\begin{matrix}-n, a-ix, a+ix\\ a+b, a+c\end{matrix}\right| 1\right).$$

(d) The Continuous Hahn polynomials [30, P. 200]

$$p_n(x;a,b,c,d) = i^n\frac{(a+c)_n(a+d)_n}{n!}{}_3F_2\left(\left.\begin{matrix}-n, n+a+b+c+d-1, a+ix\\ a+c, a+d\end{matrix}\right| 1\right). \tag{2.31}$$

(e) The Dual Hahn polynomials [30, P. 208]

$$R_n(\lambda(x);\gamma,\delta,N) = {}_3F_2\left(\left.\begin{matrix}-n, -x, x+\gamma+\delta+1\\ \gamma+1, -N\end{matrix}\right| 1\right), \quad n=0,1,2,\ldots,N, \tag{2.32}$$

where

$$\lambda(x) = x(x+\gamma+\delta+1).$$

(f) The Meixner-Pollaczek polynomials [30, P. 209]

$$P_n^{(\lambda)}(x;\phi) = \frac{(2\lambda)_n}{n!}e^{in\phi}{}_2F_1\left(\left.\begin{matrix}-n, \lambda+ix\\ 2\lambda\end{matrix}\right| 1-e^{-2i\phi}\right). \tag{2.33}$$

2.4.5 Classical orthogonal polynomials on a q-quadratic lattice

A family $p_n(x)$ of polynomials of degree n, given by (5.27), is a family of classical q-quadratic orthogonal polynomials (also known as orthogonal polynomials on non uniform lattices) if it is the solution of a divided difference equation of the type

$$\phi(x(s))\mathbb{D}_x^2 y(x(s)) + \psi(x(s))\mathbb{S}_x\mathbb{D}_x y(x(s)) + \lambda_n y(x(s)) = 0, \tag{2.34}$$

where ϕ is a polynomial of maximal degree two and ψ is a polynomial of exact degree one, λ_n is a constant depending on the integer n and the leading coefficients ϕ_2 and ψ_1 of ϕ and ψ:

$$\lambda_n = -\gamma_n(\gamma_{n-1}\phi_2 + \alpha_n\phi_1)$$

and $x(s)$ is a non uniform lattice defined by

$$x(s) = \begin{cases} c_1q^s + c_2q^{-s} + c_3 \\ c_4s^2 + c_5s + c_6 \end{cases}. \tag{2.35}$$

These polynomials can be classified as:

(a) The Askey-Wilson polynomials [30, P. 415]

$$\frac{a^n p_n(x;a,b,c,d|q)}{(ab,ac,ad;q)_n} = {}_4\phi_3\left(\left.\begin{matrix}q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta}\\ ab, ac, ad\end{matrix}\right| q;q\right), \quad x=\cos\theta.$$

(b) The q-Racah polynomials [30, P. 422]

$$R_n(\mu(x);\alpha,\beta,\gamma,\delta|q) = {}_4\phi_3\left(\left.\begin{matrix} q^{-n},\alpha\beta q^{n+1},q^{-x},\delta\gamma q^{x+1} \\ \alpha q,\beta\delta q,\gamma q\end{matrix}\right| q;q\right),\ \ n=0,1,2,\ldots,N$$

where

$$\mu(x) := q^{-x}+\delta\gamma q^{x+1}$$

and

$$\alpha q = q^{-N} \quad \text{or} \quad \beta\delta q = q^{-N} \quad \text{or} \quad \gamma q = q^{-N},$$

with N a non-negative integer.

(c) The Continuous Dual q-Hahn polynomials [30, P. 429]

$$\frac{a^n p_n(x;a,b,c|q)}{(ab,ac;q)_n} = {}_3\phi_2\left(\left.\begin{matrix} q^{-n},ae^{i\theta},ae^{-i\theta} \\ ab,ac\end{matrix}\right| q,q\right), \quad x=\cos\theta.$$

(d) The Continuous q-Hahn polynomials [30, P. 434]

$$\frac{(ae^{i\phi})^n p_n(x;a,b,c,d|q)}{(abe^{2i\theta},ac,ad;q)_n} = {}_4\phi_3\left(\left.\begin{matrix} q^{-n},abcdq^{n-1},ae^{i(\theta+2\phi)},ae^{-i\theta} \\ abe^{2i\phi},ac,ad\end{matrix}\right| q,q\right), \quad x=\cos(\theta+\phi).$$

(e) The dual q-Hahn polynomials [30, P. 450]

$$R_n(\mu(x),\gamma,\delta,N|q) = {}_3\phi_2\left(\left.\begin{matrix} q^{-n},q^{-x},\gamma\delta q^{x+1} \\ \gamma q,q^{-N}\end{matrix}\right| q,q\right), \quad n=0,1,2,\ldots,N$$

where

$$\mu(x) = q^{-x}+\gamma\delta q^{x+1}$$

(f) The Al-Salam-Chihara polynomials [30, P. 455]

$$Q_n(x;a,b|q) = \frac{(ab;q)_n}{a^n}{}_3\phi_2\left(\left.\begin{matrix} q^{-n},ae^{i\theta},ae^{-i\theta} \\ ab,0\end{matrix}\right| q,q\right), \quad x=\cos\theta.$$

(h) The q-Meixner-Pollaczek polynomials [30, P. 460]

$$P_n(x;a|q) = a^{-n}e^{-in\phi}\frac{(a^2;q)_n}{(q;q)_n}{}_3\phi_2\left(\left.\begin{matrix} q^{-n},ae^{i(\theta+2\phi)},ae^{-i\theta} \\ a^2,0\end{matrix}\right| q,q\right), \quad x=\cos(\theta+\phi).$$

(g) The continuous q-Jacobi polynomials [30, P. 463]

$$P_n^{(\alpha,\beta)}(x|q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}{}_4\phi_3\left(\left.\begin{matrix} q^{-n},q^{n+\alpha+\beta+1},q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta},q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta} \\ q^{\alpha+1},-q^{\frac{1}{2}(\alpha+\beta+1)},-q^{\frac{1}{2}(\alpha+\beta+2)}\end{matrix}\right| q;q\right), \quad x=\cos\theta.$$

As special cases there are:

(g-1) The Continuous q-Ultraspherical (Rogers) polynomials [30, P. 469]

$$C_n(x;\beta|q) = \frac{(\beta^2;q)_n}{(q;q)_n}\beta^{-\frac{1}{2}n}{}_4\phi_3\left(\begin{matrix} q^{-n}, \beta^2 q^n, \beta^{\frac{1}{2}}e^{i\theta}, \beta^{\frac{1}{2}}e^{-i\theta} \\ \beta q^{\frac{1}{2}}, -\beta, -\beta q^{\frac{1}{2}} \end{matrix}\middle| q;q\right), \quad x = \cos\theta.$$

(g-2) The Continuous q-Legendre polynomials $\alpha = \beta = 0$ [30, P. 475]

$$P_n(x|q) = {}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n+1}, q^{\frac{1}{4}}e^{i\theta}, q^{\frac{1}{4}}e^{-i\theta} \\ q, -q^{-\frac{1}{2}}, -q \end{matrix}\middle| q;q\right), \quad x = \cos\theta.$$

(h) The dual q-Krawtchouk polynomials [30, P. 505]

$$K_n(\lambda(x);c,N|q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{-x}, cq^{x-N} \\ q^{-N}, 0 \end{matrix}\middle| q,q\right), \quad n = 0,1,2,\ldots,N,$$

where

$$\lambda(x) = q^{-x} + cq^{x-N}.$$

(i) The continuous big q-Hermite polynomials [30, P. 509]

$$H_n(x;a,|q) = a^{-n}{}_3\phi_2\left(\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ 0, 0 \end{matrix}\middle| q,q\right), \quad x = \cos\theta.$$

(j) The continuous q-Laguerre polynomials [30, P. 514]

$$P_n^{(\alpha)}(x|q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}{}_3\phi_2\left(\begin{matrix} q^{-n}, q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta}, q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta} \\ q^{\alpha+1}, 0 \end{matrix}\middle| q,q\right), \quad x = \cos\theta.$$

(k) The continuous q-Hermite polynomials [30, P. 540]

$$H_n(x|q) = e^{in\theta}{}_2\phi_0\left(\begin{matrix} q^{-n}, 0 \\ - \end{matrix}\middle| q, q^n e^{-2i\theta}\right), \quad x = \cos\theta.$$

2.5 Generating functions

Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers.

1. The generating function of the sequence $(a_n)_n$ is the function

$$F(z) = \sum_{n=0}^{\infty} a_n z^n.$$

2. The exponential generating function of the sequence $(a_n)_n$ is the function

$$G(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n.$$

3. The q-exponential generating function (of first kind) of the sequence $(a_n)_n$ is the function

$$H_1(z) = \sum_{n=0}^{\infty} \frac{a_n}{(q;q)_n} z^n.$$

4. The q-exponential generating function (of second kind) of the sequence $(a_n)_n$ is the function
$$H_2(z) = \sum_{n=0}^{\infty} a_n \frac{q^{\binom{n}{2}}}{(q;q)_n} z^n.$$

Note that the convergence of the right-hand sides of the above sums is required. Throughout this text, both q-exponential generating functions of first kind and of second kind will be called for short q-exponential generating function.
More details on generating functions are available in [52].

Chapter 3

Moments of Orthogonal Polynomials: Easy Cases

In this chapter, using various computational methods, and various well-known summation formulas, we give the canonical moments of some orthogonal polynomial families.

3.1 Classical continuous orthogonal polynomials

3.1.1 Jacobi polynomials

For $\alpha > -1$ and $\beta > -1$, the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal in the interval $(-1;1)$ and fulfil the orthogonality relation [30, P. 217]

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)dx = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}. \tag{3.1}$$

The canonical Jacobi moments are therefore defined by

$$\mu_n = \int_{-1}^{1} x^n(1-x)^{\alpha}(1+x)^{\beta}dx.$$

Proposition 19. *The canonical Jacobi moments have the representation*

$$\mu_n = \frac{\Gamma(\alpha+1)n!}{\Gamma(\alpha+n+2)}{}_2F_1\left(\left.\begin{matrix}-\beta, n+1\\ \alpha+n+2\end{matrix}\right| -1\right) + (-1)^n\frac{\Gamma(\beta+1)n!}{\Gamma(\beta+n+2)}{}_2F_1\left(\left.\begin{matrix}-\alpha, n+1\\ \beta+n+2\end{matrix}\right| -1\right), \quad n = 0,1,2,\ldots \tag{3.2}$$

Proof. We first write

$$\mu_n = \int_{0}^{1} x^n(1-x)^{\alpha}(1+x)^{\beta}dx + (-1)^n\int_{0}^{1} x^n(1+x)^{\alpha}(1-x)^{\beta}dx.$$

Next, the use of the integral representation for the Gauss hypergeometric function [30, P. 8]

$${}_2F_1\left(\left.\begin{matrix}a, b\\ c\end{matrix}\right| z\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx$$

with $z=-1$ gives the desired result. In fact, for the first integral $\int_0^1 x^n(1-x)^{\alpha}(1+x)^{\beta}dx$, using the integral representation of the Gauss hypergeometric function with $b = n+1$,

$c = \alpha + n + 2$ and $a = -\beta$, it follows that

$$\int_0^1 x^n (1-x)^\alpha (1+x)^\beta dx = \frac{\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(\alpha+n+2)} {}_2F_1 \left(\left. \begin{array}{c} -\beta, n+1 \\ \alpha+\beta+2 \end{array} \right| -1 \right).$$

The second integral is computed in the same manner. $\square$

Another form of these moments will be given in Chapter 5 (5.48)-(5.49).
For special cases of Jacobi polynomials, those moments can be further simplified.

(a) Gegenbauer polynomials

Proposition 20. *The canonical Gegenbauer moments have the representation*

$$\mu_n = \begin{cases} \sqrt{\pi}\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda+1)}\frac{(2p)!}{2^{2p}p!(\lambda+1)_p}, & \text{if } n = 2p. \\ 0 & \text{if } n = 2p+1. \end{cases} \tag{3.3}$$

Proof. By definition one has

$$\mu_n = \int_{-1}^{1} x^n (1-x^2)^{\lambda-\frac{1}{2}} dx.$$

It is straightforward to see that if n is odd, then $\mu_n = 0$. We assume that n is even and write $n = 2p$. μ_n can be rewritten as

$$\mu_{2p} = \int_{-1}^{1} (x^2)^p (1-x^2)^{\lambda-\frac{1}{2}} dx.$$

Now, we make the change of variable $X = x^2$ and it follows that:

$$\begin{aligned} \mu_{2p} &= \int_0^1 X^{p-\frac{1}{2}} (1-X)^{\lambda-\frac{1}{2}} dx \\ &= B\left(p+\frac{1}{2}, \lambda+\frac{1}{2}\right) \\ &= \frac{\Gamma\left(p+\frac{1}{2}\right)\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(p+\lambda+1)}. \end{aligned}$$

The desired results follows by simplification. $\square$

Proposition 21. *The canonical Gegenbauer moments have the following exponential generating function*

$$\sqrt{\pi}\Gamma\left(\lambda+\frac{1}{2}\right)\left(\frac{2}{z}\right)^\lambda I_\lambda(z) = \sum_{n=0}^{\infty} \frac{\mu_n}{n!} z^n. \tag{3.4}$$

where $I_\lambda(z)$ is the Bessel function of first kind (see [1], Chapter 9).

Proof. Using Algorithm 2.2 from [32, P. 20] for the conversion of sums into hypergeometric notation (command `Sumtohyper` of the `hsum` package), we get the result. This result can also be obtained by direct computation. $\square$

(b) Chebyshev polynomials of first kind

Proposition 22. *The canonical moments of the Chebyshev polynomials of the first kind have the representation:*

$$\mu_n = \begin{cases} \frac{\pi(2p)!}{2^{2p}p!^2} & \text{if } n = 2p, \\ 0 & \text{if } n = 2p+1. \end{cases} \tag{3.5}$$

Proof. If we take $\lambda = 0$ in the Gegenbauer polynomials, we get the Chebyshev polynomials of the first kind. Therefore, the canonical moments of the Chebyshev polynomials of the first kind are

$$\mu_n = \begin{cases} \frac{\Gamma(p+\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(p+1)}, & \text{if } n = 2p, \\ 0 & \text{if } n = 2p+1. \end{cases}$$

Now, using the Legendre duplication formula [3, P. 22]

$$\Gamma(2a)\Gamma\left(\frac{1}{2}\right) = 2^{2a-1}\Gamma(a)\Gamma\left(a+\frac{1}{2}\right),$$

and the relations

$$\Gamma(p+1) = p!, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

the desired result follows. □

Proposition 23. *The canonical moments of the Chebyshev polynomials of the first kind have the following generating function:*

$$\frac{\pi}{\sqrt{1-z^2}} = \sum_{n=0}^{\infty} \mu_n z^n, \quad |z| < 1. \tag{3.6}$$

Proof. Using Algorithm 2.2 from [32, P. 20] for the conversion of sums into hypergeometric notation (command `Sumtohyper` of the `hsum` package), we get

$$\sum_{n=0}^{\infty} \mu_n x^n = \pi {}_1F_0 \left(\begin{array}{c|c} \frac{1}{2} & z^2 \\ - & \end{array} \right).$$

Taking $a = \frac{1}{2}$ and $z = -z^2$ in the binomial theorem (2.3), we get:

$$\pi {}_1F_0 \left(\begin{array}{c|c} \frac{1}{2} & z^2 \\ - & \end{array} \right) = \frac{\pi}{\sqrt{1-z^2}}.$$

□

(c) Chebyshev polynomials of second kind

Proposition 24. *The canonical moments of the Chebyshev polynomials of the second kind have the representation:*

$$\mu_n = \begin{cases} \frac{\pi(2p)!}{2^{2p}p!(p+1)!} & \textit{if } n = 2p, \\ 0 & \textit{if } n = 2p+1. \end{cases} \tag{3.7}$$

Proof. Take $\lambda = 1$ in the canonical Gegenbauer moments. □

Proposition 25. *The canonical moments of the Chebyshev polynomials of the second kind have the following generating function:*

$$\frac{2\pi}{1+\sqrt{1-z^2}} = \sum_{n=0}^{\infty} \mu_n z^n, \quad |z| < 1. \tag{3.8}$$

Proof. We set

$$F(z) = \frac{1}{\mu_0} \sum_{n=0}^{\infty} \mu_n z^n = \frac{1}{\pi} \sum_{p=0}^{\infty} \mu_{2p} z^{2p} = \sum_{p=0}^{\infty} \frac{\left(\frac{1}{2}\right)_p}{(p+1)!} z^{2p}.$$

Then it follows that

$$\frac{d}{dz}(z^2F(z)) = \frac{2z}{\sqrt{1-z^2}},$$

hence

$$z^2F(z) = -2\sqrt{1-z^2} + C,$$

where C is the integration constant. Taking $z = 0$ on both sides, it happens that $C = 2$ and therefore

$$F(z) = \frac{2(1-\sqrt{1-z^2})}{z^2} = \frac{2}{1+\sqrt{1-z^2}}.$$

□

(d) Legendre polynomials

Proposition 26. *The canonical Legendre moments have the representation:*

$$\mu_n = \begin{cases} \frac{2}{2p+1} \text{ if } & n = 2p \\ 0 \quad \text{if} & n = 2p+1. \end{cases} \tag{3.9}$$

Proof. By definition, we have

$$\mu_n = \int_{-1}^{1} x^n dx = \frac{1+(-1)^n}{n+1} = \begin{cases} \frac{2}{2p+1} \text{ if } & n = 2p \\ 0 \quad \text{if} & n = 2p+1. \end{cases}$$

□

An immediate consequence is

Proposition 27. *The canonical Legendre moments have the following generating function:*

$$\frac{1}{z}\ln\left(\frac{1+z}{1-z}\right) = \sum_{n=0}^{\infty} \mu_n z^n, \quad |z| < 1. \tag{3.10}$$

3.1.2 Laguerre polynomials

The Laguerre polynomials $L_n^{(\alpha)}(x)$ are orthogonal on the interval $(0, \infty)$ with respect to the weight function $\rho(x) = x^\alpha e^{-x}$ and fulfil the following orthogonality relation [30, P. 241]

$$\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) dx = \frac{\Gamma(n+\alpha+1)}{n!}\delta_{nm}, \quad \alpha > -1. \tag{3.11}$$

The canonical moments are

$$\mu_n = \int_0^\infty \rho(x) x^n dx = \int_0^\infty x^{n+\alpha} e^{-x} dx.$$

Proposition 28. *The canonical Laguerre moments have the representation*

$$\mu_n = \Gamma(n+\alpha+1), \quad n = 0, 1, 2, \ldots \tag{3.12}$$

Proof. By the definition of the canonical moments, and the use of the Gamma function (2.1), we have

$$\mu_n = \int_0^\infty x^{n+\alpha} e^{-x} dx = \int_0^\infty x^{(n+\alpha+1)-1} e^{-x} dx = \Gamma(n+\alpha+1).$$

□

Note that the canonical Laguerre moments appeared in [13] and [26].

Proposition 29 (Exponential generating function)**.** *The canonical Laguerre moments have the following exponential generating function*

$$\frac{\Gamma(\alpha+1)}{(1-z)^{\alpha+1}} = \sum_{n=0}^{\infty} \mu_n \frac{z^n}{n!}. \tag{3.13}$$

Proof. We have, by the use of the binomial theorem (2.3):

$$\sum_{n=0}^{\infty} \frac{\Gamma\left(n+\alpha+1\right)}{n!} z^n = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(\alpha+1)_n}{n!} z^n = \Gamma(\alpha+1)\,{}_1F_0 \left(\begin{matrix} \alpha+1 \\ - \end{matrix} \middle| z \right) = \frac{\Gamma(\alpha+1)}{(1-z)^{\alpha+1}}.$$

□

Another generating function for the canonical Laguerre moments appears in [13] in the form:

$$\phi(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{\alpha+2n+1}\Gamma(n+\alpha+1)\Gamma(\alpha+1)n!} \mu_n x^n = \frac{1}{2^{\alpha+1}\Gamma(\alpha+1)} e^{-x/4}.$$

3.1.3 Bessel polynomials

Let $N > 0$ be an integer. The Bessel polynomials $B_n^{(\alpha)}(x)$, $0 \le n \le N$, fulfil the following orthogonality relation [30, P. 245]

$$\int_0^{\infty} x^{\alpha} e^{-\frac{2}{x}} B_n^{(\alpha)}(x) B_m^{(\alpha)}(x) dx = -\frac{2^{\alpha+1}}{2n+\alpha+1} \Gamma(-n-\alpha) n! \delta_{mn}, \quad \alpha < -2N-1, \quad 0 \le m,n \le N. \tag{3.14}$$

Note that, since $B_n^{(\alpha)}(x) B_m^{(\alpha)}(x)$ is a polynomial of degree $n+m$, it is enough that the integral

$$\int_0^{\infty} x^{\alpha+m+n} e^{-\frac{2}{x}} dx$$

converges.

A problem could appear in the neighbourhood of 0. For this integral to converge, it is necessary that $\lim_{x\to 0^+} x^{\alpha+m+n} e^{-\frac{2}{x}} = 0$, this implies that $m+n+\alpha < -1$ for all $0 \le m,n \le N$. The last inequality will be satisfied if $2N+\alpha < -1$, that is $\alpha < -2N-1$.

Proposition 30. *The canonical moments of the Bessel polynomials have the representation:*

$$\mu_n = 2^{n+\alpha+1}\Gamma\left(-n-\alpha-1\right); \quad n = 0,1,2,\ldots,N, \quad \alpha < -2N-1. \tag{3.15}$$

Proof. By taking $n = m = 0$ in the orthogonality relation, we get

$$\int_0^{\infty} x^{\alpha} e^{-\frac{2}{x}} dx = -\frac{2^{\alpha+1}}{\alpha+1} \Gamma(-\alpha) = 2^{\alpha+1}\Gamma(-\alpha-1),$$

and this makes sense since $\alpha < -2N-1$ reads $-\alpha-1 > 2N$.
Now replacing α by $\alpha+n$ it follows that

$$\mu_n = \int_0^{\infty} x^{\alpha+n} e^{-\frac{2}{x}} dx = 2^{n+\alpha+1}\Gamma(-n-\alpha-1),$$

and this makes sense since

$$(\alpha < -2N-1 \quad \text{and} \quad 0 \le n \le N) \Rightarrow -n-\alpha-1 > N.$$

□

Proposition 31. *The canonical Bessel moments have the following generating function*

$$\frac{\pi}{\sin\left(\pi\ (\alpha+2)\right)} \frac{2^{\alpha+1}}{(-2\,z)^{\frac{\alpha+1}{2}}} I_{\alpha+1}\left(2\sqrt{-2\,z}\right) = \sum_{n=0}^{\infty} \mu_n \frac{z^n}{n!}. \tag{3.16}$$

Proof. Using Algorithm 2.2 from [32, P. 20] for the conversion of sums into hypergeometric notation (command `Sumtohyper` of the `hsum` package), we get

$$\sum_{n=0}^{\infty} \mu_n \frac{z^n}{n!} = 2^{\alpha+1}\Gamma(-\alpha-1)\,{}_1F_1\left(\left.\begin{array}{c} - \\ \alpha+2 \end{array}\right| -2z\right).$$

Next, using the relations (see [32, Eq (1.5),(1.9)])

$$(z)_k = \frac{\Gamma(z+k)}{\Gamma(z)}, \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)},$$

we write

$$\begin{aligned}
2^{\alpha+1}\Gamma(-\alpha-1)\,{}_1F_1\left(\left.\begin{array}{c} - \\ \alpha+2 \end{array}\right| -2z\right) &= 2^{\alpha+1}\Gamma(-\alpha-1)\sum_{k=0}^{\infty}\frac{(-2z)^k}{k!(\alpha+2)_k} \\
&= 2^{\alpha+1}\Gamma(-\alpha-1)\Gamma(\alpha+2)\sum_{k=0}^{\infty}\frac{(-2z)^k}{k!\Gamma(\alpha+k+2)} \\
&= \frac{2^{\alpha+1}\pi}{\sin(\pi(\alpha+2))}\sum_{k=0}^{\infty}\frac{[\frac{1}{4}(2\sqrt{-2z})^2]^k}{k!\Gamma((\alpha+1)+k+1)} \\
&= \frac{\pi}{\sin(\pi(\alpha+2))}\frac{2^{\alpha+1}}{\left(\frac{1}{2}(2\sqrt{-2z})\right)^{\alpha+1}} \\
&\quad \times \left(\frac{1}{2}(2\sqrt{-2z})\right)^{\alpha+1}\sum_{k=0}^{\infty}\frac{[\frac{1}{4}(2\sqrt{-2z})^2]^k}{k!\Gamma((\alpha+1)+k+1)} \\
&= \frac{\pi}{\sin(\pi(\alpha+2))}\frac{2^{\alpha+1}}{(-2z)^{\frac{\alpha+1}{2}}} I_{\alpha+1}\left(2\sqrt{-2z}\right).
\end{aligned}$$

□

3.1.4 Hermite polynomials

The Hermite polynomials $H_n(x)$ are orthogonal in the interval $(-\infty,+\infty)$ with respect to the weight function $\rho(x) = e^{-x^2}$ and fulfil the following orthogonality relation [30, P. 250]

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x)H_m(x)dx = \sqrt{\pi}2^n n!\delta_{mn}. \tag{3.17}$$

Proposition 32. *The canonical moments of the Hermite polynomials have the representation:*

$$\mu_n = \frac{1+(-1)^n}{2}\Gamma\left(\frac{n+1}{2}\right) = \begin{cases} \sqrt{\pi}\frac{(2p)!}{2^{2p}p!} & \text{if } n=2p \\ 0 & \text{if } n=2p+1 \end{cases}, \quad n=0,1,2,\ldots \tag{3.18}$$

Proof. By the definition of the moments, we have $\mu_n = \int_{-\infty}^{\infty} x^n e^{-x^2}dx$. By the change of variable $t = x^2$, and the use of the Gamma function (2.1), μ_n reads:

$$\begin{aligned}
\mu_n &= \int_0^{\infty} x^n e^{-x^2}dx + \int_{-\infty}^{0} x^n e^{-x^2}dx \\
&= \int_0^{\infty} x^n e^{-x^2}dx + (-1)^n\int_0^{\infty} x^n e^{-x^2}dx \\
&= \frac{1+(-1)^n}{2}\int_0^{\infty} t^{\frac{n+1}{2}-1}e^{-t}dt \\
&= \frac{1+(-1)^n}{2}\Gamma\left(\frac{n+1}{2}\right).
\end{aligned}$$

□

The canonical moments of the Hermite polynomials were given in [13] (see also [26]).

Proposition 33 (Exponential generating function)**.** *The canonical Hermite moments have the following exponential generating function*

$$\sqrt{\pi}e^{z^2/4} = \sum_{n=0}^{\infty} \mu_n \frac{z^n}{n!}. \tag{3.19}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{\mu_n}{n!} z^n &= \sum_{n=0}^{\infty} \frac{\frac{1+(-1)^n}{2}\Gamma\left(\frac{n+1}{2}\right)}{n!} z^n \\
&= \sum_{n=0}^{\infty} \frac{\Gamma\left(\frac{2n+1}{2}\right)}{(2n)!} z^{2n} \\
&= \Gamma\left(\frac{1}{2}\right) \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n}{(2n)!} z^{2n}.
\end{aligned}$$

Since

$$\left(\frac{1}{2}\right)_n = \frac{(2n)!}{2^{2n} n!},$$

we finally have

$$\sum_{n=0}^{\infty} \frac{\mu_n}{n!} z^n = \sqrt{\pi} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{z^2}{4}\right)^n = \sqrt{\pi} e^{z^2/4}.$$

□

3.2 Classical q-discrete orthogonal polynomials

3.2.1 Little q-Jacobi polynomials

For $0 < aq < 1$ and $bq < 1$, the Little q-Jacobi polynomials $p_n(x, a, b|q)$ fulfil the following orthogonality relation [30, P. 482]

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{(bq;q)_k}{(q;q)_k} (aq)^k p_m(q^k; a, b|q) p_n(q^k; a, b|q) \\
&\qquad = \frac{(abq^2;q)_\infty}{(aq;q)_\infty} \frac{(1-abq)(aq)^n}{(1-abq^{2n+1})} \frac{(q, bq;q)_n}{(aq, abq;q)_n} \delta_{mn}.
\end{aligned}$$

Therefore, the canonical Little q-Jacobi moments are:

$$\mu_n = \sum_{k=0}^{\infty} \frac{(bq;q)_k}{(q;q)_k} (aq)^k q^{nk}.$$

Proposition 34. *The canonical Little q-Jacobi moments have the representation*

$$\mu_n = \frac{(abq^{n+2};q)_\infty}{(aq^{n+1};q)_\infty} = \frac{(abq^2;q)_\infty}{(aq;q)_\infty} \frac{(aq;q)_n}{(abq^2;q)_n}. \tag{3.20}$$

Proof. The proof follows by taking $a = bq$ and $z = aq^{n+1}$ in the q-binomial theorem (2.6). □

Proposition 35. *The canonical moments of the Little q-Jacobi polynomials have the following generating function:*

$$\frac{(ab^2, aqz;q)_\infty}{(aq, z;q)_\infty} = \sum_{n=0}^{\infty} \mu_n (abq^2;q)_n \frac{z^n}{(q;q)_n}.$$

Proof. We have

$$\sum_{n=0}^{\infty}\mu_n(abq^2;q)_n\frac{z^n}{(q;q)_n}=\frac{(abq^2;q)_\infty}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(aq;q)_n}{(q;q)_n}z^n.$$

By the q-binomial theorem (2.6), the results follows. □

3.2.2 Little q-Legendre polynomials

The Little q-Legendre polynomials $p_n(x|q)$ are special cases of the Little q-Jacobi polynomials with $a=b=1$. They fulfil the orthogonality relation [30, P. 487]

$$\int_0^1 p_m(x|q)p_n(x|q)d_qx=(1-q)\sum_{k=0}^{\infty}q^kp_m(q^k|q)p_n(q^k|q)=\frac{(1-q)q^n}{(1-q^{2n+1})}\delta_{mn}.$$

Therefore, the canonical q-Legendre moments are

$$\mu_n=\sum_{k=0}^{\infty}q^kq^{nk}.$$

Proposition 36. *The canonical Little q-Legendre moments have the representation:*

$$\mu_n=\frac{1}{1-q^{n+1}},\ n=0,1,2,\dots \tag{3.21}$$

Proof. Since $|q|<1$, we have

$$\mu_n=\sum_{k=0}^{\infty}q^kq^{nk}=\lim_{k\to\infty}\frac{1-(q^{n+1})^k}{1-q^{n+1}}=\frac{1}{1-q^{n+1}}.$$

□

Note that these moments could be deduced from the canonical Little q-Jacobi moments by setting $a=b=1$.

Proposition 37. *The canonical Little q-Legendre moments have the following q-exponential generating function*

$$\frac{e_q(z)-1}{z}=\sum_{n=0}^{\infty}\mu_n\frac{z^n}{(q;q)_n}. \tag{3.22}$$

where e_q is the q-exponential function defined by (2.12).

Proof.

$$\begin{aligned}\sum_{n=0}^{\infty}\mu_n\frac{z^n}{(q;q)_n}&=\sum_{n=0}^{\infty}\frac{z^n}{(q;q)_{n+1}}\\&=\frac{1}{z}\left[\sum_{n=0}^{\infty}\frac{z^n}{(q;q)_n}-1\right]\\&=\frac{e_q(z)-1}{z}.\end{aligned}$$

□

3.2.3 q-Krawtchouk polynomials

The q-Krawtchouk polynomials $K_n(q^{-x};p,N;q)$ fulfil the following orthogonality relation [30, P. 497]

$$\begin{aligned}&\sum_{x=0}^{N}\frac{(q^{-N};q)_x}{(q;q)_x}(-p)^{-x}K_m(q^{-x};p,N;q)K_n(q^{-x};p,N;q)\\&\quad=\frac{(q,-pq^{N+1};q)_n}{(-p,q^{-N};q)_n}\frac{1+p}{1+pq^{2n}}\\&\qquad\times(-pq;q)_Np^{-N}q^{-\binom{N+1}{2}}q^{n^2}\delta_{mn},\quad p>0.\end{aligned} \tag{3.23}$$

Therefore, the canonical q-Krawtchouk moments are

$$\mu_n = \sum_{k=0}^{N} \frac{(q^{-N};q)_k}{(q;q)_k} (-p)^{-k} q^{-kn}.$$

Proposition 38. *The canonical q-Krawtchouk moments have the representation*

$$\mu_n = \frac{(-pq;q)_N}{p^N q^{\binom{N+1}{2}}} \frac{(-pq^{N+1};q)_n}{(-pq;q)_n} \frac{1}{q^{nN}}, \quad n = 0,1,2,\ldots,N. \tag{3.24}$$

Proof. By the q-binomial theorem (2.6), it follows that

$$\mu_n = \frac{(-p^{-1}q^{-n-N};q)_\infty}{(-p^{-1}q^{-n};q)_\infty}.$$

In order to simplify this expression, we compute the ratio

$$\frac{\mu_{n+1}}{\mu_n} = \frac{1 + pq^{N+1}q^n}{(1+pqq^n)q^N}.$$

It follows that

$$\mu_n = \mu_0 \frac{(-pq^{N+1};q)_n}{(-pq;q)_n q^{nN}}.$$

μ_0 is obtained by taking $m = n = 0$ in the orthogonality relation (3.23). □

The q-Krawtchouk moments with respect to the basis $(q^{-x};q)_n$ are given in Chapter 5 and another proof of (3.24) is provided.

Proposition 39. *The canonical q-Krawtchouk moments have the following q-exponential generating function*

$$\frac{(-pq;q)_N}{p^N q^{\binom{N+1}{2}}} \frac{(-pqz;q)_\infty}{(zq^{-N};q)_\infty} = \sum_{n=0}^{\infty} \mu_n (-pq;q)_n \frac{z^n}{(q;q)_n}. \tag{3.25}$$

Proof. We have

$$\sum_{n=0}^{\infty} \mu_n (-pq;q)_n \frac{z^n}{(q;q)_n} = \frac{(-pq;q)_N}{p^N q^{\binom{N+1}{2}}} \sum_{n=0}^{\infty} \frac{(-pq^{N+1};q)_n}{(q;q)_n} \left(\frac{z}{q^N}\right)^n.$$

Then, using the q-binomial theorem (2.6), we have

$$\sum_{n=0}^{\infty} \frac{(-pq^{N+1};q)_n}{(q;q)_n} \left(\frac{z}{q^N}\right)^n = \frac{(-pqz;q)_\infty}{(zq^{-N};q)_\infty}.$$

This completes the proof. □

3.2.4 Little q-Laguerre (Wall) polynomials

The Little q-Laguerre polynomials $p_n(x;a;q)$ fulfil the orthogonality relation [30, P. 519]

$$\sum_{k=0}^{\infty} \frac{(aq)^k}{(q;q)_k} p_m(q^k;a|q) p_n(q^k;a|q) = \frac{(aq)^n}{(aq;q)_\infty} \frac{(q;q)_n}{(aq;q)_n} \delta_{mn}, \quad 0 < aq < 1. \tag{3.26}$$

Therefore, the canonical Little q-Laguerre moments are:

$$\mu_n = \sum_{k=0}^{\infty} \frac{(aq)^k}{(q;q)_k} q^{nk}.$$

Proposition 40. *The canonical Little q-Laguerre moments have the representation:*

$$\mu_n = \frac{1}{(aq^{n+1};q)_\infty} = \frac{(aq;q)_n}{(aq;q)_\infty}, \quad n = 0, 1, 2, \ldots \tag{3.27}$$

Proof. The use of the q-binomial formula (2.6) with $z = aq^{n+1}$ gives the result. □

Proposition 41. *The canonical Little q-Laguerre moments have the following q-exponential generating function*

$$\frac{(azq;q)_\infty}{(aq,z;q)_\infty} = \sum_{n=0}^{\infty} \mu_n \frac{z^n}{(q;q)_n}. \tag{3.28}$$

Proof. By the q-binomial theorem (2.6), we have

$$\sum_{n=0}^{\infty} \mu_n \frac{z^n}{(q;q)_n} = \frac{1}{(aq;q)_\infty} \sum_{n=0}^{\infty} \frac{(aq;q)_n}{(q;q)_n} z^n = \frac{(azq;q)_\infty}{(aq,z;q)_\infty}.$$

□

3.2.5 q-Laguerre polynomials

The q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$ fulfil the following orthogonality relation [30, P. 522]

$$\begin{aligned}\sum_{k=-\infty}^{\infty} &\frac{q^{(\alpha+1)k}}{(-cq^k;q)_\infty} L_m^{(\alpha)}(cq^k;q) L_n^{(\alpha)}(cq^k;q) \\ &= \frac{(q,-cq^{\alpha+1},-c^{-1}q^{-\alpha};q)_\infty}{(q^{\alpha+1},-c,-c^{-1}q;q)_\infty} \frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n} \delta_{mn}. \quad \alpha > -1, \ \ c > 0.\end{aligned} \tag{3.29}$$

Therefore, the canonical q-Laguerre moments are

$$\mu_n = \sum_{k=-\infty}^{\infty} \frac{q^{(\alpha+1)k}}{(-cq^k;q)_\infty} (cq^k)^n.$$

Proposition 42. *The canonical q-Laguerre moments have the representation:*

$$\mu_n = c^n \frac{(q,-cq^{n+\alpha+1},-c^{-1}q^{-n-\alpha};q)_\infty}{(q^{n+\alpha+1},-c,-c^{-1}q;q)_\infty} \tag{3.30}$$

$$= \frac{(q,-cq^{\alpha+1},-c^{-1}q^{-\alpha};q)_\infty}{(q^{\alpha+1},-c,-c^{-1}q;q)_\infty} \frac{(q^{\alpha+1};q)_n}{q^{(\alpha+1)n}} q^{-\binom{n}{2}}. \tag{3.31}$$

Proof. By definition, we have

$$\mu_n = \sum_{k=-\infty}^{\infty} \frac{q^{(n+\alpha+1)k} c^n}{(-cq^k;q)_\infty} = \frac{c^n}{(-c;q)_\infty} \sum_{k=-\infty}^{\infty} (-c;q)_k q^{(n+\alpha+1)k}.$$

Next using the Ramanujan identity for the bilateral sum (2.9) where we take the lower parameter equal to 0, we obtain the desired formula. Another way to get the result is to take in the orthogonality relation $m = n = 0$, and then replace α by $\alpha + n$. □

Note that the canonical q-Laguerre moments with the normalization $\mu_0 = 1$ were given in [10, P. 49].

Proposition 43. *The canonical q-Laguerre moments have the following q-exponential generating function*

$$\frac{(q,-cq^{\alpha+1},-c^{-1}q^{-\alpha};q)_\infty}{(q^{\alpha+1},-c,-c^{-1}q;q)_\infty} \frac{(z;q)_\infty}{(zq^{-(\alpha+1)};q)_\infty} = \sum_{n=0}^{\infty} \mu_n \frac{q^{\binom{n}{2}}}{(q;q)_n} z^n. \tag{3.32}$$

Proof. By using the q-binomial theorem (2.6), we have

$$\sum_{n=0}^{\infty} \mu_n \frac{q^{\binom{n}{2}}}{(q;q)_n} z^n = \frac{(q,-cq^{\alpha+1},-c^{-1}q^{-\alpha};q)_\infty}{(q^{\alpha+1},-c,-c^{-1}q;q)_\infty} \sum_{n=0}^{\infty} \frac{(q^{\alpha+1};q)_n}{(q;q)_n} \left(\frac{z}{q^{(\alpha+1)}}\right)^n$$
$$= \frac{(q,-cq^{\alpha+1},-c^{-1}q^{-\alpha};q)_\infty}{(q^{\alpha+1},-c,-c^{-1}q;q)_\infty} \frac{(z;q)_\infty}{(zq^{-(\alpha+1)};q)_\infty}.$$

□

3.2.6 q-Bessel polynomials

The q-Bessel polynomials $y_n(x;a;q)$ fulfil the following orthogonality relation [30, P. 527]

$$\sum_{k=0}^{\infty} \frac{a^k}{(q;q)_k} q^{\binom{k+1}{2}} y_m(q^k;a;q) y_n(q^k;a;q) = (q;q)_n(-aq^n;q)_\infty \frac{a^n q^{\binom{n+1}{2}}}{(1+aq^{2n})} \delta_{mn}, \quad a>0.$$

Therefore the canonical q-Bessel moments are

$$\mu_n = \sum_{k=0}^{\infty} \frac{a^k}{(q;q)_k} q^{\binom{k+1}{2}} q^{nk}.$$

Proposition 44. *The canonical q-Bessel moments have the representation:*

$$\mu_n = \frac{(-aq;q)_\infty}{(-aq;q)_n}. \tag{3.33}$$

Proof. Using the Euler summation formula (2.8), and the relation $\binom{k+1}{2} = \binom{k}{2} + k$, we get:

$$\mu_n = \sum_{k=0}^{\infty} \frac{a^k}{(q;q)_k} q^{\binom{k+1}{2}} q^{nk} = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} (-aq^{n+1})^k = (-aq^{n+1};q)_\infty = \frac{(-aq;q)_\infty}{(-aq;q)_n}.$$

□

Proposition 45. *The canonical q-Bessel moments have the following generating function*

$$(-aq;q)_\infty {}_2\phi_1 \left(\begin{matrix} q,0 \\ -aq \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \mu_n z^n. \tag{3.34}$$

Proof. Using the q-version of Algorithm 2.2 from [32] for the conversion of sums into q-hypergeometric notation (`sum2qhyper`) we get the result. □

Proposition 46. *The canonical q-Bessel moments have the following generating function*

$$\frac{(-aq;q)_\infty}{1-z} = \sum_{n=0}^{\infty} \mu_n (-aq;q)_n z^n, \quad |z|<1.$$

Proof. The proof follows by simple computation using the geometric series. □

3.2.7 q-Charlier polynomials

The q-Charlier polynomials $C_n(x;a;q)$ fulfil the following orthogonality relation [30, P. 530]

$$\sum_{k=0}^{\infty} \frac{a^k}{(q;q)_k} q^{\binom{k}{2}} C_m(q^{-k};a;q) C_n(q^{-k};a;q)$$
$$= q^{-n}(-a;q)_\infty(-a^{-1}q,q;q)_n \delta_{mn}, \quad a>0. \tag{3.35}$$

Therefore, the canonical q-Charlier moments are

$$\mu_n = \sum_{k=0}^{\infty} \frac{a^k}{(q;q)_k} q^{\binom{k}{2}} q^{-nk}.$$

Proposition 47. *The canonical q-Charlier moments have the representation*

$$\mu_n = (-a;q)_\infty \left(-a^{-1}q;q\right)_n \left(\frac{a}{q}\right)^n q^{-\binom{n}{2}}. \tag{3.36}$$

Proof. We have

$$\mu_n = \sum_{k=0}^{\infty} \frac{a^k}{(q;q)_k} q^{\binom{k}{2}} q^{-nk} = \sum_{k=0}^{\infty} \frac{q^{\binom{k}{2}}}{(q;q)_k} \left(aq^{-n}\right)^k = \sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} \left(-aq^{-n}\right)^k.$$

Now applying the Euler formula (2.8), with $x := -aq^{-n}$, we get

$$\mu_n = (-aq^{-n};q)_\infty.$$

Next combining the relations

$$(aq^\lambda;q)_\infty = \frac{(a;q)_\infty}{(a;q)_\lambda} \quad \text{and} \quad (a;q)_{-n} = \frac{(-a^{-1}q)^n}{(a^{-1}q;q)_n} q^{\binom{n}{2}},$$

it follows that

$$\mu_n = \frac{(-a;q)_\infty}{(-a;q)_{-n}} = \frac{(-a;q)_\infty}{\frac{(a^{-1}q)^n}{(-a^{-1}q;q)_n} q^{\binom{n}{2}}} = (-a;q)_\infty \left(-a^{-1}q;q\right)_n \left(\frac{a}{q}\right)^n q^{-\binom{n}{2}}.$$

□

The q-Charlier moments with respect to the basis $(x;q)_n$ are given in Chapter 5. Note that the canonical q-Charlier moments with the normalization $\mu_0 = 1$ were given in [10, P. 50].

Proposition 48. *The canonical q-Charlier moments have the following q-exponential generating function*

$$\frac{(-a,-z;q)_\infty}{(aq^{-1}z;q)_\infty} = \sum_{n=0}^{\infty} \mu_n \frac{q^{\binom{n}{2}} z^n}{(q;q)_n}, \quad |az| < |q|. \tag{3.37}$$

Proof. We have

$$\sum_{n=0}^{\infty} \mu_n \frac{q^{\binom{n}{2}} z^n}{(q;q)_n} = (-a;q)_\infty \sum_{n=0}^{\infty} \frac{(-a^{-1}q;q)_n}{(q;q)_n} \left(\frac{az}{q}\right)^n.$$

The result follows by using the q-binomial theorem (2.6). □

Chapter 4

Inversion Formulas

Let $(\theta_n(x))_n$ and $(P_n(x))_n$ be two polynomial sets such that for each n, we have the expansion

$$P_n(x) = \sum_{m=0}^{n} D_m(n)\theta_m(x).$$

The *inversion problem* is the problem of finding the coefficients $I_m(n)$ in the expansion

$$\theta_n(x) = \sum_{m=0}^{n} I_m(n)P_m(x). \tag{4.1}$$

Note that when the coefficients $D_m(n)$ and $I_m(n)$ are known, one can determine the coefficients $C_m(n)$ of the *connection problem* between two polynomial sets

$$P_n(x) = \sum_{m=0}^{n} C_m(n)Q_m(x),$$

and the coefficients of the *linearization problem*

$$P_n(x)Q_m(x) = \sum_{k=0}^{n+m} L_k(m,n)R_k(x).$$

Many methods have been used to determine the inversion coefficients in the literature, see for example [5], [6] and the references therein. In [33], Koepf and Schmersau used an algorithmic approach to determine those coefficients for the classical continuous and the classical discrete orthogonal polynomials. In [18], following this method, we solved the inversion problem for the orthogonal polynomials of the q-Hahn class, therefore recovering the results given by Area et *al*. in [5].

In this chapter, we present two methods for the determination of the inversion coefficients for all the classical orthogonal polynomial sets. The importance of the inversion coefficients appears in Theorem 50 on page 45. In what follows, the inversion coefficients are provided.

4.1 The methods

4.1.1 The algorithmic method

We assume that the polynomial $P_n(x)$ has in the basis $(\theta_n(x))_n$ the expansion

$$P_n(x) = \sum_{m=0}^{n} D_m(n)\theta_m(x).$$

It is well-known that every orthogonal polynomial set $(P_n)_n$ fulfils a three-term recurrence relation of the form (see [34],[40])

$$xP_n(x) = a_nP_{n+1}(x) + b_nP_n(x) + c_nP_{n-1}(x), \quad n \geq 1. \tag{4.2}$$

Classical orthogonal polynomials satisfy further structure equations. One of those is given by the differential / difference / q-difference rule (see e.g [34],[33],[35])

$$\sigma(x)P_n'(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x) \quad (n \geq 1), \tag{4.3}$$

$$\sigma(x)\nabla P_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x) \quad (n \geq 1), \tag{4.4}$$

or

$$\sigma(x)D_{\frac{1}{q}} P_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x) \quad (n \geq 1), \tag{4.5}$$

respectively.
Another useful structure relation used here is the three-term recurrence relation for the first derivative, that is

$$xP_n'(x) = \alpha_n^\star P_{n+1}'(x) + \beta_n^\star P_n'(x) + \gamma_n^\star P_{n-1}'(x) \quad (n \geq 1), \tag{4.6}$$

$$x\Delta P_n(x) = \alpha_n^\star \Delta P_{n+1}(x) + \beta_n^\star \Delta P_n(x) + \gamma_n^\star \Delta P_{n-1}(x) \quad (n \geq 1), \tag{4.7}$$

or

$$xD_q P_n(x) = \alpha_n^\star D_q P_{n+1}(x) + \beta_n^\star D_q P_n(x) + \gamma_n^\star D_q P_{n-1}(x) \quad (n \geq 1), \tag{4.8}$$

respectively.
When similar structure relations can be established for the basis $\theta_n(x)$, one can then use them to get two or three cross rules for the coefficients $I_m(n)$ which can be determined by linear algebra. More details on this method can be found in [18] and [33].

4.1.2 Inversion results from Verma's bibasic formula

In [6], Area et *al*. used Verma's q-extension [51] of Fields and Wimp [14] expansion of

$$
\begin{aligned}
{}_{r+t}\phi_{s+u}\left(\left.\begin{matrix}(a_r),(c_t)\\(b_s),(d_u)\end{matrix}\,\right|\, q; y\omega\right) &= \sum_{j=0}^{\infty} \frac{((c_t),(e_k);q)_j}{(q,(d_u),\gamma q^j;q)_j} y^j \left[(-1)^j q^{\binom{j}{2}}\right]^{u+3-t-k} \\
&\quad \cdot {}_{t+k}\phi_{u+1}\left(\left.\begin{matrix}(c_t q^j),(e_k q^j)\\ \gamma q^{2j+1},(d_u q^j)\end{matrix}\,\right|\, q, yq^{j(u+2-t-k)}\right) \\
&\quad \cdot {}_{r+2}\phi_{s+k}\left(\left.\begin{matrix}q^{-j},\gamma q^j,(a_r)\\(b_s),(e_k)\end{matrix}\,\right|\, q,\omega q\right)
\end{aligned} \tag{4.9}
$$

in powers of $y\omega$ as given in [19, (3.7.9)] to find the solution of the inversion problem (4.1) for polynomials of the Askey scheme and its q-analogue. Here, the notation (a_r) means r parameters of the type $a_1, a_2, \cdots, a_r$ and the notation $(a_r q^j)$ means r parameters of the form $a_1 q^j, a_2 q^j, \cdots, a_r q^j$. The method is the following.

We choose $u = t = 0$, and $k = 1$ in (4.9). Then for $\omega = x$ and $\gamma = 0$, we obtain

$$
{}_r\phi_s\left(\left.\begin{matrix}(a_r)\\(b_s)\end{matrix}\,\right|\, q; yx\right) = \sum_{j=0}^{\infty} \frac{[(-1)^j q^{\binom{j}{2}}]^2}{(q;q)_j} y^j {}_1\phi_1\left(\left.\begin{matrix}0\\0\end{matrix}\,\right|\, q; q^j y\right) {}_{r+1}\phi_s\left(\left.\begin{matrix}q^{-j},(a_r)\\(b_s)\end{matrix}\,\right|\, q; qx\right).
$$

Expanding the left-hand side, the coefficient of y^n is

$$\frac{(a_R;q)_n}{(q;q)_n (b_S;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{s-r+1} x^n. \tag{4.10}$$

Moreover, the right-hand side can be rewritten as

$$\sum_{j=0}^{\infty}\sum_{h=0}^{\infty}\left(\frac{q^{jh}}{(q;q)_j}\left[(-1)^j q^{\binom{j}{2}}\right]^2 \frac{\left[(-1)^h q^{\binom{h}{2}}\right]}{(q;q)_h} y^{h+j}\right) {}_{r+1}\phi_s \left(\left.\begin{matrix} q^{-j},(a_r) \\ (b_s)\end{matrix}\right| q;qx\right),$$

so that the coefficient of y^n in this expression is now

$$\sum_{\ell=0}^{n} \frac{(-1)^{n-\ell} q^{2\binom{\ell}{2}} q^{\binom{n-\ell}{2}} q^{(n-\ell)\ell}}{(q;q)_\ell (q;q)_{n-\ell}} {}_{r+1}\phi_s \left(\left.\begin{matrix} q^{-\ell}, a_R \\ b_S\end{matrix}\right| q;qx\right). \tag{4.11}$$

From (4.10) and (4.11) we get

$$\frac{(-1)^n q^{n(n-1)/2)^{s-r}}(a_2,\ldots,a_{r+1};q)_n}{(b_1,b_2,\ldots,b_s;q)_n} x^n = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k\end{bmatrix}_q q^{\binom{k}{2}} {}_{r+1}\phi_s \left(\left.\begin{matrix} q^{-k},a_2,\ldots,a_{r+1} \\ b_1,b_2,\ldots,b_s\end{matrix}\right| q;qx\right). \tag{4.12}$$

Application of appropriate limit relations ($q \uparrow 1$) between basic hypergeometric and hypergeometric series to (4.12) leads to the formula

$$\frac{\prod_{j=2}^{p+1}(a_j)_n}{\prod_{j=1}^{s}(b_j)_n} x^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} {}_{r+1}F_s \left(\left.\begin{matrix} -k,a_2,\ldots,a_p,a_{p+1} \\ b_1,b_2,\ldots,b_s\end{matrix}\right| x\right) \tag{4.13}$$

It should be mentioned that until now, the coefficients a_R and b_S appearing in (4.13) are independent of the summation index k. However, in some families belonging to the Askey scheme and its q-analogue, one of the numerator parameters depends on k in the form $a_2 + k$ (Askey scheme) or $a_2 q^k$ (q-analogue). In these situations and in case of polynomials belonging to the q-analogue of the Askey scheme, the following formula (see [6]) should be used:

$$\begin{aligned} &\frac{((-1)^n q^{n(n-1)/2)^{s-r}}(a_3,\ldots,a_{r+1})_n}{(b_1,b_2,\ldots,b_s;q)_n} x^n \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k\end{bmatrix}_q \frac{(-1)^k q^{\binom{k}{2}}}{(a_2q^k, a_2q^{2k+1};q)_k} {}_{r+1}\phi_s \left(\left.\begin{matrix} q^{-k},a_2q^k,a_3,\ldots,a_{r+1} \\ b_1,b_2,\ldots,b_s\end{matrix}\right| q;qx\right). \end{aligned} \tag{4.14}$$

Once again, application of appropriate limit relations but now to (4.14) leads to the formula

$$\frac{\prod_{j=3}^{p+1}(a_j)_n}{\prod_{j=1}^{s}(b_j)_n} x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(a_2+k)_k (a_2+2k+1)_{n-k}} {}_{p+1}F_s \left(\left.\begin{matrix} -k,a_2+k,a_3,\ldots,a_p,a_{p+1} \\ b_1,b_2,\ldots,b_s\end{matrix}\right| x\right) \tag{4.15}$$

4.2 Explicit representations of the inversion coefficients for the classical orthogonal polynomials

4.2.1 The classical continuous case

The following results are from [33].

The Jacobi polynomials

$$\begin{aligned}(1-x)^n &= 2^n\Gamma(\alpha+n+1)\\ &\quad\times\sum_{m=0}^{n}\frac{(\alpha+\beta+2m+1)\Gamma(\alpha+\beta+m+1)}{\Gamma(\alpha+m+1)\Gamma(\alpha+\beta+n+m+2)}(-n)_m P_m^{(\alpha,\beta)}(x), \qquad (4.16)\\ (1+x)^n &= 2^n\Gamma(\beta+n+1)\\ &\quad\times\sum_{m=0}^{n}(-1)^m\frac{(\alpha+\beta+2m+1)\Gamma(\alpha+\beta+m+1)}{\Gamma(\beta+m+1)\Gamma(\alpha+\beta+n+m+2)}(-n)_m P_m^{(\alpha,\beta)}(x). \qquad (4.17)\end{aligned}$$

The Laguerre polynomials

$$x^n = (1+\alpha)_n\sum_{m=0}^{n}\frac{(-n)_m}{(1+\alpha)_m}L_m^{(\alpha)}(x). \qquad (4.18)$$

The Hermite polynomials

$$x^n = \frac{n!}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1}{k!(n-2k)!}H_{n-2k}(x) \qquad (4.19)$$

The Bessel polynomials

$$x^n = (-2)^n\sum_{m=0}^{n}(2m+\alpha+1)\frac{(-n)_m\Gamma(\alpha+m+1)}{m!\Gamma(n+m+\alpha+2)}B_m^{(\alpha)}(x). \qquad (4.20)$$

4.2.2 The classical discrete case

The following results are from [33].

The Hahn polynomials

$$x^{\underline{n}} = (-1)^n\frac{(1+\alpha)_n(-N)_n}{(\alpha+\beta+2)_n}\sum_{m=0}^{n}\frac{(\alpha+\beta+1+2m)}{(\alpha+\beta+1)}\frac{(-n)_m(1+\alpha+\beta)_m}{(n+2+\alpha+\beta)_m m!}Q_m(x;\alpha,\beta,N). \qquad (4.21)$$

The Krawtchouk polynomials

$$x^{\underline{n}} = (-1)^n(-N)_n\sum_{m=0}^{n}\frac{p^{n-m}(-n)_m}{(-N)_m}k_m^{(p)}(x,N). \qquad (4.22)$$

The Meixner polynomials

$$x^{\underline{n}} = (-1)^n(\gamma)_n\left(\frac{\mu}{\mu-1}\right)^n\sum_{m=0}^{n}\frac{(-n)_m}{(\gamma)_n m!}m_m^{(\gamma,\mu)}(x). \qquad (4.23)$$

The Charlier polynomials

$$x^{\underline{n}} = \mu^n\sum_{m=0}^{n}\frac{(-n)_m}{m!}c_m^{(\mu)}(x). \qquad (4.24)$$

4.2.3 The classical q-discrete case

Part of the following results are from [5] and [18] and have been converted following the standardization of this work. The results for the Quantum q-Krawtchouk, the q-Krawtchouk and the Affine q-Krawtchouk polynomials are obtained using (4.12) and (4.14).

The Big q-Jacobi polynomials

$$(x;q)_n = \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{q^{\binom{m}{2}}(aq,cq;q)_n}{(abq^{m+1},abq^{2m+2};q)_{n-m}} P_m(x;a,b,c;q). \tag{4.25}$$

The q-Hahn polynomials

$$(q^{-x};q)_n = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-1)^m q^{\binom{m}{2}}(\alpha q,q^{-N};q)_n}{(\alpha\beta q^{m+1};q)_m(\alpha\beta q^{2m+2};q)_{n-m}} Q_m(q^{-x};\alpha,\beta,N|q). \tag{4.26}$$

The Big q-Laguerre polynomials

$$(x;q)_n = \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}}(aq,bq;q)_n P_m(x;a,b;q). \tag{4.27}$$

The Little q-Jacobi polynomials

$$x^n = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-1)^m q^{\binom{m}{2}}(aq;q)_n}{(abq^{m+1};q)_m(abq^{2m+2};q)_{n-m}} p_m(x;a,b|q). \tag{4.28}$$

The Little q-Legendre polynomials

$$x^n = \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-1)^m q^{\binom{m}{2}}(q;q)_n}{(q^{m+1};q)_m(q^{2m+2};q)_{n-m}} P_m(x|q). \tag{4.29}$$

The q-Meixner polynomials

$$(q^{-x};q)_n = \sum_{m=0}^{n} (-1)^{n-m} q^{\frac{m(5m+1)}{2}-n(m+1)} c^n \begin{bmatrix} n \\ m \end{bmatrix}_q (bq;q)_n M_m(q^{-x};b,c;q). \tag{4.30}$$

The Quantum q-Krawtchouk polynomials

$$(q^{-x};q)_n = \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{q^{\binom{m}{2}}}{p^{m(n+1)}} (q^{-N};q)_n (pq)^{-n} K_m^{qtm}(q^{-x};p,N|q). \tag{4.31}$$

The q-Krawtchouk polynomials

$$(q^{-x};q)_n = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-1)^m q^{\binom{m}{2}}(q^{-N};q)_n}{(-pq^m;q)_m(-pq^{2m+1};q)_{n-m}} K_m(q^{-x};p,N;q). \tag{4.32}$$

The Affine q-Krawtchouk polynomials

$$(q^{-x};q)_n = \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}}(pq,q^{-N};q)_n K_m^{\mathrm{Aff}}(q^{-x};p,N;q). \tag{4.33}$$

$$(q^{-x})^n(q^{x-N};q)_n = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q (pq)^{n-m}(q^{-N};q)_n(pq;q)_m K_m^{\mathrm{Aff}}(q^{-x};p,N;q). \tag{4.34}$$

The Little q-Laguerre polynomials

$$x^n = \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}} (aq;q)_n p_m(x;a|q), \tag{4.35}$$

$$(x \ominus 1)_q^n = \sum_{m=0}^{n} (-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{n}{2}-\binom{m}{2}} (aq)^n (a^{-1}q^{-m};q)_m p_m(x;a|q). \tag{4.36}$$

The q-Laguerre polynomials

$$x^n = \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{(m-n)(2\alpha+3m+n+1)}{2} - m(m+\alpha)} (q;q)_m (q^{m+\alpha+1};q)_{n-m} L_m^{(\alpha)}(x;q), \tag{4.37}$$

The Alternative q-Charlier/q-Bessel polynomials

$$x^n = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-1)^m q^{\binom{m}{2}}}{(-aq^m;q)_m (-aq^{2m+1};q)_{n-m}} y_m(x;a|q). \tag{4.38}$$

The q-Charlier polynomials

$$(q^{-x};q)_n = \sum_{m=0}^{n} (-1)^{n-m} a^n \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{m(m+1)}{2} - n(m+1)} C_m(q^{-x};a;q). \tag{4.39}$$

The Al Salam-Carlitz I polynomials

$$x^n = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \left(\sum_{i=0}^{n-m} \begin{bmatrix} n-m \\ i \end{bmatrix}_q a^i \right) U_m^{(a)}(x;q) \tag{4.40}$$

$$(x \ominus 1)_q^n = \sum_{m=0}^{n} a^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q U_m^{(a)}(x;q). \tag{4.41}$$

The Al Salam-Carlitz II polynomials

$$(x;q)_n = \sum_{m=0}^{n} (-1)^n \begin{bmatrix} n \\ m \end{bmatrix}_q a^{n-m} q^{m(m-n)+\binom{n}{2}} V_m^{(a)}(x;q). \tag{4.42}$$

The Stieltjes-Wigert polynomials

$$x^n = \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{(m-n)(3m+n+1)}{2} - m^2} (q;q)_m S_m(x;q). \tag{4.43}$$

The Discrete q-Hermite I polynomials

$$x^n = \sum_{m=0}^{n} \frac{1+(-1)^{n-m}}{2} \begin{bmatrix} n \\ m \end{bmatrix}_q (q;q^2)_{(n-m)/2} h_m(x;q) \tag{4.44}$$

$$(x \ominus 1)_q^n = \sum_{m=0}^{n} (-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q h_m(x;q). \tag{4.45}$$

The Discrete q-Hermite II polynomials

$$(x;q)_n = \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{m(m-n)+\binom{n}{2}} \tilde{h}_m(x;q). \tag{4.46}$$

4.2.4 The classical quadratic case

The following results are obtained using the formulas (4.13) and (4.15). We provide the proof for the Wilson case, the other cases being similar.

The Wilson polynomials

$$\theta_n(x) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m (a+b+m)_{n-m}(a+c+m)_{n-m}(a+d+m)_{n-m}}{(a+b+c+d+m-1)_m(a+b+c+d+2m)_{n-m}} W_m(x^2;a,b,c,d), \tag{4.47}$$

where

$$\theta_n(x) = (a-ix)_n(a+ix)_n.$$

Proof. In order to derive this result, we recall that the Wilson polynomials [30, P. 185] have the hypergeometric representation

$$\frac{W_n(x^2;a,b,c,d)}{(a+b)_n(a+c)_n(a+d)_n} = {}_4F_3 \left(\left. \begin{matrix} -n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix} \right| 1 \right).$$

Therefore, by (4.15) with $a_2 = a+b+c+d-1$, it follows that

$$\frac{\theta_n(x)}{(a+b)_n(a+c)_n(a+d)_n} = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{(a+b+c+d-1+m)_m(a+b+c+d+2m)_{n-m}} \times {}_4F_3 \left(\left. \begin{matrix} -m, m+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix} \right| 1 \right).$$

This leads to

$$\theta_n(x) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m (a+b)_n(a+c)_n(a+d)_n}{(a+b+c+d-1+m)_m(a+b+c+d+2m)_{n-m}} \times \frac{(a+b)_m(a+c)_m(a+d)_m}{(a+b)_m(a+c)_m(a+d)_m} {}_4F_3 \left(\left. \begin{matrix} -m, m+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{matrix} \right| 1 \right),$$

and this last relation reads

$$\theta_n(x) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m (a+b+m)_{n-m}(a+c+m)_{n-m}(a+d+m)_{n-m}}{(a+b+c+d-1+m)_m(a+b+c+d+2m)_{n-m}} W_m(x^2;a,b,c,d).$$

□

Remark 49. *It should be noted that we recover this result in [36] using the algorithm method described in section 4.1.1.*

The Racah polynomials

$$\theta_n(\lambda(x)) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m (\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}{(\alpha+\beta+m+1)_m(\alpha+\beta+2m+2)_{n-m}} R_m(\lambda(x);\alpha,\beta,\gamma,\delta), \tag{4.48}$$

where

$$\theta_n(x) = (-x)_n(x+\gamma+\delta+1)_n.$$

The Continuous Dual Hahn polynomials

$$\theta_n(x) = \sum_{m=0}^{n} (-1)^m \binom{n}{m} (a+b+m)_{n-m}(a+c+m)_{n-m} S_m(x^2;a,b,c). \tag{4.49}$$

where

$$\theta_n(x) = (a-ix)_n(a+ix)_n.$$

The Continuous Hahn polynomials

$$\theta_n(x) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m m!(a+c+m)_{n-m}(a+d+m)_{n-m}}{(a+b+c+d+1+m)_m(a+b+c+d+2m)_{n-m}} p_m(x;a,b,c,d). \tag{4.50}$$

where

$$\theta_n(x) = (a+ix)_n$$

The Dual Hahn polynomials

$$\theta_n(x) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m(\gamma+1)_n(-N)_n}{(\delta+\gamma+m+1)_m(\delta+\gamma+2m+2)_{n-m}} R_m(\lambda(x);\gamma,\delta,N). \tag{4.51}$$

where

$$\theta_n(x) = (-x)_n(x+\gamma+\delta+1)_n.$$

The Meixner-Pollaczek polynomials

$$\theta_n(x) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m m!(2\lambda+m)_{n-m}}{(1-e^{-2i\phi})^n e^{im\phi}} P_m^{(\lambda)}(x;\phi). \tag{4.52}$$

where

$$\theta_n(x) = (\lambda+ix)_n.$$

4.2.5 The classical q-quadratic case

In this part, since θ will denote an angle, we will denote the basis involved in the inversion formula (4.1) by $\mathcal{B}_n$ instead of θ_n.
The following results are obtained using the formulas (4.12) and (4.14).

The Askey-Wilson polynomials

$$\mathcal{B}_n(x) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}} \frac{(-a)^m(abq^m,acq^m,adq^m;q)_{n-m}}{(abcdq^{m-1};q)_m(abcdq^{2m};q)_{n-m}} p_m(x;a,b,c,d), \tag{4.53}$$

where

$$\mathcal{B}_n(x) = (ae^{i\theta},ae^{-i\theta};q)_n, \quad x = \cos\theta.$$

The q-Racah polynomials

$$\mathcal{B}_n(\mu(x)) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-1)^m(\alpha q,\beta\delta q,\gamma q;q)_n}{(\alpha\beta q^{m+1};q)_m(\alpha\beta q^{2m+2};q)_{n-m}} R_m(\mu(x);\alpha,\beta,\delta,\gamma|q), \tag{4.54}$$

where

$$\mathcal{B}_n(\mu(x)) = (q^{-x},\gamma\delta q^{x+1};q)_n, \quad \mu(x) = q^{-x}+\delta\gamma q^{x+1}.$$

The Continuous Dual q-Hahn polynomials

$$\mathcal{B}_n(x) = \sum_{m=0}^{n} (-a)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{n}{m}} (abq^m, acq^m; q)_{n-m} p_m(x; a, b, c|q), \tag{4.55}$$

where

$$\mathcal{B}_n(x) = (ae^{i\theta}, ae^{-i\theta}; q)_n, \quad x = \cos\theta.$$

The Continuous q-Hahn polynomials

$$\mathcal{B}_n(x) = \sum_{m=0}^{n} \frac{(-ae^{i\phi})^m q^{\binom{m}{2}} (abq^m e^{2i\phi}, acq^m, adq^m; q)_{n-m}}{(abcdq^m; q)_m (abcdq^{2m}; q)_{n-m}} P_m(x; a, b, c, d|q), \tag{4.56}$$

where

$$\mathcal{B}_n(x) = (ae^{i(\theta+2\phi)}, ae^{-i\theta}; q)_n, \quad x = \cos(\theta+\phi).$$

The Dual q-Hahn polynomials

$$\mathcal{B}_n(\mu(x)) = \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}} (\gamma q, q^{-N}; q)_n R_m(\mu(x); \gamma, \delta, N|q), \tag{4.57}$$

where

$$\mathcal{B}_n(\mu(x)) = (q^{-x}, \gamma\delta q^{x+1}; q)_n, \quad \mu(x) = q^{-x} + \delta\gamma q^{x+1}.$$

The Al-Salam-Chihara polynomials

$$\mathcal{B}_n(x) = \sum_{m=0}^{n} (-a)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}} (abq^m; q)_{n-m} Q_m(x; a, b|q), \tag{4.58}$$

where

$$\mathcal{B}_n(x) = (ae^{i\theta}, ae^{-i\theta}; q)_n, \quad x = \cos\theta.$$

The q-Meixner-Pollaczek polynomials

$$\mathcal{B}_n(x) = \sum_{m=0}^{n} (-ae^{i\phi})^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{n}{m}} (q; q)_n (a^2 q^m; q)_{n-m} P_m(x; a|q), \tag{4.59}$$

where

$$\mathcal{B}_n(x) = (ae^{i(\theta+2\phi)}, ae^{-i\theta}; q)_n, \quad x = \cos(\theta+\phi).$$

The Continuous q-Jacobi polynomials

$$\mathcal{B}_n(x) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-1)^m (q; q)_m (q^{\alpha+1+m}; q)_{n-m} (-q^{\frac{1}{2}(\alpha+\beta+1)}, -q^{\frac{1}{2}(\alpha+\beta+2)}; q)_n}{(q^{m+\alpha+\beta+1}; q)_m (q^{2m+\alpha+\beta+1}; q)_{n-m}} P_m^{(\alpha,\beta)}(x|q), \tag{4.60}$$

where

$$\mathcal{B}_n(x) = (q^{\frac{1}{2}\alpha+\frac{1}{4}} e^{i\theta}, q^{\frac{1}{2}\alpha+\frac{1}{4}} e^{-i\theta}; q)_n, \quad x = \cos\theta.$$

The Continuous q-Ultraspherical (Rogers) polynomials

$$\mathcal{B}_n(x) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-\beta^{\frac{1}{2}})^m (q; q)_m (\beta q^{\frac{1}{2}}, -\beta, -\beta q^{\frac{1}{2}}; q)_n}{(\beta^2 q^m; q)_m (\beta^2 q^{2m+1}; q)_{n-m} (\beta^2; q)_m} C_m(x; \beta|q) \tag{4.61}$$

where

$$\mathcal{B}_n(x) = (\beta^{\frac{1}{2}} e^{i\theta}, \beta^{\frac{1}{2}} e^{-i\theta}; q)_n, \quad x = \cos\theta.$$

The Continuous q-Legendre polynomials

$$\mathcal{B}_n(x) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-1)^m (q, -q^{\frac{1}{2}}, -q; q)_n}{(q^{m+1}; q)_m (q^{2m+2}; q)_{n-m}} P_m(x|q), \tag{4.62}$$

where

$$\mathcal{B}_n(x) = (q^{\frac{1}{4}} e^{i\theta}, q^{\frac{1}{4}} e^{-i\theta}; q)_n, \quad x = \cos\theta.$$

The Dual q-Krawtchouk polynomials

$$\mathcal{B}_n(\lambda(x)) = \sum_{m=0}^{n} (-1)^n \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}} (q^{-N}; q)_n K_m(\lambda(x); c, N|q), \tag{4.63}$$

where

$$\mathcal{B}_n(\lambda(x)) = (q^{-x}, cq^{x-N}; q)_n, \quad \lambda(x) = q^{-x} + cq^{x-N}.$$

The Continuous big q-Hermite polynomials

$$\mathcal{B}_n(x) = \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}} H_m(x; a|q), \tag{4.64}$$

where

$$\mathcal{B}_n(x) = (ae^{i\theta}, ae^{-i\theta}; q)_n, \quad x = \cos\theta.$$

The Continuous q-Laguerre polynomials

$$\mathcal{B}_n(x) = \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{n}{m}} (q^{\alpha+1+m}; q)_{n-m} P_m^{(\alpha)}(x|q), \tag{4.65}$$

where

$$\mathcal{B}_n(x) = (q^{\frac{1}{2}\alpha+\frac{1}{4}} e^{i\theta}, q^{\frac{1}{2}\alpha+\frac{1}{4}} e^{-i\theta}; q)_n, \quad x = \cos\theta.$$

Chapter 5

Moments of Orthogonal Polynomials: Complicated Cases

5.1 Introduction

In Chapter 3, we have computed some moments. The computations were easy and we could do them directly by using some well-known results in the literature. But, we were not able to get the moments of all the classical families listed in Chapter 2. In this chapter, we establish a powerful link between the inversion formula for a family (see Chapter 4) and the moments of this family. This enables us to deduce the moments of the families mentioned earlier.

5.2 Inversion formula and moments of orthogonal polynomials

In Chapter 4, using previous works by Koepf and Schmersau (see [33]), Area, Godoy, Ronveaux and Zarzo (see [5],[6]), Foupouagnigni, Koepf, Tcheutia, Njionou (see [18]), we have given explicit expressions of $I_m(n)$ (for a suitable choice of $\theta_n(x)$) in the expansion

$$\theta_n(x) = \sum_{m=0}^{n} I_m(n) P_m(x). \tag{5.1}$$

The following theorem establishes a link between the inversion problem for a family and the generalized moments of this family.

Theorem 50. *For all $n \in \mathbb{N}$, the generalized moments of the family $(P_n)_n$ with respect to the basis $\theta_n(x)$ can be computed by the formula*

$$\mu_n(\theta_k(x)) = I_0(n) P_0 \mu_0. \tag{5.2}$$

Proof. Using the expansion (5.1), we have

$$\mu_n(\theta_k(x)) = \frac{1}{P_0}(\theta_n(x), P_0) = \frac{1}{P_0}\sum_{k=0}^{n} I_k(n)(P_n, P_0) = \frac{1}{P_0} I_0(n)(P_0, P_0) = I_0(n) P_0 \mu_0,$$

where (f, g) is the inner product defined by

$$(f, g) = \int_{-\infty}^{\infty} f(x) g(x) d\alpha(x).$$

□

It should be mentioned that the term μ_0 is easily obtained by taking $m = n = 0$ in the orthogonality relation for each family and therefore does not depend on the chosen basis. Note also that this result was announced in [22].

5.3 Some connection formulas between some bases

In order to obtain canonical moments from generalized moments, we need some connection formulas as pointed out in Section 2.4. First, we introduce some famous numbers.

5.3.1 Elementary symmetric polynomials

Definition 51. *(see [37, P. 159]) The elementary symmetric polynomials $e_k(a_1, \ldots, a_n)$ in n variables $a_1, \ldots, a_n$ for $k = 0, 1, \ldots, n$ can be defined as*

$$e_0(a_1, a_2, \ldots, a_n) = 1,$$

and

$$e_k(a_1, a_2, \ldots, a_n) = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le n} a_{j_1} a_{j_2} \cdots a_{j_k}, \quad 1 \le k \le n. \tag{5.3}$$

For example, we have

$$\begin{aligned} e_1(a_1, a_2, \ldots, a_n) &= a_1 + a_2 + \cdots + a_n, \\ e_2(a_1, a_2, \ldots, a_n) &= \sum_{1 \le i < j \le n} a_i a_j, \\ e_n(a_1, a_2, \ldots, a_n) &= a_1 a_2 \cdots a_n. \end{aligned}$$

Proposition 52. *Let $a_1, a_2, \ldots, a_n$ be n complex numbers. Then, the following expansion is valid.*

$$\prod_{k=1}^{n} (\lambda - a_k) = \lambda^n + \sum_{k=1}^{n} (-1)^k e_k(a_1, a_2, \ldots, a_n) \lambda^{n-k}. \tag{5.4}$$

Definition 53. *Let $a_1, a_2, \ldots, a_n$ be n complex numbers. We define the elementary symmetric polynomials of second kind as the coefficients $E_k(a_1, a_2, \ldots, a_n)$ in the expansion*

$$\lambda^n = \sum_{k=0}^{n} E_k(a_1, a_2, \ldots, a_n) P_k(\lambda), \tag{5.5}$$

where the polynomials $P_k(\lambda)$ are defined as

$$\begin{aligned} P_0(\lambda) &= 1 \\ P_k(\lambda) &= (\lambda - a_k) P_{k-1}(\lambda) = \prod_{j=1}^{k} (\lambda - a_j), \quad k = 1, \ldots, n. \end{aligned}$$

Proposition 54. *If $a_i \neq a_j$ for $i \neq j$, then the elementary symmetric polynomials of the second kind in the variables $a_1, \ldots, a_n$ can be computed by induction using the following algorithm*

$$E_0(a_1, \ldots, a_n) = a_1^n, \tag{5.6}$$

$$E_j(a_1, \ldots, a_n) = \frac{1}{P_j(a_{j+1})} \left[a_{j+1}^n - \sum_{k=0}^{j-1} E_k(a_1, \ldots, a_n) P_k(a_{j+1}) \right], \quad j = 1, \ldots, n-1. \tag{5.7}$$

$$E_n(a_1, \ldots, a_n) = 1.$$

Proof. We have

$$\lambda^n = \sum_{k=0}^{n} E_k(a_1, \ldots, a_n) P_k(\lambda) = E_0(a_1, \ldots, a_n) + \sum_{k=1}^{n} E_k(a_1, \ldots, a_n) P_k(\lambda).$$

Taking $\lambda = a_1$ on both sides of the previous equation gives

$$E_0(a_1, \ldots, a_n) = a_1^n.$$

Next, taking $\lambda = a_2$ provides the relation

$$a_2^n = E_0(a_1, \ldots, a_n) + E_1(a_1, \ldots, a_n)P_1(a_2),$$

and therefore we get

$$E_1(a_1, \ldots, a_n) = \frac{1}{P_1(a_2)} \left[a_2^n - E_0(a_1, \ldots, a_n)\right].$$

Now let us assume that we have found $E_0(a_1, \ldots, a_n)$,...,$E_{j-1}(a_1, \ldots, a_n)$, then, taking $\lambda = a_{j+1}$, it follows that

$$a_{j+1}^n = \sum_{k=0}^{j-1} E_k(a_1, \ldots, a_n)P_k(\lambda) + E_j(a_1, \ldots, a_n)P_j(a_{j+1}).$$

Thus, the relation (5.7) follows by a simple computation. □

We have for example

$$E_1(a_1, \ldots, a_n) = \frac{a_2^n - a_1^n}{a_2 - a_1} = \sum_{k=1}^{n} a^{n-k} b^{k-1},$$
$$E_{n-1}(a_1, \ldots, a_n) = a_1 + a_2 + \cdots + a_n.$$

5.3.2 Connection between x^n and $x^{\underline{n}}$

Definition 55. [1, P. 824]

1. *The Stirling numbers of first kind are the coefficients $S_m(n)$ in the expansion*

$$x^{\underline{n}} = x(x-1)(x-2)\cdots(x-n+1) = \sum_{m=0}^{n} S_m(n)x^m. \tag{5.8}$$

2. *The Stirling numbers of second kind are the coefficients $\mathcal{S}_m(n)$ in the expansion*

$$x^n = \sum_{m=0}^{n} \mathcal{S}_m(n)x^{\underline{n}}. \tag{5.9}$$

Those numbers fulfil several interesting properties. Here we recall some of them.

Proposition 56. [1, P. 824] *The Stirling numbers of first kind fulfil the following recurrence*

$$S_m(n+1) = S_{m-1}(n) - nS_m(n) \quad n \geq m \geq 1.$$

Some special values are

$$S_0(n) = \delta_{0n}, \quad S_1(n) = (-1)^{n-1}(n-1)!, \quad S_{n-1}(n) = -\binom{n}{2}, \quad S_n(n) = 1.$$

Proposition 57. *The Stirling numbers of first kind can be expressed in terms of the elementary symmetric polynomials as follows*

$$S_k(n) = (-1)^k e_k(0, 1, 2, \ldots, n-1), \quad 0 \leq k \leq n. \tag{5.10}$$

Proposition 58. [1, P. 824] *The Stirling numbers of second kind fulfil the following recurrence*

$$\mathcal{S}_m(n+1) = m\mathcal{S}_m(n) + \mathcal{S}_{m-1}(n), \quad n \geq m \geq 1,$$

and have the following representation

$$\mathcal{S}_m(n) = \frac{1}{m!} \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} k^n.$$

5.3.3 Connection between x^n and $(x;q)_n$

Lemma 59. [5, 30] *The following q-derivative formulas are valid.*

$$D_q(x;q)_n = -[n]_q(xq;q)_{n-1}; \tag{5.11}$$
$$D_{q^{-1}}(x;q)_n = [n]_q(x;q)_{n-1}. \tag{5.12}$$

Proof. The proof of these relations follows by direct computations. □

Lemma 60. *Let k and n be two non-negative integers such that $0 \le k \le n$. Then, the following derivative rules are valid.*

$$D_q^k(x;q)_n = (-1)^k \frac{[n]_q!}{[n-k]_q!} q^{\binom{k}{2}} (xq^k;q)_{n-k}; \tag{5.13}$$
$$D_q^k x^n = \frac{[n]_q!}{[n-k]_q!} x^{n-k}. \tag{5.14}$$

Proof. The proof is obtained by induction with respect to n. □

Proposition 61. *The following connection formulas are valid.*

$$(x;q)_n = \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}} x^m; \tag{5.15}$$

$$x^n = \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{-mn+\binom{m+1}{2}} (x;q)_m. \tag{5.16}$$

Proof. Many proofs of these two relations can be found in the literature. We give here a proof, which is based on Lemma 60.
For the relation (5.15), we first write

$$(x;q)_n = \sum_{m=0}^{n} D_m(n) x^m.$$

Taking k times the q-derivative in this equation and using Lemma 60, it follows that

$$(-1)^k \frac{[n]_q!}{[n-k]_q!} q^{\binom{k}{2}} (xq^k;q)_{n-k} = \sum_{m=k}^{n} D_m(n) \frac{[m]_q!}{[m-k]_q!} x^{m-k}.$$

Now we substitute $x = 0$ and obtain

$$(-1)^k \frac{[n]_q!}{[n-k]_q!} q^{\binom{k}{2}} = D_k(n)[k]_q!.$$

This reads

$$D_k(n) = (-1)^k \frac{[n]_q!}{[k]_q![n-k]_q!} q^{\binom{k}{2}} = (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}}.$$

For the relation (5.16), we write

$$x^n = \sum_{m=0}^{n} G_m(n)(x;q)_m.$$

As previously, taking k times the q-derivative in this equation and using once more Lemma (60), it follows that

$$\frac{[n]_q!}{[n-k]_q!} x^{n-k} = \sum_{m=k}^{n} (-1)^k G_m(n) \frac{[m]_q!}{[m-k]_q!} q^{\binom{k}{2}} (xq^k;q)_{m-k}.$$

Next, for $x = q^{-k}$, this equation reduces to

$$\frac{[n]_q!}{[n-k]_q!} q^{-k(n-k)} = (-1)^k [k]_q! q^{\binom{k}{2}} G_k(n).$$

The desired representation follows by simplification. □

Remark 62. *Once equations (5.15) and (5.16) are known, they can easily be established automatically applying q-Zeilberger's algorithm (see [32]) to the right-hand sides. These computations are contained in the Maple file attached to this work.*

5.3.4 Connection formula between x^n and $(x \ominus 1)_q^n$

Lemma 63. [47, Table 2] *The following derivative rule is valid.*

$$D_q(x \ominus y)_q^n = [n]_q (x \ominus y)_q^{n-1}, \quad n \geq 1, \tag{5.17}$$

where D_q acts on the variable x.

Proof. The proof follows by direct computation. □

Lemma 64. *The following derivative rule is valid.*

$$D_q^k (x \ominus y)_q^n = \frac{[n]_q!}{[n-k]_q!} (x \ominus y)_q^{n-k}, \;\; 0 \leq k \leq n. \tag{5.18}$$

Proof. The proof is obtained by induction with respect to n using (5.17). □

Proposition 65. *The bases $(x \ominus y)_q^n$ and x^n fulfil the following connection formulas*

$$(x \ominus y)_q^n = \sum_{m=0}^{n} (-y)^{n-m} q^{\binom{n-m}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q x^m; \tag{5.19}$$

$$x^n = \sum_{m=0}^{n} y^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q (x \ominus y)_q^m. \tag{5.20}$$

Proof. For relation (5.19) we write

$$(x \ominus y)_q^n = \sum_{m=0}^{n} C_m(n) x^n.$$

Next, we apply D_q^k to both sides of this relation and use (5.18) to get

$$\frac{[n]_q!}{[n-k]_q!} (x \ominus y)_q^{n-k} = \sum_{m=k}^{n} C_m(n) \frac{[m]_q!}{[m-k]_q!} x^{m-k} = C_k(n)[k]_q + \sum_{m=k+1}^{n} C_m(n) \frac{[m]_q!}{[m-k]_q!} x^{m-k}.$$

Now, substituting $x = 0$, it follows that

$$\frac{[n]_q!}{[n-k]_q!} (-a)^{n-k} q^{\binom{n-k}{2}} = C_k(n)[k]_q!.$$

Simplification gives the desired result. Relation (5.20) follows in the same manner. □

Remark 66. *Once equations (5.19) and (5.20) are known, they can easily be established automatically applying q-Zeilberger's algorithm (see [32]) to the right-hand sides. These computations are contained in the Maple file attached to this work.*

5.3.5 Connections between $(x \ominus y)_q^n$ and $(x;q)_n$

Proposition 67. *The bases $(x \ominus y)_q^n$ and $(x;q)_n$ fulfil the following connection formulas*

$$(x \ominus y)_q = \sum_{m=0}^{n} (-1)^m q^{-\binom{m}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q (q^{-m} \ominus y)_q^{n-m} (x;q)_m; \tag{5.21}$$

$$(x;q)_n = \sum_{m}^{n} (-1)^m q^{\binom{m}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q (yq^m;q)_{n-m} (x \ominus y)_q^m. \tag{5.22}$$

Proof. The proof is done as the proof of (5.19) using the relation

$$D_q^k (x;q)_n = \frac{[n]_q!}{[n-k]_q!} (xq^k;q)_{n-k}.$$

□

Remark 68. *Once equations (5.21) and (5.22) are known, they can easily be established automatically applying q-Zeilberger's algorithm (see [32]) to the right-hand sides. These computations are contained in the Maple file attached to this work.*

5.3.6 Connection between $(x^2)^n$ and $(a - ix)_n (a + ix)_n$

Note that

$$\vartheta_n(a,x) = (a - ix)_n (a + ix)_n = \prod_{k=0}^{n-1} (x^2 + (a+k)^2), \quad \theta_0(a,x) = 1.$$

The following proposition holds.

Proposition 69. *The following connections are valid.*

$$(a - ix)_n (a + ix)_n = \sum_{k=0}^{n} (-1)^k e_k \left(-a^2, -(a+1)^2, \cdots, -(a+n-1)^2 \right) (x^2)^{n-k}, \tag{5.23}$$

$$(x^2)^n = \sum_{k=0}^{n} E_k \left(-a^2, -(a+1)^2, \cdots, -(a+n-1)^2 \right) (a - ix)_k (a + ix)_k. \tag{5.24}$$

Proof. The proof follows from the definition of the elementary symmetric polynomials and the elementary symmetric polynomials of the second kind. □

In order to get explicit formula for $E_k \left(-a^2, -(a+1)^2, \cdots, -(a+n-1)^2 \right)$, we state the following results.

Proposition 70 (see [36])**.** *The basis $\vartheta_n(a,x)$ fulfills the following relations*

$$\mathbf{D}\vartheta_n(a,x) = n\vartheta_{n-1}\left(a + \frac{1}{2}, x\right), \tag{5.25}$$

$$\mathbf{D}^\ell \vartheta_n(a,x) = \frac{n!}{(n-l)!} \vartheta_{n-\ell}\left(a + \frac{\ell}{2}, x\right), 0 \le \ell \le n. \tag{5.26}$$

Theorem 71. *If f is a polynomial of degree n in x^2, then*

$$f(x) = \sum_{k=0}^{n} f_k \vartheta_k(a,x),$$

where

$$f_k = \frac{\mathbf{D}^k f(i(a + \frac{k}{2}))}{k!}.$$

Proof. First remark that $\vartheta_k(a, ai) = 0$ for all $k > 0$. Hence

$$\mathbf{D}^j f(x) = \sum_{k=j}^{n} f_k \frac{k!}{(k-j)!}\vartheta_{k-j}(a+\frac{j}{2}, x) = f_j j! + \sum_{k=j+1}^{n} f_k \frac{k!}{(k-j)!}\vartheta_{k-j}(a+\frac{j}{2}, x)$$

and for $x = i\left(a+\frac{j}{2}\right)$, we get

$$\mathbf{D}^j f\left(i\left(a+\frac{j}{2}\right)\right) = f_j j!.$$

This proves the proposition. □

Theorem 72 (see [11]). *Let k be a nonnegative integer. Then*

$$\mathbf{D}^k f(x) = \sum_{l=0}^{k} \frac{(-k)_l}{l!} \frac{(2ix-k-2l)}{(2ix-k+l)_{k+1}} f\left(x+\frac{k-2l}{2}i\right). \tag{5.27}$$

Corollary 73. *The following result is valid.*

$$\mathbf{D}^k x^{2n} = \sum_{l=0}^{k} \frac{(-k)_l}{l!} \frac{(2ix-k+2l)}{(2ix-k+l)_{k+1}} \left(x+\frac{k-2l}{2}i\right)^{2n}. \tag{5.28}$$

Proof. Take $f(x) = x^{2n}$ in (5.27) to get the result. □

Corollary 74. *The following connection formula is valid.*

$$x^{2n} = (-1)^n \sum_{k=0}^{n} \frac{1}{k!} \sum_{l=0}^{k} \frac{(-k)_l}{l!} \frac{(-2a-2k+2l)}{(-2a-2k+l)_{k+1}} (a+k-l)^{2n}\, \vartheta_k(a,x). \tag{5.29}$$

Proof. The proof follows from Theorem 71 and Theorem 72 with $f(x) = x^{2n}$. □

5.3.7 Connection between $(x(x+\varepsilon))^n$ and $(-x)_n(x+\varepsilon)_n$.

We recall the definition of the difference operator D_ε

$$\mathsf{D}_\varepsilon f(u(x)) = \frac{\Delta f(u(x))}{\Delta u(x)} = \frac{f(u(x)) - f(u(x+1))}{2x+1+\varepsilon},$$

and define the polynomial basis (for the Racah and the Dual Hahn polynomials) $\xi_n(x,\varepsilon)$ by

$$\xi_n(x,\varepsilon) = (-x)_n(x+\varepsilon)_n$$

which are the appropriate basis to consider for the operators D_ε.

Proposition 75. *The basis $\xi_n(x,\varepsilon)$ fulfils the following relations.*

$$\mathsf{D}_\varepsilon \xi_n(x,\varepsilon) = n\xi_{n-1}(x,\varepsilon+1) \tag{5.30}$$

$$\mathsf{D}_\varepsilon^k \xi_n(x,\varepsilon) = \frac{n!}{(n-k)!}\xi_{n-k}(x,\varepsilon+k). \tag{5.31}$$

Proof. We prove the first relation. The second one is obtained by induction. First remark that

$$\xi_n(x+1,\varepsilon) = (-x-1)_n(x+1+\varepsilon)_n = -(x+1)(x+\varepsilon+n)\xi_{n-1}(x,\varepsilon+1)$$

$$\xi_n(x,\varepsilon) = (-x+n-1)(x+\varepsilon+n-1)\xi_{n-1}(x,\varepsilon+1).$$

Thus

$$\xi_n(x+1,\varepsilon) - \xi_n(x,\varepsilon) = -n(2x+1+\varepsilon)\xi_{n-1}(x,\varepsilon+1).$$

The result follows by dividing by $-(2x+1+\varepsilon)$. □

Theorem 76. *If f is a polynomial of degree $2n$ in x, then*

$$f(x) = \sum_{k=0}^{n} \frac{(\mathsf{D}_\varepsilon^k f)(0)}{k!} \xi_k(x,\varepsilon).$$

Proof. Since f is a polynomial of degree $2n$, we can write

$$f(x) = \sum_{k=0}^{n} c_k \xi_k(x,\varepsilon).$$

Clearly,

$$\mathsf{D}_\varepsilon^j f(x) = \sum_{k=j}^{n} c_j \frac{k!}{(k-j)!} \xi_{k-j}(x,\varepsilon+j) = c_j j! + \sum_{k=j+1}^{n} c_j \frac{k!}{(k-j)!} \xi_{k-j}(x,\varepsilon+j).$$

Taking $x = 0$, it happens that

$$(\mathsf{D}_\varepsilon^j f)(0) = c_j j!.$$

This proves the theorem. □

Corollary 77. *The following connection formula is valid.*

$$(x(x+\varepsilon))^n = \sum_{k=0}^{n} \frac{\mathsf{D}_\varepsilon^k [x(x+\varepsilon)]^n_{|x=0}}{k!} \xi_k(x,\varepsilon). \tag{5.32}$$

5.3.8 Connection between x^n and $(a+ix)_n$

We recall the difference operator $\mathcal{D}$ defined as follows:

$$\mathcal{D}f(x) = f\left(x + \frac{i}{2}\right) - f\left(x - \frac{i}{2}\right),$$

and define the polynomial basis

$$\eta_n(a,x) = (a+ix)_n$$

which is the appropriate basis to consider for the operators $\mathcal{D}$.

Proposition 78. *The basis $\eta_n(a,x)$ fulfils the following relations.*

$$\mathcal{D}\eta_n(a,x) = -n\eta_{n-1}\left(a + \frac{1}{2}, x\right) \tag{5.33}$$

$$\mathcal{D}^k\eta_n(a,x) = (-1)^k \frac{n!}{(n-k)!} \eta_{n-k}\left(a + \frac{k}{2}, x\right). \tag{5.34}$$

Proof. We prove the first relation. The second one is obtained by induction.
By definition, we have

$$\begin{aligned}
\mathcal{D}\eta_n(a,x) &= \eta_n\left(a, x + \frac{i}{2}\right) - \eta_n\left(a, x - \frac{i}{2}\right) \\
&= \left(a + i\left(x + \frac{i}{2}\right)\right)_n - \left(a + i\left(x - \frac{i}{2}\right)\right)_n \\
&= \left(a - \frac{1}{2} + ix\right)_n - \left(a + \frac{1}{2} + ix\right)_n \\
&= \left(a - \frac{1}{2} + ix\right)\left(a + \frac{1}{2} + ix\right)_{n-1} - \left(a + \frac{1}{2} + ix + n - 1\right)\left(a + \frac{1}{2} + ix\right)_{n-1} \\
&= -n\left(a + \frac{1}{2} + ix\right)_{n-1} \\
&= -n\eta_{n-1}\left(a + \frac{1}{2}, x\right).
\end{aligned}$$

□

Proposition 79 (Power of $\mathcal{D}$). *Let k be a nonnegative integer, then the following relation holds.*

$$\mathcal{D}^k f(x) = \sum_{l=0}^{k} (-1)^l \binom{k}{l} f\left(x + \frac{k-2l}{2} i\right). \tag{5.35}$$

Proof. The proof is done by induction. The relation is obvious for $k = 1$. Assume it is true for a fix integer $k > 0$. Then, we have

$$\begin{aligned}
\mathcal{D}^{k+1} f(x) &= \mathcal{D}(\mathcal{D}^k f(x)) \\
&= \sum_{l=0}^{k} (-1)^l \binom{k}{l} \mathcal{D} f\left(x + \frac{k-2l}{2} i\right) \\
&= \sum_{l=0}^{k} (-1)^l \binom{k}{l} \left(f\left(x + \frac{k-2l+1}{2} i\right) - f\left(x + \frac{k-2l-1}{2} i\right)\right) \\
&= \sum_{l=0}^{k} (-1)^l \binom{k}{l} f\left(x + \frac{k-2l+1}{2} i\right) + \sum_{l=1}^{k+1} (-1)^l \binom{k}{l-1} f\left(x + \frac{k-2l+1}{2} i\right) \\
&= \sum_{l=0}^{k+1} (-1)^l \binom{k+1}{l} f\left(x + \frac{k-2l+1}{2} i\right).
\end{aligned}$$

□

Theorem 80. *If f is a polynomial of degree n in x, then*

$$f(x) = \sum_{k=0}^{n} f_k \eta_k(a, x),$$

where

$$f_k = \frac{(-1)^k}{k!} \mathcal{D}^k f\left(i\left(a + \frac{k}{2}\right)\right).$$

Proof. First remark that $\eta_k(a, ai) = 0$ for all $k > 0$. Hence

$$\mathcal{D}^j f(x) = \sum_{k=j}^{n} (-1)^k f_k \frac{k!}{(k-j)!} \eta_{k-j}(a + \frac{j}{2}, x) = (-1)^j f_j j! + \sum_{k=j+1}^{n} f_k \frac{k!}{(k-j)!} \eta_{k-j}(a + \frac{j}{2}, x)$$

and for $x = i\left(a + \frac{j}{2}\right)$, we get

$$\mathcal{D}^j f\left(i\left(a + \frac{j}{2}\right)\right) = (-1)^j j! f_j.$$

This proves the proposition. □

Corollary 81. *The following connection formula is valid.*

$$x^n = \sum_{k=0}^{n} \frac{1}{k!} \sum_{l=0}^{k} (-1)^l \binom{k}{l} ((a+l)i)^n \eta_k(a, x). \tag{5.36}$$

Proof. First, we apply theorem 80 with $f(x) = x^n$ to get

$$x^n = \sum_{k=0}^{n} \left(\frac{(-1)^k}{k!} \mathcal{D}^k x^n_{\big|x = i\left(a + \frac{k}{2}\right)} \right) \eta_k(a, x).$$

Next, using proposition 79, we have

$$\mathcal{D}^k x^n = \sum_{l=0}^{k} (-1)^l \binom{k}{l} \left(x + \frac{k-2l}{2} i\right)^n.$$

Then, we have

$$\begin{aligned}\mathcal{D}^k x^n_{\big|i\left(a+\frac{k}{2}\right)} &= \sum_{l=0}^{k}(-1)^l\binom{k}{l}\left((a+k-l)i\right)^n \\ &= (-1)^k\sum_{l=0}^{k}(-1)^l\binom{k}{l}\left((a+l)i\right)^n.\end{aligned}$$

This completes the proof. □

5.3.9 Connection between $\cos^n\theta$ and $(ae^{i\theta}, ae^{-i\theta}; q)_n$.

We first make the following remark

$$(ae^{i\theta}, ae^{-i\theta}; q)_n = (-2a)^n q^{\binom{n}{2}}\prod_{k=0}^{n-1}\left(\cos\theta - x_k\right),$$

where

$$x_k = \frac{1+a^2q^{2k}}{2aq^k}.$$

The following proposition follows.

Proposition 82. *The following connection formulas are valid.*

$$(ae^{i\theta}, ae^{-i\theta}; q)_n = (-2a)^n q^{\binom{n}{2}}\sum_{k=0}^{n}(-1)^k e_k(x_0, x_1, \cdots, x_{n-1})\cos^{n-k}\theta, \tag{5.37}$$

$$\cos^n\theta = \sum_{k=0}^{n}(-2a)^{-k}q^{-\binom{k}{2}}E_k(x_0, x_1, \cdots, x_{n-1})(ae^{i\theta}, ae^{-i\theta}; q)_k \tag{5.38}$$

with

$$x_k = \frac{1+a^2q^{2k}}{2aq^k}, \quad k = 0, 1, \ldots, n-1. \tag{5.39}$$

In what follows, we give explicit formula for $E_k(x_0, x_1, \cdots, x_{n-1})$.

Proposition 83 (see [11])**.** *The following q-derivative rule is valid.*

$$(\mathcal{D}_q^n f)(x) = \frac{2^n q^{\frac{n(1-n)}{4}}}{(q^{1/2}-q^{-1/2})^n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q \frac{q^{k(n-k)}z^{2k-n}\check{f}(q^{(n-2k)/2}z^2)}{(q^{1+n-2k}z^2; q)_k (q^{2k-n+1}z^{-2}; q)_{n-k}}, \tag{5.40}$$

where $\check{f}(z) = f((z+1/z)/2)$, $z = e^{i\theta}$, $x = \cos\theta$.

Proposition 84 (see [23])**.** *If $f(x)$ is a polynomial in $x = \cos\theta$ of degree n, then*

$$f(x) = \sum_{k=0}^{n} f_k (ae^{i\theta}, ae^{-i\theta}; q)_k; \tag{5.41}$$

where

$$f_k = \frac{(q-1)^k}{(2a)^k (q;q)_k} q^{-\frac{k(k-1)}{4}}(\mathcal{D}_q^k f)(x_k),$$

with

$$x_k = \frac{1}{2}(aq^{k/2} + q^{-k/2}/a).$$

Corollary 85. *If $f(x)$ is a polynomial of degree n in $x = \cos\theta$, then*

$$f(x) = \sum_{k=0}^{n} f_k (ae^{i\theta}, ae^{-i\theta}; q)_k,$$

with

$$f_k = q^k \sum_{j=0}^{k} \frac{q^{-(k-j)^2} a^{2(j-k)} \check{f}(aq^{k-j})}{(q, q^{1+2(k-j)}a^2; q)_j (q, q^{-1-2(k-j)}a^{-2}; q)_{k-j}}. \tag{5.42}$$

Remark 86. *Note that, by a change of variable $j := k - j$, the p_k's in corollary 85 can be written as*

$$f_k = q^k \sum_{j=0}^{k} \frac{q^{-j^2} a^{-2j} \check{f}(aq^j)}{(q, q^{1+2j}a^2; q)_{k-j} (q, q^{-1-2j}a^{-2}; q)_j}.$$

Corollary 87. *The following connection formula is valid.*

$$x^n = \sum_{k=0}^{n} q^k \sum_{j=0}^{k} \frac{q^{-j^2} a^{-2j} (aq^j + a^{-1}q^{-j})^k}{(q, q^{1+2j}a^2; q)_{k-j} (q, q^{-1-2j}a^{-2}; q)_j} (ae^{i\theta}, ae^{-i\theta}; q)_k, \quad x = \cos\theta. \tag{5.43}$$

5.4 Moments and generating functions

As previously announced, we now use the inversion formula to compute the moments of orthogonal polynomials (see Theorem 50). Connections between the bases enable us to get the canonical moments from the generalized ones.

5.4.1 The continuous case

The Jacobi polynomials

From the orthogonality relation (3.1), it follows that

$$\mu_0 = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.$$

- For $\theta_n(x) = (1-x)^n$, we have from (4.16)

$$I_0(n) = 2^n \frac{\Gamma(\alpha+1+n)\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+\beta+n+2)\Gamma(\alpha+1)} = 2^n \frac{(\alpha+1)_n}{(\alpha+\beta+2)_n}.$$

- For $\theta_n(x) = (1+x)^n$, we have from (4.17)

$$I_0(n) = 2^n \frac{\Gamma(\beta+1+n)\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+\beta+n+2)\Gamma(\beta+1)} = 2^n \frac{(\beta+1)_n}{(\alpha+\beta+2)_n}.$$

Therefore, the following proposition is valid.

Proposition 88.

1. *The generalized Jacobi moments with respect to the basis $(1-x)^n$ have the representation*

$$\mu_n((1-x)^k) = 2^{n+\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \frac{(\alpha+1)_n}{(\alpha+\beta+2)_n}. \tag{5.44}$$

2. *The generalized Jacobi moments with respect to the basis $(1+x)^n$ have the representation*

$$\mu_n((1+x)^k) = 2^{n+\alpha+\beta+1} \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \frac{(\beta+1)_n}{(\alpha+\beta+2)_n}. \tag{5.45}$$

Remark 89. *Note that these moments could be computed directly as follows. For (5.44), we start by remarking that*

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}dx = \mu_0 = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.$$

Next, we replace α by $\alpha+n$ to get

$$\begin{aligned}\mu_n((1-x)^k) &= \int_{-1}^{1}(1-x)^{\alpha+n}(1+x)^{\beta}dx = 2^{\alpha+\beta+n+1}\frac{\Gamma(\alpha+n+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+n+2)}\\ &= 2^{\alpha+\beta+n+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\frac{(\alpha+1)_n}{(\alpha+\beta+2)_n}.\end{aligned}$$

For the relation (5.45), just replace β by $\beta+n$ and proceed as previously.

Proposition 90. *The generalized Jacobi moments have the following exponential generating functions:*

$$\begin{aligned}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\frac{2^{\alpha+\beta+1}}{(1-z)^{\alpha+1}} &= \sum_{n=0}^{\infty}\mu_n((1-x)^k)\frac{(\alpha+\beta+2)_n}{n!}(2z)^n,\\ \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\frac{2^{\alpha+\beta+1}}{(1-z)^{\beta+1}} &= \sum_{n=0}^{\infty}\mu_n((1+x)^k)\frac{(\alpha+\beta+2)_n}{n!}(2z)^n.\end{aligned}$$

Proof. Using the binomial theorem (2.3), we get the results. □

In Chapter 3, we gave a representation of the canonical Jacobi moments involving the sum of two hypergeometric functions. Here, using the inversion formula, we derive another representation of those moments. We first recall the following relations which are another ways to write the binomial theorem.

$$x^n = \sum_{m=0}^{n}(-1)^m\binom{n}{m}(1-x)^m \tag{5.46}$$

$$x^n = \sum_{m=0}^{n}(-1)^{n-m}\binom{n}{m}(1+x)^m. \tag{5.47}$$

From (5.44), (5.45), (5.46) and (5.47), we have the following proposition.

Proposition 91. *The canonical Jacobi moments have the following representations*

$$\mu_n = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}{}_2F_1\left(\left.\begin{matrix}-n,\alpha+1\\ \alpha+\beta+2\end{matrix}\right|2\right) \quad \textit{(compare [13])}, \tag{5.48}$$

$$= (-1)^n 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}{}_2F_1\left(\left.\begin{matrix}-n,\beta+1\\ \alpha+\beta+2\end{matrix}\right|2\right) \tag{5.49}$$

Note that these two representations are simpler than the one we obtained in Chapter 3.

Remark 92. *In [13], the formula (5.48) is written as*

$$\mu_n = \sum_{m=0}^{n}\binom{n}{m}(-1)^m 2^m\frac{\Gamma(b+m)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b+m)} = {}_2F_1\left(\left.\begin{matrix}-n,b\\ a+b\end{matrix}\right|2\right), n\geq 0\ \ a,b>0.$$

Duran seems to set $a=\alpha+1$, $b=\beta+1$ and he uses a different standardization.

Remark 93. *It should be mentioned that the Laguerre moments, the Hermite moments and the Bessel moments computed in Chapter 3 can be recovered by this method.*

5.4.2 The discrete case

For the classical discrete orthogonal polynomials, the measure $d\alpha(x)$ in the definition 17 is a discrete measure. Therefore the moments with respect to the basis $x^{\underline{n}}$ are given by

$$\mu_n\left(x^{\underline{k}}\right) = \sum_{k=0}^{\infty} \rho(k) k^{\underline{n}}, \tag{5.50}$$

and the canonical moments are given by

$$\mu_n = \sum_{k=0}^{\infty} \rho(k) k^n, \tag{5.51}$$

where $\rho(x)$ is the discrete weight function associated to the family. These sums can be finite (as in the Hahn and the Krawtchouk cases) or infinite (as in the Meixner and the Charlier cases).

The Hahn polynomials

The Hahn polynomials $Q_n(x;\alpha,\beta,N)$ fulfil the following orthogonality relation [30, P. 204]

$$\begin{aligned}\sum_{x=0}^{N} \binom{\alpha+x}{x}\binom{\beta+N-x}{N-x} & Q_n(x;\alpha,\beta,N) Q_m(x;\alpha,\beta,N) \\ &= \frac{(-1)^n (n+\alpha+\beta+1)_{N+1} (\beta+1)_n n!}{(2n+\alpha+\beta+1)(\alpha+1)_n(-N)_n N!} \delta_{mn}, \end{aligned} \tag{5.52}$$

$$\text{for } \alpha > -1 \text{ and } \beta > -1 \text{ or } \alpha < -N \text{ and } \beta < -N.$$

With $m = n = 0$, it follows that

$$\mu_0 = \frac{(\alpha+\beta+1)_{N+1}}{(\alpha+\beta+1)N!}.$$

From the inversion formula (4.21), for $\theta_n(x) = x^{\underline{n}}$, we have

$$I_0(n) = (-1)^n \frac{(\alpha+1)_n(-N)_n}{(\alpha+\beta+2)_n}.$$

Therefore, the following proposition is valid.

Proposition 94. *The generalized Hahn moments with respect to the basis $x^{\underline{n}}$ have the representation*

$$\mu_n\left(x^{\underline{k}}\right) = (-1)^n \frac{(\alpha+\beta+1)_{N+1}}{(\alpha+\beta+1)N!} \frac{(\alpha+1)_n(-N)_n}{(\alpha+\beta+2)_n}. \tag{5.53}$$

Proposition 95. *The generalized Hahn moments with respect to $x^{\underline{n}}$ have the following generating function:*

$$\frac{(\alpha+\beta+1)_{N+1}}{(\alpha+\beta+1)N!}(1+z)^N = \sum_{n=0}^{\infty} \mu_n\left(x^{\underline{k}}\right) \frac{(\alpha+\beta+2)_n}{(\alpha+1)_n} \frac{z^n}{n!}. \tag{5.54}$$

Proof. We have

$$\sum_{n=0}^{\infty} \mu_n\left(x^{\underline{k}}\right) \frac{(\alpha+\beta+2)_n}{(\alpha+1)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{(-N)_n}{n!}(-z)^n.$$

Using the binomial theorem (2.3), the result follows. □

Proposition 96. *The canonical Hahn moments have the following representation*

$$\mu_n = \frac{(\alpha+\beta+1)_{N+1}}{(\alpha+\beta+1)N!} \sum_{m=0}^{n} (-1)^m \mathcal{S}_m(n) \frac{(\alpha+1)_m(-N)_m}{(\alpha+\beta+2)_m} \tag{5.55}$$

$$= \frac{(\alpha+\beta+1)_{N+1}}{(\alpha+\beta+1)N!} \sum_{m=0}^{n} \sum_{k=0}^{m} \frac{(-1)^{2m-k} k^n}{k!(m-k)!} \frac{(\alpha+1)_m(-N)_m}{(\alpha+\beta+2)_m} \tag{5.56}$$

Proof. The proof follows from (2.21), (2.23), (5.9) and (5.53). □

The Krawtchouk polynomials

The Krawtchouk polynomials $K_n(x;p,N)$ fulfil the following orthogonality relation [30, P. 237]

$$\sum_{k=0}^{N}\binom{N}{k}p^k(1-p)^{N-k}K_n(k;p,N)K_m(k;p,N)=\frac{(-1)^n n!}{(-N)_n}\left(\frac{1-p}{p}\right)^n\delta_{nm},\quad 0<p<1. \tag{5.57}$$

With $m=n=0$, it follows that

$$\mu_0=1.$$

From the inversion formula (4.22), for $\theta_n(x)=x^{\underline{n}}$, we have

$$I_0(n)=(-N)_n(-p)^n.$$

Therefore, the following proposition is valid.

Proposition 97. *The generalized Krawtchouk moments with respect to the basis $x^{\underline{n}}$ have the representation*

$$\mu_n\left(x^{\underline{k}}\right)=(-N)_n(-p)^n. \tag{5.58}$$

Proposition 98. *The generalized Krawtchouk moments with respect to the basis $x^{\underline{n}}$ have the following exponential generating function:*

$$(1+pz)^N=\sum_{n=0}^{\infty}\frac{\mu_n\left(x^{\underline{k}}\right)}{n!}z^n,\quad |pz|<1. \tag{5.59}$$

Proof. Using the binomial theorem (2.3), we have

$$\sum_{n=0}^{\infty}\frac{\mu_n}{n!}z^n=\sum_{n=0}^{\infty}\frac{(-N)_n}{n!}(-pz)^n={}_1F_0\left(\left.\begin{matrix}-N\\0\end{matrix}\right|-pz\right)=(1+pz)^N.$$

□

Proposition 99. *The canonical Krawtchouk moments have the representation*

$$\mu_n=\sum_{m=0}^{n}\mathcal{S}_m(n)(-N)_m(-p)^m. \tag{5.60}$$

Proof. The proof follows from (2.21), (2.23), (5.9) and (5.58). □

Proposition 100. *The canonical Krawtchouk moments have the following exponential generating function:*

$$(pe^z+1-p)^N=\sum_{n=0}^{\infty}\mu_n\frac{z^n}{n!}. \tag{5.61}$$

Proof. By definition, the canonical Krawtchouk moments are given by

$$\mu_n=\sum_{k=0}^{N}k^n\binom{N}{k}p^k(1-p)^{N-k}.$$

Therefore, it follows that

$$\begin{aligned}\sum_{n=0}^{\infty}\mu_n\frac{z^n}{n!}&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{N}k^n\binom{N}{k}p^k(1-p)^{N-k}\right)\frac{z^n}{n!}\\&=\sum_{k=0}^{N}\left(\binom{N}{k}p^k(1-p)^{N-k}\sum_{n=0}^{\infty}\frac{(kz)^n}{n!}\right)\\&=\sum_{k=0}^{N}\binom{N}{k}(pe^z)^k(1-p)^{N-k}\\&=(pe^z+1-p)^N.\end{aligned}$$

□

Proposition 101. *The canonical Krawtchouk moments μ_n can be computed by the formula $\mu_n = f_n(1)$ where $(f_n)_n$ is the sequence of functions defined by*

$$f_0(t) = (tp + 1 - p)^N, \quad f_{n+1}(t) = t f_n'(t) \quad n \geq 0.$$

Proof. We prove by induction that $f_n(t) = \sum\limits_{k=0}^{N} k^n \binom{N}{k} p^k (1-p)^{N-k} t^k$. Finally, we have the proposition by computing $f_n(1) = \mu_n$. □

The Meixner polynomials

The Meixner polynomials $M_n(x;\beta,c)$ fulfil the following orthogonality relation [30, P. 234]

$$\sum_{x=0}^{\infty} \frac{(\beta)_x}{x!} c^x M_n(x;\beta,c) M_m(x;\beta,c) = \frac{n!}{c^n (\beta)_n (1-c)^\beta} \delta_{nm}, \quad \beta > 0, \ \ 0 < c < 1. \tag{5.62}$$

For $m = n = 0$, it follows that

$$\mu_0 = \frac{1}{(1-c)^\beta}.$$

From the inversion formula (4.23), we have

$$I_0(n) = (-1)^n (\beta)_n \left(\frac{c}{c-1}\right)^n.$$

Therefore, the following proposition is valid.

Proposition 102. *The generalized Meixner moments with respect to the basis $x^{\underline{n}}$ have the representation*

$$\mu_n\left(x^{\underline{k}}\right) = \frac{(-1)^n (\beta)_n}{(1-c)^\beta} \left(\frac{c}{c-1}\right)^n. \tag{5.63}$$

Proposition 103. *The generalized Meixner moments with respect to the basis $x^{\underline{n}}$ have the following exponential generating function:*

$$\frac{1}{(1-c-cz)^\beta} = \sum_{n=0}^{\infty} \frac{\mu_n\left(x^{\underline{k}}\right)}{n!} z^n, \quad \left|\frac{cz}{1-c}\right| < 1. \tag{5.64}$$

Proof. The proof follows from the binomial theorem (2.3). □

Proposition 104. *The canonical Meixner moments have the representation*

$$\mu_n = \frac{1}{(1-c)^\beta} \sum_{m=0}^{n} (-1)^m \mathcal{S}_m(n) (\beta)_m \left(\frac{c}{c-1}\right)^m. \tag{5.65}$$

Proof. The proof follows from (2.21), (2.23), (5.9) and (5.63). □

Note that the canonical Meixner moments appear in [26].

Proposition 105. *The canonical Meixner moments have the following exponential generating function:*

$$\frac{1}{(1-ce^z)^\beta} = \sum_{n=0}^{\infty} \mu_n \frac{z^n}{n!}, \quad |ce^z| < 1. \tag{5.66}$$

Proof. By definition, the canonical Meixner moments are given by

$$\mu_n = \sum_{k=0}^{\infty} \frac{(\beta)_k c^k}{k!} k^n$$

It therefore follows that

$$\sum_{n=0}^{\infty}\frac{\mu_n}{n!}z^n = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{(\beta)_k c^k}{k!}k^n\right)\frac{z^n}{n!}$$
$$= \sum_{k=0}^{\infty}\left(\frac{(\beta)_k c^k}{k!}\sum_{n=0}^{\infty}\frac{(kz)^n}{n!}\right)$$
$$= \sum_{k=0}^{\infty}\frac{(\beta)_k}{k!}(ce^z)^k = \frac{1}{(1-ce^z)^{\beta}}.$$

□

Remark 106. *From the definition of the canonical Meixner moments*

$$\mu_n = \mu_n(\beta,c) = \sum_{k=0}^{\infty}\frac{(\beta)_k c^k}{k!}k^n,$$

it follows that

$$\frac{\partial}{\partial c}\mu_n(\beta,c) = \frac{1}{c}\mu_{n+1}(\beta,c). \tag{5.67}$$

Therefore, the moments $\mu_n(\beta,c)$ have can be represented by

$$\mu_n(\beta,c) = \frac{P_n(\beta,c)}{(1-c)^{\beta+n}},$$

where P_n is a polynomial in two variables c and β and can be computed recursively by the recurrence

$$P_{n+1}(\beta,c) = c\left[(1-c)\frac{\partial}{\partial c}P_n(\beta,c) + (\beta+n)P_n(\beta,c)\right],\quad P_0(\beta,c) = 1. \tag{5.68}$$

The Charlier polynomials

The Charlier polynomials $C_n(x;a)$ fulfil the following orthogonality relation [30, P. 247]

$$\sum_{x=0}^{\infty}\frac{a^x}{x!}C_n(x;a)C_m(x;a) = a^{-n}e^a n!\delta_{mn},\quad a>0. \tag{5.69}$$

With $m=n=0$, it follows that

$$\mu_0 = e^a.$$

From the inversion formula (4.24), we have

$$I_0(n) = a^n.$$

Therefore, the following proposition is valid.

Proposition 107. *The generalized Charlier moments with respect to the basis $x^{\underline{n}}$ have the representation*

$$\mu_n\left(x^{\underline{k}}\right) = e^a a^n. \tag{5.70}$$

Proposition 108. *The generalized Charlier moments with respect to the basis $x^{\underline{n}}$ have the following generating functions:*

$$\frac{e^a}{1-az} = \sum_{n=0}^{\infty}\mu_n\left(x^{\underline{k}}\right)z^n,\quad |az|<1. \tag{5.71}$$

$$e^{az+a} = \sum_{n=0}^{\infty}\frac{\mu_n\left(x^{\underline{k}}\right)}{n!}z^n. \tag{5.72}$$

Proposition 109. *The canonical Charlier moments have the representation*

$$\mu_n = e^a \sum_{m=0}^{n} \mathcal{S}_m(n) a^m. \tag{5.73}$$

Proof. The proof follows from (2.21), (2.23), (5.9) and (5.70). □

Note that (5.73) appears in [44] and [26] without the constant $\mu_0 = e^a$.

Proposition 110. *The canonical Charlier moments have the following exponential generating function*

$$e^{ae^z} = \sum_{n=0}^{\infty} \mu_n \frac{z^n}{n!}. \tag{5.74}$$

Proof. By definition, the canonical Charlier moments are given by

$$\mu_n = \sum_{k=0}^{\infty} \frac{a^k}{k!} k^n.$$

Therefore, we have:

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{\mu_n}{n!} z^n &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{a^k}{k!} k^n \right) \frac{z^n}{n!} \\ &= \sum_{k=0}^{\infty} \left(\frac{a^k}{k!} \sum_{n=0}^{\infty} \frac{(kz)^n}{n!} \right) \\ &= \sum_{k=0}^{\infty} \frac{a^k e^{kz}}{k!} = \sum_{k=0}^{\infty} \frac{(ae^z)^k}{k!} \\ &= e^{ae^z}.\end{aligned}$$

□

5.4.3 The q-discrete case

The Big q-Jacobi polynomials

The Big q-Jacobi polynomials $P_n(x; a, b, c; q)$ fulfil the following orthogonality relation [30, P. 438]

$$\begin{aligned}&\int_{cq}^{aq} \frac{(a^{-1}x, c^{-1}x; q)_\infty}{(x, bc^{-1}x; q)_\infty} P_m(x; a, b, c; q) P_n(x; a, b, c; q) d_q x \\ &\quad = aq(1-q) \frac{(abq^2, a^{-1}c, ac^{-1}q; q)_\infty}{(aq, bq, cq, abc^{-1}q; q)_\infty} \\ &\qquad \times \frac{1-abq}{1-abq^{2n+1}} \frac{(q, bq, abc^{-1}q; q)_n}{(aq, abq, cq; q)_n} (-caq^2)^n q^{\binom{n}{2}} \delta_{mn}.\end{aligned} \tag{5.75}$$

Let us write

$$\rho(x; q) = \frac{(a^{-1}x, c^{-1}x; q)_\infty}{(x, bc^{-1}x; q)_\infty}.$$

The q-integral in (5.75) can be written as

$$\begin{aligned}&\int_{cq}^{aq} \rho(x; q) P_m(x; a, b, c; q) P_n(x; a, b, c; q) d_q x \\ &\quad = aq(1-q) \sum_{k=0}^{\infty} \rho(aqq^k; q) P_m(aqq^k; a, b, c; q) P_n(aqq^k; a, b, c; q) \\ &\qquad - cq(1-q) \sum_{k=0}^{\infty} \rho(cqq^k; q) P_m(cqq^k; a, b, c; q) P_n(cqq^k; a, b, c; q)\end{aligned}$$

We have:

$$\int_{cq}^{aq} \rho(x;q)(x;q)_n d_q x = aq(1-q)\sum_{k=0}^{\infty} \rho(aq^{k+1};q)(aq^{k+1};q)_n - cq(1-q)\sum_{k=0}^{\infty} \rho(cq^{k+1};q)(cq^{k+1};q)_n.$$

Define the discrete measures μ_a and μ_c as

$$\mu_a = aq(1-q)\sum_{k=0}^{\infty} \rho(aq^{k+1};q)q^k \varepsilon_{aq^{k+1}}$$
$$\mu_c = cq(q-1)\sum_{k=0}^{\infty} \rho(cq^{k+1};q)q^k \varepsilon_{cq^{k+1}},$$

and put

$$\mu = \mu_a + \mu_c.$$

We then have

$$\int_{cq}^{aq} \rho(x;q)(x;q)_n d_q x = \int_{-\infty}^{\infty} (x;q)_n d\mu(x).$$

It follows by taking $m = n = 0$ in the orthogonality relation that

$$\mu_0 = aq(1-q)\frac{(abq^2, a^{-1}cac^{-1}q;q)_\infty}{(aq, bq, cq, abc^{-1}q;q)_\infty}.$$

From the inversion formula (4.25), for $\theta_n(x) = (x;q)_n$, we have

$$I_0(n) = \frac{(aq, cq;q)_n}{(abq^2;q)_n}.$$

Therefore, the following proposition is valid.

Proposition 111. *The generalized Big q-Jacobi moments with respect to the basis $(x;q)_n$ are given by*

$$\mu_n((x;q)_k) = aq(1-q)\frac{(abq^2, a^{-1}c, ac^{-1}q;q)_\infty}{(aq, bq, cq, abc^{-1}q;q)_\infty}\frac{(aq, cq;q)_n}{(abq^2;q)_n}. \tag{5.76}$$

Note that the Big q-Jacobi moments with respect to $(x;q)_n$ were given in [4, P. 91] with the normalization $\mu_0 = 1$.

Proposition 112. *The generalized Big q-Jacobi moments with respect to $(x;q)_n$ have the following generating functions*

$$aq\frac{(abq^2, a^{-1}cac^{-1}q;q)_\infty}{(aq, bq, cq, abc^{-1}q;q)_\infty}\frac{(cqz;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \mu_n((x;q)_k)\frac{(abq^2;q)_n}{(aq;q)_n}\frac{z^n}{(q;q)_n}, \quad |z|<1 \tag{5.77}$$
$$aq\frac{(abq^2, a^{-1}cac^{-1}q;q)_\infty}{(aq, bq, cq, abc^{-1}q;q)_\infty}\frac{(aqz;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \mu_n((x;q)_k)\frac{(abq^2;q)_n}{(cq;q)_n}\frac{z^n}{(q;q)_n}, \quad |z|<1. \tag{5.78}$$

Proof. The proof follows by the use of the q-binomial theorem (2.6). □

Proposition 113. *The canonical Big q-Jacobi moments have the representation*

$$\mu_n = aq\frac{(abq^2, a^{-1}cac^{-1}q;q)_\infty}{(aq, bq, cq, abc^{-1}q;q)_\infty}\sum_{m=0}^{n}(-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{-nm+\binom{m+1}{2}}\frac{(aq, cq;q)_m}{(abq^2;q)_m}. \tag{5.79}$$

Proof. Using (2.21), (2.23), (5.16) and (5.76), we get the result. □

The q-Hahn polynomials

For $0 < \alpha q < 1$ and $0 < \beta q$, or for $\alpha > q^{-N}$ and $\beta > q^{-N}$ the q-Hahn polynomials $Q_n(q^{-x};\alpha,\beta,N|q)$ fulfil the following orthogonality relation [30, P. 445]

$$\sum_{x=0}^{N} \frac{(\alpha q, q^{-N};q)_x}{(q,\beta^{-1}q^{-N};q)_x}(\alpha\beta q)^{-x} Q_m(q^{-x};\alpha,\beta,N|q) Q_n(q^{-x};\alpha,\beta,N|q)$$
$$= \frac{(\alpha\beta q^2;q)_N}{(\beta q;q)_N(\alpha q)^N} \frac{(q,\alpha\beta q^{N+2},\beta q;q)_n}{(\alpha q,\alpha\beta q,q^{-N};q)_n} \frac{(1-\alpha\beta q)(-\alpha q)^n}{1-\alpha\beta q^{2n+1}} q^{\binom{n}{2}-Nn}\delta_{mn}. \tag{5.80}$$

From the relation (5.80), with $m = n = 0$, it follows that

$$\mu_0 = \frac{(\alpha\beta q^2;q)_N}{(\beta q;q)_N(\alpha q)^N}.$$

From the inversion formula (4.26), for $\theta_n(x) = (q^{-x};q)_n$, we have

$$I_0(n) = \frac{(\alpha q,q^{-N};q)_n}{(\alpha\beta q^2;q)_n}.$$

Therefore, the following proposition is valid.

Proposition 114. *The generalized q-Hahn moments with respect to the basis $(q^{-x};q)_n$ have the representation*

$$\mu_n((q^{-x};q)_k) = \frac{(\alpha\beta q^2;q)_N}{(\beta q;q)_N(\alpha q)^N} \frac{(\alpha q,q^{-N};q)_n}{(\alpha\beta q^2;q)_n}, \quad n = 0,1,\ldots,N. \tag{5.81}$$

Proposition 115. *The generalized q-Hahn moments with respect to $(q^{-x};q)_n$ have the following generating function*

$$\frac{(\alpha\beta q^2;q)_N}{(\beta q;q)_N(\alpha q)^N} \frac{(zq^{-N};q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \mu_n((q^{-x};q)_k) \frac{(\alpha\beta q^2;q)_n}{(\alpha q;q)_n} \frac{z^n}{(q;q)_n}. \tag{5.82}$$

Proof. We have

$$\sum_{n=0}^{\infty} \mu_n((q^{-x};q)_k) \frac{(\alpha\beta q^2;q)_n}{(\alpha q;q)_n} \frac{z^n}{(q;q)_n} = \frac{(\alpha\beta q^2;q)_N}{(\beta q;q)_N(\alpha q)^N} \sum_{n=0}^{\infty} \frac{(q^{-N};q)_n}{(q;q)_n} z^n.$$

By the q-binomial theorem (2.6), the result follows. □

Proposition 116. *The canonical q-Hahn moments have the following representation*

$$\mu_n = \frac{(\alpha\beta q^2;q)_N}{(\beta q;q)_N(\alpha q)^N} \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{-mn+\binom{m+1}{2}} \frac{(\alpha q,q^{-N};q)_m}{(\alpha\beta q^2;q)_m}. \tag{5.83}$$

Proof. The proof follows by using (2.21), (2.23), (5.16) and (5.81). □

The Big q-Laguerre polynomials

For $0 < aq < 1$ and $b < 0$, the Big q-Laguerre polynomials $P_n(x;a,b;q)$ fulfil the following orthogonality relation [30, P. 479]

$$\int_{bq}^{aq} \frac{(a^{-1}x,b^{-1}x;q)_\infty}{(x;q)_\infty} P_m(x;a,b;q) P_n(x;a,b;q) d_qx$$
$$= aq\frac{(q,a^{-1}b,ab^{-1}q;q)_\infty}{(aq,bq;q)_\infty} \frac{(q;q)_n}{(aq,bq;q)_n} (-abq^2)^n q^{\binom{n}{2}} \delta_{mn}. \tag{5.84}$$

Let us write

$$\rho(x;q) = \frac{(a^{-1}x, b^{-1}x;q)_\infty}{(x;q)_\infty}.$$

The q-integral in (5.84) can be written as

$$\begin{aligned}\int_{bq}^{aq} \rho(x;q) P_m(x;a,b;q) P_n(x;a,b;q) d_q x \\ &= aq(1-q)\sum_{k=0}^{\infty} \rho(aqq^k;q) P_m(aqq^k;a,b;q) P_n(aqq^k;a,b;q) \\ &\quad - bq(1-q)\sum_{k=0}^{\infty} \rho(bqq^k;q) P_m(bqq^k;a,b;q) P_n(bqq^k;a,b;q).\end{aligned}$$

We have:

$$\int_{bq}^{aq} \rho(x;q)(x;q)_n d_q x = aq(1-q)\sum_{k=0}^{\infty} \rho(aq^{k+1};q)(aq^{k+1};q)_n - bq(1-q)\sum_{k=0}^{\infty} \rho(bq^{k+1};q)(bq^{k+1};q)_n.$$

Define the discrete measures μ_a and μ_b as

$$\begin{aligned}\mu_a &= aq(1-q)\sum_{k=0}^{\infty} \rho(aq^{k+1};q) q^k \varepsilon_{aq^{k+1}}, \\ \mu_b &= cq(q-1)\sum_{k=0}^{\infty} \rho(bq^{k+1};q) q^k \varepsilon_{bq^{k+1}},\end{aligned}$$

and put

$$\mu = \mu_a + \mu_b.$$

We then have

$$\int_{bq}^{aq} \rho(x;q)(x;q)_n d_q x = \int_{-\infty}^{\infty} (x;q)_n d\mu(x).$$

With $m = n = 0$, it follows that

$$\mu_0 = aq\frac{(q, a^{-1}b, ab^{-1}q;q)_\infty}{(aq, bq;q)_\infty}.$$

From the inversion formula (4.27), we have

$$I_0(n) = (aq, bq;q)_n.$$

Therefore, the following proposition is valid.

Proposition 117. *The generalized Big q-Laguerre moments with respect to the basis $(x;q)_n$ have the representation*

$$\mu_n((x;q)_k) = aq\frac{(q, a^{-1}b, ab^{-1}q;q)_\infty}{(aq, bq;q)_\infty}(aq, bq;q)_n. \tag{5.85}$$

Note that the generalized Big q-Laguerre moments with respect to $(x;q)_n$ were given in [4, P. 91] with the normalization $\mu_0 = 1$.

Proposition 118. *The generalized Big q-Laguerre moments with respect to $(x;q)_n$ have the following q-exponential generating function*

$$aq\frac{(q, a^{-1}b, ab^{-1}q;q)_\infty}{(aq, bq;q)_\infty}\frac{(aqz;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \mu_n((x;q)_k)\frac{z^n}{(bq, q;q)_n}, \quad |z| < 1, \tag{5.86}$$

$$aq\frac{(q, a^{-1}b, ab^{-1}q;q)_\infty}{(aq, bq;q)_\infty}\frac{(bqz;q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \mu_n((x;q)_k)\frac{z^n}{(aq, q;q)_n}, \quad |z| < 1. \tag{5.87}$$

Proof. The proof follows by the use of the q-binomial theorem (2.6). □

Proposition 119. *The canonical Big q-Laguerre moments have the representation*

$$\mu_n = aq\frac{(abq^2, a^{-1}cac^{-1}q;q)_\infty}{(aq,bq,cq,abc^{-1}q;q)_\infty}\sum_{m=0}^{n}(-1)^m\begin{bmatrix} n \\ m \end{bmatrix}_q q^{-nm+\binom{m+1}{2}}(aq,bq;q)_m. \tag{5.88}$$

Proof. Using (2.21), (2.23), (5.16) and (5.85), we get the desired result. □

The q-Meixner polynomials

For $0 \le bq < 1$ and $c > 0$, the q-Meixner polynomials $M_n(x;b,c;q)$ fulfil the following orthogonality relation [30, P. 489]

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(bq;q)_k}{(q,-bcq;q)_k}c^k q^{\binom{k}{2}}M_m(q^{-k};b,c;q)M_n(q^{-k};b,c;q)\\ = \frac{(-c;q)_\infty}{(-bcq;q)_\infty}\frac{(q,-c^{-1}q;q)_n}{(bq;q)_n}q^{-n}\delta_{mn}.\end{aligned} \tag{5.89}$$

From (5.89), with $m = n = 0$ it follows that

$$\mu_0 = \frac{(-c;q)_\infty}{(-bcq;q)_\infty}.$$

From the inversion formula (4.30), we have

$$I_0(n) = \left(-\frac{c}{q}\right)^n (bq;q)_n.$$

Therefore, the following proposition is valid.

Proposition 120. *The generalized q-Meixner moments with respect to the basis $(q^{-x};q)_n$ have the representation*

$$\mu_n((q^{-x};q)_k) = \frac{(-c;q)_\infty}{(-bcq;q)_\infty}\left(-\frac{c}{q}\right)^n (bq;q)_n. \tag{5.90}$$

Note that the q-Meixner moments with respect to $(q^{-x};q)_n$ were given in [4, P. 91] with the normalization $\mu_0 = 1$.

Proposition 121. *The generalized q-Meixner moments with respect to $(q^{-x};q)_n$ have the following q-exponential generating function*

$$\frac{(-c,-bcz;q)_\infty}{(-bcq,-cq^{-1}z;q)_\infty} = \sum_{m=0}^{\infty}\mu_n((q^{-x};q)_k)\frac{z^n}{(q;q)_n}. \tag{5.91}$$

Proof. First we write

$$\begin{aligned}\sum_{m=0}^{\infty}\mu_n((q^{-x};q)_k)\frac{z^n}{(q;q)_n} &= \frac{(-c;q)_\infty}{(-bcq;q)_\infty}\sum_{n=0}^{\infty}\frac{(bq;q)_n}{(q;q)_n}(-cq^{-1}z)^n\\ &= \frac{(-c;q)_\infty}{(-bcq;q)_\infty}{}_1\phi_0\left(\begin{matrix} bq \\ - \end{matrix}\middle| -cq^{-1}z\right).\end{aligned}$$

Then, we use the q-binomial theorem (2.6) to get

$$\frac{(-c;q)_\infty}{(-bcq;q)_\infty}{}_1\phi_0\left(\begin{matrix} bq \\ - \end{matrix}\middle| -cq^{-1}z\right) = \frac{(-c,-bcz;q)_\infty}{(-bcq,-cq^{-1}z;q)_\infty}.$$

□

Proposition 122. *The canonical q-Meixner moments have the representation*

$$\mu_n = \frac{(-c;q)_\infty}{(-bcq;q)_\infty} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{-nm+\binom{m}{2}} c^m (bq;q)_m. \tag{5.92}$$

Proof. The proof follows by using (2.21), (2.23), (5.16) and (5.90). □

The Quantum q-Krawtchouk polynomials

The Quantum q-Krawtchouk polynomials $K_n^{qtm}(q^{-x};p,N;q)$ fulfil the following orthogonality relation [30, P. 493]

$$\begin{aligned}\sum_{k=0}^{N} \frac{(pq;q)_{N-k}}{(q;q)_k(q;q)_{N-k}} (-1)^{N-k} q^{\binom{k}{2}} K_m^{qtm}(q^{-k};p,N;q) K_n^{qtm}(q^{-k};p,N;q) \\ = \frac{(-1)^n p^N (q;q)_{N-n}(q,pq;q)_n}{(q,q;q)_N} q^{\binom{N+1}{2}-\binom{n+1}{2}+Nn} \delta_{mn}, \quad p > q^{-N}.\end{aligned} \tag{5.93}$$

From (5.93), with $m = n = 0$ it follows that

$$\mu_0 = \frac{p^N (q;q)_N}{(q,q;q)_N} q^{\binom{N+1}{2}}.$$

From the inversion formula (4.31), we have

$$I_0(n) = \frac{(q^{-N};q)_n}{(pq)^n}.$$

Therefore, the following proposition is valid.

Proposition 123. *The generalized Quantum q-Krawtchouk moments with respect to the basis $(q^{-x};q)_n$ have the representation*

$$\mu_n((q^{-x};q)_k) = \frac{p^N (q;q)_N}{(q,q;q)_N} q^{\binom{N+1}{2}} \frac{(q^{-N};q)_n}{(pq)^n}. \tag{5.94}$$

Proposition 124. *The generalized Quantum q-Krawtchouk moments with respect to $(q^{-x};q)_n$ have the following q-exponential generating function*

$$q^{\binom{N+1}{2}} \frac{p^N (q;q)_N}{(q,q;q)_N} \frac{(p^{-1}q^{-N-1}z;q)_\infty}{((pq)^{-1}z;q)_\infty} = \sum_{m=0}^{\infty} \mu_n((q^{-x};q)_k) \frac{z^n}{(q;q)_n}. \tag{5.95}$$

Proof. First we write

$$\begin{aligned}\sum_{m=0}^{\infty} \mu_n((q^{-x};q)_k) \frac{z^n}{(q;q)_n} &= q^{\binom{N+1}{2}} \frac{p^N (q;q)_N}{(q,q;q)_N} \sum_{n=0}^{\infty} \frac{(q^{-N};q)_n}{(q;q)_n} \left(\frac{z}{pq}\right)^n \\ &= q^{\binom{N+1}{2}} \frac{p^N (q;q)_N}{(q,q;q)_N} {}_1\phi_0 \left(\left. \begin{matrix} q^{-N} \\ - \end{matrix} \right| \frac{z}{pq} \right).\end{aligned}$$

Then, we use the q-binomial theorem (2.6) to get

$$q^{\binom{N+1}{2}} \frac{p^N (q;q)_N}{(q,q;q)_N} {}_1\phi_0 \left(\left. \begin{matrix} q^{-N} \\ - \end{matrix} \right| \frac{z}{pq} \right) = q^{\binom{N+1}{2}} \frac{p^N (q;q)_N}{(q,q;q)_N} \frac{(p^{-1}q^{-N-1}z;q)_\infty}{((pq)^{-1}z;q)_\infty}.$$

□

Proposition 125. *The canonical Quantum q-Krawtchouk moments have the following representation*

$$\mu_n = \frac{p^N (q;q)_N}{(q,q;q)_N} q^{\binom{N+1}{2}} \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{-nm+\binom{m+1}{2}} \frac{(q^{-N};q)_m}{(pq)^m} \tag{5.96}$$

Proof. The proof follows from (2.21), (2.23), (5.16) and (5.94). □

The q-Krawtchouk polynomials

From the q-Krawtchouk orthogonality relation (3.23), with $m = n = 0$, it follows that

$$\mu_0 = (-pq;q)_N p^{-N} q^{-\binom{N+1}{2}}.$$

From the inversion formula (4.32), we have

$$I_0(n) = \frac{(q^{-N};q)_n}{(-pq;q)_n}.$$

Therefore, the following proposition is valid.

Proposition 126. *The generalized q-Krawtchouk moments with respect to the basis* $(q^{-x};q)_n$ *have the representation*

$$\mu_n((q^{-x};q)_k) = p^{-N} q^{-\binom{N+1}{2}} (-pq;q)_N \frac{(q^{-N};q)_n}{(-pq;q)_n}. \tag{5.97}$$

Remark 127. *The canonical q-Krawtchouk moments are already given by (3.24). These moments can be recovered by using (2.21), (2.23), (5.16) and (5.97), combined with q-Zeilberger's algorithm [32] implemented the* `qsum` *package.*

Proposition 128. *The q-Krawtchouk moments with respect to* $(q^{-x};q)_n$ *have the following q-exponential generating function*

$$(-pq;q)_N p^{-N} q^{-\binom{N+1}{2}} \frac{(zq^{-N};q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty} \mu_n((q^{-x};q)_k)(-pq;q)_n \frac{z^n}{(q;q)_n}. \tag{5.98}$$

Proof. We have

$$\sum_{m=0}^{\infty} (-pq;q)_n \mu_n((q^{-x};q)_k) \frac{z^n}{(q;q)_n} = p^{-N} q^{-\binom{N+1}{2}} (-pq;q)_N \sum_{n=0}^{\infty} \frac{(q^{-N};q)_n}{(q;q)_n} z^n.$$

By the q-binomial theorem (2.6), it follows that

$$\sum_{n=0}^{\infty} \frac{(q^{-N};q)_n}{(q;q)_n} z^n = \frac{(zq^{-N};q)_\infty}{(z;q)_\infty}.$$

This completes the proof. □

The Affine q-Krawtchouk polynomials

The Affine q-Krawtchouk polynomials $K_n^{Aff}(q^{-x};p,N;q)$ fulfil the following orthogonality relation [30, P. 501]

$$\begin{aligned}\sum_{k=0}^{N} \frac{(pq;q)_k (q;q)_N}{(q;q)_k (q;q)_{N-k}} (pq)^{-k} K_m^{Aff}(q^{-k};p,N;q) K_n^{Aff}(q^{-k};p,N;q) \\ = (pq)^{n-N} \frac{(q;q)_n (q;q)_{N-n}}{(pq;q)_n (q;q)_N} \delta_{mn}, \quad 0 < pq < 1.\end{aligned} \tag{5.99}$$

From (5.99), with $m = n = 0$ it follows that

$$\mu_0 = (pq)^{-N}.$$

From the inversion formulas (4.33) and (4.34), we have:

- for $\theta_n(x) = (q^{-x};q)_n$,

$$I_0(n) = (pq, q^{-N};q)_n.$$

- for $\theta_n(x) = (q^{-x})^n (q^{x-N}; q)_n = (q^{-x} \ominus q^{-N})_q^n$,

$$I_0(n) = (pq)^n (q^{-N}; q)_n.$$

Therefore, the following proposition is valid.

Proposition 129. *The generalized Affine q-Krawtchouk moments*

1. *with respect to the basis $(q^{-x}; q)_n$ have the representation*

$$\mu_n((q^{-x}; q)_k) = (pq)^{-N} (pq, q^{-N}; q)_n, \tag{5.100}$$

2. *with respect to the basis $\theta_n(x) = (q^{-x} \ominus q^{-N})_q^n$ have the representation*

$$\mu_n\left((q^{-x} \ominus q^{-N})_q^k\right) = (pq)^{n-N} (q^{-N}; q)_n. \tag{5.101}$$

Proposition 130. *The generalized Affine q-Krawtchouk moments have the following q-exponential generating functions*

$$(pq)^{-N} \frac{(zq^{-N}; q)_\infty}{(z; q)_\infty} = \sum_{m=0}^{\infty} \frac{\mu_n((q^{-x}; q)_k)}{(pq; q)_n} \frac{z^n}{(q; q)_n}, \tag{5.102}$$

$$(pq)^{-N} \frac{(pq^{1-N} z; q)_\infty}{(pqz; q)_\infty} = \sum_{n=0}^{\infty} \mu_n\left((q^{-x} \ominus q^{-N})_q^k\right) \frac{z^n}{(q; q)_n}. \tag{5.103}$$

Proof. The proof of (5.103) follows from the q-binomial theorem (2.6). □

Proposition 131. *The canonical Affine q-Krawtchouk moments have the following representation*

$$\mu_n = (pq)^{-N} \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{-nm+\binom{m+1}{2}} (pq, q^{-N}; q)_m. \tag{5.104}$$

Proof. The proof follows from (2.21), (2.23), (5.16) and (5.100). □

The Little q-Laguerre polynomials

The Little q-Laguerre polynomials fulfil the orthogonality relation (3.26). With $m = n = 0$, it follows that

$$\mu_0 = \frac{1}{(aq; q)_\infty}.$$

From the inversion formulas (4.35) and (4.36), we have

- for $\theta_n(x) = x^n$,

$$I_0(n) = (aq; q)_n$$

- for $\theta_n(x) = (x \ominus 1)_q^n$,

$$I_0(n) = (-aq)^n q^{\binom{n}{2}}.$$

Therefore, the following proposition is valid.

Proposition 132. *The Little q-Laguerre moments*

1. *with respect to the basis x^n have the representation (compare to (3.27)),*

$$\mu_n = \frac{(aq; q)_n}{(aq; q)_\infty}, \quad \textit{compare with [4, P. 91]} \tag{5.105}$$

2. *with respect to the basis* $(x\ominus 1)_q^n$ *have the representation*

$$\mu_n((x\ominus 1)_q^k) = \frac{(-aq)^n q^{\binom{n}{2}}}{(aq;q)_\infty}. \tag{5.106}$$

The canonical moment μ_n is given in [4, P. 91] with the normalization $\mu_0 = 1$.

Proposition 133. *The generalized Little q-Laguerre moments with respect to the basis* $(x\ominus 1)_q^n$ *have the following generating function*

$$(aqz;q)_\infty = \sum_{n=0}^{\infty} \frac{\mu_n((x\ominus 1)_q^k)}{(q;q)_n} z^n. \tag{5.107}$$

Proof. The result is obtained by the use of the Euler formula (2.8). □

The q-Laguerre polynomials

The q-Laguerre polynomials $L_n^{(\alpha)}(x;q)$ fulfil the following orthogonality relations

Discrete orthogonality For the discrete orthogonality, see Equation (3.29).

Continuous orthogonality [30, P. 522]

$$\begin{aligned}\int_0^\infty & \frac{x^\alpha}{(-x;q)_\infty} L_m^{(\alpha)}(x;q) L_n^{(\alpha)}(x;q)dx \\ &= \frac{(q^{-\alpha};q)_\infty}{(q;q)_\infty}\frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n}\Gamma(-\alpha)\Gamma(\alpha+1)\delta_{mn}, \quad \alpha > -1.\end{aligned} \tag{5.108}$$

With $m = n = 0$, it follows that

- For the discrete orthogonality

$$\mu_0^{(d)} = \frac{(q, -cq^{\alpha+1}, -c^{-1}q^{-\alpha};q)_\infty}{(q^{\alpha+1}, -c, -c^{-1}q;q)_\infty}$$

- For the continuous orthogonality

$$\mu_0^{(c)} = \frac{(q^{-\alpha};q)_\infty}{(q;q)_\infty}\Gamma(-\alpha)\Gamma(\alpha+1).$$

From the inversion formulas (4.37), for $\theta_n(x) = x^n$, we have:

$$I_0(n) = q^{-\binom{n}{2}-n(\alpha+1)}(q^{\alpha+1};q)_n.$$

The following proposition is therefore valid.

Proposition 134. *The canonical q-Laguerre moments have the representation*

$$\mu_n^{(d)} = \frac{(q, -cq^{\alpha+1}, -c^{-1}q^{-\alpha};q)_\infty}{(q^{\alpha+1}, -c, -c^{-1}q;q)_\infty} q^{-\binom{n}{2}-n(\alpha+1)}(q^{\alpha+1};q)_n, \tag{5.109}$$

for the discrete orthogonality (compare to (3.30)), and

$$\mu_n^{(c)} = \frac{(q^{-\alpha};q)_\infty}{(q;q)_\infty}\Gamma(-\alpha)\Gamma(\alpha+1)q^{-\binom{n}{2}-n(\alpha+1)}(q^{\alpha+1};q)_n, \tag{5.110}$$

for the continuous orthogonality.

Remark 135. *The moments (5.109) obtained using the inversion formula for the discrete orthogonality are of course the same as the ones obtained by direct computations in Chapter 3. The generating function is already given.*

The q-Charlier polynomials

The q-Charlier polynomials fulfil the orthogonality relation (3.35). With $m = n = 0$, it follows that

$$\mu_0 = (-a;q)_\infty.$$

From the inversion formula (4.39), for $\theta_n(x) = (q^{-x};q)_n$, we have

$$I_0(n) = \left(-\frac{a}{q}\right)^n.$$

Therefore, the following proposition is valid.

Proposition 136. *The generalized q-Charlier moments with respect to the basis $(q^{-x};q)_n$ have the representation*

$$\mu_n((q^{-x};q)_k) = (-a;q)_\infty \left(-\frac{a}{q}\right)^n. \tag{5.111}$$

Remark 137. *The canonical q-Charlier moments are already given by (3.36). These moments can be recovered by using (2.21), (2.23), (5.16) and (5.111), combined with the q-Zeilberger's algorithm [32] implemented in the `qsum` package.*

Proposition 138. *The generalized q-Charlier moments with respect to $(q^{-x};q)_n$ have the following q-exponential generating function:*

$$\frac{(-a;q)_\infty}{(-aq^{-1}z;q)_\infty} = \sum_{n=0}^{\infty} \mu_n((q^{-x};q)_k)\frac{z^n}{(q;q)_n}. \tag{5.112}$$

Proof. The proof follows from Euler's formula (2.7). □

The Al Salam-Carlitz I polynomials

The Al-Salam-Carlitz I polynomials $U_n^{(a)}(x;q)$ fulfil the following orthogonality relation [30, P. 534]

$$\begin{aligned}&\int_a^1 (qx, a^{-1}qx;q)_\infty U_m^{(a)}(x;q)U_n^{(a)}(x;q)d_qx\\ &\quad = (-a)^n(1-q)(q;q)_n(q,a,a^{-1}q;q)_\infty q^{\binom{n}{2}}\delta_{mn}, \quad a<0.\end{aligned} \tag{5.113}$$

Let us write

$$\rho(x;q) = (qx, a^{-1}qx;q)_\infty.$$

The q-integral in (5.113) can be written as

$$\begin{aligned}&\int_a^1 \rho(x;q)U_m^{(a)}(x;q)U_n^{(a)}(x;q)d_qx\\ &\quad = (1-q)\sum_{k=0}^{\infty} q^k\rho(q^k;q)U_m^{(a)}(q^k;q)U_n^{(a)}(q^k;q)\\ &\qquad - a(1-q)\sum_{k=0}^{\infty} q^k\rho(aq^k;q)U_m^{(a)}(aq^k;q)U_n^{(a)}(aq^k;q).\end{aligned}$$

Define the discrete measures μ_1 and μ_a as

$$\mu_1 = (1-q)\sum_{k=0}^{\infty}\rho(q^k;q)q^k\varepsilon_{q^k},$$

$$\mu_a = aq(q-1)\sum_{k=0}^{\infty}\rho(q^k;q)q^k\varepsilon_{aq^k},$$

and put

$$\mu = \mu_1 + \mu_a.$$

We then have

$$\int_a^1 \rho(x;q)(x\ominus 1)_q^n d_q x = \int_{-\infty}^{\infty}(x\ominus 1)_q^n d\mu(x).$$

With $m = n = 0$, it follows that

$$\mu_0 = (1-q)(q,a,a^{-1}q;q)_\infty.$$

From the inversion formulas (4.41) and (4.40),

- for $\theta_n(x) = (x\ominus 1)_q^n$, we have
$$I_0(n) = a^n;$$
- for $\theta_n(x) = x^n$, we have
$$I_0(n) = \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q a^i.$$

Therefore, the following proposition is valid.

Proposition 139. *The Al-Salam-Carlitz I moments*

1. *with respect to the basis $(x\ominus 1)_q^n$ (generalized moments) have the representation*
$$\mu_n((x\ominus 1)_q^k) = (1-q)(q,a,a^{-1}q;q)_\infty a^n \tag{5.114}$$
2. *with respect to the basis x^n (canonical moments) have the representation*
$$\mu_n = (1-q)(q,a,a^{-1}q;q)_\infty \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q a^i. \tag{5.115}$$

Note that these canonical moments appear in [9, Eq. (10.8), P. 197]

Proposition 140. *The generalized Al-Salam-Carlitz I moments with respect to $(x\ominus 1)_q^n$ have the following q-exponential generating function*

$$(q,a,a^{-1}q;q)_\infty \frac{1-q}{1-az} = \sum_{n=0}^{\infty} \mu_n((x\ominus 1)_q^k) z^n, \quad |az|<1. \tag{5.116}$$

The Al-Salam-Carlitz II polynomials

The Al-Salam-Carlitz II polynomials $V_n^{(a)}(q^{-x};q)$ fulfil the following orthogonality relation [30, P. 537]

$$\sum_{k=0}^{\infty} \frac{q^{k^2}a^k}{(q;q)_k(aq;q)_k} V_m^{(a)}(q^{-k};q)V_n^{(a)}(q^{-k};q)$$
$$= \frac{(q;q)_n a^n}{(aq;q)_\infty q^{n^2}}\delta_{mn}, \quad 0<aq<1. \tag{5.117}$$

Therefore, the canonical Al-Salam-Carlitz II moments are

$$\mu_n = \sum_{k=0}^{\infty} \frac{q^{k^2}a^k}{(q;q)_k(aq;q)_k} q^{-kn}.$$

From (5.117), with $m = n = 0$, it follows that

$$\mu_0 = \frac{1}{(aq;q)_\infty}.$$

From the inversion formula (4.42), for $\theta_n(x) = (q^{-x};q)_n$, we have

$$I_0(n) = (-a)^n q^{\binom{n}{2}}.$$

Therefore, the following proposition is valid.

Proposition 141. *The generalized Al-Salam-Carlitz II moments with respect to the basis $(q^{-x};q)_n$ have the representation*

$$\mu_n((q^{-x};q)_k) = \frac{(-a)^n q^{\binom{n}{2}}}{(aq;q)_\infty}. \tag{5.118}$$

Note that the Al-Salam-Carlitz II moments with respect to $(q^{-x};q)_n$ are given in [4, P. 91] without the term μ_0.

Proposition 142. *The generalized Al-Salam-Carlitz II moments with respect to $(q^{-x};q)_n$ have the following q-exponential generating function*

$$\frac{(az;q)_\infty}{(aq;q)_\infty} = \sum_{n=0}^{\infty} \mu_n((x;q)_k)\frac{z^n}{(q;q)_n}. \tag{5.119}$$

Proof. The proof follows from Euler's formula (2.8). □

Proposition 143. *The canonical Al-Salam-Carlitz II moments have the representation*

$$\mu_n = \frac{1}{(aq;q)_\infty}\sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q a^m q^{m(m-n)}. \tag{5.120}$$

Proof. The proof follows from (2.21), (2.23), (5.16) and (5.118). □

Note that these moments appear in [9, Eq. (10.10), P. 197]

The Stieltjes-Wigert polynomials

The Stieltjes-Wigert polynomials $S_n(x;q)$ fulfil the following orthogonality relation [30, P. 544]

$$\int_0^\infty \frac{S_m(x;q)S_n(x;q)}{(-x,-qx^{-1};q)_\infty}dx = -\frac{\ln q}{q^n}\frac{(q;q)_\infty}{(q;q)_n}\delta_{mn}. \tag{5.121}$$

With $m = n = 0$, it follows that

$$\mu_0 = -\ln q(q;q)_\infty.$$

From the inversion formula (4.43), for $\theta_n(x) = x^n$, we have

$$I_0(n) = q^{-\binom{n+1}{2}}.$$

Therefore, the following proposition is valid.

Proposition 144. *The canonical Stieltjes-Wigert moments have the representation*

$$\mu_n = -\ln q(q;q)_\infty q^{-\binom{n+1}{2}}. \tag{5.122}$$

Note that these moments appeared in [4, P. 91] and [10, P. 223].

Proposition 145. *The canonical Stieltjes-Wigert moments have the following q-exponential generating function:*

$$\frac{\ln q^{-1}(q;q)_\infty}{(q^{-1}z;q)_\infty} = \sum_{n=0}^{\infty}\mu_n\frac{q^{\binom{n}{2}}z^n}{(q;q)_n}. \tag{5.123}$$

Proof. First we remark that $\binom{n+1}{2} = \binom{n}{2} + n$ and then we apply Euler's formula (2.7). □

The Discrete q-Hermite I polynomials

The Discrete q-Hermite I polynomials $h_n(x;q)$ fulfil the following orthogonality relation [30, P. 547]

$$\int_{-1}^{1}(qx,-qx;q)_\infty h_m(x;q)h_n(x;q)d_qx$$
$$= (1-q)(q;q)_n(q,-1,-q;q)_\infty q^{\binom{n}{2}}\delta_{mn}. \tag{5.124}$$

Let us write

$$\rho(x;q) = (qx,-qx;q)_\infty.$$

The q-integral in (5.124) can be written as

$$\int_{-1}^{1}\rho(x;q)h_m(x;q)h_n(x;q)d_qx$$
$$= (1-q)\sum_{k=0}^{\infty}q^k\rho(q^k;q)h_m(q^k;q)h_n(q^k;q)$$
$$+ (1-q)\sum_{k=0}^{\infty}q^k\rho(-q^k;q)h_m(-q^k;q)h_n(-q^k;q).$$

Define the discrete measures μ_1 and μ_{-1} as

$$\mu_1 = (1-q)\sum_{k=0}^{\infty}\rho(q^k;q)q^k\varepsilon_{q^k}$$
$$\mu_{-1} = (1-q)\sum_{k=0}^{\infty}\rho(-q^k;q)q^k\varepsilon_{-q^k},$$

and put

$$\mu = \mu_1 + \mu_{-1}.$$

We then have

$$\int_{-1}^{1}\rho(x;q)(x\ominus 1)_q^n d_qx = \int_{-\infty}^{\infty}(x\ominus 1)_q^n d\mu(x).$$

With $m = n = 0$, it follows that

$$\mu_0 = (1-q)(q,a,a^{-1}q;q)_\infty.$$

From the inversion formulas (4.44) and (4.45),

- for $\theta_n(x) = (x\ominus 1)_q^n$, we have
$$I_0(n) = (-1)^n;$$
- for $\theta_n(x) = x^n$, we have
$$I_0(n) = \frac{1+(-1)^n}{2}(q;q^2)_{n/2}.$$

Therefore, the following proposition is valid.

Proposition 146. *The Discrete q-Hermite I moments*

1. *with respect to the basis $(x\ominus 1)_q^n$ (generalized moments) have the representation*
$$\mu_n((x\ominus 1)_q^k) = (1-q)(q,-1,-q;q)_\infty(-1)^n \tag{5.125}$$
2. *with respect to the basis x^n (canonical moments) have the representation*
$$\mu_n = (1-q)(q,-1,-q;q)_\infty\frac{1+(-1)^n}{2}(q;q^2)_{n/2}, \quad \textit{compare with [4, P. 91].} \tag{5.126}$$

Proposition 147. *The generalized Discrete q-Hermite I moments with respect to $(x\ominus 1)_q^n$ have the following generating function*

$$(q,-1,-q;q)_\infty\frac{1-q}{1+z} = \sum_{n=0}^{\infty}\mu_n((x\ominus 1)_q^k)z^n, \quad |z|<1. \tag{5.127}$$

The Discrete q-Hermite II polynomials

The Discrete q-Hermite II polynomials $\tilde{h}_n(x;q)$ fulfil the following orthogonality relation [30, P. 550]

$$\int_{-\infty}^{\infty} \frac{\tilde{h}_m(x;q)\tilde{h}_n(x;q)}{(-x^2;q^2)_\infty} d_qx = \frac{(q^2,-q,-q;q^2)_\infty}{(q^3,-q^2,-q^2;q^2)_\infty} \frac{(q;q)_n}{q^{n^2}} \delta_{mn}. \tag{5.128}$$

Let us write

$$\rho(x;q) = \frac{1}{(-x^2;q^2)_\infty}.$$

The q-integral in (5.128) can be written as

$$\begin{aligned}\int_{-\infty}^{\infty} \rho(x;q)\tilde{h}_m(x;q)\tilde{h}_n(x;q)d_qx = \\ = (1-q)\sum_{k=-\infty}^{\infty} q^k\rho(q^k;q)h_m(q^k;q)h_n(q^k;q) \\ + (1-q)\sum_{k=-\infty}^{\infty} q^k\rho(-q^k;q)h_m(-q^k;q)h_n(-q^k;q).\end{aligned}$$

Define the discrete measures μ_1 and μ_2 as

$$\mu_1 = (1-q)\sum_{k=-\infty}^{\infty} \rho(q^k;q)q^k\varepsilon_{q^k}$$

$$\mu_2 = (1-q)\sum_{k=-\infty}^{\infty} \rho(-q^k;q)q^k\varepsilon_{-q^k},$$

and put

$$\mu = \mu_1 + \mu_2.$$

We then have

$$\int_{-\infty}^{\infty} \rho(x;q)(x;q)_n d_qx = \int_{-\infty}^{\infty} (x;q)_n d\mu(x).$$

It follows that

$$\mu_0 = \frac{(q^2,-q,-q;q^2)_\infty}{(q^3,-q^2,-q^2;q^2)_\infty}.$$

From the inversion formula (4.46), for $\theta_n(x) = (x;q)_n$, we have

$$I_0(n) = q^{\binom{n}{2}}.$$

Therefore, the following proposition is valid.

Proposition 148. *The generalized Discrete q-Hermite II moments with respect to the basis $(x;q)_n$ have the representation*

$$\mu_n((x;q)_k) = \frac{(q^2,-q,-q;q^2)_\infty}{(q^3,-q^2,-q^2;q^2)_\infty} q^{\binom{n}{2}}. \tag{5.129}$$

Note that these moments appeared in [4, P. 91] with the normalization $\mu_0 = 1$.

Proposition 149. *The generalized Discrete q-Hermite II moments with respect to $(x;q)_n$ have the following q-exponential generating function:*

$$\frac{(-z,q^2,-q,-q;q^2)_\infty}{(q^3,-q^2,-q^2;q^2)_\infty} = \sum_{n=0}^{\infty} \mu_n((x;q)_k)\frac{z^n}{(q;q)_n}. \tag{5.130}$$

Proof. The proof follows from the Euler formula (2.8). □

Proposition 150. *The canonical Discrete q-Hermite II moments have the representation*

$$\mu_n = \frac{(q^2,-q,-q;q^2)_\infty}{(q^3,-q^2,-q^2;q^2)_\infty}\sum_{m=0}^{n}(-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{m(m-n)}. \tag{5.131}$$

Proof. The proof follows from (2.21), (2.23), (5.16) and (5.129). □

5.4.4 The quadratic case

The Wilson polynomials

The Wilson polynomials $W_n(x^2;a,b,c,d)$ fulfil the following orthogonality relation [30, P. 186]

$$\int_0^\infty \left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right|^2 W_m(x^2;a,b,c,d)W_n(x^2;a,b,c,d)dx$$
$$= \frac{2\pi\Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+b+c)\Gamma(n+b+d)\Gamma(n+c+d)n!}{\Gamma(2n+a+b+c+d)(n+a+b+c+d-1)_n^{-1}}\delta_{mn}, \quad (5.132)$$

With $m=n=0$, it follows that

$$\mu_0 = 2\pi\frac{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)}{\Gamma(a+b+c+d)}.$$

From the inversion formula (4.47), for $\theta_n(x)=(a+ix)_n(a-ix)_n$, we have

$$I_0(n) = \frac{(a+b)_n(a+c)_n(a+d)_n}{(a+b+c+d)_n}.$$

The following proposition is therefore valid.

Proposition 151. *The generalized Wilson moments with respect to the basis $\vartheta_n(a,x)$ have the representation*

$$\mu_n(\vartheta_n(a,x)) = 2\pi\frac{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)}{\Gamma(a+b+c+d)}\frac{(a+b)_n(a+c)_n(a+d)_n}{(a+b+c+d)_n}. \quad (5.133)$$

Proposition 152. *The generalized Wilson moments with respect to $\vartheta_n(a,x)$ have the following generating function*

$$\mu_0(1-z)^{a+b} = \sum_{n=0}^\infty \mu_n(\vartheta_n(a,x))\frac{(a+b+c+d)_n}{(a+c)_n(a+d)_n}\frac{z^n}{n!}, \quad (5.134)$$
$$\mu_0(1-z)^{a+c} = \sum_{n=0}^\infty \mu_n(\vartheta_n(a,x))\frac{(a+b+c+d)_n}{(a+b)_n(a+d)_n}\frac{z^n}{n!}, \quad (5.135)$$
$$\mu_0(1-z)^{a+d} = \sum_{n=0}^\infty \mu_n(\vartheta_n(a,x))\frac{(a+b+c+d)_n}{(a+b)_n(a+c)_n}\frac{z^n}{n!}. \quad (5.136)$$

with

$$\mu_0 = 2\pi\frac{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)}{\Gamma(a+b+c+d)}.$$

Proof. Using the binomial formula (2.3) we get the result. □

Proposition 153. *The canonical Wilson moments have the representation*

$$\mu_n = 2\pi\frac{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)}{\Gamma(a+b+c+d)}$$
$$\times\sum_{m=0}^n E_m(-a^2,-(a+1)^2,\cdots,-(a+n-1)^2)\frac{(a+b)_m(a+c)_m(a+d)_m}{(a+b+c+d)_m}.$$

Proof. Combining

$$\mu_n(\vartheta_n(a,x)) = 2\pi\frac{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)}{\Gamma(a+b+c+d)}\frac{(a+b)_n(a+c)_n(a+d)_n}{(a+b+c+d)_n}$$

with

$$(x^2)^n = \sum_{m=0}^n E_m(-a^2,-(a+1)^2,\cdots,-(a+n-1)^2)\vartheta_m(a,x),$$

we get the result. □

Proposition 154. *The canonical Wilson moments have the following representation*

$$\mu_n = \mu_0 \sum_{k=0}^{n} \sum_{l=0}^{k} \frac{(-k)_l}{k!l!} \frac{(a+b)_k(a+c)_k(a+d)_k}{(a+b+c+d)_k} \frac{(-2a-2k+2l)}{(-2a-2k+l)_{k+1}} (a+k-l)^{2n}. \quad (5.137)$$

Proof. The result is obtained using relations (2.21), (2.23), (5.29) and (5.133). □

The Racah polynomials

The Racah polynomials $R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$ fulfil the following orthogonality relation [30, P. 191]

$$\begin{aligned}\sum_{x=0}^{N} &\frac{(\alpha+1)_x(\beta+\delta+1)_x(\gamma+1)_x(\gamma+\delta+1)_x((\gamma+\delta+3)/2)_x}{(-\alpha+\gamma+\delta+1)_x(-\beta+\gamma+1)_x((\gamma+\delta+1)/2)_x(\delta+1)_x x!} R_m(\lambda(x))R_n(\lambda(x)) \\ &= M\frac{(n+\alpha+\beta+1)_n(\alpha+\beta-\gamma+1)_n(\alpha-\delta+1)_n(\beta+1)_n n!}{(\alpha+\beta+1)_{2n}(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}\delta_{mn}, \quad (5.138)\end{aligned}$$

where

$$R_n(\lambda(x)) = R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$$

and

$$M = \begin{cases} \dfrac{(-\beta)_N(\gamma+\delta+2)_N}{(-\beta+\gamma+1)_N(\delta+1)_N} & \text{if } \alpha+1=-N \\ \dfrac{(-\alpha+\delta)_N(\gamma+\delta+2)_N}{(-\alpha+\gamma+\delta+1)_N(\delta+1)_N} & \text{if } \beta+\delta+1=-N \\ \dfrac{(\alpha+\beta+2)_N(-\delta)_N}{(\alpha-\delta+1)_N(\beta+1)_N} & \text{if } \gamma+1=-N. \end{cases}$$

It follows that

$$\mu_0 = \begin{cases} \dfrac{(-\beta)_N(\gamma+\delta+2)_N}{(-\beta+\gamma+1)_N(\delta+1)_N} & \text{if } \alpha+1=-N \\ \dfrac{(-\alpha+\delta)_N(\gamma+\delta+2)_N}{(-\alpha+\gamma+\delta+1)_N(\delta+1)_N} & \text{if } \beta+\delta+1=-N \\ \dfrac{(\alpha+\beta+2)_N(-\delta)_N}{(\alpha-\delta+1)_N(\beta+1)_N} & \text{if } \gamma+1=-N. \end{cases}$$

From the inversion formula (4.48), for $\theta_n(x) = (-x)_n(x+\gamma+\delta+1)_n$, we have

$$I_0(n) = \frac{(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}{(\alpha+\beta+2)_n}.$$

Therefore, the following proposition is valid.

Proposition 155. *The generalized Racah moments with respect to the basis $\theta_n(x) = (-x)_n(x+\gamma+\delta+1)_n$ have the representation*

$$\mu_n(\theta_n(x)) = \begin{cases} \dfrac{(-\beta)_N(\gamma+\delta+2)_N}{(-\beta+\gamma+1)_N(\delta+1)_N}\dfrac{(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}{(\alpha+\beta+2)_n} & \textit{if } \alpha+1=-N \\ \dfrac{(-\alpha+\delta)_N(\gamma+\delta+2)_N}{(-\alpha+\gamma+\delta+1)_N(\delta+1)_N}\dfrac{(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}{(\alpha+\beta+2)_n} & \textit{if } \beta+\delta+1=-N \\ \dfrac{(\alpha+\beta+2)_N(-\delta)_N}{(\alpha-\delta+1)_N(\beta+1)_N}\dfrac{(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}{(\alpha+\beta+2)_n} & \textit{if } \gamma+1=-N. \end{cases} \quad (5.139)$$

Proposition 156. *The canonical Racah moments have the following representation*

$$\mu_n = \mu_0 \sum_{k=0}^{n} \frac{\mathrm{D}_\varepsilon^k[x(x+\varepsilon)]^n_{|x=0}}{k!} \frac{(\alpha+1)_k(\beta+\delta+1)_k(\gamma+1)_k}{(\alpha+\beta+2)_k} \quad (5.140)$$

where $\varepsilon = \gamma+\delta+1$.

Proof. The result is obtained using relations (2.21), (2.23), (5.32) and (5.139). □

The Continuous Dual Hahn polynomials

The Continuous Dual Hahn polynomials $S_n(x^2;a,b,c)$ fulfil the following orthogonality relation [30, P. 196]

$$\int_0^\infty \left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|^2 S_m(x^2;a,b,c)S_n(x^2;a,b,c)dx$$
$$= \Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+b+c)n!\delta_{mn}. \quad (5.141)$$

With $m=n=0$, it follows that

$$\mu_0 = \Gamma(a+b)\Gamma(a+c)\Gamma(b+c).$$

From the inversion formula (4.49), for $\vartheta_n(a,x) = (a-ix)_n(a+ix)_n$, we have

$$I_0(n) = (a+c)_n(a+d)_n.$$

Therefore, the following proposition is valid.

Proposition 157. *The generalized Continuous Dual Hahn moments with respect to the basis $\vartheta_n(a,x) = (a-ix)_n(a+ix)_n$ have the following representation*

$$\mu_n(\vartheta_k(a,x)) = \Gamma(a+b)\Gamma(a+c)\Gamma(b+c)(a+c)_n(a+d)_n. \quad (5.142)$$

Proposition 158. *The generalized Continuous Dual Hahn moments with respect to $\vartheta_n(a,x)$ have the following generating functions:*

$$\mu_0(1-z)^{a+c} = \sum_{n=0}^{\infty} \mu_n(\vartheta_k(a,x))\frac{z^n}{(a+d)_n n!}, \quad (5.143)$$
$$\mu_0(1-z)^{a+d} = \sum_{n=0}^{\infty} \mu_n(\vartheta_k(a,x))\frac{z^n}{(a+c)_n n!}. \quad (5.144)$$

with

$$\mu_0 = \Gamma(a+b)\Gamma(a+c)\Gamma(b+c).$$

Proof. Using the binomial theorem (2.3) we get the result. □

Proposition 159. *The canonical Continuous Dual Hahn moments have the following representation*

$$\mu_n = \mu_0 \sum_{m=0}^{n} E_m(-a^2, -(a+1)^2, \ldots, -(a+n-1)^2)(a+c)_m(a+d)_m. \quad (5.145)$$

with

$$\mu_0 = \Gamma(a+b)\Gamma(a+c)\Gamma(b+c).$$

Proof. Since

$$\mu_n(\vartheta_k(a,x)) = \Gamma(a+b)\Gamma(a+c)\Gamma(b+c)(a+c)_n(a+d)_n$$

and

$$(x^2)^n = \sum_{m=0}^{n} E_m(-a^2, -(a+1)^2, \ldots, -(a+n-1)^2)\vartheta_m(a,x),$$

the result follows. □

Proposition 160. *The canonical Continuous Dual Hahn moments have the following representation*

$$\mu_n = \mu_0 \sum_{k=0}^{n}\sum_{l=0}^{k} \frac{(-k)_l}{k!l!}\frac{(-2a-2k+2l)(a+c)_k(a+d)_k}{(-2a-2k+l)_{k+1}}(a+k-l)^{2n}. \quad (5.146)$$

Proof. The result is obtained using relations (2.21), (2.23), (5.29) and (5.142). □

The Continuous Hahn polynomials

The Continuous Hahn polynomials $p_n(x;a,b,c,d)$ fulfil the following orthogonality relation [30, P. 200]

$$\int_{-\infty}^{\infty}\Gamma(a+ix)\Gamma(b+ix)\Gamma(c-ix)\Gamma(d-ix)p_m(x;a,b,c,d)p_n(x;a,b,c,d)dx$$
$$=\frac{\Gamma(n+a+c)\Gamma(n+a+d)\Gamma(n+b+c)\Gamma(n+b+d)}{(2n+a+b+c+d-1)\Gamma(n+a+b+c+d-1)n!}\delta_{mn}. \tag{5.147}$$

With $m=n=0$, it follows that

$$\mu_0=\frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}.$$

From the inversion formula (4.49), with $\eta_n(x)=(a+ix)_n$, we have:

$$I_0(n)=\frac{(a+c)_n(a+d)_n}{(a+b+c+d)_n}.$$

Therefore, the following proposition is valid.

Proposition 161. *The generalized Continuous Hahn moments with respect to the basis $(a+ix)_n$ have the representation*

$$\mu_n((a+ix)_k)=\frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}\frac{(a+c)_n(a+d)_n}{(a+b+c+d)_n}. \tag{5.148}$$

Proposition 162. *The generalized moments of the Continuous Hahn polynomials with respect to $(a+ix)_n$ have the following exponential generating function:*

$$\mu_0(1-x)^{a+c}=\sum_{n=0}^{\infty}\mu_n((a+ix)_k)\frac{(a+b+c+d)_n}{(a+d)_n}\frac{z^n}{n!}, \tag{5.149}$$

$$\mu_0(1-x)^{a+d}=\sum_{n=0}^{\infty}\mu_n((a+ix)_k)\frac{(a+b+c+d)_n}{(a+d)_n}\frac{z^n}{n!}. \tag{5.150}$$

with

$$\mu_0=\frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}.$$

Proof. The proof uses the binomial theorem (2.3). □

Proposition 163. *The canonical Continuous Hahn moments have the following representation*

$$\mu_n=\mu_0\sum_{m=0}^{n}(-i)^m E_m(ai,(a+1)i,\dots,(a+n-1)i)\frac{(a+c)_n(a+d)_m}{(a+b+c+d)_m}. \tag{5.151}$$

with

$$\mu_0=\frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}.$$

Proof. Using

$$(a+ix)_n=i^n\prod_{k=0}^{n-1}(x-(a+k)i),$$

we get

$$x^n=\sum_{m=0}^{n}(-i)^m E_m(ai,(a+1)i,\dots,(a+n-1)i)(a+ix)_m.$$

The proposition is proved using the fact that

$$\mu_n((a+ix)_k)=\frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}\frac{(a+c)_n(a+d)_n}{(a+b+c+d)_n}.$$

□

Proposition 164. *The canonical Continuous Hahn moments have the following representation*

$$\mu_n = \mu_0 \sum_{k=0}^{n} \sum_{l=0}^{k} \frac{(-1)^l}{k!} \binom{k}{l} \frac{(a+c)_k (a+d)_k}{(a+b+c+d)_k} ((a+l)i)^n, \tag{5.152}$$

with

$$\mu_0 = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)}.$$

Proof. The result is obtained using relations (2.21), (2.23), (5.36) and (5.148). □

The Dual Hahn polynomials

For $\gamma > -1$ and $\delta > -1$, or for $\gamma < -N$ and $\delta < -N$, the Dual Hahn polynomials $R_n(\lambda(x); \gamma, \delta, N)$ fulfil the following orthogonality relation [30, P. 209]

$$\sum_{x=0}^{N} \frac{(2x+\gamma+\delta+1)(\gamma+1)_x(-N)_x N!}{(-1)^x (x+\gamma+\delta+1)_{N+1}(\delta+1)_x} R_m(\lambda(x); \gamma, \delta, N) R_n(\lambda(x); \gamma, \delta, N)$$
$$= \frac{\delta_{mn}}{\binom{\gamma+n}{n}\binom{\delta+N-n}{N-n}}. \tag{5.153}$$

With $m = n = 0$, it follows that

$$\mu_0 = \frac{1}{\binom{\delta+N}{N}}.$$

From the inversion formula (4.51), we have

$$I_0(n) = \frac{(\gamma+1)_n(-N)_n}{(\gamma+\delta+2)_n}.$$

Therefore, the following proposition is valid.

Proposition 165. *The generalized Dual Hahn moments with respect to the basis $\theta_n(x) = (-x)_n(x+\gamma+\delta+1)_n$ have the representation*

$$\mu_n(\theta_k(x)) = \frac{1}{\binom{\delta+N}{N}} \frac{(\gamma+1)_n(-N)_n}{(\gamma+\delta+2)_n}. \tag{5.154}$$

Proposition 166. *The generalized Dual Hahn moments with respect to $\theta_n(x) = (-x)_n(x+\gamma+\delta+1)_n$ have the following exponential generating function:*

$$\frac{1}{\binom{\delta+N}{N}} {}_2F_1 \left(\left. \begin{matrix} -N, \alpha+1 \\ \gamma+\delta+2 \end{matrix} \right| z \right) = \sum_{n=0}^{\infty} \mu_n \frac{z^n}{n!}. \tag{5.155}$$

Proposition 167. *The canonical Dual Hahn moments have the following representation*

$$\mu_n = \frac{1}{\binom{\delta+N}{N}} \sum_{k=0}^{n} \frac{\mathsf{D}_\varepsilon^k [x(x+\varepsilon)]^n_{|x=0}}{k!} \frac{(\gamma+1)_k(-N)_k}{(\gamma+\delta+2)_k}. \tag{5.156}$$

where $\varepsilon = \gamma + \delta + 1$.

Proof. The result is obtained using relations (2.21), (2.23), (5.32) and (5.154). □

The Meixner-Pollaczek polynomials

The Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x;\phi)$ fulfil the following orthogonality relation [30, P. 213]

$$\int_{-\infty}^{\infty} e^{(2\phi-\pi)x}|\Gamma(\lambda+ix)|^2 P_m^{(\lambda)}(x;\phi)P_n^{(\lambda)}(x;\phi)dx$$
$$= 2\pi\frac{\Gamma(n+2\lambda)}{(2\sin\phi)^{2\lambda}n!}\delta_{mn}, \quad \lambda>0 \quad \text{and} \quad 0<\phi<\pi. \tag{5.157}$$

With $m=n=0$, it follows that

$$\mu_0 = 2\pi\frac{\Gamma(2\lambda)}{(2\sin\phi)^{2\lambda}}.$$

From the inversion formula (4.52), for $\theta_n(x)=(\lambda+ix)_n$, we have

$$I_0(n) = \frac{(2\lambda)_n}{(1-e^{-2i\phi})^n}.$$

Therefore, the following proposition is valid.

Proposition 168. *The generalized Meixner-Pollaczek moments with respect to the basis $(\lambda+ix)_n$ have the representation*

$$\mu_n((\lambda+ix)_k) = 2\pi\frac{\Gamma(2\lambda)}{(2\sin\phi)^{2\lambda}}\frac{(2\lambda)_n}{(1-e^{-2i\phi})^n}. \tag{5.158}$$

Proposition 169. *The generalized Meixner-Pollaczek moments with respect to the basis $(\lambda+ix)_n$ have the following exponential generating function*

$$2\pi\frac{\Gamma(2\lambda)}{(2\sin\phi)^{2\lambda}}\left(1-\frac{z}{1-e^{-2i\phi}}\right)^{-2\lambda} = \sum_{n=0}^{\infty}\mu_n((\lambda+ix)_k)\frac{z^n}{n!}, \quad \left|\frac{z}{1-e^{-2i\phi}}\right|<1. \tag{5.159}$$

Proof. We have

$$\sum_{n=0}^{\infty}\mu_n\frac{z^n}{n!} = 2\pi\frac{\Gamma(2\lambda)}{(2\sin\phi)^{2\lambda}}\sum_{n=0}^{\infty}\frac{(2\lambda)_n}{n!}\left(\frac{z}{1-e^{-2i\theta}}\right)^n = 2\pi\frac{\Gamma(2\lambda)}{(2\sin\phi)^{2\lambda}}{}_1F_0\left(\left.\begin{matrix}2\lambda\\ -\end{matrix}\right|\frac{z}{1-e^{-2i\phi}}\right).$$

Using the binomial theorem (2.3), we get the result. ☐

Proposition 170. *The canonical Meixner-Pollaczek moments have the following representation*

$$\mu_n = 2\pi\frac{\Gamma(2\lambda)}{(2\sin\phi)^{2\lambda}}\sum_{m=0}^{n}(-i)^m E_m(ai,(a+1)i,\ldots,(a+n-1)i)\frac{(2\lambda)_m}{(1-e^{-2i\phi})^m}. \tag{5.160}$$

Proof. The proof is similar to the proof of Proposition 163. ☐

Proposition 171. *The canonical Meixner-Pollaczek moments have the following representation*

$$\mu_n = \frac{2\pi\Gamma(2\lambda)}{(2\sin\phi)^{2\lambda}}\sum_{k=0}^{n}\sum_{l=0}^{k}\frac{(-1)^l}{k!}\binom{k}{l}\frac{(2\lambda)_k((a+l)i)^n}{(1-e^{-2i\phi})^k}. \tag{5.161}$$

Proof. The result is obtained using relations (2.21), (2.23), (5.36) and (5.161). ☐

5.4.5 The q-quadratic case

In this part, since θ will denote an angle, we will denote the basis involved in the inversion formula (5.1) by $\mathcal{B}_n$ instead of θ_n.

The Askey-Wilson polynomials

If a, b, c, d are real, or occur in complex conjugate pairs if complex, and $\max(|a|, |b|, |c|, |d|) < 1$, then the Askey-Wilson polynomials $p_n(x; a, b, c, d|q)$ fulfil the following orthogonality relation

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{w(x)}{\sqrt{1-x^2}}p_n(x;a,b,c,d|q)p_m(x;a,b,c,d|q)dx = h_n\delta_{mn}, \quad x=\cos\theta, \tag{5.162}$$

where

$$w(x) = w(x;a,b,c,d|q) = \left|\frac{(e^{2i\theta};q)_\infty}{(ae^{i\theta},be^{i\theta},ce^{i\theta},de^{i\theta};q)_\infty}\right|^2 = \frac{h(x,1)h(x,q^{1/2})h(x,-1)h(x,-q^{1/2})}{h(x,a)h(x,b)h(x,c)h(x,d)}.$$

with

$$h(x,a) = \prod_{k=0}^{\infty}\left(1-2axq^k+a^2q^{2k}\right) = \left(ae^{i\theta},ae^{-i\theta};q\right)_\infty;$$

and

$$h_n = \frac{(abcdq^{2n};q)_\infty(abcdq^{n-1};q)_n}{(q^{n+1},abq^n,acq^n,adq^n,bcq^n,bdq^n,cdq^n;q)_\infty}.$$

With $m = n = 0$, it follows that

$$\mu_0 = \frac{2\pi(abcd;q)_\infty}{(q;ab,ac,ad,bc,bd,cd;q)_\infty}.$$

From the inversion formula (4.53), for $\mathcal{B}_n(x) = (ae^{i\theta}, ae^{-i\theta}; q)_n$, we have

$$I_0(n) = \frac{(ab,ac,ad;q)_n}{(abcd;q)_n}.$$

Therefore, the following proposition is valid.

Proposition 172. *The generalized Askey-Wilson moments with respect to $\mathcal{B}_n(x) = (ae^{i\theta}, ae^{-i\theta}; q)_n$ have the representation*

$$\mu_n(\mathcal{B}_n(x)) = \frac{2\pi(abcd;q)_\infty}{(q;ab,ac,ad,bc,bd,cd;q)_\infty}\frac{(ab,ac,ad;q)_n}{(abcd;q)_n}. \tag{5.163}$$

Proposition 173. *The generalized Askey-Wilson moments with respect to $\mathcal{B}_n(x) = (ae^{i\theta}, ae^{-i\theta}; q)_n$ have the following generating functions*

$$\frac{2\pi(abcd,abz;q)_\infty}{(q;ab,ac,ad,bc,bd,cd,z;q)_\infty} = \sum_{m=0}^{\infty}\mu_n(\mathcal{B}_n(x))\frac{(abcd;q)_n}{(ac,ad;q)_n}\frac{z^n}{(q;q)_n}, \quad |z|<1, \tag{5.164}$$

$$\frac{2\pi(abcd,acz;q)_\infty}{(q;ab,ac,ad,bc,bd,cd,z;q)_\infty} = \sum_{m=0}^{\infty}\mu_n(\mathcal{B}_n(x))\frac{(abcd;q)_n}{(ab,ad;q)_n}\frac{z^n}{(q;q)_n}, \quad |z|<1, \tag{5.165}$$

$$\frac{2\pi(abcd,adz;q)_\infty}{(q;ab,ac,ad,bc,bd,cd,z;q)_\infty} = \sum_{m=0}^{\infty}\mu_n(\mathcal{B}_n(x))\frac{(abcd;q)_n}{(ab,ac;q)_n}\frac{z^n}{(q;q)_n}, \quad |z|<1. \tag{5.166}$$

Proof. The results are obtained using the q-binomial theorem (2.6). □

Proposition 174. *The canonical Askey-Wilson moments have the following representation*

$$\mu_n = \frac{2\pi(abcd;q)_\infty}{(q;ab,ac,ad,bc,bd,cd;q)_\infty}\sum_{m=0}^{n}(-2a)^{-m}q^{-\binom{m}{2}}E_m(x_0,\ldots,x_{n-1})\frac{(ab,ac,ad;q)_m}{(abcd;q)_m}, \tag{5.167}$$

where the numbers x_k, $k = 0, \ldots, n-1$ are defined by (5.39).

Proposition 175. *The canonical Askey-Wilson moments have the following representation*

$$\mu_n = \frac{2\pi(abcd;q)_\infty}{(q;ab,ac,ad,bc,bd,cd;q)_\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{(ab,ac,ad;q)_k}{(abcd;q)_k}\frac{q^k q^{-j^2}a^{-2j}(aq^j+a^{-1}q^{-j})^k}{(q,q^{1+2j}a^2;q)_{k-j}(q,q^{-1-2j}a^{-2};q)_j}. \tag{5.168}$$

Proof. The result is obtained using relations (2.21), (2.23), (5.43) and (5.163). □

Note that formula (5.168) appears in [12] and the proof is done using a direct computation.

The q-Racah polynomials

The q-Racah polynomials $R_n(\mu(x);\alpha,\beta,\gamma,\delta|q)$ fulfil the following orthogonality relation

$$\sum_{x=0}^{N}\frac{(\alpha q,\beta\delta q,\gamma q,\gamma\delta q;q)_x}{(q,\alpha^{-1}\gamma\delta q,\beta^{-1}\gamma q,\delta q;q)_x}\frac{(1-\gamma\delta q^{2x+1})}{(\alpha\beta q)^x(1-\gamma\delta q)}R_m(\mu(x))R_n(\mu(x)) = h_n\delta_{mn}, \tag{5.169}$$

where

$$R_n(\mu(x)) = R_n(\mu(x);\alpha,\beta,\gamma,\delta|q)$$

and

$$h_n = \frac{\left(\alpha^{-1}\beta^{-1}\gamma,\alpha^{-1}\delta,\beta^{-1},\gamma\delta q^2;q\right)_\infty}{(\alpha^{-1}\beta^{-1}q^{-1},\alpha^{-1}\gamma\delta q,\beta^{-1}\gamma q,\delta q;q)_\infty}\frac{(1-\alpha\beta q)(\gamma\delta q)^n}{(1-\alpha\beta q^{2n+1})}\frac{\left(q,\alpha\beta\gamma^{-1}q,\alpha\delta^{-1}q,\beta q;q\right)_n}{(\alpha q,\alpha\beta q,\beta\delta q,\gamma q;q)_n}.$$

With $m=n=0$, it follows that

$$\mu_0 = \frac{\left(\alpha^{-1}\beta^{-1}\gamma,\alpha^{-1}\delta,\beta^{-1},\gamma\delta q^2;q\right)_\infty}{(\alpha^{-1}\beta^{-1}q^{-1},\alpha^{-1}\gamma\delta q,\beta^{-1}\gamma q,\delta q;q)_\infty}.$$

From the inversion formula (4.54), for $\mathcal{B}_n(\mu(x)) = (q^{-x},\gamma\delta q^{x+1};q)_n$, we have

$$I_0(n) = \frac{(\alpha q,\beta\delta q,\gamma q)_n}{(\alpha\beta q^2;q)_n}.$$

Therefore, the following proposition is valid.

Proposition 176. *The generalized q-Racah moments with respect to the basis $\mathcal{B}_n(\mu(x)) = (q^{-x},\gamma\delta q^{x+1};q)_n$ have the representation*

$$\mu_n(\mathcal{B}_k(\mu(x))) = \frac{\left(\alpha^{-1}\beta^{-1}\gamma,\alpha^{-1}\delta,\beta^{-1},\gamma\delta q^2;q\right)_\infty}{(\alpha^{-1}\beta^{-1}q^{-1},\alpha^{-1}\gamma\delta q,\beta^{-1}\gamma q,\delta q;q)_\infty}\frac{(\alpha q,\beta\delta q,\gamma q)_n}{(\alpha\beta q^2;q)_n}. \tag{5.170}$$

Proposition 177. *The generalized q-Racah moments with respect to $\mathcal{B}_n(\mu(x)) = (q^{-x},\gamma\delta q^{x+1};q)_n$ have the following generating function:*

$$\frac{\left(\alpha^{-1}\beta^{-1}\gamma,\alpha^{-1}\delta,\beta^{-1},\gamma\delta q^2,\gamma qz;q\right)_\infty}{(\alpha^{-1}\beta^{-1}q^{-1},\alpha^{-1}\gamma\delta q,\beta^{-1}\gamma q,\delta q,z;q)_\infty} = \sum_{n=0}^{\infty}\frac{\mu_n(\mathcal{B}_k(\mu(x)))(\alpha\beta q^2;q)_n}{(\alpha q,\beta\delta q;q)_n}\frac{z^n}{(q;q)_n} \tag{5.171}$$

(5.172)

Now, we give the canonical q-Racah moments in terms of the elementary symmetric polynomials of second kind.

Proposition 178. *The canonical q-Racah moments have the following representation*

$$\mu_n = \frac{\left(\alpha^{-1}\beta^{-1}\gamma,\alpha^{-1}\delta,\beta^{-1},\gamma\delta q^2;q\right)_\infty}{(\alpha^{-1}\beta^{-1}q^{-1},\alpha^{-1}\gamma\delta q,\beta^{-1}\gamma q,\delta q;q)_\infty}\sum_{m=0}^{n}(-1)^m q^{-\binom{m}{2}}E_m(g_0,\ldots,g_{n-1})\frac{(\alpha q,\beta\delta q,\gamma q)_m}{(\alpha\beta q^2;q)_m}, \tag{5.173}$$

with

$$g_m = q^{-m}+\gamma\delta q^{m+1},\quad m=0,\ldots,n-1. \tag{5.174}$$

Proof. We first observe that

$$\mathcal{B}_n(\mu(x)) = (q^{-x}, \gamma\delta q^{x+1}; q)_n = (-1)^n q^{\binom{n}{2}} \prod_{k=0}^{n-1} \left(\mu(x) - (q^{-k} + \gamma\delta q^{k+1})\right),$$

where $\mu(x) = q^{-x} + \gamma\delta q^{x+1}$ (see [30, P. 422]). It follows that

$$(\mu(x))^n = \sum_{m=0}^{n} (-1)^m q^{-\binom{m}{2}} E_m(x_0, \ldots, x_{n-1}) \mathcal{B}_m(\mu(x)).$$

The proof of the proposition follows using (5.170). □

The Continuous Dual q-Hahn polynomials

If a, b, c are real or one is real and the other two are complex conjugates, and $\max(|a|, |b|, |c|) < 1$, the Continuous Dual Hahn polynomials $p_n(x; a, b, c|q)$ fulfil the following orthogonality relation [30, P. 429]

$$\frac{1}{2\pi} \int_{-1}^{1} \frac{w(x)}{\sqrt{1-x^2}} p_n(x; a, b, c|q) p_m(x; a, b, c|q) dx = h_n \delta_{mn}, \quad x = \cos\theta, \tag{5.175}$$

where

$$w(x) = w(x; a, b, c|q) = \left| \frac{(e^{2i\theta}; q)_\infty}{(ae^{i\theta}, be^{i\theta}, ce^{i\theta}; q)_\infty} \right|^2 = \frac{h(x,1)h(x,q^{1/2})h(x,-1)h(x,-q^{1/2})}{h(x,a)h(x,b)h(x,c)}.$$

with

$$h(x,a) = \prod_{k=0}^{\infty} \left(1 - 2axq^k + a^2 q^{2k}\right) = \left(ae^{i\theta}, ae^{-i\theta}; q\right)_\infty;$$

and

$$h_n = \frac{1}{(q^{n+1}, abq^n, acq^n, bcq^n; q)_\infty}.$$

With $m = n = 0$, it follows that

$$\mu_0 = \frac{2\pi}{(q, ab, ac, bc; q)_\infty}.$$

From the inversion formula (4.55), for $\mathcal{B}_n(x) = (ae^{i\theta}, ae^{-i\theta}; q)_n$, we have

$$I_0(n) = (ab, ac; q)_n.$$

Therefore, the following proposition is valid.

Proposition 179. *The generalized Continuous Dual q-Hahn moments with respect to the basis $\mathcal{B}_n(x) = (ae^{i\theta}, ae^{-i\theta}; q)_n$ is given by*

$$\mu_n(\mathcal{B}_k(x)) = \frac{2\pi(ab, ac; q)_n}{(q, ab, ac, bc; q)_\infty}. \tag{5.176}$$

Proposition 180. *The generalized Continuous Dual q-Hahn moments with respect to $\mathcal{B}_n(x) = (ae^{i\theta}, ae^{-i\theta}; q)_n$ have the following generating functions*

$$\frac{2\pi(abz; q)_\infty}{(z, q, ab, ac, bc; q)_\infty} = \sum_{n=0}^{\infty} \mu_n(\mathcal{B}_k(x)) \frac{z^n}{(ac, q; q)_n} \tag{5.177}$$

$$\frac{2\pi(acz; q)_\infty}{(z, q, ab, ac, bc; q)_\infty} = \sum_{n=0}^{\infty} \mu_n(\mathcal{B}_k(x)) \frac{z^n}{(ab, q; q)_n}. \tag{5.178}$$

Proof. The results are obtained using the q-binomial theorem (2.6). □

Proposition 181. *The canonical Continuous Dual q-Hahn moments have the following representation*

$$\mu_n = \frac{2\pi}{(q, ab, ac, bc; q)_\infty} \sum_{m=0}^{n} (-1)^m q^{-\binom{m}{2}} E_m(x_0, \ldots, x_{n-1})(ab, ac; q)_m, \tag{5.179}$$

where the numbers x_k are defined by (5.39).

Proof. The proof is similar to the proof of (5.167). □

Proposition 182. *The canonical Continuous Dual q-Hahn moments have the following representation*

$$\mu_n = \frac{2\pi}{(q, ab, ac, bc; q)_\infty} \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{q^k q^{-j^2} a^{-2j} (aq^j + a^{-1}q^{-j})^k}{(q, q^{1+2j}a^2; q)_{k-j} (q, q^{-1-2j}a^{-2}; q)_j} (ab, ac; q)_k. \tag{5.180}$$

Proof. The result is obtained using relations (2.21), (2.23), (5.43) and (5.176). □

The Continuous q-Hahn polynomials

If $c = a$ and $d = b$, if a and b are real and $\max(|a|, |b|) < 1$, or if $b = \bar{a}$ and $|a| < 1$, then the Continuous q-Hahn polynomials $p_n(\cos(\theta + \phi); a, b, c, d|q)$ fulfil the following orthogonality relation

$$\frac{1}{4\pi} \int_{-\pi}^{\pi} w(\cos(\theta+\phi)) p_n(\cos(\theta+\phi); a, b, c, d|q) p_m(\cos(\theta+\phi); a, b, c, d|q) dx = h_n \delta_{mn}, \tag{5.181}$$

where

$$\begin{aligned} w(x) = w(x; a, b, c, d|q) &= \left| \frac{\left(e^{2i(\theta+\phi))}; q\right)_\infty}{\left(ae^{i(\theta+2\phi)}, be^{i(\theta+2\phi)}, ce^{i\theta}, de^{i\theta}; q\right)_\infty} \right|^2 \\ &= \frac{h(x,1)h(x,q^{1/2})h(x,-1)h(x,-q^{1/2})}{h(x,ae^{i\phi})h(x,be^{i\phi})h(x,ce^{-i\phi})h(x,de^{-i\phi})}. \end{aligned}$$

with $\quad x = \cos(\theta + \phi)$,

$$h(x, a) = \prod_{k=0}^{\infty} \left(1 - 2axq^k + a^2 q^{2k}\right) = \left(ae^{i(\theta+\phi)}, ae^{-i(\theta+\phi)}; q\right)_\infty;$$

and

$$h_n = \frac{(abcdq^{2n}; q)_\infty (abcdq^{n-1}; q)_n}{(q^{n+1}, abq^n e^{2i\phi}, acq^n, adq^n, bcq^n, bdq^n, cdq^n e^{-2i\phi}; q)_\infty}.$$

With $m = n = 0$, It follows that

$$\mu_0 = \frac{4\pi (abcd; q)_\infty}{(q, abe^{2i\phi}, ac, ad, bc, bd, cde^{-2i\phi}; q)_\infty}.$$

From the inversion formula (4.56), with $\mathcal{B}_n(x) = (ae^{i(\theta+2\phi)}, ae^{-i\theta}; q)_n$, we have

$$I_0(n) = \frac{(abe^{2i\phi}, ac, ad; q)_n}{(abcd; q)_n}.$$

Therefore, the following proposition is valid.

Proposition 183. *The generalized Continuous q-Hahn moments with respect to the basis $\mathcal{B}_n(x) = (ae^{i(\theta+2\phi)}, ae^{-i\theta}; q)_n$ have the representation*

$$\mu_n(\mathcal{B}_k(x)) = \frac{4\pi (abcd; q)_\infty}{(q, abe^{2i\phi}, ac, ad, bc, bd, cde^{-2i\phi}; q)_\infty} \frac{(abe^{2i\phi}, ac, ad; q)_n}{(abcd; q)_n}. \tag{5.182}$$

Proposition 184. *The canonical Continuous q-Hahn moments have the representation*

$$\mu_n = \mu_0 \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{(abe^{2i\phi}, ac, ad;q)_k}{(abcd;q)_k} \frac{q^k q^{-j^2} a^{-2j} e^{2i\phi}(ae^{-i\phi}q^j + a^{-1}e^{i\phi}q^{-j})^k}{(q,q^{1+2j}a^2e^{-2i\phi};q)_{k-j}(q,q^{-1-2j}a^{-2}e^{2i\phi};q)_j}, \quad (5.183)$$

where

$$\mu_0 = \frac{4\pi(abcd;q)_\infty}{(q, abe^{2i\phi}, ac, ad, bc, bd, cde^{-2i\phi};q)_\infty}.$$

Proof. From (5.43), we have

$$x^n = \sum_{k=0}^{n} q^k \sum_{j=0}^{k} \frac{q^{-j^2}a^{-2j}(aq^j + a^{-1}q^{-j})^k}{(q,q^{1+2j}a^2;q)_{k-j}(q,q^{-1-2j}a^{-2};q)_j}(ae^{i\theta}, ae^{-i\theta};q)_k.$$

Replacing θ by $\theta+\phi$, it follows that

$$x^n = \sum_{k=0}^{n} q^k \sum_{j=0}^{k} \frac{q^{-j^2}a^{-2j}(aq^j + a^{-1}q^{-j})^k}{(q,q^{1+2j}a^2;q)_{k-j}(q,q^{-1-2j}a^{-2};q)_j}(ae^{i(\theta+\phi)}, ae^{-i(\theta+\phi)};q)_k.$$

Next taking $a = ae^{-i\phi}$ we get

$$x^n = \sum_{k=0}^{n} q^k \sum_{j=0}^{k} \frac{q^{-j^2}a^{-2j}e^{2i\phi}(ae^{-i\phi}q^j + a^{-1}e^{i\phi}q^{-j})^k}{(q,q^{1+2j}a^2e^{-2i\phi};q)_{k-j}(q,q^{-1-2j}a^{-2}e^{2i\phi};q)_j}(ae^{i(\theta+2\phi)}, ae^{-i\theta};q)_k. \quad (5.184)$$

(5.183) is obtained using relations (2.21), (2.23), (5.182) and (5.184). □

The Dual q-Hahn polynomials

For $0 < \gamma q < 1$ and $0 < \delta q < 1$, or for $\gamma > q - N$ and $\delta > q - N$, the Dual q-Hahn polynomials $R_n(\mu(x);\gamma,\delta,N|q)$ fulfil the following orthogonality relation [30, P. 451]

$$\sum_{x=0}^{N} \frac{(\gamma q, \gamma\delta q, q^{-N};q)_x}{(q,\gamma\delta q^{N+2},\delta q;q)_x} \frac{(1-\gamma\delta q^{2x+1})}{(1-\gamma\delta q)(-\gamma q)^x} q^{Nx-\binom{x}{2}} R_n(\mu(x);\gamma,\delta,N|q) R_m(\mu(x);\gamma,\delta,N|q) = h_n\delta_{mn}, \quad (5.185)$$

with

$$h_n = \frac{(\gamma\delta q^2;q)_N}{(\gamma q;q)_N}(\gamma q)^{-N}\frac{(q,\delta^{-1}q^{-N};q)_n}{(\gamma q, q^{-N};q)_n}(\gamma\delta q)^n.$$

For $m = n = 0$, it follows that

$$\mu_0 = \frac{(\gamma\delta q^2;q)_N}{(\gamma q;q)_N}(\gamma q)^{-N}.$$

From the inversion formula (4.57), for $\mathcal{B}_n(\mu(x)) = (q^{-x}, \gamma\delta q^{x+1};q)_n$, we have

$$I_0(n) = (\gamma q, q^{-N};q)_n.$$

Therefore, the following proposition is valid

Proposition 185. *The generalized Dual q-Hahn moments with respect to the basis $\mathcal{B}_n(\mu(x)) = (q^{-x}, \gamma\delta q^{x+1};q)_n$ have the representation*

$$\mu_n(\mathcal{B}_k(\mu(x))) = (\gamma q)^{-N}\frac{(\gamma\delta q^2;q)_N}{(\gamma q;q)_N}(\gamma q, q^{-N};q)_n. \quad (5.186)$$

Proposition 186. *The generalized Dual q-Hahn moments with respect to $\mathcal{B}_n(\mu(x)) = (q^{-x}, \gamma\delta q^{x+1};q)_n$ have the following generating function:*

$$(\gamma q)^{-N}\frac{(\gamma\delta q^2;q)_N}{(\gamma q;q)_N}\frac{(zq^{-N};q)_\infty}{(z;q)_\infty} = \sum_{n=0}^{\infty}\mu_n(\mathcal{B}_k(\mu(x)))\frac{z^n}{(\gamma q, q;q)_n}. \quad (5.187)$$

Proof. Using the q-binomial theorem (2.6) we get the result. □

Proposition 187. *The canonical Dual q-Hahn moments have the following representation*

$$\mu_n = (\gamma q)^{-N}\frac{(\gamma\delta q^2;q)_N}{(\gamma q;q)_N}\sum_{m=0}^{n}(-1)^m q^{-\binom{m}{2}}E_m(g_0,\ldots,g_{n-1})(\gamma q,q^{-N};q)_m, \tag{5.188}$$

with the numbers g_k defined by (5.174).

Proof. The proof is similar to the proof of (5.173). □

The Al-Salam-Chihara polynomials

If a and b are real or complex conjugates and $\max(|a|,|b|) < 1$, then the Al-Salam-Chihara polynomials fulfil the following orthogonality relation [30, P. 455]

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{w(x)}{\sqrt{1-x^2}}Q_n(x;a,b|q)Q_m(x;a,b|q)dx = \frac{\delta_{mn}}{(q^{n+1},abq^n;q)_n}, \quad x=\cos\theta, \tag{5.189}$$

where

$$w(x) := w(x;a,b|q) = \left|\frac{(e^{2i\theta};q)_\infty}{(ae^{i\theta},ae^{-i\theta};q)_\infty}\right|^2 = \frac{h(x,1)h(x,q^{1/2})h(x,-1)h(x,-q^{1/2})}{h(x,a)h(x,b)},$$

with

$$h(x,a) = \prod_{k=0}^{\infty}\left(1-2axq^k+a^2q^{2k}\right) = \left(ae^{i\theta},ae^{-i\theta};q\right)_\infty.$$

For $m=n=0$, it follows that

$$\mu_0 = \frac{2\pi}{(q,ab;q)_\infty}.$$

From the inversion formula (4.58), for $\mathcal{B}_n(x) = (ae^{i\theta},ae^{-i\theta};q)_n$, we have

$$I_0(n) = (ab;q)_n.$$

Therefore, the following proposition is valid.

Proposition 188. *The generalized Al-Salam-Chihara moments with respect to the basis $\mathcal{B}_n(x) = (ae^{i\theta},ae^{-i\theta};q)_n$ are given by*

$$\mu_n(\mathcal{B}_k(x)) = \frac{2\pi(ab;q)_n}{(q,ab;q)_\infty}. \tag{5.190}$$

Proposition 189. *The generalized Al-Salam-Chihara moments with respect to $\mathcal{B}_n(x) = (ae^{i\theta},ae^{-i\theta};q)_n$ have the following q-exponential generating function:*

$$\frac{2\pi(abz;q)_\infty}{(z,ab,q;q)_\infty} = \sum_{n=0}^{\infty}\mu_n(\mathcal{B}_k(x))\frac{z^n}{(q;q)_n}. \tag{5.191}$$

Proof. Using the relation (5.190), we have

$$\sum_{n=0}^{\infty}\mu_n\frac{z^n}{(q;q)_n} = \frac{2\pi}{(q,qb;q)_\infty}\sum_{n=0}^{\infty}\frac{(ab;q)_n}{(q;q)_n}z^n.$$

By the q-binomial theorem (2.6), we get the result. □

Proposition 190. *The canonical Al-Salam-Chihara moments have the following representation*

$$\mu_n = \frac{2\pi}{(q,ab;q)_\infty}\sum_{m=0}^{n}(-2a)^{-m}q^{-\binom{m}{2}}E_m(x_0,\ldots,x_{n-1})(ab;q)_m, \tag{5.192}$$

with x_k defined by

$$x_k = \frac{1+a^2q^{2k}}{2aq^k}, \quad k=0,1,\ldots,n-1.$$

.

Proof. The proof is similar to the proof of (5.167). □

Proposition 191. *The canonical Al-Salam-Chihara moments have the following representation*

$$\mu_n = \frac{2\pi}{(q,ab;q)_\infty} \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{q^k q^{-j^2} a^{-2j} (aq^j + a^{-1}q^{-j})^k}{(q,q^{1+2j}a^2;q)_{k-j}(q,q^{-1-2j}a^{-2};q)_j}(ab;q)_k. \tag{5.193}$$

Proof. The result is obtained using relations (2.21), (2.23), (5.43) and (5.190). □

The q-Meixner-Pollaczek polynomials

The q-Meixner-Pollaczek polynomials $P_n(\cos(\theta+\phi);a|q)$ fulfil the following orthogonality relation [30, P. 460]

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} w(\cos(\theta+\phi))P_n(\cos(\theta+\phi);a|q)P_m(\cos(\theta+\phi);a|q)dx = \frac{\delta_{mn}}{(q;q)_n(q,a^2q^n;q)_\infty}, \quad 0<a<1, \tag{5.194}$$

where

$$w(x;a|q) = \left|\frac{(e^{2i(\theta+\phi)};q)_\infty}{(ae^{i(\theta+2\phi)},ae^{i\theta};q)_\infty}\right|^2 = \frac{h(x,1)h(x,q^{1/2})h(x,-1)h(x,-q^{1/2})}{h(x,ae^{i\phi})h(x,ae^{-i\phi})},$$

with

$$h(x,a) = \prod_{k=0}^{\infty}\left(1-2axq^k+a^2q^{2k}\right) = \left(ae^{i(\theta+\phi)},ae^{-i(\theta+\phi)};q\right)_\infty, \quad x=\cos(\theta+\phi).$$

With $m=n=0$, it follows that

$$\mu_0 = \frac{2\pi}{(a^2,q;q)_\infty}.$$

From the inversion formula (4.59), for $\mathcal{B}_n(x) = (ae^{i(\theta+2\phi)},ae^{-i\theta};q)_n$ we have

$$I_0(n) = (a^2,q;q)_n.$$

Therefore, the following proposition is valid.

Proposition 192. *The generalized q-Meixner-Pollaczek moments with respect to the basis* $\mathcal{B}_n(x) = (ae^{i(\theta+2\phi)},ae^{-i\theta};q)_n$ *have the representation*

$$\mu_n(\mathcal{B}_k(x)) = 2\pi\frac{(a^2,q;q)_n}{(a^2,q;q)_\infty}. \tag{5.195}$$

Proposition 193. *The generalized q-Meixner-Pollaczek moments with respect to* $(ae^{i(\vartheta+2\phi)},ae^{-i\vartheta};q)_n$ *have the following generating function*

$$\frac{2\pi(a^2z;q)_\infty}{(z,a^2,q;q)_\infty} = \sum_{n=0}^{\infty}\mu_n(\mathcal{B}_k(x))\frac{z^n}{(q,q;q)_n} \tag{5.196}$$

Proof. The proof of (5.196) uses the q-binomial theorem (2.6). □

Proposition 194. *The canonical q-Meixner-Pollaczek moments have the following representation*

$$\mu_n = \frac{2\pi}{(a^2,q;q)_\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{q^k q^{-j^2}a^{-2j}e^{2i\phi}(ae^{-i\phi}q^j + a^{-1}e^{i\phi}q^{-j})^k}{(q,q^{1+2j}a^2e^{-2i\phi};q)_{k-j}(q,q^{-1-2j}a^{-2}e^{2i\phi};q)_j}(a^2,q;q)_k. \tag{5.197}$$

Proof. The result is obtained using relations (2.21), (2.23), (5.184) and (5.195).

□

The Continuous q-Jacobi polynomials

For $\alpha > -\frac{1}{2}$ and $\beta > -\frac{1}{2}$ the Continuous q-Jacobi polynomials $P_n^{(\alpha,\beta)}(x|q)$ fulfil the orthogonality relation [30, P. 464]

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{w(x)}{\sqrt{1-x^2}}P_n^{(\alpha,\beta)}(x|q)P_m^{(\alpha,\beta)}(x|q)dx = h_n\delta_{mn},$$

where

$$h_n = \frac{(q^{\frac{1}{2}(\alpha+\beta+2)},q^{\frac{1}{2}(\alpha+\beta+3)};q)_\infty}{(q,q^{\alpha+1},q^{\beta+1},-q^{\frac{1}{2}(\alpha+\beta+1)},-q^{\frac{1}{2}(\alpha+\beta+2)};q)_\infty}\frac{1-q^{\alpha+\beta+1}}{1-q^{2n+\alpha+\beta+1}}\frac{(q^{\alpha+1},q^{\beta+1},-q^{\frac{1}{2}(\alpha+\beta+3;q)_n})}{(q,q^{\alpha+\beta+1},-q^{\frac{1}{2}(\alpha+\beta+1)};q)_n}q^{(\alpha+\frac{1}{2})n},$$

$$w(x) = w(x;q^\alpha,q^\beta|q) = \frac{h(x,1)h(x,q^{1/2})h(x,-1)h(x,-q^{1/2})}{h(x,q^{\frac{1}{2}\alpha+\frac{1}{4}})h(x,q^{\frac{1}{2}\alpha+\frac{3}{4}})h(x,-q^{\frac{1}{2}\beta+\frac{1}{4}})h(x,-q^{\frac{1}{2}\beta+\frac{3}{4}})},$$

with

$$h(x,a) = \prod_{k=0}^{\infty}\left(1-2axq^k+a^2q^{2k}\right) = \left(ae^{i\theta},ae^{-i\theta};q\right)_\infty, \quad x=\cos\theta.$$

With $m=n=0$, it follows that

$$\mu_0 = 2\pi\frac{(q^{\frac{1}{2}(\alpha+\beta+2)},q^{\frac{1}{2}(\alpha+\beta+3)};q)_\infty}{(q,q^{\alpha+1},q^{\beta+1},-q^{\frac{1}{2}(\alpha+\beta+1)},-q^{\frac{1}{2}(\alpha+\beta+2)};q)_\infty}.$$

From the inversion formula (4.60), for $\mathcal{B}_n(x) = (q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta},q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta};q)_n$, we have

$$I_0(n) = \frac{(q^{\alpha+1},-q^{\frac{1}{2}(\alpha+\beta+1)},-q^{\frac{1}{2}(\alpha+\beta+2)};q)_n}{(q^{\alpha+\beta+1};q)_n}.$$

Therefore, the following proposition is valid.

Proposition 195. *The generalized Continuous q-Jacobi moments with respect to the basis $\mathcal{B}_n(x) = (q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta},q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta};q)_n$ have the representation*

$$\mu_n(\mathcal{B}_k(x)) = \frac{2\pi(q^{\frac{1}{2}(\alpha+\beta+2)},q^{\frac{1}{2}(\alpha+\beta+3)};q)_\infty}{(q,q^{\alpha+1},q^{\beta+1},-q^{\frac{1}{2}(\alpha+\beta+1)},-q^{\frac{1}{2}(\alpha+\beta+2)};q)_\infty}\frac{(q^{\alpha+1},-q^{\frac{1}{2}(\alpha+\beta+1)},-q^{\frac{1}{2}(\alpha+\beta+2)};q)_n}{(q^{\alpha+\beta+1};q)_n}. \tag{5.198}$$

Proposition 196. *The canonical Continuous q-Jacobi moments have the following representation*

$$\begin{aligned}\mu_n = &\frac{2\pi(q^{\frac{1}{2}(\alpha+\beta+2)},q^{\frac{1}{2}(\alpha+\beta+3)};q)_\infty}{(q,q^{\alpha+1},q^{\beta+1},-q^{\frac{1}{2}(\alpha+\beta+1)},-q^{\frac{1}{2}(\alpha+\beta+2)};q)_\infty}\\ &\sum_{m=0}^{n}\left(-2q^{\frac{1}{2}\alpha+\frac{1}{4}}\right)^{-m}E_m(x_0,\ldots,x_{n-1})\frac{(q^{\alpha+1},-q^{\frac{1}{2}(\alpha+\beta+1)},-q^{\frac{1}{2}(\alpha+\beta+2)};q)_m}{(q^{\alpha+\beta+1};q)_m}\end{aligned} \tag{5.199}$$

where the numbers x_k are given by (5.39).

Proof. The proof is similar to the proof of (5.167) where we take $a = q^{\frac{1}{2}\alpha+\frac{1}{4}}$. □

Proposition 197. *The canonical Continuous q-Jacobi moments have the following representation*

$$\begin{aligned}\mu_n = &\frac{2\pi(q^{\frac{1}{2}(\alpha+\beta+2)},q^{\frac{1}{2}(\alpha+\beta+3)};q)_\infty}{(q,q^{\alpha+1},q^{\beta+1},-q^{\frac{1}{2}(\alpha+\beta+1)},-q^{\frac{1}{2}(\alpha+\beta+2)};q)_\infty}\\ &\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{q^{k-j^2-(\alpha+\frac{1}{2})j}(q^{j+\frac{\alpha}{2}+\frac{1}{4}}+q^{-j-\frac{\alpha}{2}-\frac{1}{4}})^k}{(q,q^{2j+\alpha+\frac{3}{2}};q)_{k-j}(q,q^{-2j-\alpha-\frac{3}{2}};q)_j}\frac{(q^{\alpha+1},-q^{\frac{1}{2}(\alpha+\beta+1)},-q^{\frac{1}{2}(\alpha+\beta+2)};q)_k}{(q^{\alpha+\beta+1};q)_k}.\end{aligned} \tag{5.200}$$

Proof. The result is obtained using relations (2.21), (2.23), (5.43) with $a = q^{\frac{1}{2}\alpha+\frac{1}{4}}$ and (5.198). □

The Continuous q-Ultraspherical (Rogers) polynomials

The generalized Continuous q-Ultraspherical polynomials $C_n(x;\beta|q)$ fulfil the following orthogonality relation [30, P. 469]

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{w(x)}{\sqrt{1-x^2}}C_n(x;\beta|q)C_m(x;\beta|q)dx=\frac{(\beta,\beta q,q)_\infty}{(\beta^2,q;q)_\infty}\frac{(\beta^2;q)_n}{(q;q)_n}\frac{1-\beta}{1-\beta q^n}\delta_{mn},\quad |\beta|<1,$$

where

$$w(x)=\left|\frac{(e^{2i\theta};q)_\infty}{(\beta e^{2i\theta}q)_\infty}\right|^2=\frac{h(x,1)h(x,q^{1/2})h(x,-1)h(x,-q^{1/2})}{h(x,\beta^{\frac12})h(x,\beta^{\frac12}q^{\frac12})h(x,-\beta^{\frac12})h(x,-\beta^{\frac12}q^{\frac12})}$$

with

$$h(x,a)=\prod_{k=0}^{\infty}\left(1-2axq^k+a^2q^{2k}\right)=\left(ae^{i\theta},ae^{-i\theta};q\right)_\infty,\quad x=\cos\theta.$$

For $m=n=0$, it follows that

$$\mu_0=2\pi\frac{(\beta,\beta q;q)_\infty}{(\beta^2,q;q)_\infty}.$$

From the inversion formula (4.61), for $\mathcal{B}_n(x)=(\beta^{\frac12}e^{i\theta},\beta^{\frac12}e^{-i\theta};q)_n$, we have

$$I_0(n)=\frac{(\beta q^{\frac12},-\beta,-\beta q^{\frac12};q)_n}{(\beta^2q;q)_n}.$$

Therefore, the following proposition is valid.

Proposition 198. *The generalized Continuous q-Ultraspherical moments with respect to the basis $\mathcal{B}_n(x)=(\beta^{\frac12}e^{i\theta},\beta^{\frac12}e^{-i\theta};q)_n$ have the representation*

$$\mu_n(\mathcal{B}_k(x))=2\pi\frac{(\beta,\beta q;q)_\infty}{(\beta^2,q;q)_\infty}\frac{(\beta q^{\frac12},-\beta,-\beta q^{\frac12};q)_n}{(\beta^2q;q)_n}.\tag{5.201}$$

Proposition 199. *The canonical Continuous q-Ultraspherical moments have the representation*

$$\mu_n=2\pi\frac{(\beta,\beta q;q)_\infty}{(\beta^2,q;q)_\infty}\sum_{m=0}^{n}\left(-2q^{\frac12}\right)^{-m}q^{-\binom{m}{2}}E_m(x_0,\dots,x_{n-1})\frac{(\beta q^{\frac12},-\beta,-\beta q^{\frac12};q)_m}{(\beta^2q;q)_m},\tag{5.202}$$

where the numbers x_k are given by (5.39).

Proof. The proof is similar to the proof of (5.167) where we take $a=\beta q^{\frac12}$. □

Proposition 200. *The canonical Continuous q-Ultraspherical moments have the representation*

$$\mu_n=2\pi\frac{(\beta,\beta q;q)_\infty}{(\beta^2,q;q)_\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{(\beta q^{\frac12},-\beta,-\beta q^{\frac12};q)_k}{(\beta^2q;q)_k}\frac{q^k\beta^{-j}q^{-j^2}(\beta^{\frac12}q^j+\beta^{-\frac12}q^{-j})^k}{(q,\beta q^{1+2j};q)_{k-j}(q,\beta^{-1}q^{-1-2j};q)_j}\tag{5.203}$$

Proof. The result is obtained using relations (2.21), (2.23), (5.43) with $a=\beta q^{\frac12}$ and (5.201). □

The Continuous q-Legendre polynomials

The continuous q-Legendre polynomials $P_n(x|q)$ fulfil the following orthogonality relation [30, P. 475]

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{w(x;1|q)}{\sqrt{1-x^2}}P_n(x|q)P_m(x|q)dx=\frac{(q^{\frac12};q)_\infty}{(q,q,-q^{\frac12},-q;q)_\infty}\frac{q^{\frac12 n}}{1-q^{n+\frac12}}\delta_{mn}$$

where

$$w(x;a|q) = \left| \frac{(e^{2i\theta};q)_\infty}{(a^2 q^{\frac{1}{2}} e^{2i\theta};q)_\infty} \right|^2 = \frac{h(x,1)h(x,q^{1/2})h(x,-1)h(x,-q^{1/2})}{h(x,a^q \frac{1}{4})h(x,aq^{\frac{3}{4}}q^{\frac{1}{2}})h(x,-\beta^{\frac{1}{2}})h(x,-\beta^{\frac{1}{2}}q^{\frac{1}{2}})}$$

with

$$h(x,a) = \prod_{k=0}^{\infty} \left(1 - 2axq^k + a^2 q^{2k}\right) = \left(ae^{i\theta}, ae^{-i\theta}; q\right)_\infty, \quad x = \cos\theta.$$

With $m = n = 0$, it follows that

$$\mu_0 = \frac{2\pi}{1-q^{\frac{1}{2}}} \frac{(q^{\frac{1}{2}};q)_\infty}{(q,q,-q^{\frac{1}{2}},-q;q)_\infty}$$

From the inversion formula (4.62), for $\mathcal{B}_n(x) = (q^{\frac{1}{4}}e^{i\theta}, q^{\frac{1}{4}}e^{-i\theta};q)_n$, we have

$$I_0(n) = \frac{(-q^{\frac{1}{2}},-q;q)_n}{(q^2;q)_n}.$$

Therefore, the following proposition is valid.

Proposition 201. *The generalized Continuous q-Legendre moments with respect to the basis $\mathcal{B}_n(x) = (q^{\frac{1}{4}}e^{i\theta}, q^{\frac{1}{4}}e^{-i\theta};q)_n$ have the representation*

$$\mu_n(\mathcal{B}_k(x)) = \frac{2\pi}{1-q^{\frac{1}{2}}} \frac{(q^{\frac{1}{2}};q)_\infty}{(q,q,-q^{\frac{1}{2}},-q;q)_\infty} \frac{(-q^{\frac{1}{2}},-q;q)_n}{(q^2;q)_n}. \tag{5.204}$$

Proposition 202. *The canonical Continuous q-Legendre moments have the representation*

$$\mu_n = \frac{2\pi}{1-q^{\frac{1}{2}}} \frac{(q^{\frac{1}{2}};q)_\infty}{(q,q,-q^{\frac{1}{2}},-q;q)_\infty} \sum_{m=0}^{n} \left(-2q^{\frac{1}{4}}\right)^{-m} q^{-\binom{m}{2}} E_m(x_0,\ldots,x_{n-1}) \frac{(-q^{\frac{1}{2}},-q;q)_m}{(q^2;q)_m}, \tag{5.205}$$

where the numbers x_k are given by (5.39).

Proof. The proof is similar to the proof of (5.167) where we take $a = q^{\frac{1}{4}}$. □

Proposition 203. *The canonical Continuous q-Legendre moments have the representation*

$$\mu_n = \frac{2\pi}{1-q^{\frac{1}{2}}} \frac{(q^{\frac{1}{2}};q)_\infty}{(q,q,-q^{\frac{1}{2}},-q;q)_\infty} \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{(-q^{\frac{1}{2}},-q;q)_k}{(q^2;q)_k} \frac{q^k q^{-j^2} q^{-\frac{j}{2}} (q^{j+\frac{1}{4}} + q^{-j-\frac{1}{4}})^k}{(q,q^{2j+\frac{3}{2}};q)_{k-j} (q,q^{-2j-\frac{3}{2}};q)_j}. \tag{5.206}$$

Proof. The result is obtained using relations (2.21), (2.23), (5.43) with $a = q^{\frac{1}{2}}$ and (5.204). □

The Dual q-Krawtchouk polynomials

The Dual q-Krawtchouk polynomials $K_m(\lambda(x);c,N|q)$ fulfil the following orthogonality relation [30, P. 505]

$$\begin{aligned}\sum_{x=0}^{N} \frac{(cq^{-N},q^{-N};q)_x}{(q,cq;q)_x} \frac{(1-cq^{2x-N})}{(1-cq^{-N})} c^{-x} q^{x(2N-x)} K_m(\lambda(x)) K_n(\lambda(x)) \\ = (c^{-1};q)_N \frac{(q;q)_n}{(q^{-N};q)_n} (cq^{-N})^n \delta_{mn}, \quad c<0,\end{aligned} \tag{5.207}$$

where

$$K_m(\lambda(x)) := K_m(\lambda(x);c,N|q), \quad \lambda(x) = q^{-x} + cq^{x-N}.$$

For $m = n = 0$, it follows that

$$\mu_0 = (c^{-1};q)_N.$$

From the inversion formula (4.63), with $\mathcal{B}_n(\lambda(x)) = (q^{-x}, cq^{x-N}; q)_n$, we have

$$I_0(n) = (q^{-N}; q)_n.$$

Therefore, the following proposition is valid.

Proposition 204. *The generalized Dual q-Krawtchouk moments with respect to the basis $\mathcal{B}_n(\lambda(x)) = (q^{-x}, cq^{x-N}; q)_n$ have the representation*

$$\mu_n(\mathcal{B}_k(\lambda(x))) = (c^{-1}; q)_N (q^{-N}; q)_n. \tag{5.208}$$

Proposition 205. *The generalized Dual q-Krawtchouk moments with respect to $\mathcal{B}_n(\lambda(x)) = (q^{-x}, cq^{x-N}; q)_n$ have the following q-exponential generating function.*

$$(c^{-1}; q)_N \frac{(zq^{-N}; q)_\infty}{(z; q)_\infty} = \sum_{n=0}^{\infty} \mu_n(\mathcal{B}_k(\lambda(x))) \frac{z^n}{(q; q)_n}. \tag{5.209}$$

Proof. We have, by the q-binomial theorem (2.6):

$$\sum_{n=0}^{\infty} \mu_n \frac{z^n}{(q;q)_n} = (c^{-1}; q)_N \sum_{n=0}^{\infty} \frac{(q^{-N}; q)_n}{(q;q)_n} z^n = (c^{-1}; q)_N \frac{(zq^{-N}; q)_\infty}{(z; q)_\infty}.$$

□

Proposition 206. *The canonical Dual q-Krawtchouk moments have the representation*

$$\mu_n = (c^{-1}; q)_N \sum_{m=0}^{n} (-1)^m q^{-\binom{m}{2}} E_m(\ell_0, \ldots, \ell_{n-1}) (q^{-N}; q)_m, \tag{5.210}$$

where

$$\ell_k = q^{-k} + cq^{-N} q^k, \quad k = 0, \ldots, n-1. \tag{5.211}$$

Proof. First, we remark that

$$\mathcal{B}_n(\lambda(x)) = (q^{-x}, cq^{x-N}; q)_n = (-1)^n q^{\binom{n}{2}} \prod_{k=0}^{n-1} \left(\lambda(x) - (q^{-k} + cq^{-N} q^k) \right).$$

This implies

$$\lambda(x)^n = \sum_{m=0}^{n} (-1)^m q^{-\binom{m}{2}} E_m(\ell_0, \ldots, \ell_{n-1}) \mathcal{B}_m(\lambda(x)).$$

Therefore, the proposition follows. □

The Continuous Big q-Hermite polynomials

The Continuous big q-Hermite polynomials $H_n(x; a|q)$ fulfil the following orthogonality relation [30, P. 510]

$$\frac{1}{2\pi} \int_{-1}^{1} \frac{w(x)}{\sqrt{1-x^2}} H_m(x; a|q) H_n(x; a|q) dx = \frac{\delta_{mn}}{(q^{n+1}; q)_\infty}, \tag{5.212}$$

where

$$w(x) := w(x; a|q) = \left| \frac{(e^{2i\theta}; q)_\infty}{(ae^{i\theta}; q)_\infty} \right|^2 = \frac{h(x,1) h(x,-1) h(x, q^{\frac{1}{2}}) h(x, -q^{\frac{1}{2}})}{h(x, a)},$$

with

$$h(x, \alpha) := \prod_{k=0}^{\infty} \left(1 - 2\alpha x q^k + \alpha^2 q^{2k} \right) = \left(\alpha e^{i\theta}, \alpha e^{-i\theta}; q \right)_\infty, \quad x = \cos\theta.$$

With $m = n = 0$, it follows that

$$\mu_0 = \frac{2\pi}{(q;q)_\infty}.$$

From the inversion formula (4.64), for $\mathcal{B}_n(x) = (ae^{i\theta}, ae^{-i\theta}; q)_n$, we have

$$I_0(n) = 1.$$

Therefore, the following proposition is valid.

Proposition 207. *The generalized Continuous Big q-Hermite moments with respect to the basis $\mathcal{B}_n(x) = (ae^{i\theta}, ae^{-i\theta}; q)_n$ have the representation*

$$\mu_n(\mathcal{B}_k(x)) = \frac{2\pi}{(q;q)_\infty}. \tag{5.213}$$

Proposition 208. *The generalized Continuous Big q-Hermite moments have the following generating function:*

$$\frac{2\pi}{(q;q)_\infty}\frac{1}{1-z} = \sum_{n=0}^{\infty} \mu_n(\mathcal{B}_k(x)) z^n, \quad |z| < 1, \tag{5.214}$$

$$\frac{2\pi}{(z,q;q)_\infty} = \sum_{n=0}^{\infty} \mu_n(\mathcal{B}_k(x)) \frac{z^n}{(q;q)_n}. \tag{5.215}$$

Proof. The proof of (5.214) follows from the binomial theorem (2.3) and the proof of (5.215) follows from the q-binomial theorem (2.6). □

Proposition 209. *The canonical Continuous q-Hermite moments have the representation*

$$\mu_n = \frac{2\pi}{(q;q)_\infty} \sum_{m=0}^{n} (-2a)^{-m} q^{-\binom{m}{2}} E_m(x_0, \ldots, x_{n-1}), \tag{5.216}$$

where the numbers x_k are given by (5.39).

Proof. The proof is similar to the proof of (5.167). □

Proposition 210. *The canonical Continuous q-Hermite moments have the representation*

$$\mu_n = \frac{2\pi}{(q;q)_\infty} \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{q^k q^{-j^2} a^{-2j} (aq^j + a^{-1}q^{-j})^k}{(q, q^{1+2j}a^2; q)_{k-j} (q, q^{-1-2j}a^{-2}; q)_j}. \tag{5.217}$$

Proof. The result is obtained using relations (2.21), (2.23), (5.43) and (5.213). □

The Continuous q-Laguerre polynomials

The Continuous q-Laguerre polynomials $P_n^{(\alpha)}(x|q)$ fulfil the following orthogonality relation [30, P. 514]

$$\frac{1}{2\pi}\int_{-1}^{1} \frac{w(x)}{\sqrt{1-x^2}} P_m^{(\alpha)}(x|q) P_n^{(\alpha)}(x|q) dx = \frac{1}{(q, q^{\alpha+1}; q)_\infty} \frac{(q^{(\alpha+1)}; q)_n}{(q;q)_n} q^{(\alpha+\frac{1}{2})n} \delta_{mn}. \tag{5.218}$$

where

$$w(x) := w(x; q^\alpha | q) = \left| \frac{(e^{2i\theta}; q)_\infty}{(q^{\frac{1}{2}\alpha+\frac{1}{4}} e^{i\theta}, q^{\frac{1}{2}\alpha+\frac{3}{4}} e^{i\theta}; q)_\infty} \right|^2 = \left| \frac{(e^{i\theta}, -e^{i\theta}; q^{\frac{1}{2}})_\infty}{(q^{\frac{1}{2}\alpha+\frac{1}{4}} e^{i\theta}; q^{\frac{1}{2}})_\infty} \right|^2$$
$$= \frac{h(x,1)h(x,-1)h(x,q^{\frac{1}{2}})h(x,-q^{\frac{1}{2}})}{h(x,q^{\frac{1}{2}\alpha+\frac{1}{4}})h(x,q^{\frac{1}{2}\alpha+\frac{4}{4}})},$$

where

$$h(x,\alpha) := \prod_{k=0}^{\infty}\left(1-2\alpha x q^k+\alpha^2 q^{2k}\right) = \left(\alpha e^{i\theta},\alpha e^{-i\theta};q\right)_\infty, \quad x=\cos\theta.$$

With $m=n=0$, it follows that

$$\mu_0 = \frac{2\pi}{(q,q^{\alpha+1};q)_\infty}.$$

From the inversion formula (4.65), for $\mathcal{B}_n(x) = (q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta}, q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta};q)_n$, we have

$$I_0(n) = (q^{\alpha+1};q)_n.$$

Therefore, the following proposition is valid.

Proposition 211. *The generalized Continuous q-Laguerre moments with respect to the basis $\mathcal{B}_n(x) = (q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta}, q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta};q)_n$ have the representation*

$$\mu_n(\mathcal{B}_k(x)) = \frac{2\pi(q^{\alpha+1};q)_n}{(q,q^{\alpha+1};q)_\infty}. \tag{5.219}$$

Proposition 212. *The generalized Continuous q-Laguerre moments with respect to $\mathcal{B}_n(x) = (q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta}, q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta};q)_n$ have the following q-exponential generating function:*

$$\frac{2\pi(q^{\alpha+1}z;q)_\infty}{(z,q,q^{\alpha+1};q)_\infty} = \sum_{n=0}^{\infty}\mu_n(\mathcal{B}_k(x))\frac{z^n}{(q;q)_n}. \tag{5.220}$$

Proof. The proof follows from the q-binomial theorem (2.6). □

Proposition 213. *The canonical Continuous q-Laguerre moments have the representation*

$$\mu_n = \frac{2\pi}{(q,q^{\alpha+1};q)_\infty}\sum_{m=0}^{n}\left(-2q^{\frac{1}{2}\alpha+\frac{1}{4}}\right)^{-m} q^{-\binom{m}{2}}E_m(x_0,\ldots,x_{n-1})(q^{\alpha+1};q)_m, \tag{5.221}$$

where the numbers x_k are given by (5.39).

Proof. The proof is similar to the proof of (5.167) where we take $a = q^{\frac{1}{2}\alpha+\frac{1}{4}}$. □

Proposition 214. *The canonical Continuous q-Laguerre moments have the representation*

$$\mu_n = \frac{2\pi}{(q,q^{\alpha+1};q)_\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{q^k q^{-j^2}q^{-(\alpha+\frac{1}{2})j}(q^{j+\frac{\alpha}{2}+\frac{1}{4}}+q^{-j-\frac{\alpha}{2}-\frac{1}{4}})^k(q^{\alpha+1};q)_k}{(q,q^{2j+\alpha+\frac{3}{2}};q)_{k-j}(q,q^{-2j-\alpha-\frac{3}{2}};q)_j}. \tag{5.222}$$

Proof. The result is obtained using relations (2.21), (2.23), (5.43) with $a = q^{\frac{1}{2}\alpha+\frac{1}{2}}$ and (5.219). □

The Continuous q-Hermite polynomials

The continuous q-Hermite polynomials $H_n(x|q)$ fulfil the following orthogonality relation [30, P. 541]

$$\frac{1}{2\pi}\int_{-1}^{1}\frac{w(x|q)}{\sqrt{1-x^2}}H_m(x|q)H_n(x|q)dx = \frac{\delta_{mn}}{(q^{n+1};q)_\infty}, \tag{5.223}$$

where

$$w(x|q) = \left|\left(e^{2i\theta};q\right)_\infty\right|^2 = h(x,1)h(x,-1)h(x,q^{\frac{1}{2}})h(x,-q^{\frac{1}{2}}),$$

with

$$h(x,a) = \prod_{k=0}^{\infty}\left(1-2axq^k+a^2q^{2k}\right) = \left(ae^{i\theta},ae^{-i\theta};q\right)_\infty, \quad x=\cos\theta.$$

The canonical moments of the Continuous q-Hermite polynomials are given for every non-negative integer n by

$$\mu_n = \int_{-1}^{1}\frac{x^n w(x|q)}{\sqrt{1-x^2}}dx = \int_0^{\pi}(\cos\theta)^n(e^{2i\theta},e^{-2i\theta};q)_\infty d\theta. \tag{5.224}$$

Proposition 215 (See Lemma 13.1.4 in [25]). *The following relation is valid*

$$\int_0^\pi e^{2ij\theta}(e^{2i\theta},e^{-2i\theta};q)_\infty d\theta = \frac{\pi(-1)^j}{(q;q)_\infty}(1+q^j)q^{j(j-1)/2}. \tag{5.225}$$

Proof. Let

$$I_j = \int_0^\pi e^{2ij\theta}(e^{2i\theta},e^{-2i\theta};q)_\infty d\theta.$$

The Jacobi triple product identity (2.10) gives

$$\begin{aligned}
I_j &= \int_0^\pi e^{2ij\theta}(1-e^{2i\theta})(qe^{2i\theta},e^{-2i\theta};q)_\infty d\theta \\
&= \int_0^\pi \frac{e^{2ij\theta}(1-e^{2i\theta})}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n+1)/2}e^{2in\theta}d\theta \\
&= \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{2(q;q)_\infty}\int_{-\pi}^{\pi}(1-e^{i\theta})e^{i(j+n)\theta}d\theta.
\end{aligned}$$

The result follows from the orthogonality of the trigonometric functions on $[-\pi;\pi]$. ☐

Proposition 216. *The canonical Continuous q-Hermite moments have the following representation*

$$\mu_{2n+1}=0, \quad \mu_{2n} = \frac{\pi(-1)^n}{(q;q)_\infty}\sum_{k=0}^{n}(-1)^k\binom{2n}{k}(1+q^{n-k})q^{\frac{(n-k)(n-k-1)}{2}}, \quad n=0,1,2,\cdots \tag{5.226}$$

Proof. Note that $\mu_n=0$ when n is odd. We start by writing

$$\begin{aligned}
\cos^n\theta &= \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^n \\
&= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{ik\theta}e^{-i(n-k)\theta} \\
&= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(2k-n)} \\
&= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-2k)}.
\end{aligned}$$

Next, we use the the relation (5.225) to get:

$$\begin{aligned}
\mu_{2n} &= \int_0^\pi(\cos\theta)^{2n}(e^{2i\theta},e^{-2i\theta};q)_\infty d\theta \\
&= \frac{1}{2^{2n}}\sum_{k=0}^{2n}\binom{2n}{k}\int_0^\pi e^{2i(n-k)}(e^{2i\theta},e^{-2i\theta};q)_\infty d\theta \\
&= \frac{\pi(-1)^n}{(q;q)_\infty}\sum_{k=0}^{n}(-1)^k\binom{2n}{k}(1+q^{n-k})q^{\frac{(n-k)(n-k-1)}{2}}.
\end{aligned}$$

☐

Conclusion and Perspectives

We have provided in this thesis representations for the moments (canonical and generalized) of *all* classical orthogonal polynomials listed in [30]. Next, interesting generating functions for those moments are given. In order to obtain those moments, we have stated the inversion formulas (see Chapter 4) for all those families, also, we have developed many connection formulas between specific polynomial bases.

Some of those moments (canonical and generalized) were already known, however as mentioned in the introduction, many of them appear for the first time.

Note that only the classical orthogonal polynomials listed in [30] have been studied. There are other classes of orthogonal polynomials that are obtained by a modification of the three term recurrence relations of the classical orthogonal polynomials listed in [30], we have for example [16]: the associated orthogonal polynomials, the co-recursive and the generalized co-recursive orthogonal polynomials, the co-recursive associated and the generalized co-recursive associated orthogonal polynomials, the co-dilated and the generalized co-dilated orthogonal polynomials, the generalized co-modified orthogonal polynomials. The next step of this work could consist to find the corresponding moments for these orthogonal polynomials.

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Eidesstattliche Erklärung

Hiermit versichere ich, dass ich die vorliegende Dissertation selbständig und ohne unerlaubte Hilfe angefertigt und andere als die in der Dissertation angegebenen Hilfsmittel nicht benutzt habe. Alle Stellen, die wörtlich oder sinngemäß aus veröffentlichten oder unveröffentlichten Schriften entnommen sind, habe ich als solche kenntlich gemacht. Kein Teil dieser Arbeit ist in einem anderen Promotions- oder Habilitationsverfahren verwendet worden.