Algorithmic Methods for Mixed Recurrence Equations, Zeros of Classical Orthogonal Polynomials and Classical Orthogonal Polynomial Solutions of Three-Term Recurrence Equations

By

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To my wife Charlotte Gilberbine Mazeufouo and my children

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Contents

Abstract

Using an algorithmic approach, we derive classes of mixed recurrence equations satisfied by classical orthogonal polynomials. Starting from certain structure relations satisfied by classical orthogonal polynomials or their connection formulae, we show that our mixed recurrence equations are structurally valid. However, they couldn't be easily obtained with classical methods and for this reason, our algorithmic approach is important. The main algorithmic tool used here is an extended version of Zeilberger's algorithm. As application of the mixed recurrence equations,

1. we investigate interlacing properties of zeros of sequences of classical orthogonal polynomials;
2. we prove quasi-orthogonality of certain classes of polynomials and determine the location of the extreme zeros of the quasi-orthogonal polynomials with respect to the end points of the interval of orthogonality of the polynomial sequence, where possible;
3. we find bounds for the extreme zeros of classical orthogonal polynomials.

Every orthogonal polynomial system $\{p_n(x)\}_{n\geq 0}$ satisfies a three-term recurrence relation of the type

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x) \; (n = 0, 1, 2, \ldots, p_{-1} \equiv 0),$$

with $C_n A_n A_{n-1} > 0$. Moreover, Favard's theorem states that the converse is also true. A general method to derive the coefficients A_n, B_n, C_n in terms of the polynomial coefficients of the divided-difference equations satisfied by orthogonal polynomials on a quadratic or q-quadratic lattice is revisited. The Maple implementations `rec2ortho` of Koornwinder and Swarttouw [1996–1998] or `retode` of Koepf and Schmersau [2002] were developed to identify classical orthogonal polynomials knowing their three-term recurrence relations. The two implementations `rec2ortho` and `retode` do not handle classical orthogonal polynomials on a quadratic or q-quadratic lattice. We extend the Maple implementation `retode` of Koepf and Schmersau [2002] to cover classical orthogonal polynomials on quadratic or q-quadratic lattices and to answer as application an open problem submitted by Alhaidari [2017] during the 14th International Symposium on Orthogonal Polynomials, Special Functions and Applications.

Chapter 0

General Introduction

We say that a polynomial set $\{y_n(x)\}_{n\geq 0}$, where $y_n(x)$ is of exact degree n in the variable x, is orthogonal with respect to the measure $d\mu(x)$ defined on the interval (c,d) (with $-\infty \leq c < d \leq +\infty$) if the following orthogonality relation is valid

$$\langle y_n(x), y_m(x)\rangle := \int_c^d y_n(x)y_m(x)d\mu(x) \begin{cases} = 0 & \text{if} \quad n \neq m \\ \neq 0 & \text{if} \quad n = m. \end{cases}$$

If the nondecreasing, real valued, bounded function $\mu(x)$ is absolutely continuous with $d\mu(x) = \rho(x)dx$, $\rho(x) \geq 0$, then the orthogonality relation reduces to

$$\langle y_n(x), y_m(x)\rangle = \int_c^d y_n(x)y_m(x)\rho(x)dx \begin{cases} = 0 & \text{if} \quad n \neq m \\ = h_n \neq 0 & \text{if} \quad n = m. \end{cases}$$

The sequence $\{y_n\}_{n\geq 0}$ is said to be orthogonal with respect to the weight function $\rho(x)$ defined on the interval (c,d). We refer to the sequence $\{y_n\}_{n\geq 0}$ as orthogonal polynomials of a continuous variable.

However, if $\mu(x)$ is a step-function with jumps $\rho(x) = \rho_j$ at the points $x = x_j = j$, $j = 0, 1, 2, \ldots$, then the orthogonality relation takes the form

$$\langle y_n(x), y_m(x)\rangle = \sum_{x=0}^{\infty} y_n(x)y_m(x)\rho(x) \begin{cases} = 0 & \text{if} \quad n \neq m \\ = h_n \neq 0 & \text{if} \quad n = m. \end{cases}$$

In this case, the variable $x = x_j$ is discrete instead of being continuous and we refer to the sequence $\{y_n\}_{n\geq 0}$ as orthogonal polynomials of a discrete variable.

A family $\{y_n\}_{n\geq 0}$ of orthogonal polynomials of a continuous variable is said to be classical if the weight function $\rho(x)$ is solution of the Pearson equation

$$(\sigma(x)\rho(x))' = \tau(x)\rho(x),$$

where $\sigma(x) = ax^2 + bx + c$ is a polynomial of at most second order and $\tau(x) = dx + e$ is a polynomial of first order, with $\sigma(x) > 0$ on (c,d) and $\lim\limits_{x\to c,d} x^n\sigma(x)\rho(x) = 0$.

It is known that classical orthogonal polynomials (in short COP) of a continuous variable satisfy a second-order differential equation of the type

$$\sigma(x)y_n''(x) + \tau(x)y_n'(x) + \lambda_n y_n(x) = 0,$$

where λ_n is a constant depending on the leading coefficients of σ and τ.

On the other hand, COP of a discrete variable satisfy (depending on the type of the discrete variable) three types of difference equations. COP of a discrete variable on a linear lattice satisfy a second-order difference equation

$$\sigma(x)\Delta\nabla y_n(x) + \tau(x)\Delta y_n(x) + \lambda_n y_n(x) = 0,$$

if the variable is of the form $x = x_j = j$, $j = 0, 1, \ldots$, where Δ and ∇ are, respectively, the forward and the backward difference operators defined by

$$\Delta f(x) = f(x+1) - f(x),\ \nabla f(x) = f(x) - f(x-1).$$

Classical q-orthogonal polynomials satisfy a second-order q-difference equation

$$\sigma(x)D_q D_{\frac{1}{q}} y_n(x) + \tau(x)D_q y_n(x) + \lambda_{n,q} y_n(x) = 0,$$

if the variable is of the form $x = x_j = q^j$, $j = 0, 1, \ldots$ or $j = \ldots, -2, -1, 0, 1, 2, \ldots$, where D_q is the Hahn operator defined by

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x}.$$

COP of a discrete variable on a quadratic or a q-quadratic lattice satisfy a second-order divided-difference equation

$$\sigma(x(s))\mathbb{D}_x^2 y_n(x(s)) + \tau(x(s))\mathbb{S}_x\mathbb{D}_x y_n(x(s)) + \lambda_n y_n(x(s)) = 0,$$

if the variable is of the form

$$x = x(s) = \begin{cases} c_4 s^2 + c_5 s + c_6 & \text{if } q = 1, \\ c_1 q^s + c_2 q^{-s} + c_3 & \text{if } q \neq 1. \end{cases}$$

Here the operators $\mathbb{D}_x$ and $\mathbb{S}_x$ are defined by Foupouagnigni [2008]

$$\mathbb{D}_x f(x(s)) = \frac{f(x(s+\frac{1}{2})) - f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})},$$

$$\mathbb{S}_x f(x(s)) = \frac{f(x(s+\frac{1}{2})) + f(x(s-\frac{1}{2}))}{2}.$$

The classical orthogonal polynomials considered in the sequel (see e.g. [Chihara, 1978], [Nikiforov and Uvarov, 1988], [Nikiforov et al., 1991], [Koekoek et al., 2010] and references therein) are defined in terms of the generalized hypergeometric series

$${}_pF_q\left(\left.\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right| x\right) = \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_q)_m} \frac{x^m}{m!},$$

where $(a)_m$ denotes the Pochhammer symbol (or shifted factorial) defined by

$$(a)_m = \begin{cases} 1 & \text{if } m = 0 \\ a(a+1)(a+2)\cdots(a+m-1) & \text{if } m \in \mathbb{N}. \end{cases}$$

Their q-orthogonal analogues, $0 < q < 1$, are given in terms of basic hypergeometric series (see e.g. [Gasper and Rahman, 1990], [Koekoek et al., 2010] and references therein)

$$ {}_r\phi_s\left(\begin{array}{c} a_1,\ldots,a_r \\ \\ b_1,\ldots,b_s \end{array}\middle| q;z\right) = \sum_{k=0}^{\infty} \frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_s;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r}\frac{z^k}{(q;q)_k}, $$

where the q-Pochhammer symbol $(a_1,a_2,\ldots,a_k;q)_n$ is defined by

$$ (a_1,\ldots,a_r;q)_k := (a_1;q)_k\cdots(a_r;q)_k, \quad \text{with} \quad (a_i;q)_k = \begin{cases} \prod_{j=0}^{k-1}(1-a_iq^j) & \text{if} \quad k\in\{1,2,3,\ldots\} \\ 1 & \text{if} \quad k=0. \end{cases} $$

If $\{p_n\}_{n\geq 0}$ is a sequence of polynomials orthogonal on (c,d), with respect to the weight function $w(x)$, then the polynomial $p_n(x)$ has exactly n simple zeros in (c,d) and the zeros of $p_n(x)$ and $p_{n+1}(x)$ separate each other. That is, if $c < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < d$ and $c < x_{n+1,1} < x_{n+1,2} < \cdots < x_{n+1,n+1} < d$ are the zeros of p_n and p_{n+1}, respectively, then

$$ x_{n+1,1} < x_{n,1} < x_{n+1,2} < x_{n,2} < \cdots < x_{n+1,n} < x_{n,n} < x_{n+1,n+1}. $$

The zeros of orthogonal polynomials are used for example in the Gauss quadrature formula, in polynomial interpolation as interpolation nodes (see e. g. [Szegő, 1975], [Nikiforov and Uvarov, 1988], [Mason and Handscomb, 2002], [Ismail, 2005] and references therein). The zeros of the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are the unique location of n unit charges distributed in $(-1,1)$ in the logarithmic field generated by two fixed charges with strengths $(\beta+1)/2$ and $(\alpha+1)/2$ fixed at -1 and 1 (see e.g. [Valent and Van Assche, 1995], [Ismail, 2005, Remark 3.5.1]).

Definition 0.1 (see e. g. Driver and Muldoon [2016], Driver and Jordaan [2018]). *Let $n \in \mathbb{N}$. If $x_{n,1} < x_{n,2} < \ldots < x_{n,n}$ are the zeros of p_n and $y_{n,1} < y_{n,2} < \ldots < y_{n,n}$ are the zeros of q_n, then the zeros of p_n and q_n are interlacing if*

$$ x_{n,1} < y_{n,1} < x_{n,2} < y_{n,2} < \ldots < x_{n,n} < y_{n,n} \tag{1} $$

or if

$$ y_{n,1} < x_{n,1} < y_{n,2} < x_{n,2} < \ldots < y_{n,n} < x_{n,n}. $$

In case p_n is replaced by p_{n+1}, (1) is replaced by

$$ x_{n+1,1} < y_{n,1} < x_{n+1,2} < y_{n,2} < \ldots < x_{n+1,n} < y_{n,n} < x_{n+1,n+1}. $$

According to results by Peherstorfer [1990], interlacing properties of the zeros of orthogonal polynomial expansions are responsible for the existence of positive interpolatory quadrature formulas (see also [Locher, 1993], [Criscuolo et al., 1990]). Starting from interlacing properties of the zeros of the orthogonal polynomials, Mastroianni and Occorsio [1995] proposed a method to approximate the finite Hilbert transform. Interlacing also happened to be crucial in [Bender et al., 2000].

Definition 0.2. *Let $\{p_n\}_{n\geq 0}$ be a sequence of polynomials with degree $p_n = n$ for each $n \in \mathbb{N}$. For a positive integer $r < n$, the sequence $\{p_n\}_{n\geq 0}$ is quasi-orthogonal of order r with respect to a positive Borel measure μ if*

$$ \int x^k p_n(x)d\mu(x) = 0 \text{ for } k = 0,1,\ldots,n-1-r, \ \forall n\in\mathbb{N}. \tag{2} $$

It is clear that if (2) holds for $r = 0$, the sequence $\{p_n\}_{n\geq 0}$ is orthogonal with respect to the measure μ.

Using certain structure relations satisfied by classical orthogonal polynomials or their connection formulae, we show that some classes of mixed recurrence equations satisfied by classical orthogonal polynomials (with shifted parameters) are structurally valid. However, they cannot be easily obtained with classical methods. To solve this problem, we use an algorithmic approach to find these mixed recurrence equations. A list of some problems that motivates the consideration of special linear combinations of polynomials, orthogonal with respect to a given weight on a given interval, is given in [Grinshpun, 2004]. The major algorithmic tool for our development is an extended version of Zeilberger's algorithm (see [Koepf, 2014] and reference therein). Without this preprocessing the relevant recurrence equations are not easily accessible. Using our mixed recurrence equations,

1. we investigate interlacing properties of zeros of sequences of orthogonal polynomials. In the cases when the zeros do not interlace, we give numerical examples to illustrate this;

2. we prove quasi-orthogonality of certain classes of polynomials and determine the location of the extreme zeros of the quasi-orthogonal polynomials with respect to the end points of the interval of orthogonality of the polynomial sequence, where possible;

3. we find bounds for the extreme zeros of the classical orthogonal polynomials.

Every orthogonal polynomial system $(p_n(x))_{n\geq 0}$ satisfies a three-term recurrence relation of the type

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x) \; (n = 0, 1, 2, \ldots, p_{-1} \equiv 0),$$

with $C_n A_n A_{n-1} > 0$. Moreover, Favard's theorem [Chihara, 1978, Section 4] states that the converse is also true. Alhaidari [2017] submitted (as open problem during the 14th International Symposium on Orthogonal Polynomials, Special Functions and Applications) two polynomial systems defined by their three-term recurrence relations and initial values. He was interested in the derivation of their weight functions, generating functions, orthogonality relations, etc.. In order to solve this problem as suggested in the comments by W. Van Assche in [Van Assche, 2019], we use the computer algebra system Maple to identify the polynomials from their recurrence relations, similar as in the Maple implementation `rec2ortho` of Koornwinder and Swarttouw [1996–1998] or `retode` of Koepf and Schmersau [2002]. The two implementations `rec2ortho` and `retode` do not handle classical orthogonal polynomials on a quadratic or a q-quadratic lattice. We extend the Maple implementation `retode` of Koepf and Schmersau [2002] to cover classical orthogonal polynomials on a quadratic or a q-quadratic lattice and to answer the problem by Alhaidari [2017] as application.

The plan of the work is as follows: Chapter 1 is devoted to the preliminary results and the derivation of the mixed recurrence equations using Zeilberger's algorithm and its q-version. In Chapter 2, we use our algorithms to recover known results for classical orthogonal polynomials of a continuous and discrete variable, and moreover, we improve some bounds in these cases. We also collect the scattered results in one place which will make them accessible. In Chapter 3 and 4, the interlacing properties, the quasi-orthogonality as well as the bounds for the extreme zeros for classical q-orthogonal polynomials and orthogonal polynomials on a quadratic or a q-quadratic lattices are studied, respectively. Finally, in addition, in Chapter 4, we implement the algorithm to identify classical orthogonal polynomials on a quadratic or a q-quadratic lattice from their three-term recurrence relations.

Chapter 1

Preliminary results

The purpose of this chapter is to give the main results which will be applied to derive the interlacing properties and the bounds of the extreme zeros of the classical orthogonal polynomials consider in the sequel, and to study quasi-orthogonal polynomials. Moreover, we show how the mixed recurrence equations involved in the main results will be derived, using an algorithmic approach.

1.1 Interlacing properties for zeros of sequences of classical orthogonal polynomials

The separation of the zeros of different sequences of Hahn polynomials of the same or adjacent degree was first studied by Levit [1967], and similar interlacing results followed for Jacobi polynomials ([Askey, 1990], [Driver et al., 2008]), Krawtchouk polynomials ([Chihara and Stanton, 1990], [Jordaan and Toókos, 2009]), Meixner and Meixner-Pollaczek polynomials [Jordaan and Toókos, 2009]. The different sequences were obtained by integer shifts of the parameters, and in order to prove these results, recurrence equations, following from the contiguous relations for hypergeometric polynomials [Rainville, 1960, p. 71], [Prudnikov et al., 1990] or basic hypergeometric series [Swarttouw, 1990], [Gupta et al., 1992], were used. In the case of Gauss' hypergeometric function (cf. [Szegő, 1975, Eq. (4.21.3)]), a useful algorithm in this regard is available as a computer package [Vidunas and Koornwinder, 2000].

Interlacing results for the zeros of different sequences of q-orthogonal sequences with shifted parameters are given for q-Laguerre polynomials in [Jordaan and Toókos, 2010], [Moak, 1981], for Al-Salam-Chihara, q-Meixner-Pollaczek and q-ultraspherical polynomials in [Jordaan and Toókos, 2010] and for ${}_2\phi_1$ hypergeometric polynomials, associated with the little q-Jacobi polynomials, in [Gochhayat et al., 2016]. The recurrence equations necessary to prove these results were obtained, respectively, from relationships between polynomials orthogonal w.r.t. a positive measure $d\Psi(x)$ and those orthogonal to $xd\Psi(x)$ (cf. [Karlin and McGregor, 1957]), from the generating functions of the appropriate polynomials and from the contiguous relations satisfied by the basic hypergeometric function (cf. [Heine, 1847]). In order to determine the specific order of the interlacing zeros, Markov's monotonicity theorem (or a consequence of it), is used (cf. [Szegő, 1975, Theorems 6.12.1, 6.12.2] or [Ismail, 2005]):

Theorem 1.1 (see [Szegő, 1975]). *Let $w(x)$ and $W(x)$ be two weight functions on (c, d),* *both positive and continuous for* $c < x < d$. *Let* $\frac{W(x)}{w(x)}$ *be increasing. Then if* $\{x_v\}$ *and* $\{X_v\}$ *denote the zeros of the corresponding orthogonal polynomials of degree* n *in decreasing order, we have*

$$x_v < X_v, \quad v = 1, 2, \ldots, n.$$

In this section, we show how mixed recurrence equations, satisfied by different sequences of orthogonal polynomial systems, are used to study interlacing properties of the zeros of sequences of orthogonal systems.

Lemma 1.2 (cf. Brezinski et al. [2004], Jordaan and Toókos [2010], Gochhayat et al. [2016]). *Let* (c, d) *be a finite or infinite interval and* p_n *and* q_n *polynomials (not necessarily orthogonal) of degree* n, *with zeros* $c < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < d$ *and* $c < y_{n,1} < y_{n,2} < \cdots < y_{n,n} < d$, *respectively, satisfying the interlacing property*

$$x_{n,1} < y_{n-1,1} < x_{n,2} < y_{n-1,2} < \cdots < x_{n,n-1} < y_{n-1,n-1} < x_{n,n}. \tag{1.1}$$

Let a *and* b *be continuous functions on* (c, d) *and assume that* f_n *is a polynomial of degree* n, *with zeros* $c < z_{n,1} < z_{n,2} < \cdots < z_{n,n} < d$, *satisfying the equation*

$$f_n(x) = a(x)p_n(x) + b(x)q_{n-1}(x). \tag{1.2}$$

Then,

(a) if b *has constant sign on* (c, d), *the zeros of* f_n *and* p_n *interlace;*

(b) if a *has constant sign on* (c, d), f_n *has an odd number of zeros between any two consecutive zeros of* q_{n-1}.

Proof. Assume that f_n has degree n with zeros $z_{n,1} < z_{n,2} < \cdots < z_{n,n}$.

(a) We evaluate (1.2) at $x_{n,k}$ and $x_{n,k+1}$, $k \in \{1, 2, \ldots, n-1\}$, two consecutive zeros of $p_n(x)$. Then

$$f_n(x_{n,k})f_n(x_{n,k+1}) = b(x_{n,k})b(x_{n,k+1})q_{n-1}(x_{n,k})q_{n-1}(x_{n,k+1}).$$

By (1.1) the zeros of p_n and q_{n-1} interlace, therefore q_{n-1} will differ in sign at $x_{n,k}$ and $x_{n,k+1}$, $k \in \{1, 2, \ldots, n-1\}$, which implies $q_{n-1}(x_{n,k})q_{n-1}(x_{n,k+1}) < 0$. Since $b(x)$ has constant sign on (c, d), we have $b(x_{n,k})b(x_{n,k+1}) > 0$ and therefore $f_n(x_{n,k})f_n(x_{n,k+1}) < 0$. f_n must therefore have an odd number of zeros in each interval with endpoints $x_{n,k}$ and $x_{n,k+1}$, $k \in \{1, 2, \ldots, n-1\}$, and the interlacing result follows.

(b) We evaluate (1.2) at $y_{n-1,k}$ and $y_{n-1,k+1}$, $k \in \{1, 2, \ldots, n-2\}$, two consecutive zeros of $q_{n-1}(x)$. Then

$$f_n(y_{n-1,k})f_n(y_{n-1,k+1}) = a(y_{n-1,k})a(y_{n-1,k+1})p_n(y_{n-1,k})p_n(y_{n-1,k+1}).$$

From (1.1) we know that the zeros of p_n and q_{n-1} interlace, therefore p_n will differ in sign at $y_{n-1,k}$ and $y_{n-1,k+1}$, $k \in \{1, 2, \ldots, n-2\}$, and $p_n(y_{n-1,k})p_n(y_{n-1,k+1}) < 0$. Since $a(x)$ has constant sign on (c, d), we have $a(y_{n-1,k})a(y_{n-1,k+1}) > 0$ and therefore $f_n(y_{n-1,k})f_n(y_{n-1,k+1}) < 0$, which implies that f_n must have an odd number of zeros in each interval with endpoints $y_{n-1,k}$ and $y_{n-1,k+1}$, $k \in \{1, 2, \ldots, n-2\}$.

□

In the following result, which follows from Lemma 1.2, we assume that the polynomials p_n and q_n are monic. In fact if p_n is a polynomial of degree n with leading coefficient $k_n \neq 0$, then the monic polynomial $\tilde{p}_n = p_n/k_n$ and p_n have the same zeros.

Corollary 1.3 (cf. Brezinski et al. [2004], Joulak [2005], Jordaan and Toókos [2010], Gochhayat et al. [2016]). *Let (c,d) be a finite or infinite interval and assume that p_n and q_n are monic polynomials (not necessarily orthogonal) of degree n, with zeros $c < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < d$ and $c < y_{n,1} < y_{n,2} < \cdots < y_{n,n} < d$, respectively, satisfying the interlacing property (1.1). Assume that a and b are continuous and have constant sign on (c,d) and that f_n is a polynomial of degree n with zeros $c < z_{n,1} < z_{n,2} < \cdots < z_{n,n} < d$, satisfying (1.2). Then, for each $k \in \{1,2,\ldots,n-1\}$,*

(a) if $a(x)$ and $b(x)$ have the same sign on (c,d), $z_{n,k} < x_{n,k} < y_{n-1,k} < z_{n,k+1} < x_{n,k+1}$;

(b) if $a(x)$ and $b(x)$ differ in sign on (c,d), $x_{n,k} < z_{n,k} < y_{n-1,k} < x_{n,k+1} < z_{n,k+1}$.

Proof. Assume that f_n has degree n and both a and b have constant sign on (c,d). Then both results of Lemma 1.2 are true. From Lemma 1.2(a), the zeros of f_n and p_n interlace and either $z_{n,k} < x_{n,k}$ or $x_{n,k} < z_{n,k}$ for each $k \in \{1,2,\ldots,n\}$.

Evaluating (1.2) at $y_{n-1,n-1}$ and $x_{n,n}$, we obtain

$$f_n(x_{n,n})f_n(y_{n-1,n-1}) = a(y_{n-1,n-1})b(x_{n,n})p_n(y_{n-1,n-1})q_{n-1}(x_{n,n}). \tag{1.3}$$

Since, by assumption, p_n and q_{n-1} are monic polynomials with interlacing zeros, $p_n(y_{n-1,n-1}) < 0$ and $q_{n-1}(x_{n,n}) > 0$.

(a) Assume a and b have the same sign on (c,d). Then $a(y_{n-1,n-1})b(x_{n,n}) > 0$ and, since $p_n(y_{n-1,n-1})q_{n-1}(x_{n,n}) < 0$, we deduce from (1.3) that $f_n(x_{n,n})f_n(y_{n-1,n-1}) < 0$. This implies f_n has an odd number of zeros in the interval $(y_{n-1,n-1}, x_{n,n})$.

Suppose $z_{n,k} < x_{n,k}$, $k \in \{1,2,\ldots,n\}$. From (1.1) we deduce that $z_{n,1} < x_{n,1} < y_{n-1,1}$ and thus one zero of f_n lies to the left of $y_{n-1,1}$. From Lemma 1.2(b), we know there is an odd number of zeros of f_n in each of the $n-2$ intervals $(y_{n-1,k}, y_{n-1,k+1})$, $k \in \{1,2,\ldots,n-2\}$. If each of the $n-2$ intervals between the first and the last zero of q_{n-1} has exactly one zero of f_n, we have $n-1$ zeros accounted for. There is only one zero remaining (since f_n has n zeros), and we deduce that only one zero of f_n lies in $(y_{n-1,n-1}, x_{n,n})$, which leads to the configuration

$$z_{n,1} < x_{n,1} < y_{n-1,1} < z_{n,2} < \cdots < x_{n,n-1} < y_{n-1,n-1} < z_{n,n} < x_{n,n}.$$

Suppose $x_{n,k} < z_{n,k}$, $k \in \{1,2,\ldots,n\}$. From (1.1), we deduce that $y_{n-1,n-1} < x_{n,n} < z_{n,n}$. This contradicts the fact that f_n must have an odd number of zeros in the interval $(y_{n-1,n-1}, x_{n,n})$.

(b) Assume a and b have different signs on (c,d). Then $a(y_{n-1,n-1})b(x_{n,n}) < 0$ and we deduce from (1.3) that $f_n(x_{n,n})f_n(y_{n-1,n-1}) > 0$, thus f_n has either 0 or an even number of zeros in the interval $(y_{n-1,n-1}, x_{n,n})$.

Suppose $x_{n,k} < z_{n,k}$, $k \in \{1,2,\ldots,n\}$. From (1.1) we deduce that $y_{n-1,n-1} < x_{n,n} < z_{n,n}$ and the only option, counting the zeros, is that

$$x_{n,1} < z_{n,1} < y_{n-1,1} < x_{n,2} < \cdots < z_{n,n-1} < y_{n-1,n-1} < x_{n,n} < z_{n,n}.$$

Suppose $z_{n,k} < x_{n,k}, k \in \{1, 2, \ldots, n\}$. From (1.1) we deduce that $z_{n,1} < x_{n,1} < y_{n-1,1}$ and thus one zero of f_n lies to the left of $y_{n-1,1}$. From Lemma 1.2(b), we know there is an odd number of zeros of f_n in each of the $n-2$ intervals $(y_{n-1,k}, y_{n-1,k+1}), k \in \{1, 2, \ldots, n-2\}$. If each of the $n-2$ intervals between the first and the last zero of q_{n-1} has exactly one zero of f_n, we have $n-1$ zeros accounted for. There is only one zero remaining (since f_n has n zeros). The one remaining zero therefore must lie to the right of $y_{n-1,n-1}$, such that $y_{n-1,n-1} < z_{n,n} < x_{n,n}$, which contradicts the fact that f_n must have either 0 or an even number of zeros in the interval $(y_{n-1,n-1}, x_{n,n})$.

□

From Corollary 1.3 we remark that, once we have a relation of type (1.2), it is sufficient to know the sign of $a(x)$ and $b(x)$ in (1.2) to prove our interlacing results.

1.2 Quasi-orthogonal polynomials

We recall that a sequence of polynomials $\{p_n\}_{n\geq 0}$, where each polynomial p_n has degree n, is orthogonal with respect to the weight function $w(x) > 0$ on the finite (or infinite) interval (c, d) if

$$\int_c^d x^m p_n(x)w(x)dx = 0, m \in \{0, 1, \ldots, n-1\}, \ \forall n \in \mathbb{N}.$$

In order for orthogonality conditions to hold, we often need restrictions on the parameters of the classical orthogonal polynomials and when the parameters deviate from these restricted values in an orderly way, the zeros may depart from the interval of orthogonality in a predictable way. This phenomenon can be explained in terms of the concept of quasi-orthogonality. The sequence of polynomials $\{q_{n,k}\}_{n\geq 0}$, where each polynomial $q_{n,k}$ has degree n, is quasi-orthogonal of order $k \in \{1, 2, \ldots, n-1\}$ with respect to the weight function $w(x) > 0$ on (c, d) if

$$\int_c^d x^m q_{n,k}(x)w(x)dx = 0, m \in \{0, 1, \ldots, n-k-1\}, \ \forall n \in \mathbb{N}. \tag{1.4}$$

Quasi-orthogonality was first studied by Riesz [1923], followed by Fejér [1933], Shohat [1937], Chihara [1957], Dickinson [1961], Draux [1990], Maroni [1991], Joulak [2005], ... The quasi-orthogonality of Jacobi, Gegenbauer and Laguerre sequences is discussed in [Brezinski et al., 2004], the quasi-orthogonality of Meixner sequences in [Jooste et al., 2013] and of Meixner-Pollaczek, Hahn, Dual-Hahn and Continuous Dual-Hahn sequences in [Johnston et al., 2016]. More recently, interlacing properties of zeros of quasi-orthogonal Meixner, Jacobi, Laguerre and Gegenbauer polynomials were studied in [Driver and Jooste, 2017], [Driver and Jordaan, 2016], [Driver and Muldoon, 2016], [Driver and Muldoon, 2015] and in [Bultheel et al., 2010] interlacing properties of zeros of quasi-orthogonal polynomials were used to prove results on Gaussian-type quadrature. Ismail and Wang [2019] developed a general theory of quasi-orthogonal polynomials. They first derive three-term recurrence relation and second-order differential equations for quasi-orthogonal polynomials. They also give an expression for their discriminants in terms of the recursion coefficients of the corresponding orthogonal polynomials. In addition, they investigate an electrostatic equilibrium problem where the equilibrium position of movable charges is attained at the zeros of the quasi-orthogonal polynomials. Quasi-orthogonal polynomials are characterized by the following property:

Lemma 1.4 (Brezinski et al. [2004], Chihara [1957]). *Let $\{p_n\}_{n\geq 0}$ be a family of orthogonal polynomials on (c,d) with respect to the weight function $w(x) > 0$. A necessary and sufficient condition for a polynomial sequence $\{q_{n,k}\}_{n\geq 0}$ with $\deg(q_{n,k}) = n$ to be quasi-orthogonal of order $k \leq n-1$ with respect to w on (c,d), is that*

$$q_{n,k}(x) = \sum_{i=0}^{k} a_{n,i} p_{n-i}(x), \;\; a_{n,0}a_{n,k} \neq 0, \;\; n > k. \tag{1.5}$$

Remark 1.5. *Bracciali et al. [2018] established necessary and sufficient conditions so that the quasi-orthogonal polynomials $\{q_{n,k}\}_{n\geq 0}$ defined by (1.5) also constitute a sequence of orthogonal polynomials.*

Lemma 1.6 (Brezinski et al. [2004], Shohat [1937]). *If a sequence $\{q_{n,k}\}_{n\geq 0}$ is quasi-orthogonal of order $k \geq 1$ on (c,d) with respect to $w(x) > 0$, then at least $(n-k)$ real, distinct zeros of $q_{n,k}$ lie in the interval (c,d).*

Lemma 1.7 (Brezinski et al. [2004], Joulak [2005]). *Suppose $q_{n,1}(x) = p_n(x)+a_n p_{n-1}(x), a_n \neq 0$. Let $y_{n,j}, j \in \{1,2,\ldots,n\}$, be the zeros of $q_{n,1}(x)$ and let $f_n(x) = \frac{p_n(x)}{p_{n-1}(x)}$. We have*

(i) $y_{n,1} < c$ *if and only if* $-a_n < f_n(c) < 0$;

(ii) $d < y_{n,n}$ *if and only if* $-a_n > f_n(d) > 0$;

(iii) $q_{n,1}$ *has all its zeros in* (c,d) *if and only if* $f_n(c) < -a_n < f_n(d)$.

Lemma 1.8 (Brezinski et al. [2004], Joulak [2005]). *Suppose $q_{n,1}(x) = p_n(x)+a_n p_{n-1}(x), a_n \neq 0$. Let $x_{n,j}, j \in \{1,2,\ldots,n\}$, denote the zeros of $p_n(x)$ and $y_{n,j}, j \in \{1,2,\ldots,n\}$, the zeros of $q_{n,1}(x)$. Then*

(i) $a_n < 0$ *if and only if* $x_{n,1} < y_{n,1} < x_{n-1,1} < x_{n,2} < y_{n,2} < \cdots < x_{n-1,n-1} < x_{n,n} < y_{n,n}$;

(ii) $a_n > 0$ *if and only if* $y_{n,1} < x_{n,1} < x_{n-1,1} < y_{n,2} < x_{n,2} < \cdots < x_{n-1,n-1} < y_{n,n} < x_{n,n}$.

1.3 Bounds of extreme zeros of classical orthogonal polynomials

Let $\{p_n\}_{n\geq 0}$ be a sequence of monic orthogonal polynomials with zeros $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$. It is well known that p_n satisfies a three-term recurrence equation

$$p_n(x) = (x - B_n)p_{n-1}(x) - C_n p_{n-2}(x), \tag{1.6}$$

where B_n and C_n do not depend on x, $p_{-1} \equiv 0$, $p_0 \equiv 1$ and $C_n > 0$, and that the zeros of p_n and p_{n-1} interlace. It is also known that, if p_n and p_{n-2} do not have a common zero, then the $n-1$ zeros of $(x - B_n)p_{n-2}(x)$ interlace with the n zeros of p_n [Beardon, 2011, Theorem 3], therefore $x_{n,1} < B_n < x_{n,n}$ and the point B_n is a natural inner bound for the extreme zeros of p_n. Beardon generalised this result in [Beardon, 2011, Theorem 4] and we state it here as a lemma:

Lemma 1.9. *Suppose $\{p_n\}_{n\geq 0}$ is a sequence of polynomials, satisfying (1.6). Then, given n, there exist real polynomials S_m of degree m, where $m < n-2$, such that*

$$C_n C_{n-1} \ldots C_{n-m+2} p_{n-m}(x) = S_{m-1}(x) p_{n-1}(x) + S_{m-2}(x) p_n(x) \tag{1.7}$$

and if p_{n-m} and p_n do not have any common zeros, their zeros interlace in the Stieltjes sense. More-over, the $n-1$ zeros of $S_{m-1}p_{n-m}$ interlace with the n zeros of p_n.

An important feature of the polynomials S_{m-1} is that they are completely determined by the coefficients in (1.6) (cf. [Segura, 2008, Theorem 1]). A natural consequence of Lemma 1.9 is

Corollary 1.10. *(cf. [Driver and Jordaan, 2012, Corollary 2.2]) Suppose (1.7) holds for $m, n \in \mathbb{N}$ fixed, $m < n-2$. The smallest and largest zeros of S_{m-1} are inner bounds for the extreme zeros of p_n.*

Equations similar to (1.7), involving polynomials p_n, p_{n-1}, and $g_{n-m}, m \in \{2, 3, \ldots\}$, where the polynomial g_{n-m} belongs to a related orthogonal sequence, obtained by integer shifts of the appropriate parameters, are used to obtain (more accurate) inner bounds for the extreme zeros of orthogonal sequences (cf. [Driver and Jordaan, 2012]). However, as in (1.7), the coefficient of $p_{n-1}(x)$, which will be denoted by G_{m-1} in (1.9), needs to be a polynomial of exact degree $m-1$ in order to have full interlacing between the n zeros of p_n and the $n-1$ zeros of $G_{m-1}(x)g_{n-m}(x)$. In [Jooste and Jordaan, 2014, Theorem 2.1], conditions necessary for the existence of such mixed three-term recurrence equations are given for $m = 2$:

If, for $k \in \mathbb{N}_0$ fixed and $\{g_{n,k}\}_{n\geq 0}$ a sequence of polynomials orthogonal with respect to $c_k(x)w(x) > 0$ on (c, d), where $c_k(x)$ is a polynomial of degree k in x, the sequence $\{p_n\}_{n\geq 0}$ satisfies

$$A_n c_k(x) g_{n-2,k}(x) = a_{k-2}(x) p_n(x) - (x - B_n) p_{n-1}(x), \ n \in \{2, 3, \ldots\}, \tag{1.8}$$

with A_n, B_n, a_{-1}, a_{-2} constants and a_{k-2} a polynomial of degree $k-2$ defined on (c, d) whenever $k \in \{2, 3, \ldots\}$, then $k \in \{0, 1, 2, 3, 4\}$.
We generalise the result in [Jooste and Jordaan, 2014, Theorem 2.1] by providing conditions necessary for equations, similar to (1.8), involving the polynomials $g_{n-m,k}, m \in \{2, 3, \ldots, n-1\}$, p_n and p_{n-1}, to exist.

Theorem 1.11 (cf. [Jooste et al.]). *Let $\{p_n\}_{n\geq 0}$ be a sequence of polynomials orthogonal on the (finite or infinite) interval (c, d) with respect to the weight function $w(x) > 0$. Let $k \in \mathbb{N}_0$ and $m \in \{2, 3, \ldots, n-1\}$ be fixed and suppose $\{g_{n,k}\}_{n\geq 0}$ is a sequence of polynomials orthogonal with respect to $c_k(x)w(x) > 0$ on (c, d), where $c_k(x)$ is a polynomial of degree k, that satisfies*

$$A_n c_k(x) g_{n-m,k}(x) = a_{k-m}(x) p_n(x) - G_{m-1}(x) p_{n-1}(x), \ n \in \{2, 3, \ldots\}, \tag{1.9}$$

with A_n, B_n, a_{-1} and a_{-2} constants, a_{k-m} a polynomial defined on (c, d) and of degree $m-2$ when $k - m \in \{-m, -m+1, \ldots, m-2\}$ and of degree $k-m$ whenever $k - m \in \{m-1, m, m+1, \ldots\}$, and $G_m(x)$ a polynomial of degree m. Then

(i) $k \in \{0, 1, 2, \ldots, 2m\}$;

(ii) *if $g_{n-m,k}$ and p_n are co-prime, the $n-1$ real, simple zeros of $G_{m-1}(x)g_{n-m,k}$ interlace with the zeros of p_n, the smallest zero of G_{m-1} is an upper bound for the smallest zero of p_n, and the largest zero of G_{m-1} is a lower bound for the largest zero of p_n;*

(iii) *if $g_{n-m,k}$ and p_n are not co-prime and have r common zeros counting multiplicity, then*
a) $r \leq \min\{m, n-m-1\}$;
b) these r common zeros are simple zeros of G_{m-1};
c) no two successive zeros of p_n, nor its largest or smallest zero can be a zero of G_{m-1};
d) the $n-2r-1$ zeros of $G_{m-1}g_{n-m,k}(x)$, none of which is a zero of p_n, together with the r common zeros of $g_{n-m,k}$ and p_n, interlace with the $n-r$ non-common zeros of p_n;
e) the smallest zero of G_{m-1} is an upper bound for the smallest zero of p_n, and the largest zero of G_{m-1} is a lower bound for the largest zero of p_n.

Proof. The proof of this theorem is explicitly given in [Jooste et al.] □

The bounds obtained in this way are more accurate than the inner bounds obtained using mixed recurrence equations in the specific case when $m = 2$, as was done for the extreme zeros of the Jacobi, Laguerre and Gegenbauer polynomials in [Driver and Jordaan, 2012], Meixner and Krawtchouk polynomials in [Jooste and Jordaan, 2014] and Hahn polynomials in [Jooste et al., 2017]. In our applications, the polynomials $g_{n-m,k}, m \in \{2, 3, \ldots, n-1\}$ are typically obtained from the polynomials of the orthogonal sequence $\{p_n\}_{n\geq 0}$, by making appropriate parameter shifts of (in total) k units. Inner bounds for the extreme zeros of Gegenbauer, Laguerre and Jacobi polynomials were given in [Neumann, 1921], [Bottema, 1931], [Szegő, 1975], [Krasikov, 2006], [Gupta and Muldoon, 2007], [Area et al., 2012]; bounds for the extreme zeros of the discrete orthogonal Charlier, Meixner, Krawtchouk and Hahn polynomials in [Krasikov and Zarkh, 2009], [Area et al., 2013], for the extreme zeros of the q-Jacobi and q-Laguerre polynomials in [Gupta and Muldoon, 2007] and for the little q-Jacobi polynomials in [Gochhayat et al., 2016]. Lower bounds for $x_{n,1}$ and upper bounds for $x_{n,n}$ can be found in the case of classical continuous and discrete orthogonal polynomials in [Szegő, 1975], [Ismail and Li, 1992], [Krasikov, 2002], [Area et al., 2004], [Krasikov, 2006], [Dimitrov and Rafaeli, 2009], [Krasikov and Zarkh, 2009], [Dimitrov and Nikolov, 2010], [Area et al., 2013] and in [Krasikov, 2005], bounds of the extreme zeros of (symmetric) orthogonal polynomials are given in terms of the coefficients of their three-term recurrence equations.

In the next section, we use an algebraic method to obtain mixed three-term recurrence equations involving polynomials $p_n(x; \alpha, \beta)$ and $p_{n-1}(x; \alpha, \beta)$, belonging to the same sequence that is orthogonal on an interval (c, d) with respect to a measure $w(x; \alpha, \beta)$, and a polynomial from a related sequence, obtained by integer shifts of the parameters α and β, namely $p_{n-m}(x; \alpha + s, \beta + t), m \in \{2, 3, \ldots, n-1\}$, which is orthogonal with respect to

$$w(x; \alpha + s, \beta + t) = c_{s+t}(x; \alpha, \beta)w(x; \alpha, \beta) > 0$$

on (c, d), where $c_k(x; \alpha, \beta)$ is a polynomial of degree k in x. If the sequence is q-orthogonal with respect to the weight $w(x; \alpha, \beta)$, the equations involve the polynomials $p_n(x; \alpha, \beta)$ and $p_{n-1}(x; \alpha, \beta)$, and $p_{n-m}(x; \alpha q^s, \beta q^t), m \in \{2, 3, \ldots, n-1\}$ and the latter polynomial is orthogonal with respect to

$$w(x; \alpha q^s, \beta q^t) = c_{s+t}(x; \alpha, \beta)w(x; \alpha, \beta) > 0$$

on (c, d). From Theorem 1.11(i) it follows that such equations only exist for the values of s and t such that $s + t \in \{0, 1, \dots, 2m\}$. We note that the polynomial coefficient of the polynomial $p_{n-1}(x; \alpha, \beta)$ in the mixed recurrence equation involving polynomials $p_n(x; \alpha, \beta)$, $p_{n-1}(x; \alpha, \beta)$ and $p_{n-m}(x; \alpha + s, \beta + t), m \in \{2, 3, \dots, n-1\}$, will be denoted by $G_{m-1,s,t}(x)$. From Theorem 1.11(ii) and (iii) we deduce that the smallest and largest zeros of $G_{m-1,s,t}(x)$ are (inner) bounds for the extreme zeros of p_n.

1.4 Mixed recurrence equations satisfied by different sequences of orthogonal polynomial systems

As seen in the previous sections, we are interested by equations of type (1.2), (1.5) and (1.9). We will first show that such equations are structurally valid and then we provide an algorithmic approach to derive them.

In order to find for example equations of type (1.5) used to prove quasi-orthogonality, we can use the structure relation (cf. [Koepf and Schmersau, 1998], [Medem et al., 2001], [Foupouagnigni et al., 2012])

$$p_n(x) = a_n \mathcal{D} p_{n+1}(x) + b_n \mathcal{D} p_n(x) + c_n \mathcal{D} p_{n-1}(x), \tag{1.10}$$

where the constants a_n, b_n and c_n are explicitly given and $\mathcal{D}$ is a derivative or difference operator. Most of the classical orthogonal polynomial systems considered in the sequel (see [Koekoek et al., 2010, Chapters 9 and 14]) satisfy

$$\mathcal{D} p_n(x) = S(n) p_{n-1,k}(x), \;\; k \in \{-1, 0, 1, 2\}, \tag{1.11}$$

where $S(n)$ does not depend on x and $p_{n-1,k}(x)$ denotes the polynomial obtained when each of the parameters on which the polynomial $p_n(x)$ depends, can be shifted by k units in the case of the classical systems, or, in the case of the q-classical systems, when the parameters can each be multiplied by q^k. Substituting (1.11) in (1.10) yields

$$p_n(x) = a_n S(n+1) p_{n,k}(x) + b_n S(n) p_{n-1,k}(x) + c_n S(n-1) p_{n-2,k}(x)$$

or, by making a parameter shift,

$$p_{n,-k}(x) = a'_n S'(n+1) p_n(x) + b'_n S'(n) p_{n-1}(x) + c'_n S'(n-1) p_{n-2}(x),$$

where a'_n, b'_n, c'_n and $S'(n)$ are the values of the coefficients taking into consideration the parameter shift. We therefore get a linear combination of polynomials in an orthogonal sequence as in (1.5). For the so-called very classical orthogonal polynomials, the general expression for the parameters $a_{n,i}$, $i \in \{0, 1, 2\}$, in (1.5) for $k = 2$, i. e.,

$$q_{n,2}(x) = p_n(x) + a_{n,1} p_{n-1}(x) + a_{n,2} p_{n-2}(x)$$

were given in [Marcellán and Petronilho, 1994, Eq. (76)] in terms of the coefficients of the differential equations they satisfy.

We can also apply the operator $\mathcal{D}$ to (1.10) to obtain

$$\mathcal{D} p_n(x) = a_n \mathcal{D}^2 p_{n+1}(x) + b_n \mathcal{D}^2 p_n(x) + c_n \mathcal{D}^2 p_{n-1}(x). \tag{1.12}$$

Replacing (1.12) in (1.10) and using (1.11) twice, yields

$$\begin{aligned}p_n(x) = &a_n a_{n+1} S(n+2)S(n+1)p_{n,2k}(x) + a_n(b_n+b_{n+1})S(n+1)S(n)p_{n-1,2k}(x)\\ &+(a_n c_{n+1}+a_{n-1}c_n+b_n^2)S(n)S(n-1)p_{n-2,2k}(x)\\ &+c_n(b_n+b_{n-1})S(n-1)S(n-2)p_{n-3,2k}+c_n c_{n-1}S(n-2)S(n-3)p_{n-4,2k}(x).\end{aligned}$$

By applying a parameter shift again, we obtain

$$\begin{aligned}p_{n,-2k}(x) = &a'_n a'_{n+1} S'(n+2)S'(n+1)p_n(x) + a'_n(b'_n+b'_{n+1})S'(n+1)S'(n)p_{n-1}(x)\\ &+(a'_n c'_{n+1}+a'_{n-1}c'_n+(b'_n)^2)S'(n)S'(n-1)p_{n-2}(x)\\ &+c'_n(b'_n+b'_{n-1})S'(n-1)S'(n-2)p_{n-3}+c'_n c'_{n-1}S'(n-2)S'(n-3)p_{n-4}(x).\end{aligned}$$

These induction arguments show that equations of type (1.5) are structurally valid.

Classical orthogonal polynomials $p_n(x)$ satisfy the three-term recurrence equation

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x), n \in \{1,2,3,\ldots\}, \tag{1.13}$$

as well as a derivative rule [Koepf and Schmersau, 1998], [Medem et al., 2001], [Foupouagnigni et al., 2012] of the form

$$\tilde{\sigma}(x)\mathcal{D}p_n(x) = \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), n \in \{1,2,\ldots\}, \tag{1.14}$$

where $\mathcal{D} \in \{\frac{d}{dx}, \Delta, \nabla, D_q, D_{\frac{1}{q}}\}$. The coefficients $A_n, B_n, C_n, \alpha_n, \beta_n$ and γ_n are explicitly given in terms of the coefficients of the differential equations they satisfy.

Proposition 1.12. *Let $\{p_n(x)\}_{n\geq 0}$ be a system of classical orthogonal polynomials of a continuous, discrete or q-discrete variable, that satisfies (1.11). Then, for $k \in \{-1,0,1,2\}$, there exist polynomials $f_j(x), h_j(n,x), g_j(n,x)$ such that*

$$f_j(x)p_{n-j,jk}(x) = h_j(n,x)p_n(x) + g_{j-1}(n,x)p_{n-1}(x), j \in \{1,2,\ldots\}. \tag{1.15}$$

Proof. The proof is done by induction on $j \in \{1,2,\ldots\}$.

Step 1: Let $j=1$. If we substitute p_{n+1} from (1.13) into (1.14), we obtain

$$\tilde{\sigma}(x)\mathcal{D}p_n(x) = ((A_n x + B_n)\alpha_n + \beta_n)p_n(x) + (\gamma_n - C_n\alpha_n)p_{n-1}(x).$$

Application of (1.11) leads to

$$\tilde{\sigma}(x)S(n)p_{n-1,k}(x) = ((A_n x + B_n)\alpha_n + \beta_n)p_n(x) + (\gamma_n - C_n\alpha_n)p_{n-1}(x),$$

and (1.15) is valid for $j=1$ with

$$f_1(x) = \tilde{\sigma}(x), h_1(n,x) = \frac{1}{S(n)}((A_n x + B_n)\alpha_n + \beta_n), g_0(n,x) = \frac{\gamma_n - C_n\alpha_n}{S(n)}.$$

Step 2: Fix $j \geq 2$ and suppose that (1.15) is valid for j. We need to prove that (1.15) is also valid for $j+1$. We know that, for $Y_{n,j}(x) = \mathcal{D}^j p_n(x), j \in \{1,2,\ldots\}$, the relations

$$Y_{n+1,j}(x) = (A_{n,j}x + B_{n,j})Y_{n,j}(x) - C_{n,j}Y_{n-1,j}(x), n \in \{1,2,3,\ldots\}, \tag{1.16}$$

and

$$\tilde{\sigma}_j(x)\mathcal{D}Y_{n,j}(x) = \alpha_{n,j}Y_{n+1,j}(x) + \beta_{n,j}Y_{n,j}(x) + \gamma_{n,j}Y_{n-1,j}(x), n \in \{1,2,3,\ldots\}, \tag{1.17}$$

are valid since $\{Y_{n,j}(x)\}_{n\geq 0}$ is also a classical orthogonal polynomial system for $j = 1, 2, \ldots$. If we substitute (1.16) in (1.17), we obtain

$$\tilde{\sigma}_j(x)\mathcal{D}\left(\mathcal{D}^j p_n(x)\right) = ((A_{n,j}x+B_{n,j})\alpha_{n,j}+\beta_{n,j})\mathcal{D}^j p_n(x)+(\gamma_{n,j}-C_{n,j}\alpha_{n,j})\mathcal{D}^j p_{n-1}(x). \quad (1.18)$$

By iterating (1.11), it follows that

$$\mathcal{D}^j p_n(x) = p_{n-j,jk}(x)\prod_{l=0}^{j-1} S(n-l), j \leq n. \quad (1.19)$$

We substitute (1.19) in (1.18), multiply by $f_j(x)$ and use the induction hypothesis to get

$$\begin{aligned}\tilde{\sigma}_j(x)f_j(x)\mathcal{D}^{j+1}p_n(x) = \big((A_{n,j}x + B_{n,j})\alpha_{n,j} + \beta_{n,j}\big)\Big(&h_j(n,x)p_n(x)\\ &+ g_{j-1}(n,x)p_{n-1}(x)\Big)\prod_{l=0}^{j-1} S(n-l) + (\gamma_{n,j} - C_{n,j}\alpha_{n,j})\Big(h_j(n-1,x)p_{n-1}(x)\\ &+ g_{j-1}(n-1,x)p_{n-2}(x)\Big)\prod_{l=0}^{j-1} S(n-1-l).\end{aligned}$$

Replacing n by $n-1$ in (1.13), we obtain p_{n-2} that we substitute in the right hand side of the above equation and using once more (1.19) for j replaced by $j+1$, yields the equation

$$f_{j+1}(x)p_{n-(j+1),(j+1)k}(x) = h_{j+1}(n,x)p_n(x) + g_j(n,x)p_{n-1}(x),$$

where

$$\begin{aligned}f_{j+1}(x) &= \tilde{\sigma}_j(x)f_j(x),\\ h_{j+1}(n,x) &= \frac{(A_{n,j}x + B_{n,j})\alpha_{n,j} + \beta_{n,j}}{S(n-j+1)}h_j(n,x) - \frac{\gamma_{n,j} - C_{n,j}\alpha_{n,j}}{S(n)C_{n-1}}g_{j-1}(n-1,x),\\ g_j(n,x) &= \frac{(A_{n,j}x + B_{n,j})\alpha_{n,j} + \beta_{n,j}}{S(n-j+1)}g_{j-1}(n,x)\\ &\quad + \frac{\gamma_{n,j} - C_{n,j}\alpha_{n,j}}{S(n)}\left(h_j(n-1,x) + \frac{g_{j-1}(n-1,x)}{C_{n-1}}(A_{n-1}x + B_{n-1})\right).\end{aligned}$$

□

This proof shows how one can iteratively get equations of type (1.9) for classical orthogonal polynomials of a continuous, a discrete or a q-discrete variable. We also refer the reader to [Koepf and Schmersau, 1998], [Foupouagnigni et al., 2012], [Tcheutia, 2014], where we have the so-called connection formulae for classical orthogonal polynomials from which one can deduce certain equations of type (1.2), (1.5) and (1.9). One may also use contiguous relations for the hypergeometric and basic hypergeometric series (see e.g. [Heine, 1847], [Swarttouw, 1990], [Jordaan and Toókos, 2009], [Gochhayat et al., 2016]) to get some of these recurrence equations, as well as the generating functions of classical orthogonal polynomials (see e.g. [Jordaan and Toókos, 2009], [Tcheutia et al., 2018b]). Another option to get equations of type (1.9) is the following.

Lemma 1.13. *(Christoffel's formula, cf. [Szegő, 1975, Theorem 2.5]) Let $\{p_n(x)\}_{n\geq 0}$ be the orthonormal polynomials associated with the distribution $d\alpha(x)$ on the interval $[c,d]$. Also let*

$$\rho(x) = a(x-x_1)(x-x_2)\cdots(x-x_k),\ \ a \neq 0, \quad (1.20)$$

be a polynomial of degree k which is non-negative in this interval. Then the orthogonal polynomials $\{q_n(x)\}_{n\geq 0}$, associated with the distribution $\rho(x)d\alpha(x)$, can be represented in terms of the polynomials $p_n(x)$ as follows:

$$\rho(x)q_n(x) = \begin{vmatrix} p_n(x) & p_{n+1}(x) & \dots & p_{n+k}(x) \\ p_n(x_1) & p_{n+1}(x_1) & \dots & p_{n+k}(x_1) \\ \dots & \dots & \dots & \dots \\ p_n(x_k) & p_{n+1}(x_k) & \dots & p_{n+k}(x_k) \end{vmatrix}. \tag{1.21}$$

In case of a zero x_j, of multiplicity m, $m > 1$, we replace the corresponding rows of (1.21) by the derivatives of order $0, 1, 2, \dots, m-1$ of the polynomials $p_n(x), p_{n+1}(x), \dots, p_{n+k}(x)$ at $x = x_j$.

Using Christoffel's formula for $\rho(x) = c_k(x)$ in (1.20) and $q_n(x) = g_{n-m,k}(x)$, we get

$$c_k(x)g_{n-m,k}(x) = \sum_{j=0}^{k} u_j p_{n-m+j}(x).$$

By using Beardon's theorem (cf. Lemma 1.9), the latter equation can be reduced to a mixed three-term recurrence equation of type (1.9).

However, due to the complexity of classical methods, it is not really easy for example to get equations of type (1.2), (1.5) and (1.9) satisfied by classical orthogonal polynomials on quadratic or q-quadratic lattices, or in general to get in a unified approach equations of type (1.2), (1.5) and (1.9). As a consequence, our algorithmic approach to derive such equations is welcome. Using an extended version of Zeilberger's algorithm (see e.g. [Petkovšek et al., 1996], [Koepf, 2014]) and following the approach in [Chen et al., 2012], [Koepf, 2014], we write, using the Computer Algebra System Maple, procedures to find mixed recurrence equations of type (1.2), (1.5) and (1.9) satisfied by all the classical orthogonal polynomials. We also use an adaption of the q-version of Zeilberger's algorithm which is an extension of Gosper's algorithm. Gosper's algorithm deals with the question how to find an anti-difference s_k for given a_k, i.e., a sequence s_k for which $a_k = \Delta s_k = s_{k+1} - s_k$, in a particular case that s_k is a hypergeometric term, i.e., $\dfrac{s_{k+1}}{s_k} \in \mathbb{Q}(k)$.

Given $F(n,k)$, Zeilberger's algorithm provides a recurrence equation for

$$s_n = \sum_{k=-\infty}^{\infty} F(n,k),$$

where $F(n,k)$ is a hypergeometric term with respect to both n and k. We set

$$a_k = F(n,k) + \sum_{j=1}^{J} \sigma_j(n) F(n+j,k) \tag{1.22}$$

with undetermined variables $\sigma_j(n)$ and apply Gosper's algorithm to a_k. If successful, Gosper's algorithm finds $g(n,k)$ with

$$g(n,k+1) - g(n,k) = a_k \tag{1.23}$$

and at the same time $\sigma_j(n)$, $j \in \{1, 2, \dots, J\}$. By summation, we have from (1.23)

$$0 = \sum_{k=-\infty}^{\infty} a_k$$

as a telescoping sum and from (1.22)

$$s_n + \sum_{j=1}^{J} \sigma_j(n) s_{n+j} = \sum_{k=-\infty}^{\infty} a_k.$$

We deduce that

$$s_n + \sum_{j=1}^{J} \sigma_j(n) s_{n+j} = 0,$$

which is a recurrence equation satisfied by s_n. We refer the reader to [Koepf, 2014, Chapters 5–7] and references therein for more details about the algorithms of Gosper and Zeilberger and their q-analogues. The q-analogues of Gosper's and Zeilberger's algorithms are implemented in the Maple `qsum` package [Koepf, 2014] which can be downloaded at http://www.mathematik.uni-kassel.de/~koepf/Publikationen. By applying an adaption of the `sumdiffeq` [Koepf, 2014, p. 210] and the `qsumdiffeq` [Koepf, 2014, p. 219] procedures of the `hsum` and the `qsum` packages, we wrote codes to derive recurrence equations of type (1.2), (1.5) and (1.9) for the classical orthogonal polynomial systems considered in the sequel. Our Maple codes can be downloaded from http://www.mathematik.uni-kassel.de/~tcheutia/.

The first program called `Mixedrec1`$(F, k, S(n), s_0, a, s_1, s_2, r)$ finds a recurrence equation of the form

$$S(n - s_0, a + s_1) = \sum_{j=0}^{J} \sigma_j S(n - j, a + s_2 + rj), J \in \{1, 2, \dots\},\ r \in \{0, 1\},$$

where $S(n, a) = \sum_{k=-\infty}^{\infty} F$, F is a hypergeometric term w.r.t. k, n and a, and s_0, s_1, s_2, are integers and the second one, denoted by `qMixedrec1`$(F, q, k, S(n), s_0, a, s_1, s_2, r)$, is the q-analogue of the first one and finds a recurrence equation of the form

$$S(n - s_0, aq^{s_1}) = \sum_{j=0}^{J} \sigma_j S(n - j, aq^{s_2+rj}), J \in \{1, 2, \dots\},\ r \in \{0, 1\}.$$

These first two programs can be used when we want to play with at most one parameter of the polynomial. When dealing with two parameters, the following ones generalizing the first ones are more suitable.

The program called `Mixedrec2`$(F, k, S(n), n_0, a, s_0, b, s_1, s_2, s_3, r_1, r_2)$ finds a recurrence equation of the form

$$S(n-n_0, a+s_0, b+s_1) = \sum_{j=0}^{J} \sigma_j S(n-j, a+s_2+r_1 j, b+s_3+r_2 j), J \in \{1, 2, \dots\},\ r_1, r_2 \in \{0, 1\},$$

where $S(n,a,b) = \sum_{k=-\infty}^{\infty} F$, F is a hypergeometric term w.r.t. k, n, a and b, and n_0, s_i, $i = 0,1,2,3$, are integers and the one denoted by `qMixedrec2`$(F, q, k, S(n), n_0, a, s_0, b, s_1, s_2, s_3, r_1, r_2)$ is its q-analogue and finds a recurrence equation of the form

$$S(n-n_0, aq^{s_0}, bq^{s_1}) = \sum_{j=0}^{J} \sigma_j S(n-j, aq^{s_2+r_1 j}, bq^{s_3+r_2 j}), J \in \{1,2,\ldots\},\ r_1, r_2 \in \{0,1\}.$$

Note that these procedures can be extended to as many parameters as possible.

Chapter 2

Classical orthogonal polynomials of a continuous and a discrete variable

In this chapter we use our algorithms to recover known mixed recurrence equations from which the interlacing properties of classical orthogonal polynomials of a continuous and a discrete variable were derived. We also recover equations which characterize quasi-orthogonal polynomials of a continuous and a discrete variable. With our implementations, we can derive more mixed recurrence equations which cannot easily be obtained using contiguous relations of hypergeometric functions or their generating functions. Moreover, we show that using our implementations, we get mixed recurrence equations which can improve the existing bounds of the extreme zeros of the polynomials considered in this chapter. These bounds can be found in our joint work [Jooste et al.]. Let us recall that Jooste et al. [2017] were the first to use this algorithmic approach to find bounds of the Hahn polynomials and this paper was the starting point of this work. Finally, the existing results on interlacing properties, quasi-orthogonality or bounds of the extreme zeros are collected in this chapter to make them accessible for the readers. We cite the references for each result and the proofs can be found in the cited references. In the sequel, we will denote the monic polynomials associated to p_n by $\tilde{p}_n$.

2.1 The Jacobi polynomials

The Jacobi polynomials defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\left.\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}\right|\frac{1-x}{2}\right),$$

are orthogonal on $(-1,1)$ with respect to $w(x) = (1-x)^\alpha(1+x)^\beta$ provided that $\alpha > -1,\ \beta > -1$. We denote the monic Jacobi family by $\tilde{P}_n^{(\alpha,\beta)}(x) = \frac{2^n n!}{(n+\alpha+\beta+1)_n} P_n^{(\alpha,\beta)}(x)$.

2.1.1 Interlacing of zeros

Our implementations (see http://www.mathematik.uni-kassel.de/~tcheutia/), using

```
> FJac:=pochhammer(alpha+1, n)/n!
> *(hyperterm([-n,n+alpha+beta+1],[alpha+1],(1-x)/2,k))
```

```
> Mixedrec1(FJac,k,P(n),0,alpha,0,1,0);
> Mixedrec1(FJac,k,P(n),0,beta,0,1,0);
> Mixedrec2(FJac,k,P(n),0,alpha,0,beta,1,1,0,0,1);
> Mixedrec1(FJac,k,P(n),1,beta,2,0,0);
> Mixedrec1(FJac,k,P(n),1,alpha,2,0,0);
> Mixedrec2(FJac,k,P(n),0,alpha,0,beta,2,0,0,0,1);
```

give the following equations (see e.g. [Driver et al., 2008]), respectively,

$$P_n^{(\alpha,\beta)}(x) = \frac{(n+\alpha+\beta+1)}{2\,n+\alpha+\beta+1}P_n^{(\alpha+1,\beta)}(x) - \frac{(n+\beta)}{2\,n+\alpha+\beta+1}P_{n-1}^{(\alpha+1,\beta)}(x); \tag{2.1}$$

$$P_n^{(\alpha,\beta)}(x) = \frac{(n+\alpha+\beta+1)}{2\,n+\alpha+\beta+1}P_n^{(\alpha,\beta+1)}(x) + \frac{(n+\alpha)}{2\,n+\alpha+\beta+1}P_{n-1}^{(\alpha,\beta+1)}(x); \tag{2.2}$$

$$P_n^{(\alpha,\beta+1)}(x) = P_n^{(\alpha+1,\beta)}(x) - P_{n-1}^{(\alpha+1,\beta+1)}(x);$$

$$\begin{aligned}
&(x+1)^2\,(n+\alpha+\beta+1)\,(2\,n+\alpha+\beta)\,P_{n-1}^{(\alpha,\beta+2)}(x)\\
&= 2\,n\,((2\,n+\alpha+\beta)\,x+(\alpha+3\,\beta+2\,n+2))\,P_n^{(\alpha,\beta)}(x)+4\,(\beta+1)\,(n+\beta)\,P_{n-1}^{(\alpha,\beta)}(x);\\
&(x-1)^2\,(n+\alpha+\beta+1)\,(2\,n+\alpha+\beta)\,P_{n-1}^{(\alpha+2,\beta)}(x)\\
&= 2\,n\,((2\,n+\alpha+\beta)\,x-(3\,\alpha+\beta+2\,n+2))\,P_n^{(\alpha,\beta)}(x)+4\,(\alpha+1)\,(\alpha+n)\,P_{n-1}^{(\alpha,\beta)}(x);\\
&(n+\alpha+\beta+1)\,(n+\alpha+\beta+2)\,(x+1)\,P_n^{(\alpha,\beta+2)}(x)\\
&= ((2\,n+\alpha+\beta+1)\,(2\,n+\alpha+\beta+2)\,(x+1)-2\,n\,(\alpha+n))\,P_n^{(\alpha,\beta)}(x)\\
&-(\alpha+n)\,((2\,n+\alpha+\beta+2)\,x+\alpha+3\,\beta+2\,n+4)\,P_{n-1}^{(\alpha,\beta+1)}(x).
\end{aligned}$$

From the latter equations, the following interlacing results are deduced. Their proofs can be found in [Driver et al., 2008].

Theorem 2.1 (see [Driver et al., 2008]). *Let $\alpha > -1$, $\beta > -1$, $t \in (0,2)$ and $k \in (0,2)$. Let $-1 < x_{n,1} < \ldots < x_{n,n} < 1$ be the zeros of $\tilde{P}_n^{(\alpha,\beta)}(x)$, $-1 < y_{n,1} < \ldots < y_{n,n} < 1$ be the zeros of $P_n^{(\alpha+t,\beta)}(x)$, $-1 < z_{n,1} < \ldots < z_{n,n} < 1$ be the zeros of $P_n^{(\alpha,\beta+k)}(x)$, $-1 < Y_{n,1} < \ldots < Y_{n,n} < 1$ be the zeros of $P_n^{(\alpha+2,\beta)}(x)$, $-1 < Z_{n,1} < \ldots < Z_{n,n} < 1$ be the zeros of $P_n^{(\alpha,\beta+2)}(x)$, $-1 < t_{n,1} < \ldots < t_{n,n} < 1$ be the zeros of $P_n^{(\alpha+t,\beta+k)}(x)$. Then for $i = 1,2,\ldots,n-1$,*

- *(a)* $y_{n,i} < x_{n,i} < y_{n-1,i} < y_{n,i+1} < x_{n,i+1}$,
- *(b)* $x_{n,i} < z_{n,i} < z_{n-1,i} < x_{n,i+1} < z_{n,i+1}$,
- *(c)* $y_{n,i} < x_{n,i} < z_{n,i} < t_{n-1,i} < y_{n,i+1} < x_{n,i+1} < z_{n,i+1}$,
- *(d)* $x_{n,i} < x_{n-1,i} < z_{n-1,i} < Z_{n-1,i} < x_{n,i+1}$,
- *(e)* $x_{n,i} < Y_{n-1,i} < y_{n-1,i} < x_{n-1,i} < x_{n,i+1}$.

Remark 2.2. *The interlacing properties in Theorem 2.1 are not retained in general (see [Driver et al., 2008]):*

- *(a) when one or both of the parameters α, β are increased by more than 2,*

(b) *for the zeros of* $P_n^{(\alpha,\beta)}(x)$ *and those of* $P_{n-1}^{(\alpha-t,\beta)}(x)$ *or* $P_{n-1}^{(\alpha,\beta-k)}(x)$ *or* $P_{n-1}^{(\alpha-t,\beta-k)}(x)$ *where* $t,k>0$.

Using the counterexample $\alpha=1.266,\ \beta=1.85,\ \alpha'=\alpha+0.2,\ \beta'=\beta+0.2,\ n=4,$ *Driver et al. [2008] remarked that the zeros of* $P_n^{(\alpha,\beta)}(x)$ *and* $P_n^{(\alpha',\beta')}(x)$ *do not interlace in general when both parameters are increased simultaneously.*

2.1.2 Quasi-orthogonality

We substitute α by $\alpha-1$ in (2.1) and β by $\beta-1$ in (2.2) to get

$$P_n^{(\alpha-1,\beta)}(x)=\frac{n+\alpha+\beta}{2\,n+\alpha+\beta}P_n^{(\alpha,\beta)}(x)-\frac{n+\beta}{2\,n+\alpha+\beta}P_{n-1}^{(\alpha,\beta)}(x); \tag{2.3}$$
$$P_n^{(\alpha,\beta-1)}(x)=\frac{n+\alpha+\beta}{2\,n+\alpha+\beta}P_n^{(\alpha,\beta)}(x)+\frac{n+\alpha}{2\,n+\alpha+\beta}P_{n-1}^{(\alpha,\beta)}(x). \tag{2.4}$$

Substituting β by $\beta-1$ in (2.3) and using (2.4) yields

$$P_n^{(\alpha-1,\beta-1)}(x)=\frac{(n+\alpha+\beta-1)(n+\alpha+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta-1)}P_n^{(\alpha,\beta)}(x)$$
$$+\frac{(\alpha-\beta)(n+\alpha+\beta-1)}{(2n+\alpha+\beta)(2n+\alpha+\beta-2)}P_{n-1}^{(\alpha,\beta)}(x)-\frac{(n+\beta-1)(n+\alpha-1)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta-2)}P_{n-2}^{(\alpha,\beta)}(x).$$

Using our implementations, we also recover the following mixed recurrence equations given in [Driver and Jordaan, 2018].

$$P_{n-1}^{(\alpha,\beta)}(x)=\frac{2\left((\alpha+\beta+2n-2)\,x-\alpha+\beta\right)n}{(x^2-1)\,(n+\alpha+\beta-1)\,(2\,n+\alpha+\beta-2)}P_n^{(\alpha-1,\beta-1)}(x)$$
$$-\frac{4\,(n+\beta-1)\,(n+\alpha-1)}{(x^2-1)\,(n+\alpha+\beta-1)\,(2\,n+\alpha+\beta-2)}P_{n-1}^{(\alpha-1,\beta-1)}(x);$$
$$P_n^{(\alpha-1,\beta-1)}(x)=P_n^{(\alpha-1,\beta)}(x)+\frac{1}{2}\,(1-x)\,P_{n-1}^{(\alpha,\beta)}(x);$$
$$P_n^{(\alpha-1,\beta-1)}(x)=\frac{(n+\alpha+\beta)}{\alpha+\beta+2\,n}P_n^{(\alpha,\beta)}(x)-\frac{((\alpha+\beta+2\,n)\,x-\alpha+\beta)}{2(\alpha+\beta+2\,n)}P_{n-1}^{(\alpha,\beta)}(x).$$

Theorem 2.3 (see [Brezinski et al., 2004], [Driver and Jordaan, 2018]). *(i) The Jacobi polynomials* $P_n^{(\alpha-k,\beta-l)}(x)$ *where* $-1<\alpha<0,\ -1<\beta<0,$ *and* $k,l\in\mathbb{N}$ *with* $k+l<n$, *are quasi-orthogonal of order* $k+l$ *with respect to the weight function* $(1-x)^\alpha(1+x)^\beta$ *on the interval* $(-1,1)$. $P_n^{(\alpha-k,\beta-l)}(x)$, *with* $k+l<n$, *has at least* $n-(k+l)$ *zeros in the interval* $(-1,1)$.

(ii) Let $x_{n,1}<\ldots<x_{n,n}$ *be the zeros of* $P_n^{(\alpha,\beta)}(x)$, $y_{n,1}<\ldots<y_{n,n}$ *be the zeros of* $P_n^{(\alpha-1,\beta)}(x)$, $z_{n,1}<\ldots<z_{n,n}$ *be the zeros of* $P_n^{(\alpha,\beta-1)}(x)$ *and* $t_{n,1}<\ldots<t_{n,n}$ *be the zeros of* $P_n^{(\alpha-1,\beta-1)}(x)$:

(a) *If* $-1<\alpha$ *and* $-1<\beta<0$, *then*

$$z_{n,1}<-1<x_{n,1}<x_{n-1,1}<z_{n,2}<x_{n,2}<\ldots<x_{n-1,n-1}<z_{n,n}<x_{n,n};$$

(b) *If* $-1 < \beta$ *and* $-1 < \alpha < 0$, *then*

$$x_{n,1} < y_{n,1} < x_{n-1,1} < x_{n,2} < y_{n,2} < \ldots < x_{n-1,n-1} < x_{n,n} < 1 < y_{n,n};$$

(c) *If* $-1 < \alpha < 0$, $-1 < \beta < 0$, *then*

- *for* $n \geq 2$,

$$t_{n,1} < -1 < x_{n-1,1} < t_{n,2} < x_{n-1,2} < \ldots < t_{n,n-1} < x_{n-1,n-1} < 1 < t_{n,n};$$

- *for* $n \geq 3$,

$$t_{n-1,1} < t_{n,1} < -1 < t_{n,2} < t_{n-1,2} < \ldots < t_{n-1,n-2} < t_{n,n-1} < 1 < t_{n,n} < t_{n-1,n-1};$$

- *for* $n \geq 1$,

$$t_{n,1} < -1 < y_{n,1} < t_{n,2} < \ldots < t_{n,n-1} < y_{n,n-1} < 1 < y_{n,n} < t_{n,n};$$

- *for* $n \geq 2$, *the zeros of* $P_n^{(\alpha-1,\beta-1)}$ *and the zeros of* $P_n^{(\alpha,\beta)}$ *are not interlacing. However, if* $P_n^{(\alpha-1,\beta-1)}(\gamma) \neq 0$, *where* $\gamma := \frac{\alpha-\beta}{\alpha+\beta+2n}$, *the zeros of* $P_n^{(\alpha,\beta)}(x)$ *interlace with the zeros of* $(x-\gamma)P_n^{(\alpha-1,\beta-1)}(x)$.

2.1.3 Bounds of the extreme zeros

We provide, using our code, equations of the form

$$f(x)P_{n-k}^{(\alpha+s_1,\ \beta+s_2)}(x) = H(x)P_n^{(\alpha,\beta)}(x) + G_{k-1,s_1,s_2}(x)P_{n-1}^{(\alpha,\beta)}(x),\ s_1+s_2 \in \{0,1,\ldots,2k\}. \tag{2.5}$$

If we denote by $B^{(1)}_{k,s_1,s_2}$, $B^{(2)}_{k,s_1,s_2}$ the smallest and the largest zero of G_{k-1,s_1,s_2}, respectively, then for the smallest zero $x_{n,1}$ and the largest zero $x_{n,n}$ of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, the inequality

$$x_{n,1} < B^{(1)}_{k,s_1,s_2} < B^{(2)}_{k,s_1,s_2} < x_{n,n},\ \ k \leq n, \tag{2.6}$$

is valid. For $k = 2$, $s_1 = 2$ and $s_2 = 0$, we recover the bound given by [Szegő, 1975, Eq. (6.2.11)]

$$B^{(2)}_{2,2,0} = 1 - \frac{2(\alpha+1)}{2n+\alpha+\beta} < x_{n,n}.$$

For $k = 2,\ s_1 = 4,\ s_2 = 0$, the bound (cf. [Driver and Jordaan, 2012, Eq. (8)]) for $x_{n,n}$

$$B^{(2)}_{2,4,0} = 1 - \frac{2(\alpha+1)(\alpha+3)}{2(n-1)(n+\alpha+\beta+2)+(\alpha+3)(\alpha+\beta+2)} < x_{n,n}, \tag{2.7}$$

which is already sharper than [Szegő, 1975, Eq. (6.2.11)] and the bound obtained in [Area et al., 2012, Cor. 3]), is recovered. For $k = 2,\ s_1 = 0,\ s_2 = 4$, we also recover a bound (cf. [Driver and Jordaan, 2012, Section 2.2]) for $x_{n,1}$

$$x_{n,1} < B^{(1)}_{2,0,4} = -1 + \frac{2(\beta+1)(\beta+3)}{2(n-1)(n+\alpha+\beta+2)+(\beta+3)(\alpha+\beta+2)}. \tag{2.8}$$

For $k=3$, $s_1=0$ and $s_2=6$, we have (2.5) with $G_{2,0,6}(x)=a_2x^2+a_1x+a_0$, where

$$a_2=a_0(\alpha,\beta)+K(\alpha,\beta),\ \ a_1=2a_0(\alpha,\beta)+K(\alpha,\beta)-4\frac{(\beta+1)_5}{\beta+3},$$
$$a_0:=a_0(\alpha,\beta)=3\,n^4+(6\,\alpha+6\,\beta+6)\,n^3+\left(3\,\alpha^2+9\,\alpha\,\beta-\beta^2+9\,\alpha-24\,\beta-41\right)n^2$$
$$+\,(\alpha+\beta+1)\left(3\,\alpha\,\beta-4\,\beta^2+3\,\alpha-30\,\beta-44\right)n+(\beta+1)_2\left((\alpha-\beta)^2-11\,\alpha+13\,\beta+38\right),$$
$$K(\alpha,\beta)=8\,(\beta+4)\,(\beta+2)\left(n^2+(\alpha+\beta+1)\,n+\frac{1}{2}\,(\beta+1)\,(\alpha-2)\right).$$

The smallest zero

$$B^{(1)}_{3,0,6}=\frac{1}{2a_2}(-a_1-\sqrt{a_1^2-4a_0a_2})$$

of $G_{2,0,6}$ is an upper bound of $x_{n,1}$ and is sharper than the bound (2.8). Let us note here and everywhere else in the sequel that we say "sharper or more accurate" in general according to simulations.

For $k=3$, $s_1=6$ and $s_2=0$, we have (2.5) with $G_{2,6,0}(x)=b_2x^2+b_1x+b_0$, where

$$b_2=a_0(\beta,\alpha)+K(\beta,\alpha),\ b_1=-2a_0(\beta,\alpha)-K(\beta,\alpha)+4\frac{(\alpha+1)_5}{\alpha+3},\ b_0=a_0(\beta,\alpha).$$

The largest zero

$$B^{(2)}_{3,6,0}=\frac{1}{2b_2}(-b_1+\sqrt{b_1^2-4b_0b_2})$$

of $G_{2,6,0}$ is a lower bound of $x_{n,n}$ and is sharper than the bound (2.7). Some numerical simulations are done in Table 2.1 to illustrate how sharp are the bounds derived from our recurrence equations.

n	$n=4$	$n=12$	$n=19$	$n=100$
α	$\alpha=-0.9$	$\alpha=30.9$	$\alpha=30.9$	$\alpha=-0.5$
β	$\beta=-0.8$	$\beta=-0.8$	$\beta=32.8$	$\beta=30$
zero $x_{n,1}$	-0.9668157248842541	-0.999156791323282	-0.682	-0.951
bound $B^{(1)}_{4,0,8}$	-0.966815724842536	-0.999156791323269	-0.677	-0.94998
bound $B^{(1)}_{3,0,6}$	-0.966815719	-0.9991567909	-0.657	-0.946
bound (2.8)	-0.96674	-0.9991545	-0.55	-0.92
bound (2.7)	0.984109	-0.05	0.59	0.99990427
bound $B^{(2)}_{3,6,0}$	0.98411889115	0.1083	0.687	0.9999055189
bound $B^{(2)}_{4,8,0}$	0.98411889130462334	0.1369	0.7058	0.99990552024133
zero $x_{n,n}$	0.98411889130462342	0.1414	0.7102	0.99990552024165

Table 2.1: Comparison of the bounds for the extreme zeros of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$

Krasikov [2006] proved that if $\alpha \geq \beta > -1$, then for $n \geq 5$

$$x_{n,1} < A + 9(1-A^2)^{2/3}(2R)^{-1/3}, \tag{2.9}$$

and for $n \geq 56$,

$$x_{n,n} > B - 9(1-B^2)^{2/3}(2R)^{-1/3}, \tag{2.10}$$

where

$$s = \alpha+\beta+1,\ q = \alpha-\beta,\ r = 2n+\alpha+\beta+1,\ R = \sqrt{(r^2-q^2+2s+1)(r^2-s^2)},$$

and

$$A = -\frac{R+q(s+1)}{r^2+2s+1},\ B = \frac{R-q(s+1)}{r^2+2s+1}.$$

In Table 2.2, we compare the bounds (2.9), (2.10), $B^{(1)}_{3,0,6}, B^{(2)}_{3,6,0}$ under the hypothesis of (2.9), (2.10).

n	$n=10$	$n=56$	$n=75$	$n=100$
α	$\alpha=0.5$	$\alpha=75$	$\alpha=50$	$\alpha=-0.5$
β	$\beta=-0.8$	$\beta=13$	$\beta=-0.1$	$\beta=-0.8$
$x_{n,1}$	-0.995905277982168	-0.98178	-0.9997329012	-0.999955987905
bound $B^{(1)}_{4,0,8}$	-0.995905277982118	-0.98168	-0.999732900	-0.9999559833
bound $B^{(1)}_{3,0,6}$	-0.995905276236077	-0.9809	-0.99973287	-0.99995598788
bound (2.9)	-0.994901948730266	-0.971	-0.99954	-0.999945
bound (2.10)	n/a	0.577	0.789	0.999808
bound $B^{(2)}_{3,6,0}$	0.954222147582673	0.616	0.819	0.9998762585
bound $B^{(2)}_{4,8,0}$	0.954244065608416	0.6468	0.8338	0.9998762603419
$x_{n,n}$	0.954244105748079	0.660	0.8394	0.9998762603423

Table 2.2: Comparison of the bounds (2.9), (2.10), $B^{(1)}_{3,0,6}$, $B^{(2)}_{3,6,0}$ of $P_n^{(\alpha,\beta)}(x)$

We note from various numerical simulations that for the Jacobi polynomials, sharpest bounds for $x_{n,1}$ are the smallest zero of $G_{k-1,0,2k}$, $k \geq 2$ in (2.5) and we get sharpest bounds for $x_{n,n}$ by taking the largest zero of $G_{k-1,2k,0}$, $k \geq 2$ in (2.5).

2.2 The Laguerre polynomials

The Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{array}{c|c} -n & \\ & x \\ \alpha+1 & \end{array}\right),\ \alpha > -1,$$

are orthogonal w.r.t. $w(x) = x^\alpha e^{-x}$ on $(0,\infty)$. The monic Laguerre polynomials will be denoted by $\tilde{L}_n^{(\alpha)}(x) = (-1)^n n! L_n^{(\alpha)}(x)$.

2.2.1 Interlacing of the zeros

The following mixed recurrence equations (that we recover using our implementations) are valid (see e.g. [Driver and Jordaan, 2007], [Driver and Jordaan, 2011, Eq. (4)])

$$\begin{aligned}
&\tilde{L}_n^{(\alpha)}(x) = \tilde{L}_n^{(\alpha+1)}(x) + n\tilde{L}_{n-1}^{(\alpha+1)}(x); && (2.11)\\
&(\alpha+n+1)\tilde{L}_n^{(\alpha)}(x) = (\alpha+1)\tilde{L}_n^{(\alpha+2)}(x) + n(x+\alpha+1)\tilde{L}_{n-1}^{(\alpha+2)}(x);\\
&x\tilde{L}_n^{(\alpha+1)}(x) = -(n-x)\tilde{L}_n^{(\alpha)}(x) - n(n+\alpha)\tilde{L}_{n-1}^{(\alpha)}(x);\\
&x\tilde{L}_{n-1}^{(\alpha+1)}(x) = \tilde{L}_n^{(\alpha)}(x) + (n+\alpha)\tilde{L}_{n-1}^{(\alpha)}(x); && (2.12)\\
&x^2\tilde{L}_{n-1}^{(\alpha+2)}(x) = (x+\alpha+1)\tilde{L}_n^{(\alpha)}(x) + (\alpha+1)(n+\alpha)\tilde{L}_{n-1}^{(\alpha)}(x);\\
&(n+\alpha+1)\tilde{L}_n^{(\alpha)}(x) = (\alpha+1)\tilde{L}_n^{(\alpha+1)}(x) + nx\tilde{L}_{n-1}^{(\alpha+2)}(x).
\end{aligned}$$

The following interlacing properties are derived from the first two preceding equations.

Theorem 2.4 (see [Driver and Jordaan, 2007]). *Let $\alpha > -1$, and let $0 < x_{n,1} < \ldots < x_{n,n}$ be the zeros of $L_n^{(\alpha)}(x)$, let $0 < y_{n,1} < \ldots < y_{n,n}$ be the zeros of $L_n^{(\alpha+1)}(x)$ while $0 < Y_{n,1} < \ldots < Y_{n,n}$ are the zeros of $L_n^{(\alpha+2)}(x)$. Then for $i \in \{1, 2, \ldots, n-1\}$,*

(a) $x_{n,i} < y_{n,i} < y_{n-1,i} < x_{n,i+1} < y_{n,i+1}$,

(b) $x_{n,i} < Y_{n,i} < Y_{n-1,i} < x_{n,i+1} < Y_{n,i+1}$,

(c) $x_{n,i} < x_{n-1,i} < y_{n-1,i} < Y_{n-1,i} < x_{n,i+1}$,

(d) $x_{n,i} < y_{n,i} < Y_{n,i} < Y_{n-1,i} < x_{n,i+1} < y_{n,i+1} < Y_{n,i+1}$.

Let $0 < z_{n,1} < \ldots < z_{n,n}$ be the zeros of $L_n^{(\alpha+t)}(x)$ where $0 < t < 2$, then $0 < x_{n,i} < z_{n,i} < Y_{n,i}$, $i = 1, 2, \ldots, n$. However, as shown by Driver and Jordaan [2007], the zeros of $L_n^{(\alpha+3)}(x)$ and $L_n^{(\alpha)}(x)$, as well as the zeros of $L_n^{(\alpha+2)}(x)$ and $L_{n-1}^{(\alpha)}(x)$ do not interlace in general for $\alpha > -1$.

Using our implementations, we recover the equations obtained in [Driver and Jordaan, 2011, Eqs. (5), (9), (12)]:

$$\begin{aligned}
L_{n+1}^{(\alpha)}(x) &= \frac{(n+\alpha+1-x)\,L_n^{(\alpha)}(x)}{n+1} - \frac{xL_{n-1}^{(\alpha+1)}(x)}{n+1};\\
L_{n+1}^{(\alpha)}(x) &= \frac{(\alpha-x+1)\,(\alpha+1+n)\,L_n^{(\alpha)}(x)}{(n+1)\,(\alpha+1)} - \frac{x^2L_{n-1}^{(\alpha+2)}(x)}{(n+1)\,(\alpha+1)};\\
L_{n-2}^{(\alpha+3)}(x) &= -\frac{n\,((-n+1)\,x+(\alpha+2)\,(\alpha+1))\,L_n^{(\alpha)}(x)}{x^3}\\
&+\frac{(\alpha+n)\,((-\alpha-n-1)\,x+(\alpha+2)\,(\alpha+1))\,L_{n-1}^{(\alpha)}(x)}{x^3};\\
L_{n-2}^{(\alpha+4)}(x) &= -\frac{n\,((-n+1)\,x^2-2\,(n-1)\,(\alpha+2)\,x+(\alpha+3)\,(\alpha+2)\,(\alpha+1))\,L_n^{(\alpha)}(x)}{x^4}\\
&+\frac{(\alpha+2)\,(\alpha+n)\,((-\alpha-2\,n-1)\,x+(\alpha+3)\,(\alpha+1))\,L_{n-1}^{(\alpha)}(x)}{x^4}; \qquad (2.13)
\end{aligned}$$

$$(n+1)L_{n+1}^{(\alpha)}(x) = (\alpha+2n+1-x)L_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x). \qquad (2.14)$$

Driver and Jordaan [2011] derived from the latter equations the following results:

Theorem 2.5 (see [Driver and Jordaan, 2011]). *(a) The zeros of $L_{n-1}^{(\alpha)}$, together with the point $\alpha+1+2n$, interlace with the zeros of $L_{n+1}^{(\alpha)}$;*

(b) The zeros of $L_{n-1}^{(\alpha+1)}$, together with the point $\alpha+1+n$, interlace with the zeros of $L_{n+1}^{(\alpha)}$;

(c) The zeros of $L_{n-1}^{(\alpha+2)}$, together with the point $\alpha+1$, interlace with the zeros of $L_{n+1}^{(\alpha)}$;

(d) The zeros of $L_{n-1}^{(\alpha+3)}$, together with the point $\frac{(\alpha+1)(\alpha+2)}{n+\alpha+2}$, interlace with the zeros of $L_{n+1}^{(\alpha)}$;

(e) The zeros of $L_{n-1}^{(\alpha+4)}$, together with the point $\frac{(\alpha+1)(\alpha+3)}{2n+\alpha+3}$, interlace with the zeros of $L_{n+1}^{(\alpha)}$.

Remark 2.6. *The extra interlacing points in (d) and (e), respectively, are the upper bounds for the smallest zero $x_{n+1,1}$ of the Laguerre polynomial $L_{n+1}^{(\alpha)}$ obtained in [Gupta and Muldoon, 2007, Eqs. (2.9) and (2.10)], namely $x_{n+1,1} < \frac{(\alpha+1)(\alpha+2)}{n+\alpha+2}$ and $x_{n+1,1} < \frac{(\alpha+1)(\alpha+3)}{2n+\alpha+3}$.*

2.2.2 Quasi-orthogonality

Substituting α by $\alpha-1$ in (2.11) and (2.12) yields, respectively,

$$\begin{aligned}\tilde{L}_n^{(\alpha-1)}(x) &= \tilde{L}_n^{(\alpha)}(x) + n\tilde{L}_{n-1}^{(\alpha)}(x);\\ x\tilde{L}_{n-1}^{(\alpha)}(x) &= \tilde{L}_n^{(\alpha-1)}(x) + (n+\alpha-1)\tilde{L}_{n-1}^{(\alpha-1)}(x).\end{aligned}$$

Therefore $\tilde{L}_n^{(\alpha-j)}(x)$ can be written as a linear combination of $\tilde{L}_n^{(\alpha)}(x), \tilde{L}_{n-1}^{(\alpha)}(x), \dots, \tilde{L}_{n-j}^{(\alpha)}(x)$.

Theorem 2.7 (see [Brezinski et al., 2004], [Driver and Muldoon, 2015]). *(i) Let $j-2 < \alpha < j-1$, $j \in \{1,\dots,n-1\}$ such that $\alpha - j < -1$. The Laguerre polynomials $\tilde{L}_n^{(\alpha-j)}(x)$ are quasi-orthogonal of order j on $(0,\infty)$ with respect to the weight function $x^\alpha e^{-x}$. $\tilde{L}_n^{(\alpha-j)}(x)$ has at least $(n-j)$ positive real zeros.*

(ii) If $-1<\alpha<0$ and $x_{n,i}, i \in \{1,\dots,n\}$ and $y_{n,i}, i\in\{1,\dots,n\}$ denote, respectively, the zeros of $\tilde{L}_n^{(\alpha)}(x)$ and $\tilde{L}_n^{(\alpha-1)}(x)$, then

$$\begin{aligned}&y_{n,1} < 0 < x_{n,1} < x_{n-1,1} < y_{n,2} < x_{n,2} < \ldots < x_{n-1,n-1} < y_{n,n} < x_{n,n},\\ &y_{n-1,1} < y_{n,1} < 0 < y_{n,2} < y_{n-1,2} < \ldots < y_{n,n-1} < y_{n-1,n-1} < y_{n,n}.\end{aligned}$$

From the equations

$$\begin{aligned}(n-1)(\alpha+n-1)\tilde{L}_{n-2}^{(\alpha)}(x) &= -\tilde{L}_n^{(\alpha)}(x) - (\alpha+2n-1-x)\tilde{L}_{n-1}^{(\alpha)}(x),\\ (n-1)x\tilde{L}_{n-2}^{(\alpha+1)}(x) &= -\tilde{L}_n^{(\alpha)}(x) - (n+\alpha-x)\tilde{L}_{n-1}^{(\alpha)}(x),\\ (n-1)x^2\tilde{L}_{n-2}^{(\alpha+2)}(x) &= -(\alpha+1)\tilde{L}_n^{(\alpha)}(x) - (\alpha+1-x)\tilde{L}_{n-1}^{(\alpha)}(x),\end{aligned}$$

Driver and Muldoon [2015] derived the following results.

Theorem 2.8 (see [Driver and Muldoon, 2015]). *Let $n-2\in\mathbb{N}$, α fixed, $-2<\alpha<-1$, and suppose $\tilde{L}_n^{(\alpha)}(x)$ is the sequence of Laguerre polynomials.*

1. *The zeros of* $x\tilde{L}_{n-2}^{(\alpha)}$*, together with the point* $\alpha + 2n - 1$*, interlace with the zeros of* $\tilde{L}_n^{(\alpha)}(x)$ *provided* $\tilde{L}_{n-2}^{(\alpha)}(x)$ *and* $\tilde{L}_n^{(\alpha)}(x)$ *are co-prime.*

2. *Stieltjes interlacing does not hold between the zeros of* $\tilde{L}_{n-2}^{(\alpha)}(x)$ *and* $\tilde{L}_n^{(\alpha)}(x)$*.*

3. *The* $n-2$ *simple, positive zeros of* $\tilde{L}_{n-2}^{(\alpha+1)}(x)$*, together with the point* $n+\alpha$*, interlace with the* $n-1$ *positive zeros of* $\tilde{L}_n^{(\alpha)}(x)$ *if* $\tilde{L}_{n-2}^{(\alpha+1)}(x)$ *and* $\tilde{L}_n^{(\alpha)}(x)$ *are co-prime.*

4. *For* $2 \le t \le 4$*, the* $n-2$ *simple, positive zeros of* $\tilde{L}_{n-2}^{(\alpha+t)}(x)$ *interlace with the* $n-1$ *positive zeros of* $\tilde{L}_n^{(\alpha)}(x)$*.*

5. *For the negative zero* $x_{n,1}$ *of* $\tilde{L}_n^{(\alpha)}(x)$*, the chain of inequalities*

$$\alpha+1 < \frac{\alpha+1}{n} < \frac{(\alpha+1)(\alpha+3)}{\alpha+2n+1} < x_{n,1} < (\alpha+1)\left(\frac{\alpha+2}{n(n+\alpha+1)}\right)^{1/2} < \frac{(\alpha+1)(\alpha+2)}{\alpha+n+1} < 0,$$

is valid.

2.2.3 Bounds of the extreme zeros

Using our code, we get equations of type

$$f(x)L_{n-k}^{(\alpha+s)}(x) = H(x)L_n^{(\alpha)}(x) + G_{k-1,s}(x)L_{n-1}^{(\alpha)}(x),\ k = 2, 3, \ldots,\ s \in \{0, 1, \ldots, 2k\}, \quad (2.15)$$

satisfied by the Laguerre polynomials. Let $B_{k,s}^{(1)}$, $B_{k,s}^{(2)}$ be the smallest and the largest zero of $G_{k-1,s}$, respectively. For $k = 2$ and $s = 3$, we recover the upper bound $B_{2,3}^{(1)} = \dfrac{(\alpha+1)(\alpha+2)}{n+\alpha+1}$ for the smallest zero $x_{n,1}$ of $L_n^{(\alpha)}(x)$ obtained by Hahn (cf. [Hahn, 1933]) and for $s = 4$ the upper bound $B_{2,4}^{(1)} = \dfrac{(\alpha+1)(\alpha+3)}{2n+\alpha+1}$ for $x_{n,1}$ given by Szegő [Szegő, 1975, Eq. (6.31.12)]. For $k = 2$ and $s = 0$, we have the natural bound $B_{2,0}^{(1)} = 2n+\alpha-1$ obtained by Szegő (cf. [Szegő, 1975]). For $k = 3$ and $s = 5$ and $s = 0$, respectively, we recover the equations given in [Driver and Jordaan, 2012, Eqs. (4) and (6)] which provided a strict upper bound $B_{3,5}^{(1)}$ for the smallest zero $x_{n,1}$ of $L_n^{(\alpha)}$ and a lower bound $B_{3,0}^{(2)}$ for the largest zero $x_{n,n}$ of $L_n^{(\alpha)}$, respectively. For $k = 3$ and $s = 6$, we get

$$\begin{aligned} x^6 L_{n-3}^{(\alpha+6)}(x) &= H(x)L_n^{(\alpha)}(x) + (n+\alpha)\Big((\alpha+3)(3n(n+\alpha+1) + (\alpha+1)_2)x^2 \\ &- 2(\alpha+2)_3(\alpha+2n+1)x + (\alpha+1)_5\Big)L_{n-1}^{(\alpha)}(x). \end{aligned}$$

This equation provides for $x_{n,1}$ the upper bound

$$\begin{aligned} x_{n,1} < B_{3,6}^{(1)} = \frac{1}{3n(n+\alpha+1)+(\alpha+1)_2}\Big((\alpha+2)(\alpha+4)(\alpha+2n+1) \\ - \sqrt{(\alpha+2)(\alpha+4)\Big(\left(\alpha^2+6\,\alpha+17\right)n^2 + (\alpha+1)\left(\alpha^2+6\,\alpha+17\right)n - (\alpha+2)\left(\alpha+1\right)^2\Big)}\Big). \end{aligned} \quad (2.16)$$

Numerical simulations indicate that (2.16) is sharper than the upper bound given in [Driver and Jordaan, 2012, Eq. (5)] but could not be compared with the bound

$$x_{n,1} < V^2 + \frac{9V^{4/3}}{(U^2 - V^2)^{1/3}(2 - 27\delta^{2/3})}, \text{ if } \delta < \frac{1}{50}, \tag{2.17}$$

given in Krasikov [2006] for

$$U = \sqrt{n+\alpha+1} + \sqrt{n},\ V = \sqrt{n+\alpha+1} - \sqrt{n},\ \delta = \frac{1}{n} + \frac{1}{\alpha+1}. \tag{2.18}$$

Note that the condition $\delta < \frac{1}{50}$ with $\alpha > -1$ is valid for $n \geq 51$. However, for $k = 4$ and $s = 8$ in (2.15), it turns out that the smallest zero $B^{(1)}_{4,8}$ of the polynomial

$$\begin{aligned} G_{3,8}(x) &= (\alpha+4)(\alpha+2n+1)\left(\alpha^2 + 2\alpha n + 2n^2 + 5\alpha + 2n + 6\right)x^3 \\ &\quad - (\alpha+3)_3\left(3\alpha^2 + 10\alpha n + 10n^2 + 9\alpha + 10n + 6\right)x^2 \\ &\quad + 3(\alpha+2)_5(\alpha+2n+1)x - (\alpha+1)_7 \end{aligned}$$

is a more accurate upper bound for $x_{n,1}$ compared to (2.16), and for $k = 4$ and $s = 0$ in (2.15), the largest zero $B^{(2)}_{4,0}$ of the polynomial

$$\begin{aligned} G_{3,0}(x) &= x^3 - 3(2n+\alpha-3)x^2 + (10n^2 + 10(\alpha-3)n + 3(\alpha-3)_2)x \\ &\quad - (\alpha+2n-3)(2n^2 + 2(\alpha-3)n + (\alpha-2)_2) \end{aligned}$$

is an accurate lower bound for $x_{n,n}$, the largest zero of $L^{(\alpha)}_n(x)$, compared to the lower bound [Driver and Jordaan, 2012, Eq. (7)].

Krasikov [2006] showed that for $n \geq 30$, and U, V given in (2.18),

$$x_{n,n} > \begin{cases} U^2 - \frac{9U^{4/3}}{2(U^2-V^2)^{1/3}}, & \text{if } \alpha \leq 2(3+2\sqrt{3})n - 1, \\ U^2 - \frac{9U^{4/3}}{(U^2-V^2)^{1/3}(2-3n^{-2/3})}, & \text{otherwise.} \end{cases} \tag{2.19}$$

This bound of $x_{n,n}$ is sharper than $B^{(2)}_{4,0}$ but if we take $(k, s) = (7, 0)$, we get from simulations that the largest zero $B^{(2)}_{7,0}$ of $G_{6,0}$ is sharper than the bound in (2.19). In Table 2.3, we compare $B^{(1)}_{4,8}$, $B^{(2)}_{4,0}$, the lower bound $3n-4$ for $x_{n,n}$ obtained by Neumann [1921], the lower bound $z_{n,0} := 4n + \alpha - 16\sqrt{2n}$ given by Bottema [1931], the upper bound

$$x_{n,1} < z_{n,1} := \frac{(\alpha+1)(\alpha+2)(\alpha+4)(2n+\alpha+1)}{(\alpha+1)^2(\alpha+2) + (5\alpha+11)n(n+\alpha+1)}$$

obtained by [Gupta and Muldoon, 2007, Eq. (2.11)], (2.19) and the bound $B^{(2)}_{7,0}$.

We observe from the simulations that the sharpest upper bounds (compared to the existing ones) for $x_{n,1}$ are obtained for $s = 2k$, $k \geq 4$ and the sharpest lower bounds for $x_{n,n}$ for $s = 0$ and $k \geq 4$. We also observe as in [Gupta and Muldoon, 2007] that the upper bounds for $x_{n,1}$ will be sharpest for α close to -1.

$n,\ \alpha$	$5,\ -0.9$	$13,\ 340.56$	$21,\ 65.3$	$101,\ -0.9$
zero $x_{n,1}$	0.020777151319288	251.82	27.677	0.00103830995555334
bound $B^{(1)}_{4,8}$	0.020777151319291	259.04	28.59	0.00103830995555361
bound $z_{n,1}$	0.020777504961963	309.59	36.616	0.0010383284039825
bound $z_{n,0}$	n/a	310.98	45.61	175.69
bound $3n-4$	11	35	59	299
bound $B^{(2)}_{4,0}$	11.1133332	453.62	161.88	338.20
bound (2.19)	7.94	443.001	166.78	370.93
bound $B^{(2)}_{7,0}$	11.1262992	469.109	172.19	371.24
zero $x_{n,n}$	11.1262992	469.74	172.77	377.13

Table 2.3: Comparison of the bounds for the extreme zeros of the Laguerre polynomials $L_n^{(\alpha)}(x)$

2.3 The Bessel polynomials

The Bessel polynomials

$$y_n(x;\alpha) = {}_2F_0\left(\begin{array}{c} -n, n+\alpha+1 \\ - \end{array}\middle| -\frac{x}{2}\right),\ n=0,1,\ldots,N,\ \alpha < -2N-1,$$

are orthogonal on $(0,\infty)$ w.r.t. $w(x) = x^\alpha e^{-\frac{2}{x}}$. The monic Bessel polynomials are given by $\tilde{y}_n(x;\alpha) = \frac{2^n}{(n+\alpha+1)_n} y_n(x;\alpha)$.

2.3.1 Interlacing of the zeros

The following mixed recurrence equations are valid.

$$\begin{aligned}
\tilde{y}_n(x;\alpha-1) &= \tilde{y}_n(x;\alpha) + \frac{2n}{(\alpha+2n)(\alpha+2n-1)}\tilde{y}_{n-1}(x;\alpha);\\
\tilde{y}_n(x;\alpha-2) &= \frac{\alpha+2n-1}{\alpha+n-1}\tilde{y}_n(x;\alpha) - \frac{n((\alpha+2n)x-2)}{(\alpha+2n)(\alpha+n-1)}\tilde{y}_{n-1}(x;\alpha);\\
\tilde{y}_n(x;\alpha-2) &= \frac{\alpha+2n-1}{\alpha+n-1}\tilde{y}_n(x;\alpha-1) - \frac{nx}{\alpha+n-1}\tilde{y}_{n-1}(x;\alpha).
\end{aligned}$$

It follows that

Theorem 2.9. *Let $n = 0,1,\ldots,N$, $\alpha < -2N-1$ and $x \in (0,\infty)$. Let $0 < x_{n,1} < \ldots < x_{n,n}$ be the zeros of $\tilde{y}_n(x;\alpha)$, $0 < y_{n,1} < \ldots < y_{n,n}$ be the zeros of $\tilde{y}_n(x;\alpha-1)$ and $0 < Y_{n,1} < \ldots < Y_{n,n}$ be the zeros of $\tilde{y}_n(x;\alpha-2)$. Then for $i \in \{1,2,\ldots,n-1\}$,*

(a) $y_{n,i} < x_{n,i} < x_{n-1,i} < y_{n,i+1} < x_{n,i+1}$,

(b) $Y_{n,i} < x_{n,i} < x_{n-1,i} < Y_{n,i+1} < x_{n,i+1}$,

(c) $Y_{n,i} < y_{n,i} < x_{n,i} < x_{n-1,i} < Y_{n,i+1} < y_{n,i+1} < x_{n,i+1}$.

Remark 2.10. *We deduce from the latter theorem that* $y_{n,i} < y_{n-1,i} < x_{n-1,i} < y_{n,i+1}$, $Y_{n,i} < Y_{n-1,i} < x_{n-1,i} < Y_{n,i+1}$ *and* $Y_{n,i} < y_{n-1,i} < x_{n-1,i} < Y_{n,i+1}$.

2.3.2 Quasi-orthogonality

For the Bessel polynomials, the following recurrence equation is valid:

$$\begin{aligned}\tilde{y}_n(x;\alpha+1) &= \frac{x\,(\alpha+1+2\,n)\,(\alpha+2\,n)-2\,n}{x\,(\alpha+1+2\,n)\,(\alpha+2\,n)}\tilde{y}_n(x;\alpha)\\ &+\frac{4n(\alpha+n)}{x(\alpha+1+2n)\,(\alpha-1+2\,n)\,(\alpha+2\,n)^2}\tilde{y}_{n-1}(x;\alpha).\end{aligned}$$

Therefore, the polynomial $\tilde{y}_n(x;\alpha+k)$ $(k\geq 1)$ is not quasi-orthogonal with respect to $x^{\alpha}e^{-\frac{2}{x}}$, on $(0,\infty)$, since it cannot be written as a linear combination of the polynomials $\tilde{y}_n(x;\alpha)$, $\tilde{y}_{n-1}(x;\alpha)$, ..., $\tilde{y}_{n-k}(x;\alpha)$.

2.3.3 Bounds of the extreme zeros

We use equations of type

$$f(x)y_{n-k}(x;\alpha+s) = H(x)y_n(x;\alpha)+G_{k-1,s}(x)y_{n-1}(x;\alpha),\ \ s\in\{0,1,\ldots,2k\}, \qquad (2.20)$$

to find the bounds of the extreme zeros $x_{n,1}$ and $x_{n,n}$ of the Bessel polynomials $y_n(x;\alpha)$. Let $B^{(1)}_{k,s}$, $B^{(2)}_{k,s}$ be the smallest and the largest zeros of $G_{k-1,s}$, respectively. For $k=3$ and $s=0$, we derive the lower bound $B^{(2)}_{3,0}$ of $x_{n,n}$ given by

$$\begin{aligned}B^{(2)}_{3,0} &= \frac{2\alpha+4n-4}{(\alpha+2n-4)_5}\Big(-\alpha\,(\alpha+2\,n-1)\,(\alpha+2\,n-3)+\Big((1-\alpha-2\,n)\,(\alpha+2\,n-3)\\ &\times\big((n-1)\,\alpha^3+\left(5\,n^2-10\,n+1\right)\alpha^2+8\,n\,(n-1)\,(n-2)\,\alpha+4\,n^2\,(n-2)^2\big)\Big)^{1/2}\Big).\end{aligned}$$

The upper bound

$$B^{(1)}_{3,6} = -2\,\frac{\alpha+3+\sqrt{-\alpha\,n-n^2+\alpha-n+3}}{\alpha^2+\alpha\,n+n^2+5\,\alpha+n+6}$$

for $x_{n,1}$ is obtained for $k=3$ and $s=6$. As we can observe from the simulations in Table 2.4 for $k=4$, the sharpest bounds for $x_{n,1}$ are obtained for $s=2k$, $k\geq 4$, and for $s=0$, $k\geq 4$, we get sharpest bounds for $x_{n,n}$.

2.4 The Hahn polynomials

The Hahn polynomials

$$Q_n(x;\alpha,\beta,N) = {}_3F_2\left(\left.\begin{matrix}-n,-x,n+1+\alpha+\beta\\ \alpha+1,-N\end{matrix}\,\right|\,1\right), n=0,1,2,\ldots,N,$$

$$\alpha>-1 \text{ and } \beta>-1, \text{ or } \alpha<-N \text{ and } \beta<-N,$$

are orthogonal w.r.t. $w(x)=\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}$, $x=0,1,\ldots,N$. The monic Hahn polynomials will be denoted by $\tilde{Q}_n(x;\alpha,\beta,N)=\frac{(\alpha+1)_n(-N)_n}{(n+\alpha+\beta+1)_n}Q_n(x;\alpha,\beta,N)$.

$n,\ \alpha$	$4,\ -20$	$10,\ -25$	$50,\ -305$	$100,\ -205$
zero $x_{n,1}$	0.080626	0.05438	0.003997	0.0052
bound $B_{4,8}^{(1)}$	0.08097	0.0558	0.00437	0.0058
bound $B_{3,6}^{(1)}$	0.084899	0.06	0.0048	0.0066
bound $B_{3,0}^{(2)}$	0.287	1.699	0.019	14.07
bound $B_{4,0}^{(2)}$	0.291999	1.71575	0.0214	14.2335
zero $x_{n,n}$	0.29213	1.7161	0.02284	14.23786

Table 2.4: Comparison of the bounds for the extreme zeros of the Bessel polynomials $y_n(x;\alpha)$

2.4.1 Interlacing of the zeros

Using our implementations, we show that the Hahn polynomials are solutions of the following recurrence equations [Jordaan and Toókos, 2009]:

$$\tilde{Q}_n(x;\alpha,\beta,N) = \tilde{Q}_n(x;\alpha+1,\beta,N) + \frac{n\,(n+\beta)\,(N-n+1)}{(2\,n+\alpha+\beta)\,(2\,n+\alpha+\beta+1)}\tilde{Q}_{n-1}(x;\alpha+1,\beta,N); \tag{2.21}$$

$$\tilde{Q}_n(x;\alpha,\beta,N) = \tilde{Q}_n(x;\alpha,\beta+1,N) - \frac{n\,(\alpha+n)\,(N-n+1)}{(2\,n+\alpha+\beta)\,(2\,n+\alpha+\beta+1)}\tilde{Q}_{n-1}(x;\alpha,\beta+1,N); \tag{2.22}$$

$$\tilde{Q}_n(x;\alpha,\beta+1,N) = \tilde{Q}_n(x;\alpha+1,\beta,N) + \frac{n\,(-n+1+N)}{2\,n+\alpha+\beta+1}\tilde{Q}_{n-1}(x;\alpha+1,\beta+1,N).$$

Some mixed recurrence relations satisfied by the Hahn polynomials are also given in [Levit, 1967] as well as some separation theorems. Jordaan and Toókos [2009] derived the following interlacing properties.

Theorem 2.11 (see [Jordaan and Toókos, 2009])**.** *Let $\alpha,\beta > -1$ and let $0 < x_{n,1} < \ldots < x_{n,n}$ be the zeros of $Q_n(x;\alpha,\beta,N)$, $0 < y_{n,1} < \ldots < y_{n,n}$ be the zeros of $Q_n(x;\alpha+k,\beta,N)$, $0 < z_{n,1} < \ldots < z_{n,n}$ be the zeros of $Q_n(x;\alpha,\beta+s,N)$, $0 < t_{n,1} < \ldots < t_{n,n}$ be the zeros of $Q_n(x;\alpha+k,\beta+s,N)$ where $0 < k,s \leq 1$. Then for $i = 1,2,\ldots,n-1$,*

(a) $x_{n,i} < y_{n,i} < y_{n-1,i} < x_{n,i+1} < y_{n,i+1}$,

(b) $z_{n,i} < x_{n,i} < z_{n-1,i} < z_{n,i+1} < x_{n,i+1}$,

(c) $z_{n,i} < x_{n,i} < y_{n,i} < t_{n-1,i} < z_{n,i+1} < x_{n,i+1} < y_{n,i+1}$.

Remark 2.12. *(i) The interlacing order given in [Jordaan and Toókos, 2009, Thm. 5.1] should read $p_i < t_i < q_i < p_{i+1} < t_{i+1}$ and not $t_i < p_i < q_i < t_{i+1} < p_{i+1}$ where t_i, p_i, q_i are, respectively, the zeros of $Q_n(x;\alpha,\beta,N)$, $Q_n(x;\alpha,\beta+s,N)$, $Q_{n-1}(x;\alpha,\beta+s,N)$ in increasing order with $0 < s \leq 1$.*

(b) From (a) and (b), we deduce, respectively, the interlacing $x_{n,i} < x_{n-1,i} < y_{n-1,i} < x_{n,i+1}$ and $x_{n,i} < z_{n-1,i} < x_{n-1,i} < x_{n,i+1}$, $i = 1,2,\ldots,n-1$.

2.4.2 Quasi-orthogonality

Substituting α by $\alpha-1$ in (2.21) and β by $\beta-1$ in (2.22) yields (see [Johnston et al., 2016])

$$\tilde{Q}_n(x;\alpha-1,\beta,N)=\tilde{Q}_n(x;\alpha,\beta,N)+\frac{n\,(n+\beta)\,(N-n+1)}{(2\,n+\alpha+\beta-1)\,(2\,n+\alpha+\beta)}\tilde{Q}_{n-1}(x;\alpha,\beta,N); \tag{2.23}$$

$$\tilde{Q}_n(x;\alpha,\beta-1,N)=\tilde{Q}_n(x;\alpha,\beta,N)-\frac{n\,(\alpha+n)\,(N-n+1)}{(2\,n+\alpha+\beta-1)\,(2\,n+\alpha+\beta)}\tilde{Q}_{n-1}(x;\alpha,\beta,N). \tag{2.24}$$

Replacing β by $\beta-1$ in (2.23) and using (2.24), we get

$$\begin{aligned}\tilde{Q}_n(x;\alpha-1,\beta-1,N)&=\tilde{Q}_n(x;\alpha,\beta,N)-\frac{n\,(\alpha-\beta)\,(N-n+1)}{(2\,n+\alpha-2+\beta)\,(\alpha+\beta+2\,n)}\tilde{Q}_{n-1}(x;\alpha,\beta,N)\\ &-\frac{n\,(n-1+\alpha)\,(N-n+1)\,(n-1)\,(n+\beta-1)\,(N-n+2)}{(2\,n-3+\alpha+\beta)\,(2\,n+\alpha-2+\beta)^2\,(2\,n+\alpha+\beta-1)}\tilde{Q}_{n-2}(x;\alpha,\beta,N).\end{aligned}$$

Johnston et al. [2016] deduced the following results.

Theorem 2.13 (cf. [Johnston et al., 2016]). *(i) Let $N\in\mathbb{N}$ with $n\in\{0,1,\dots,N\}$. Then for $\alpha>-1,\beta>-1$ and $k,l\in\mathbb{N}$ fixed with $k+l<N$ such that $\alpha-k<-1$, $\beta-l<-1$, the sequence of Hahn polynomials $\{\tilde{Q}_n(x;\alpha-k,\beta-l,N)\}$ is quasi-orthogonal of order $k+l$ with respect to $\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}$, $x=0,1,\dots,N$ and the polynomials have at least $n-(k+l)$ real, distinct zeros in the interval $(0,N)$.*

(ii) For $n\in\{2,3,\dots\}$, let $x_{n,i},i\in\{1,2,\dots,n\}$ be the zeros of $\tilde{Q}_n(x;\alpha,\beta,N)$, $y_{n,i},i\in\{1,2,\dots,n\}$ be the zeros of $\tilde{Q}_n(x;\alpha-1,\beta,N)$ and $z_{n,i},i\in\{1,2,\dots,n\}$ be the zeros of $\tilde{Q}_n(x;\alpha,\beta-1,N)$. Then
for $\beta>-1$, $-1<\alpha<0$, $y_{n,1}<0<x_{n,1}<x_{n-1,1}<y_{n,2}<x_{n,2}<\dots<x_{n-1,n-1}<y_{n,n}<x_{n,n}$;
for $\alpha>-1$, $-1<\beta<0$, $x_{n,1}<z_{n,1}<x_{n-1,1}<x_{n,2}<z_{n,2}<\dots<x_{n-1,n-1}<x_{n,n}<N<z_{n,n}$.

(iii) For $n\in\{2,3,\dots\}$ and $\alpha,\beta>-1$, the zeros of $\tilde{Q}_{n-1}(x;\alpha,\beta,N)$ interlace with the zeros of the second-order quasi-orthogonal polynomial

(a) $\tilde{Q}_n(x;\alpha-1,\beta-1,N)$ for $-1<\alpha,\beta<0$;
(b) $\tilde{Q}_n(x;\alpha-2,\beta,N)$ for $-1<\alpha<0$, $\beta>-1$;
(c) $\tilde{Q}_n(x;\alpha,\beta-2,N)$ for $\alpha>-1$, $-1<\beta<0$.

2.4.3 Bounds of the extreme zeros

For the Hahn polynomials, we use, for $k=2,3,\dots,$ $s_1+s_2\in\{0,1,\dots,2k\}$, equations of the form

$$f(x)Q_{n-k}(x;\alpha+s_1,\beta+s_2,N)=H(x)Q_n(x;\alpha,\beta,N)+G_{k-1,s_1,s_2}(x)Q_{n-1}(x;\alpha,\beta,N). \tag{2.25}$$

Let $B^{(1)}_{k,s_1,s_2}$, $B^{(2)}_{k,s_1,s_2}$ be the smallest and the largest zeros of G_{k-1,s_1,s_2}, respectively. The case $k = 2$ was already studied in [Jooste et al., 2017] and the bounds derived from that case are not accurate as the one obtained for $k = 3$. In fact, for $k \geq 3$ in (2.25), sharpest upper bounds for $x_{n,1}$ are the smallest zeros of $G_{k-1,2k,0}$ and sharpest lower bound for $x_{n,n}$ are the largest zeros of $G_{k-1,0,2k}$ as shown in Table 2.5 for $k = 3$. From the simulations, the bounds we get for $k = 3$, $s_1 = 2k$, $s_2 = 0$, are already sharper than the ones given in [Krasikov and Zarkh, 2009] for $\alpha \geq \beta > -1$ or $\alpha \leq \beta \leq -N-1$:

$$x_{n,1} < \frac{(n+\alpha)(N-n+1)}{\alpha+\beta+1} \tag{2.26}$$

$$< \frac{N(n+\alpha)+(n+\beta)(n-1)}{2n+\alpha+\beta} < x_{n,n}. \tag{2.27}$$

$n,\ \alpha$	$5,\ 10$	$5,\ 200$	$5,\ 10.5$	$100,\ 3$
$\beta,\ N$	$2,\ 5$	$2,\ 30$	$10,\ 30$	$-0.5,\ 1000$
zero $x_{n,1}$	0.1659	23.219	5.0265	0.036
bound $B^{(1)}_{3,6,0}$	0.47	23.435	5.67	2.105
bound (2.26)	1.15	26.256	18.74	n/a
bound $B^{(1)}_{2,4,0}$	1.41	24.736	9.00092	5.802
bound $B^{(2)}_{2,0,4}$	2.726	26.638	21.419	995.2478
bound $B^{(2)}_{3,0,6}$	4.01	28.417	24.699	997.649
bound (2.27)	4.68	29.14	17.21	n/a
zero $x_{n,n}$	4.9975	29.998	25.3339	999.9999

Table 2.5: Comparison of the bounds for the extreme zeros of the Hahn polynomials $Q_n(x;\alpha,\beta,N)$

2.5 The Krawtchouk polynomials

The Krawtchouk polynomials

$$K_n(x;p,N) = {}_2F_1\left(\left.\begin{matrix}-n,-x\\-N\end{matrix}\,\right|\,\frac{1}{p}\right), n = 0,1,2,\ldots,N,\ 0<p<1,$$

are orthogonal w.r.t. $w(x) = \binom{N}{x}p^x(1-p)^{N-x}$, $x = 0,1,\ldots,N$. The monic Krawtchouk polynomials will be denoted by $\tilde{K}_n(x;p,N) = (-N)_n p^n K_n(x;p,N)$.

2.5.1 Interlacing of the zeros

The following mixed recurrence equations are valid. Note that the two equations are given in [Jordaan and Toókos, 2009] but with a misprint in each equation.

$$\begin{aligned}\tilde{K}_n(x;p,N) &= \tilde{K}_n(x;p,N-1) - np\tilde{K}_{n-1}(x;p,N-1);\\ \tilde{K}_n(x;p,N) &= \frac{N-np+1-x}{N-x+1}\tilde{K}_n(x;p,N+1)\\ &\quad + \frac{np\,(1-p)\,(N-n+2)}{N+1-x)}\tilde{K}_{n-1}(x;p,N+1).\end{aligned}$$

In addition to the first equation, Chihara and Stanton [1990] also derived the equations

$$\begin{aligned}\tilde{K}_n(x;p,N) &= \tilde{K}_n(x+1;p,N) - n\tilde{K}_{n-1}(x;p,N-1);\\ \tilde{K}_n(x;p,N) &= \tilde{K}_n(x-1;p,N-1) + n(1-p)\tilde{K}_{n-1}(x-1;p,N-1).\end{aligned}$$

It follows that

Theorem 2.14 (see [Chihara and Stanton, 1990], [Jordaan and Toókos, 2009]). *Let $p \in (0,1)$ and $n = 0,1,\ldots,N$. If $0 < x_{n,1} < \ldots < x_{n,n} < N$ are the zeros of $K_n(x;p,N)$, $0 < y_{n,1} < \ldots < y_{n,n} < N+1$ are the zeros of $K_n(x;p,N+1)$ and $0 < z_{n,1} < \ldots < z_{n,n} < N-1$ are the zeros of $K_n(x;p,N-1)$ then, for $i = 1,2,\ldots,n-1$,*

(a) $z_{n,i} < x_{n,i} < z_{n-1,i} < z_{n,i+1} < x_{n,i+1} \Leftrightarrow x_{n,i} < y_{n,i} < x_{n-1,i} < x_{n,i+1} < y_{n,i+1}$,

(b) $y_{n,i} < x_{n,i} + 1 < x_{n-1,i} + 1 < y_{n,i+1} < x_{n,i+1} + 1$,

(c) $x_{n,i} < y_{n,i} < y_{n-1,i} < x_{n,i+1} < y_{n,i+1}$ *when* $n \leq \frac{1}{p}$.

With counterexamples, Jordaan and Toókos [2009] showed that the zeros of $K_n(x;p,N)$ and $K_n(x;p,N+t)$ do not interlace in general when t is an integer greater than 1 and the zeros of $K_n(x;p,N)$ and $K_{n-1}(x;p,N+1)$ or those of $K_n(x;p,N-2)$ do not also generally interlace.

2.6 The Meixner polynomials

The Meixner polynomials are defined by

$$M_n(x;\gamma,\mu) = {}_2F_1\left(\left.\begin{array}{c}-n,-x\\ \gamma\end{array}\right| 1-\frac{1}{\mu}\right),\ \gamma > 0,\ 0 < \mu < 1,$$

and are orthogonal w.r.t. $w(x) = \frac{(\gamma)_x}{x!}\mu^x$ for $x = 0,1,\ldots$. The monic Meixner family is denoted by $\tilde{M}_n(x;\gamma,\mu) = \frac{(\gamma)_n\mu^n}{(\mu-1)^n}M_n(x;\gamma,\mu)$.

2.6.1 Interlacing of the zeros

Our implementation yields the equations (cf. [Jordaan and Toókos, 2009])

$$\tilde{M}_n(x;\gamma,\mu) = \tilde{M}_n(x;\gamma+1,\mu) + \frac{n\mu}{1-\mu}\tilde{M}_{n-1}(x;\gamma+1,\mu); \tag{2.28}$$

$$\tilde{M}_n(x;\gamma,\mu) = \frac{\gamma+n(1-\mu)}{n+\gamma}\tilde{M}_n(x;\gamma+2,\mu) + \frac{\mu\, n\left((\mu-2)\,\gamma+(n+x+1)\,(\mu-1)\right)}{(\mu-1)\,(n+\gamma)}\tilde{M}_{n-1}(x;\gamma+2,\mu).$$

Jordaan and Toókos [2009] derived from the latter equations the following interlacing property.

Theorem 2.15 (see [Jordaan and Toókos, 2009])**.** *Let $\gamma>0$, $0<\mu<1$ and let $0<x_{n,1}<x_{n,2}<\ldots<x_{n,n}$ be the zeros of $M_n(x;\gamma,\mu)$ and $0<y_{n,1}<y_{n,2}<\ldots<y_{n,n}$ be the zeros of $M_n(x;\gamma+t,\mu)$ where $0<t\le 2$. Then*

$$x_{n,i}<y_{n,i}<y_{n-1,i}<x_{n,i+1}<y_{n,i+1},\quad i=1,2,\ldots,n-1.$$

Moreover, they show that this interlacing is not valid in general for shifts of $t>2$ or $t<0$.

2.6.2 Quasi-orthogonality

We substitute γ by $\gamma-1$ in (2.28) to get

$$\tilde{M}_n(x;\gamma-1,\mu) = \tilde{M}_n(x;\gamma,\mu) + \frac{n\mu}{1-\mu}\tilde{M}_{n-1}(x;\gamma,\mu),$$

from which we deduce the equation

$$\tilde{M}_n(x;\gamma-2,\mu) = \tilde{M}_n(x;\gamma,\mu) - 2n\frac{\mu}{\mu-1}\tilde{M}_{n-1}(x;\gamma,\mu) + n(n-1)\frac{\mu^2}{(\mu-1)^2}\tilde{M}_{n-2}(x;\gamma,\mu).$$

It follows that

Theorem 2.16 (see [Jooste et al., 2013])**.** *Let $0<\mu<1$ and $0<\gamma<1$.*

1. *The Meixner polynomials $M_n(x;\gamma-k,\mu)$ with $k=1,2,\ldots,n-1$ are quasi-orthogonal of order k with respect to the weight function $\frac{(\gamma)_x}{x!}\mu^x$ on $(0,\infty)$.*
2. *The Meixner polynomials $M_n(x;\gamma-k,\mu)$ with $-k<\gamma<-k+1$ have at least $n-k$ zeros in $(0,\infty)$ when $k=1,2,\ldots,n-1$.*
3. *If $0<\gamma<1$, then the smallest zero of $M_n(x;\gamma-1,\mu)$ is negative.*
4. *If $0<\gamma<1$ and $n>\frac{\gamma-2}{\mu-1}$ then all the zeros of $M_n(x;\gamma-2,\mu)$ are nonnegative and simple.*

2.6.3 Bounds of the extreme zeros

We use recurrence equations of the form

$$f(x)M_{n-k}(x;\gamma+s,\mu)=H(x)M_n(x;\gamma,\mu)+G_{k-1,s}(x)M_{n-1}(x;\gamma,\mu),\ \ s\in\{0,1,\ldots,2k\} \tag{2.29}$$

satisfied by the Meixner polynomials. For $k=2$, we recover the recurrence equations given in [Jooste and Jordaan, 2014] and the bounds derived from these equations:

$$x_{n,1}<z_{n,1}=\frac{(n+\gamma)_2\,(n-1+\mu\,(3+\gamma-3\,n))-\mu^2n\,(n-1)\,((n+\gamma+1)\,\mu-4\,\gamma-3\,n-3)}{(1-\mu)\,(\gamma\,(\gamma+2\,n+1)-n\,(\mu^2n-\mu^2-n-1))} \tag{2.30}$$

$$<z_{n,2}=\frac{\gamma\,\mu+(n-1)\,(\mu+1)}{1-\mu}<x_{n,n}. \tag{2.31}$$

Moreover, for $k=3$ and $s=0$, we deduce the bound

$$z_{n,2}<Z_{n,2}=\frac{2\,\gamma\,\mu+(2\,n-3)\,(\mu+1)+\sqrt{4\,\mu\,n\,(\gamma+n-3)-4\,\gamma\,\mu+\mu^2+10\,\mu+1}}{2(1-\mu)}<x_{n,n}.$$

$k=3$, $s=6$ yield the bound

$$x_{n,1}<Z_{n,1}=\frac{A-\sqrt{B}}{2\,(1-\mu)\,((n+\gamma)_3-(n-2)_3\,\mu^3)}<z_{n,1},$$

with

$$A=-\mu^3\,(n-2)_3\,((2\,\mu-4)\,n+(\mu-3)\,(2\,\gamma+3))+(n+\gamma)_3\,((-4\,\mu+2)\,n+2\,\gamma\,\mu+9\,\mu-3)\,,$$

$$\begin{aligned}B&=-\mu^6\,(n-2)_3^2\Big(-\mu^2+\left(-4\,\gamma\,n-4\,n^2+4\,\gamma+14\right)\mu+24\,\gamma^2+48\,\gamma\,n+24\,n^2+48\,\gamma\\&+48\,n-1\Big)+2\,\mu^3\,(n-2)_3\,(n+\gamma)_3\Big(\left(30\,n^2-60\,n+17\right)\mu^2+\left(-40\,\gamma\,n-40\,n^2+40\,\gamma+50\right)\mu\\&+30\,\gamma^2+60\,\gamma\,n+30\,n^2+60\,\gamma+60\,n+17\Big)+(n+\gamma)_3^2\Big(\left(-24\,n^2+48\,n+1\right)\mu^2\\&+\left(4\,\gamma\,n+4\,n^2-4\,\gamma-14\right)\mu+1\Big).\end{aligned}$$

We provide in Table 2.6 numerical examples to illustrate these bounds.

$n,\ \gamma,\ \mu$	$8,\ 0.09,\ 0.99$	$15,\ 20,\ 0.5$	$100,\ 20,\ 0.5$	$100,\ 0.09,\ 0.99$
zero $x_{n,1}$	1.118068	2	2.9×10^{-13}	0.0555
bound $Z_{n,1}$	1.118078	6.06	18.75	0.0756
bound $Z_{n,2}$	1959.6293	90.93	468.36	29415.91
zero $x_{n,n}$	2114.696	106.44	589.77	37133.5

Table 2.6: Comparison of the bounds for the extreme zeros of the Meixner polynomials $M_n(x;\gamma,\mu)$

Chapter 3

Classical q-orthogonal polynomials

This chapter is devoted to the interlacing properties, the quasi-orthogonality of classical q-orthogonal polynomials. The bounds of their extreme zeros are also given. The results of this chapter can be found in the joint works by Tcheutia et al. [2018a], Tcheutia et al. [2018b] and Jooste et al.. The equations from which the bounds of the extreme zeros are derived can be found in the Maple file associated to this work.

3.1 Introduction

Let $0 < q < 1$. The classical q-orthogonal polynomials were introduced by Hahn [1949] and can be written in terms of basic hypergeometric series [Gasper and Rahman, 2004], as introduced by Heine [1847]. These polynomials are associated especially to quantum groups (cf. [Koelink and Koornwinder, 1989], [Koornwinder, 1990], [Koornwinder, 1994]), as introduced in [Drinfeld, 1987], [Woronowicz, 1987]. We list the systems of monic q-orthogonal polynomials considered in this chapter (cf. [Koekoek et al., 2010]).

1. Big q-Jacobi polynomials

$$\tilde{P}_n(x;\alpha,\beta,\gamma;q) = \frac{(\alpha q;q)_n(\gamma q;q)_n}{(\alpha\beta q q^n;q)_n}\,{}_3\phi_2\left(\left.\begin{matrix} q^{-n},\alpha\beta q^{n+1},x \\ \alpha q,\gamma q\end{matrix}\,\right|\, q;q\right), \tag{3.1}$$

with $0 < \alpha q < 1$, $0 \leq \beta q < 1$ and $\gamma < 0$, $x \in (\gamma q, \alpha q)$;

2. q-Hahn polynomials

$$\tilde{Q}_n(\bar{x};\alpha,\beta,N|q) = \frac{(\alpha q;q)_n(q^{-N};q)_n}{(\alpha\beta q q^n;q)_n}\,{}_3\phi_2\left(\left.\begin{matrix} q^{-n},\alpha\beta q^{n+1},\bar{x} \\ \alpha q,q^{-N}\end{matrix}\,\right|\, q;q\right),$$

with $\bar{x} = q^{-x}$, $n \in \{0,1,\ \ldots,\ N\}$, $0 < \alpha q < 1$ and $0 < \beta q < 1$ or $\alpha > q^{-N}$ and $\beta > q^{-N}$, $\bar{x} \in (1, q^{-N})$;

3. Little q-Jacobi polynomials

$$\tilde{p}_n(x;\alpha,\beta|q) = (-1)^n q^{\binom{n}{2}}\frac{(\alpha q;q)_n}{(\alpha\beta q q^n;q)_n}\,{}_2\phi_1\left(\left.\begin{matrix} q^{-n},\alpha\beta q^{n+1} \\ \alpha q\end{matrix}\,\right|\, q;qx\right),$$

with $0 < \alpha q < 1$, $\beta q < 1, x \in (0,1)$;

4. q-Meixner polynomials

$$\tilde{M}_n(\bar{x};\beta,\gamma;q) = (-1)^n q^{-n^2}\gamma^n(\beta q;q)_n\, {}_2\phi_1\left(\begin{array}{c} q^{-n},\bar{x} \\ \beta q \end{array}\middle|\, q;-\frac{q^{n+1}}{\gamma}\right),$$

with $\bar{x} = q^{-x}$, $0 \le \beta q < 1$, $\gamma > 0$, $\bar{x} \in (1,\infty)$;

5. q-Krawtchouk polynomials

$$\tilde{K}_n(\bar{x};p,N;q) = \frac{(q^{-N};q)_n}{(-pq^n;q)_n}\, {}_3\phi_2\left(\begin{array}{c} q^{-n},\bar{x},-pq^n \\ q^{-N},0 \end{array}\middle|\, q;q\right),$$

with $\bar{x} = q^{-x}$ and $n \in \{0,1,\ \ldots,\ N\}$, $p > 0$, $\bar{x} \in (1,q^{-N})$;

6. q-Laguerre polynomials

$$\tilde{L}_n^{(\alpha)}(x;q) = \frac{(-1)^n(q^{\alpha+1};q)_n}{q^{n(n+\alpha)}}\, {}_1\phi_1\left(\begin{array}{c} q^{-n} \\ q^{\alpha+1} \end{array}\middle|\, q;-q^{n+\alpha+1}x\right),\ \alpha > -1,\ x \in (0,\infty);$$

7. Alternative q-Charlier or q-Bessel polynomials

$$\tilde{y}_n(x;\alpha;q) = \frac{(-1)^n q^{\binom{n}{2}}}{(-\alpha q^n;q)_n}\, {}_2\phi_1\left(\begin{array}{c} q^{-n},-\alpha q^n \\ 0 \end{array}\middle|\, q;qx\right),\ \alpha > 0,\ x \in (0,1);$$

8. Al-Salam-Carlitz I polynomials

$$\tilde{U}_n^{(\alpha)}(x;q) = (-\alpha)^n q^{\binom{n}{2}}\, {}_2\phi_1\left(\begin{array}{c} q^{-n},x^{-1} \\ 0 \end{array}\middle|\, q;\frac{qx}{\alpha}\right),\ \alpha < 0,\ x \in (\alpha,1);$$

9. Al-Salam-Carlitz II polynomials

$$\tilde{V}_n^{(\alpha)}(x;q) = (-\alpha)^n q^{-\binom{n}{2}}\, {}_2\phi_0\left(\begin{array}{c} q^{-n},x \\ - \end{array}\middle|\, q;\frac{q^n}{\alpha}\right),\ 0 < \alpha q < 1,\ x \in (1,\infty).$$

3.2 The big q-Jacobi polynomials

The sequence of big q-Jacobi polynomials $\{\tilde{P}_n(x;\alpha,\beta,\gamma;q)\}$ is orthogonal for $0 < \alpha q < 1$, $0 \le \beta q < 1$ and $\gamma < 0$ with respect to the continuous weight function $w(x) = \frac{(\alpha^{-1}x,\gamma^{-1}x;q)_\infty}{(x,\beta\gamma^{-1}x;q)_\infty}$, on the interval $(\gamma q,\alpha q)$. As the parameter α decreases to αq, the interval in which the zeros lie decreases to $(\gamma q,\alpha q^2)$ and we can deduce that the zeros of $\tilde{P}_n(x;\alpha,\beta,\gamma;q)$ decrease as α decreases to αq. Similarly, as γ increases to γq, the zeros will increase, since the interval in which the zeros lie reduces to $(\gamma q^2,\alpha q)$.

3.2.1 Interlacing properties

We have $\tilde{P}_n(x;\alpha,\beta,\gamma;q) = \sum_{k=-\infty}^{\infty} F$, where, by (3.1),

$$F := \frac{(\alpha q,\gamma q;q)_n (q^{-n},\alpha\beta q^{n+1},x;q)_k q^k}{(\alpha\beta q^{n+1};q)_n (\alpha q,\gamma q,q;q)_k}.$$

Using our codes with `qMixRec1(F,q,k,P(n),0,alpha,0,1,0)`, `qMixRec1(F,q,k,P(n),0,beta,0,1,0)` and `qMixRec2(F,q,k,P(n),0,alpha,0,beta,1,1,0,0,1)`, respectively, we get the following recurrence equations (see the Maple file which can be downloaded from http://www.mathematik.uni-kassel.de/~tcheutia/).

Proposition 3.1. *The following mixed recurrence equations are valid:*

$$\tilde{P}_n(x;\alpha,\beta,\gamma;q) = \tilde{P}_n(x;\alpha q,\beta,\gamma;q) + \frac{\alpha\, q\,(q^n-1)\,(\beta\, q^n-1)\,(\gamma\, q^n-1)}{(\alpha\beta\, q^{2\,n+1}-1)\,(\alpha\,\beta\, q^{2\,n}-1)}\tilde{P}_{n-1}(x;\alpha q,\beta,\gamma;q); \tag{3.2a}$$

$$\begin{aligned}\tilde{P}_n(x;\alpha,\beta,\gamma;q) &= \tilde{P}_n(x;\alpha,\beta q,\gamma;q)\\ &\quad - \frac{\alpha\,\beta\, q^{n+1}\,(\alpha\, q^n-1)\,(\gamma\, q^n-1)\,(q^n-1)}{(\alpha\beta\, q^{2\,n+1}-1)\,(\alpha\,\beta\, q^{2\,n}-1)}\tilde{P}_{n-1}(x;\alpha,\beta q,\gamma;q);\end{aligned} \tag{3.2b}$$

$$\tilde{P}_n(x;\alpha,\beta q,\gamma;q) = \tilde{P}_n(x;\alpha q,\beta,\gamma;q) + \frac{\alpha q(q^n-1)(\gamma q^n-1)}{\alpha\beta q^{2n+1}-1}\tilde{P}_{n-1}(x;\alpha q,\beta q,\gamma;q). \tag{3.2c}$$

From the latter equations, we deduce the interlacing properties of the zeros given by

Theorem 3.2. *Let* $0<\alpha q<1$, $0\le\beta q<1$, $\gamma<0$ *and denote the zeros of* $\tilde{P}_n(x;\alpha,\beta,\gamma;q)$ *by* $\gamma q < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < \alpha q$, *the zeros of* $\tilde{P}_n(x;\alpha q,\beta,\gamma;q)$ *by* $y_{n,1} < y_{n,2} < \cdots < y_{n,n}$, *the zeros of* $\tilde{P}_n(x;\alpha,\beta q,\gamma;q)$ *by* $z_{n,1} < z_{n,2} < \cdots < z_{n,n}$ *and the zeros of* $\tilde{P}_n(x;\alpha q,\beta q,\gamma;q)$ *by* $t_{n,1} < t_{n,2} < \cdots < t_{n,n}$. *Then, for each* $i \in \{1,2,\ldots,n-1\}$,

(a) $y_{n,i} < x_{n,i} < y_{n-1,i} < y_{n,i+1} < x_{n,i+1}$,

(b) $x_{n,i} < z_{n,i} < z_{n-1,i} < x_{n,i+1} < z_{n,i+1}$,

(c) $y_{n,i} < x_{n,i} < z_{n,i} < t_{n-1,i} < y_{n,i+1} < x_{n,i+1} < z_{n,i+1}$.

Proof. Let $0<\alpha q<1$, $0\le\beta q<1$, $\gamma<0$. Since $0<q<1$, it follows that $q^n-1<0$, $\alpha q^n-1<0$, $\beta q^n-1<0$, $\alpha\beta q^{2n}-1<0$, $\alpha\beta q^{2n+1}-1<0$ and $\gamma q^n-1<0$.

(a) Since $\tilde{P}_n(x;\alpha q,\beta,\gamma;q)$ and $\tilde{P}_{n-1}(x;\alpha q,\beta,\gamma;q)$ belong to the same orthogonal sequence, their zeros interlace and the interlacing property (1.1) is satisfied. (3.2a) is in the form of (1.2), i.e.,

$$f_n(x) = a(x)p_n(x) + b(x)q_{n-1}(x),$$

with $a(x)=1$ and, taking into consideration the restrictions on the parameters, $b(x)$ is a negative constant and the interlacing follows from Corollary 1.3 (b).

(b) The polynomials $\tilde{P}_n(x;\alpha,\beta q,\gamma;q)$ and $\tilde{P}_{n-1}(x;\alpha,\beta q,\gamma;q)$ belong to the same orthogonal sequence and their zeros satisfy (1.1). (3.2b) is in the form of (1.2) with $a(x)=1$ and taking into consideration the restrictions on the parameters, $b(x)$ is a positive constant. The result follows from Corollary 1.3 (a).

(c) In (b) we have proved that the zeros of $\tilde{P}_n(x;\alpha,\beta,\gamma;q)$ and $\tilde{P}_{n-1}(x;\alpha,\beta q,\gamma;q)$ interlace for all α such that $0<\alpha q<1$, from which we can deduce that the zeros of $\tilde{P}_n(x;\alpha q,\beta,\gamma;q)$ and $\tilde{P}_{n-1}(x;\alpha q,\beta q,\gamma;q)$ interlace, satisfying (1.1). Equation (3.2c) is in the form of (1.2) with $a(x)=1$ and taking into consideration the restrictions on the parameters, $b(x)$ is a negative constant. Applying Corollary 1.3 (b), we obtain $y_{n,i}<z_{n,i}<t_{n-1,i}<y_{n,i+1}<z_{n,i+1}$ for each $i\in\{1,2,\ldots,n-1\}$. Furthermore, $y_{n,i}<x_{n,i}<z_{n,i}$ for each $i\in\{1,2,\ldots,n\}$ (from (a) and (b)), and the required combined interlacing follows.

□

Corollary 3.3. *For each $i\in\{1,2,\ldots,n-1\}$,*

(a) $x_{n,i}<y_{n-1,i}<x_{n-1,i}<x_{n,i+1}$,

(b) $x_{n,i}<x_{n-1,i}<z_{n-1,i}<x_{n,i+1}$.

Proof. We obtain the results by combining the interlacing of the zeros of $\tilde{P}_n(x;\alpha,\beta,\gamma;q)$ and $\tilde{P}_{n-1}(x;\alpha,\beta,\gamma;q)$ with the results proved in Theorem 3.2 (a) and (b), respectively. □

Remark 3.4. *(i) In general, the zeros of $\tilde{P}_n(x;\alpha,\beta,\gamma;q)$ do not interlace with the zeros of $\tilde{P}_n(x;\alpha,\beta,\gamma q;q)$ or with the zeros of $\tilde{P}_{n-1}(x;\alpha,\beta,\gamma q;q)$. For example, when $n=4,\alpha=1,\beta=3,\ \gamma=-5,q=0.14$, the zeros of $\tilde{P}_n(x;\alpha,\beta,\gamma;q)$ are $\{-0.6993,-0.1066,0.0198,0.1353\}$, the zeros of $\tilde{P}_n(x;\alpha,\beta,\gamma q;q)$ are $\{-0.0992,0.0000,0.0071,0.1407\}$ and the zeros of $\tilde{P}_{n-1}(x;\alpha,\beta,\gamma q;q)$ are $\{-0.0978,0.0056,0.1399\}$;*

(ii) When $\beta=0$ in the definition of the monic big q-Jacobi polynomials, we obtain the monic big q-Laguerre polynomials, i.e., $\tilde{P}_n(x;\alpha,0,\gamma;q)=\tilde{P}_n(x;\alpha,\gamma;q)$ [Koekoek et al., 2010, Eq. (14.5.13)]. The interlacing property of the zeros of the big q-Laguerre polynomials, as α decreases to αq, can thus be obtained from the result obtained for the big q-Jacobi polynomials. Furthermore, we have $\tilde{P}_n(x;\alpha,\beta;q)=\tilde{P}_n(x;\beta,\alpha;q)$ and the interlacing property as β increases to βq follows directly. The interlacing results of Theorem 3.2 and Corollary 3.3 are therefore valid, where $x_{n,i},y_{n,i},z_{n,i},t_{n,i},i\in\{1,2,\ldots,n\}$ are the zeros of $\tilde{P}_n(x;\alpha,\gamma;q)$, $\tilde{P}_n(x;\alpha q,\gamma;q)$, $\tilde{P}_n(x;\alpha,\gamma q;q)$ and $\tilde{P}_n(x;\alpha q,\gamma q;q)$, respectively.

3.2.2 Quasi-orthogonality

The first two recurrence equations in the following proposition follow from (3.2a) and (3.2b), with α and β replaced by $\frac{\alpha}{q}$ and $\frac{\beta}{q}$, respectively. The big q-Jacobi polynomials are orthogonal for $\gamma<0$, and by replacing γ by $\frac{\gamma}{q}, 0<q<1$, we obtain the polynomial $\tilde{P}_n(x;\alpha,\beta,\frac{\gamma}{q};q)$ of which all the parameters are still in the regions where orthogonality is guaranteed and we will therefore not consider a q-shift of γ.

Proposition 3.5.

$$\tilde{P}_n(x;\frac{\alpha}{q},\beta,\gamma;q)=\tilde{P}_n(x;\alpha,\beta,\gamma;q)+\frac{\alpha q\,(q^n-1)\,(\beta\,q^n-1)\,(\gamma\,q^n-1)}{(\alpha\,\beta\,q^{2\,n}-1)\,(\alpha\,\beta\,q^{2\,n}-q)}\tilde{P}_{n-1}(x;\alpha,\beta,\gamma;q); \tag{3.3a}$$

$$\tilde{P}_n(x;\alpha,\frac{\beta}{q},\gamma;q)=\tilde{P}_n(x;\alpha,\beta,\gamma;q)-\frac{\alpha\beta q^{n+1}\,(q^n-1)\,(\alpha q^n-1)\,(\gamma q^n-1)}{(\alpha\,\beta\,q^{2\,n}-1)\,(\alpha\,\beta\,q^{2\,n}-q)}\tilde{P}_{n-1}(x;\alpha,\beta,\gamma;q); \tag{3.3b}$$

$$\tilde{P}_n(x;\alpha,\frac{\beta}{q},\frac{\gamma}{q};q)=\tilde{P}_n(x;\alpha,\beta,\gamma;q)-\frac{(q^n-1)\,(\alpha\,q^n-1)\,(-\alpha\,\beta\,q^n+\gamma)}{(\alpha\,\beta\,q^{2\,n}-1)\,(\alpha\,\beta\,q^{2\,n-1}-1)}\tilde{P}_{n-1}(x;\alpha,\beta,\gamma;q). \tag{3.3c}$$

Corollary 3.6.

$$\begin{aligned}\tilde{P}_n(x;\frac{\alpha}{q},\frac{\beta}{q},\gamma;q)&=\tilde{P}_n(x;\alpha,\beta,\gamma;q)\\&-\frac{\alpha q\,(\alpha\beta q^{2\,n}-\beta q^{n+1}-\beta q^n+q)\,(q^n-1)\,(\gamma q^n-1)}{(\alpha\,\beta\,q^{2\,n}-1)\,(\alpha\,\beta\,q^{2\,n}-q^2)}\tilde{P}_{n-1}(x;\alpha,\beta,\gamma;q)\\&-\frac{\alpha^2\beta\,(q^n-1)\,(\beta q^n-q)\,(\alpha q^n-q)\,(\gamma q^n-1)\,(\gamma q^n-q)\,(q^n-q)\,q^{n+3}}{(\alpha\,\beta\,q^{2\,n}-q^2)^2\,(\alpha\,\beta\,q^{2\,n}-q^3)\,(\alpha\,\beta\,q^{2\,n}-q)}\tilde{P}_{n-2}(x;\alpha,\beta,\gamma;q).\end{aligned} \tag{3.4}$$

Proof. By replacing β with $\frac{\beta}{q}$ in (3.3a), we obtain an equation involving polynomials $\tilde{P}_n(x;\frac{\alpha}{q},\frac{\beta}{q},\gamma;q)$, $\tilde{P}_n(x;\alpha,\frac{\beta}{q},\gamma;q)$ and $\tilde{P}_{n-1}(x;\alpha,\frac{\beta}{q},\gamma;q)$. We use (3.3b) to replace the latter two polynomials and, after simplifying, we obtain (3.4). □

We will start by proving the quasi-orthogonality of the sequence $\{\tilde{P}_n(x;\frac{\alpha}{q^k},\beta,\gamma;q)\}_{n=0}^{\infty}$. In order to ensure that the parameter $\frac{\alpha}{q^k}$ is not in the region where orthogonality is guaranteed, we fix $\alpha>1$ with $0<\alpha q<1$, such that $\frac{\alpha}{q^k}>1$, $k\in\{1,2,\ldots,n-1\}$.

Theorem 3.7. *Let $k,l,m\in\mathbb{N}_0$, $\alpha,\beta,\gamma\in\mathbb{R}$, $0<\alpha q<1$, $0\le\beta q<1$ and $\gamma<0$. The sequence of big q-Jacobi polynomials*

(i) $\{\tilde{P}_n(x;\frac{\alpha}{q^k},\beta,\gamma;q)\}_{n\ge0}$, $\alpha>1$, *is quasi-orthogonal of order $k\le n-1$ with respect to $w(x)$ on the interval $(\gamma q,\alpha q)$ and the polynomials have at least $(n-k)$ real, distinct zeros in $(\gamma q,\alpha q)$;*

(ii) $\{\tilde{P}_n(x;\alpha,\frac{\beta}{q^m},\gamma;q)\}_{n\ge0}$, $\beta>1$, *is quasi-orthogonal of order $m\le n-1$ with respect to $w(x)$ on $(\gamma q,\alpha q)$ and the polynomials have at least $(n-m)$ real, distinct zeros in $(\gamma q,\alpha q)$;*

(iii) $\{\tilde{P}_n(x;\alpha,\frac{\beta}{q^l},\frac{\gamma}{q^l};q)\}_{n\ge0}$, $\beta>1$, *is quasi-orthogonal of order $l\le n-1$ with respect to $w(x)$ on $(\gamma q,\alpha q)$ and the polynomials have at least $(n-l)$ real, distinct zeros in $(\gamma q,\alpha q)$;*

(iv) $\{\tilde{P}_n(x;\frac{\alpha}{q^k},\frac{\beta}{q^m},\gamma;q)\}_{n\ge0}$, $\alpha,\beta>1$ *is quasi-orthogonal of order $k+m\le n-1$ with respect to $w(x)$ on $(\gamma q,\alpha q)$ and the polynomials have at least $n-(k+m)$ real, distinct zeros in $(\gamma q,\alpha q)$.*

Proof.

(i) Fix $\alpha > 1$ such that $0 < \alpha q < 1$. From Lemma 1.4 and (3.3a), it follows that $\tilde{P}_n(x; \frac{\alpha}{q}, \beta, \gamma; q)$ is quasi-orthogonal of order one on $(\gamma q, \alpha q)$ and according to Lemma 1.6, at least $(n-1)$ zeros of $\tilde{P}_n(x; \frac{\alpha}{q}, \beta, \gamma; q)$ lie in the interval $(\gamma q, \alpha q)$. By iteration, we can express $\tilde{P}_n(x; \frac{\alpha}{q^k}, \beta, \gamma; q)$ as a linear combination of $\tilde{P}_n(x; \alpha, \beta, \gamma; q)$, $\tilde{P}_{n-1}(x; \alpha, \beta, \gamma; q)$, $\ldots$, $\tilde{P}_{n-k}(x; \alpha, \beta, \gamma; q)$, and from Lemma 1.4 we deduce that $\tilde{P}_n(x; \frac{\alpha}{q^k}, \beta, \gamma; q)$ is quasi-orthogonal of order k on $(\gamma q, \alpha q)$. It follows from Lemma 1.6 that at least $(n-k)$ zeros of $\tilde{P}_n(x; \frac{\alpha}{q^k}, \beta, \gamma; q)$ are in $(\gamma q, \alpha q)$.

(ii)-(iii) Fix $\beta > 1$ such that $0 < \beta q < 1$. The proofs follow in exactly the same way as the proof of (i), by using (3.3b) and (3.3c), together with Lemmas 1.4 and 1.6.

(iv) Fix $\alpha > 1, \beta > 1$ such that $0 < \alpha q < 1$ and $0 < \beta q < 1$. From (3.4), $\tilde{P}_n(x; \frac{\alpha}{q}, \frac{\beta}{q}, \gamma; q)$ can be written as a linear combination of $\tilde{P}_n(x; \alpha, \beta, \gamma; q)$, $\tilde{P}_{n-1}(x; \alpha, \beta, \gamma; q)$ and $\tilde{P}_{n-2}(x; \alpha, \beta, \gamma; q)$, and it follows from Lemma 1.4 that each polynomial $\tilde{P}_n(x; \frac{\alpha}{q}, \frac{\beta}{q}, \gamma; q)$, $n \in \{1, 2, \ldots\}$, is quasi-orthogonal of order two on $(\gamma q, \alpha q)$. From Lemma 1.6, we know that at least $(n-2)$ zeros of $\tilde{P}_n(x; \frac{\alpha}{q}, \frac{\beta}{q}, \gamma; q)$ lie in $(\gamma q, \alpha q)$. By iteration, we can express $\tilde{P}_n(x; \frac{\alpha}{q^k}, \frac{\beta}{q^m}, \gamma; q)$ as a linear combination of $\tilde{P}_n(x; \alpha, \beta, \gamma; q)$, $\tilde{P}_{n-1}(x; \alpha, \beta, \gamma; q)$, $\ldots$, $\tilde{P}_{n-(k+m)}(x; \alpha, \beta, \gamma; q)$, and the results follow directly from Lemmas 1.4 and 1.6.

□

In order to determine the location of the zeros of the order one and order two quasi-orthogonal systems, we use a q-analogue of the Vandermonde identity, namely

$$ {}_2\phi_1\left(\begin{matrix} q^{-n}, b \\ c \end{matrix} \middle| q; q\right) = \frac{(\frac{c}{b}; q)_n}{(c; q)_n} b^n \text{ [Gasper and Rahman, 2004, Eq. (1.5.3)].} \tag{3.5} $$

Theorem 3.8. *Let $n \in \mathbb{N}$, $\alpha, \beta, \gamma \in \mathbb{R}$, such that $0 < \alpha q, \beta q < 1$ and $\gamma < 0$. Suppose $x_{n,j}, j \in \{1, 2, \ldots, n\}$ denote the zeros of $\tilde{P}_n(x; \alpha, \beta, \gamma; q)$, $y_{n,j}, j \in \{1, 2, \ldots, n\}$ the zeros of $\tilde{P}_n(x; \frac{\alpha}{q}, \beta, \gamma; q)$, $z_{n,j}, j \in \{1, 2, \ldots, n\}$ the zeros of $\tilde{P}_n(x; \alpha, \frac{\beta}{q}, \gamma; q)$, $v_{n,j}, j \in \{1, 2, \ldots, n\}$ the zeros of $\tilde{P}_n(x; \frac{\alpha}{q}, \frac{\beta}{q}, \gamma; q)$ and $w_{n,j}, j \in \{1, 2, \ldots, n\}$ the zeros of $\tilde{P}_n(x; \alpha, \frac{\beta}{q}, \frac{\gamma}{q}; q)$. Then,*

(i) when we fix $\alpha > 1$, such that $0 < \alpha q < 1$,

$$ \gamma q < x_{n,1} < y_{n,1} < x_{n-1,1} < x_{n,2} < y_{n,2} < x_{n-1,2} < \cdots < x_{n-1,n-1} < x_{n,n} < y_{n,n}; $$

(ii) when we fix $\beta > 1$, such that $0 < \beta q < 1$,

$$ z_{n,1} < \gamma q < x_{n,1} < x_{n-1,1} < z_{n,2} < x_{n,2} < \cdots < x_{n-1,n-1} < z_{n,n} < x_{n,n} < \alpha q; $$

(iii) when we fix $\beta > 1$, such that $0 < \beta q < 1$, we also have

$$ w_{n,1} < x_{n,1} < x_{n-1,1} < w_{n,2} < x_{n,2} < \cdots < x_{n-1,n-1} < w_{n,n} < x_{n,n} < \alpha q; $$

(iv) when we fix $\alpha, \beta > 1$, such that $0 < \alpha q, \beta q < 1$, all the zeros of $\tilde{P}_n(x; \frac{\alpha}{q}, \frac{\beta}{q}, \gamma; q)$ are real and distinct and $v_{n,1} < \gamma q$.

Proof.

(i) From (3.3a), we obtain $a_n = \frac{\alpha q(q^n-1)(\beta q^n-1)(\gamma q^n-1)}{(\alpha\beta q^{2n}-q)(\alpha\beta q^{2n}-1)} < 0$, and the interlacing result, as well as the location of $y_{n,1}$, follows from Lemma 1.8 (i).

(ii) From (3.3b), we obtain $a_n = -\frac{\alpha\beta q^{n+1}(q^n-1)(\alpha q^n-1)(\gamma q^n-1)}{(\alpha\beta q^{2n}-1)(\alpha\beta q^{2n}-q)}$, which is positive when taking into consideration the values of the parameters. The interlacing result, as well as the location of $y_{n,n}$, follows from Lemma 1.8 (ii).

The polynomial $\tilde{P}_n(x;\alpha,\beta,\gamma;q)$ evaluated at $x=\gamma q$, can be written in terms of a ${}_2\phi_1$-hypergeometric function. We apply (3.5), and simplify, to obtain

$$f_n(\gamma q) = \frac{\tilde{P}_n(\gamma q;\alpha,\beta,\gamma;q)}{\tilde{P}_{n-1}(\gamma q;\alpha,\beta,\gamma;q)} = \frac{\alpha\,(\alpha\,\beta\,q^n-1)\,(\beta\,q^n-1)\,(\gamma\,q^n-1)\,q^{n+1}}{(\alpha\,\beta\,q^{2\,n}-q)\,(\alpha\,\beta\,q^{2\,n}-1)} \tag{3.6}$$

and by taking into account the values of the parameters, this expression is negative. Since

$$-a_n - f_n(\gamma q) = -\frac{\alpha\,(\beta-1)\,(\gamma\,q^n-1)\,q^{n+1}}{\alpha\,\beta\,q^{2\,n}-q} < 0,$$

the result follows from Lemma 1.7 (i).

(iii) From (3.3c), we obtain $a_n = -\frac{(q^n-1)(\alpha q^n-1)(-\alpha\beta q^n+\gamma)q}{(\alpha\beta q^{2n}-q)(\alpha\beta q^{2n}-1)} > 0$, and the interlacing result, as well as the location of $w_{n,n}$, follows from Lemma 1.8 (ii).

(iv) Fix $\alpha > 1$ and $\beta > 1$ such that $0 < \alpha q < 1$ and $0 < \beta q < 1$. We use (3.4), with a_n the coefficient of $\tilde{P}_{n-1}(x;\alpha,\beta,\gamma;q)$ and b_n the coefficient of $\tilde{P}_{n-2}(x;\alpha,\beta,\gamma;q)$. By taking into account the values of the parameters,

$$b_n = -\frac{\alpha^2\beta\,(q^n-1)\,(\beta q^n-q)\,(\alpha q^n-q)\,(\gamma q^n-1)\,(\gamma q^n-q)\,(q^n-q)\,q^{n+3}}{(\alpha\,\beta\,q^{2\,n}-q^2)^2\,(\alpha\,\beta\,q^{2\,n}-q^3)\,(\alpha\,\beta\,q^{2\,n}-q)} < 0,$$

and it follows from [Brezinski et al., 2004, Theorem 4] that $v_{n,j}$, $j \in \{1,2,\ldots,n\}$, are real.

In order to determine the location of $v_{n,1}$ and $v_{n,n}$, we use [Joulak, 2005, Theorem 9]. Since

$$\begin{aligned}
f_n(\gamma q) f_{n-1}(\gamma q) + a_n f_{n-1}(\gamma q) + b_n &= \frac{\tilde{P}_n(\gamma q;\alpha,\beta,\gamma;q)}{\tilde{P}_{n-1}(\gamma q;\alpha,\beta,\gamma;q)} + a_n \frac{\tilde{P}_{n-1}(\gamma q;\alpha,\beta,\gamma;q)}{\tilde{P}_{n-2}(\gamma q;\alpha,\beta,\gamma;q)} + b_n \\
&= \frac{\alpha^2\,(\beta-1)\,(\gamma q^n-1)\,(\beta q^n-q)\,(\gamma q^n-q)\,q^{n+2}}{(\alpha\,\beta\,q^{2n}-q^2)\,(\alpha\,\beta\,q^{2n}-q^3)} \\
&< 0,
\end{aligned}$$

it follows that $v_{n,1} < \gamma q$. Furthermore,

$$\begin{aligned}
&f_n(\alpha q) f_{n-1}(\alpha q) + a_n f_{n-1}(\alpha q) + b_n \\
&= -\frac{(\alpha q^n-q)\,(\alpha^2\beta q^{2n}-\alpha\gamma q^{n+1}-\alpha\beta q^n-\alpha q^{n+1}+\gamma q^{n+1}+\alpha q)\,(\gamma-\alpha\beta)\,q^{n+2}}{(\alpha\beta q^{2n}-q^3)\,(\alpha\beta q^{2n}-q^2)}
\end{aligned}$$

and since the sign of this expression varies as the parameters vary within the regions applicable, we cannot determine the position of $v_{n,n}$.

$\square$

Remark 3.9. *(i) From Theorem 3.7 (i) we know that the polynomial* $\tilde{P}_n(x;\frac{\alpha}{q},\beta,\gamma;q), \alpha > 1$, *is quasi-orthogonal of order one and an interlacing result is proved in Theorem 3.8(i), but the location of the extreme zero* $y_{n,n}$, *with respect to* $(\gamma q, \alpha q)$, *is not fixed, since the sign of*

$$\begin{aligned} -a_n - f_n(\alpha q) &= a_n - \frac{\tilde{P}_n(\alpha q;\alpha,\beta,\gamma;q)}{\tilde{P}_{n-1}(\alpha q;\alpha,\beta,\gamma;q)} \\ &= -\frac{(-\alpha^2 q^{2\,n}\beta + \alpha\,\beta\,q^n + q^n\alpha\,\gamma + \alpha\,q^n - \gamma\,q^n - \alpha)\,q}{\alpha\,\beta\,q^{2\,n} - q} \end{aligned} \tag{3.7}$$

changes as the parameters vary within the region applicable.

(ii) When $\beta = 0$ *in the definition of the big q-Jacobi polynomials (3.1), we obtain the big q-Laguerre polynomials, i.e.,* $\tilde{P}_n(x;\alpha,0,\gamma;q) = \tilde{P}_n(x;\alpha,\gamma;q)$ *[Koekoek et al., 2010, (14.5.13)] and we can use (3.3a) with* $\beta = 0$. *Let* $x_{n,j}, j \in \{1,2,\dots,n\}$, *be the zeros of* $\tilde{P}_n(x;\alpha,\gamma;q)$ *and* $y_{n,j}, j \in \{1,2,\dots,n\}$, *the zeros of* $\tilde{P}_n(x;\frac{\alpha}{q},\gamma;q)$. *When* $\beta = 0$ *in (3.7), we obtain*

$$-a_n - f_n(\alpha q) = a_n - \frac{\tilde{P}_n(\alpha q;\alpha,\gamma;q)}{\tilde{P}_{n-1}(\alpha q;\alpha,\gamma;q)} = \gamma q^n(\alpha-1) + \alpha\,(q^n-1) < 0,$$

taking into consideration that $\alpha > 1$, $0 < \alpha q < 1$ *and* $\gamma < 0$, *and*

$$f_n(\gamma q) + a_n = \frac{\tilde{P}_n(\gamma q;\alpha,\gamma;q)}{\tilde{P}_{n-1}(\gamma q;\alpha,\gamma;q)} + a_n = \alpha\ q^n\,(\gamma\,q^n - 1) + a_n = \alpha\,(\gamma\,q^n - 1) < 0.$$

We thus have $f_n(\gamma q) < -a_n < f_n(\alpha q)$ *and according to Lemma 1.7 (iii), all the zeros of the order one quasi-orthogonal polynomial* $\tilde{P}_n(x;\frac{\alpha}{q},\gamma;q), \alpha > 1$, *lie in* $(\gamma q, \alpha q)$. *Furthermore, since* $a_n < 0$, *it follows from Lemma 1.8 (ii) that*

$$\gamma q < x_{n,1} < y_{n,1} < x_{n-1,1} < x_{n,2} < y_{n,2} < x_{n-1,2} < \cdots < x_{n-1,n-1} < x_{n,n} < y_{n,n} < \alpha q.$$

3.2.3 Bounds of the extreme zeros

The big q-Jacobi polynomials $\tilde{P}_n(x;\alpha,\beta,\gamma;q)$ satisfied equations of the type

$$f(x)\tilde{P}_{n-k}(x;\alpha q^{s_1},\beta q^{s_2},\gamma;q) = H(x)\tilde{P}_n(x;\alpha,\beta,\gamma;q) + G_{k-1,s_1,s_2}(x)\tilde{P}_{n-1}(x;\alpha,\beta,\gamma;q),$$

with $s_1 + s_2 \in \{0,1,\dots,2k\}$. Let $B^{(1)}_{k,s_1,s_2}$ and $B^{(2)}_{k,s_1,s_2}$ be the smallest and the largest zero of G_{k-1,s_1,s_2}, respectively. For $k=2$, the best upper bound for $x_{n,1}$ is $B^{(1)}_{2,0,4}$, the zero of $G_{1,0,4}$, and the best lower bound for $x_{n,n}$ is $B^{(2)}_{2,4,0}$, the zero of $G_{1,4,0}$. We observe from numerical simulations that the sharpest bounds for $x_{n,1}$ are the smallest zeros $B^{(1)}_{k,0,2k}$ of $G_{k-1,0,2k}$ and the sharpest bounds of $x_{n,n}$ are the largest zeros $B^{(2)}_{k,2k,0}$ of $G_{k-1,2k,0}$.

3.3 The q-Hahn polynomials

The q-Hahn polynomials $\tilde{Q}_n(\bar{x};\alpha,\beta,N|q)$ with $\bar{x} = q^{-x}$ and $n \in \{0,1,\dots,N\}$ are orthogonal on $(1,q^{-N})$ with respect to the discrete weight $w(x) = \frac{(\alpha q,q^{-N};q)_x}{(q,\beta^{-1}q^{-N};q)_x(\alpha\beta q)^x}$ for $0 < \alpha q < 1$ and $0 < \beta q < 1$ or $\alpha > q^{-N}$ and $\beta > q^{-N}$.

3.3.1 Interlacing properties

Proposition 3.10.

$$\tilde{Q}_n(x;\alpha,\beta,N|q) = \tilde{Q}_n(x;\alpha q,\beta,N|q) + \frac{\alpha\,(q^n-1)\,(\beta\,q^n-1)\,\left(q^n-q^{N+1}\right)}{q^N\,(\alpha\beta\,q^{2\,n+1}-1)\,(\alpha\,\beta\,q^{2\,n}-1)}\tilde{Q}_{n-1}(x;\alpha q,\beta,N|q); \tag{3.8a}$$

$$\begin{aligned}\tilde{Q}_n(x;\alpha,\beta,N|q) &= \tilde{Q}_n(x;\alpha,\beta q,N|q)\\ &\quad+\frac{\alpha\,\beta\,q^{n-N}\,\left(q^{N+1}-q^n\right)\,(\alpha\,q^n-1)\,(q^n-1)}{(\alpha\beta\,q^{2\,n+1}-1)\,(\alpha\,\beta\,q^{2\,n}-1)}\tilde{Q}_{n-1}(x;\alpha,\beta q,N|q);\end{aligned} \tag{3.8b}$$

$$\tilde{Q}_n(x;\alpha,\beta q,N|q) = \tilde{Q}_n(x;\alpha q,\beta,N|q) + \frac{\alpha(q^n-1)(q^n-q^{N+1})}{q^N(\alpha\beta q^{2n+1}-1)}\tilde{Q}_{n-1}(x;\alpha q,\beta q,N|q); \tag{3.8c}$$

$$\begin{aligned}\tilde{Q}_n\,(x;\alpha,\beta,N|q) &= \frac{\left(\alpha\beta q^{2\,n+1}-1\right)\left(\alpha\beta q^{N+2}(\alpha q^{n+1}-1)-\alpha q^{N-n+2}(q^n-1)-(\alpha q-1)\right)}{(\alpha q^{n+1}-1)\,(\alpha\beta q^{n+1}-1)\,(\alpha\beta q^{n+N+2}-1)}\\ &\times\tilde{Q}_n(x;\alpha q,\beta,N|q) + \frac{\alpha q\,(\beta q^n-1)\,\left(q^n-q^{1+N}\right)\,(q^n-1)\,\left(\alpha q^2-x\right)}{q^n\,(\alpha q^{n+1}-1)\,(\alpha\beta q^{n+1}-1)\,(\alpha\beta q^{n+N+2}-1)}\tilde{Q}_{n-1}(x;\alpha q^2,\beta,N|q).\end{aligned} \tag{3.8d}$$

Theorem 3.11. *Let $0<\beta q<1$, $0<\alpha q<1$, $n\in\{0,1,\dots,N\}$. We denote the zeros of $\tilde{Q}_n(x;\alpha,\beta,N|q)$ by $1<x_{n,1}<x_{n,2}<\cdots<x_{n,n}<q^{-N}$, the zeros of $\tilde{Q}_n(x;\alpha q,\beta,N|q)$ by $y_{n,1}<y_{n,2}<\cdots<y_{n,n}$, the zeros of $\tilde{Q}_n(x;\alpha,\beta q,N|q)$ by $z_{n,1}<z_{n,2}<\cdots<z_{n,n}$ and the zeros of $\tilde{Q}_n(x;\alpha q,\beta q,N|q)$ by $t_{n,1}<t_{n,2}<\cdots<t_{n,n}$. Then, for $i\in\{1,2,\dots,n-1\}$,*

(a) $x_{n,i}<y_{n,i}<y_{n-1,i}<x_{n,i+1}<y_{n,i+1}$,

(b) $z_{n,i}<x_{n,i}<z_{n-1,i}<z_{n,i+1}<x_{n,i+1}$,

(c) $z_{n,i}<x_{n,i}<y_{n,i}<t_{n-1,i}<z_{n,i+1}<x_{n,i+1}<y_{n,i+1}$.

Proof. Let $0<\beta q<1$, $0<\alpha q<1$, $n\in\{0,1,\dots,N\}$. Since $0<q<1$, it follows that $q^n-1<0$, $\beta q^n-1<0$, $\alpha\beta q^{2n}-1<0$ and $\alpha\beta q^{2n+1}-1<0$. Furthermore, $q^m<q^n$ for $m>n$ and consequently $q^{N+1}-q^n<0$.

The two polynomials on the right-hand side of each of the equations (3.8a) and (3.8b) belong to the same orthogonal sequence, therefore their zeros interlace and satisfy the interlacing property (1.1). Each of these equations are thus in the form of (1.2) with $a(x)=1$. Furthermore,

(a) in (3.8a), $b(x)>0$ on $(1,q^{-N})$ and the required interlacing follows from Corollary 1.3 (a);

(b) $b(x)$ in (3.8b) is a negative constant and the result follows from Corollary 1.3 (b);

(c) From the interlacing of the zeros of $\tilde{Q}_n(x;\alpha,\beta q,N|q)$ and $\tilde{Q}_{n-1}(x;\alpha,\beta q,N|q)$ for all α such that $0<\alpha q<1$ (from (b)), the interlacing of the zeros of $\tilde{Q}_n(x;\alpha q,\beta,\gamma;q)$ and $\tilde{Q}_{n-1}(x;\alpha q,\beta q,\gamma;q)$ follows directly. Equation (3.8c) is in the form of (1.2) with $a(x)=1$ and taking into consideration the restrictions on the parameters, $b(x)$ is a positive constant. Applying Corollary 1.3 (a), we obtain $z_{n,i}<y_{n,i}<t_{n-1,i}<z_{n,i+1}<y_{n,i+1}$ for each $i\in\{1,2,\dots,n-1\}$. Furthermore, it follows from (a) and (b) that $z_{n,i}<x_{n,i}<y_{n,i}$ for each $i\in\{1,2,\dots,n\}$, and the required combined interlacing follows.

$\square$

Corollary 3.12. *For each $i \in \{1, 2, \ldots, n-1\}$,*

(a) $x_{n,i} < x_{n-1,i} < y_{n-1,i} < x_{n,i+1}$,

(b) $x_{n,i} < z_{n-1,i} < x_{n-1,i} < x_{n,i+1}$.

Proof. We obtain the results by combining the interlacing of the zeros of $\tilde{Q}_n(x; \alpha, \beta, N|q)$ and $\tilde{Q}_{n-1}(x; \alpha, \beta, N|q)$ with the results proved in Theorem 3.11 (a) and (b), respectively. $\square$

Remark 3.13. *(i) When we let $\beta = 0$ in the definition of the monic q-Hahn polynomials, we obtain the monic affine q-Krawtchouk polynomials [Koekoek et al., 2010, Section 14.16] $\tilde{K}_n^{Aff}(\bar{x}; \alpha, N; q)$, orthogonal on $(1, q^{-N})$ if $0 < \alpha q < 1$. The interlacing results in Theorem 3.11 (a) and Corollary 3.12 (a) follow from (3.8a) (with $\beta = 0$), where $x_{n,i}$ and $y_{n,i}, i \in \{1, 2, \ldots, n\}$ are the zeros of $\tilde{K}_n^{Aff}(\bar{x}; \alpha, N; q)$ and $\tilde{K}_n^{Aff}(\bar{x}; \alpha q, N; q)$, respectively. Furthermore, when we let $\beta = 0$ in (3.8d), we find that*

$$x_{n,i} < y_{n,i} < Y_{n-1,i} < x_{n,i+1} < y_{n,i+1},$$

for each $i \in \{1, 2, \ldots, n-1\}$, where $Y_{n,i}, i \in \{1, 2, \ldots, n\}$ are the zeros of $\tilde{K}_n^{Aff}(\bar{x}; \alpha q^2, N; q)$;

(ii) Since $\lim_{\alpha \to \infty} \tilde{Q}_n(\bar{x}; \alpha, \beta, N|q) = \tilde{K}_n^{qtm}(\bar{x}; \beta, N; q)$ [Koekoek et al., 2010, Section 14.14], we obtain from (3.8b), the equation

$$\tilde{K}_n^{qtm}(\bar{x}; \beta, N; q) = \tilde{K}_n^{qtm}(\bar{x}; \beta q, N; q) + \frac{\left(q^{N+1} - q^n\right)\left(q^n - 1\right)}{\beta q^{2n+N+1}} \tilde{K}_{n-1}^{qtm}(\bar{x}; \beta q, N; q),$$

from which the interlacing results in Theorem 3.11 (b) and Corollary 3.12 (b) follow directly, where $x_{n,i}$ and $z_{n,i}, i \in \{1, 2, \ldots, n\}$ are the zeros of the monic quantum q-Krawtchouk polynomials $\tilde{K}_n^{qtm}(\bar{x}; \beta, N; q)$ and $\tilde{K}_n^{qtm}(\bar{x}; \beta q, N; q)$, respectively.

3.3.2 Quasi-orthogonality

We will consider the case $0 < \alpha q, \beta q < 1$. The following equations follow from (3.8a) and (3.8b), with α and β replaced by $\frac{\alpha}{q}$ and $\frac{\beta}{q}$, respectively.

$$\tilde{Q}_n(\bar{x}; \frac{\alpha}{q}, \beta, N|q) = \tilde{Q}_n(\bar{x}; \alpha, \beta, N|q) + \frac{\alpha\left(q^n - 1\right)\left(\beta q^n - 1\right)\left(q^n - q^{N+1}\right)}{q^N\left(\alpha\beta\, q^{2\,n} - 1\right)\left(\alpha\,\beta\, q^{2\,n} - q\right)} \tilde{Q}_{n-1}(\bar{x}; \alpha, \beta, N|q); \tag{3.9a}$$

$$\tilde{Q}_n(\bar{x}; \alpha, \frac{\beta}{q}, N|q) = \tilde{Q}_n(\bar{x}; \alpha, \beta, N|q) - \frac{\alpha\beta\left(q^n - q^{N+1}\right)\left(\alpha q^n - 1\right)\left(q^n - 1\right)}{\left(\alpha\beta q^{2\,n} - 1\right)\left(\alpha\,\beta\, q^{2\,n} - q\right) q^{N-n}} \tilde{Q}_{n-1}(\bar{x}; \alpha, \beta, N|q). \tag{3.9b}$$

Corollary 3.14.

$$\begin{aligned}\tilde{Q}_n(\bar{x}; \frac{\alpha}{q}, \frac{\beta}{q}, N|q) &= \tilde{Q}_n(\bar{x}; \alpha, \beta, N|q) \\ &- \frac{\alpha\left(\alpha\beta q^{2\,n} - \beta q^{n+1} - \beta q^n + q\right)\left(q^n - 1\right)\left(q^n - q^{N+1}\right)}{\left(\alpha\,\beta\, q^{2\,n} - q^2\right)\left(\alpha\,\beta\, q^{2\,n} - 1\right) q^N} \tilde{Q}_{n-1}(\bar{x}; \alpha, \beta, N|q) \\ &- \frac{\alpha^2 \beta q^{n+1}\left(q^n - 1\right)\left(\beta q^n - q\right)\left(q^n - q^{N+1}\right)\left(\alpha q^n - q\right)\left(q^n - q\right)\left(q^n - q^{N+2}\right)}{\left(\alpha\,\beta\, q^{2\,n} - q^2\right)^2\left(\alpha\,\beta\, q^{2\,n} - q\right)\left(\alpha\,\beta\, q^{2\,n} - q^3\right) q^{2N}} \tilde{Q}_{n-2}(\bar{x}; \alpha, \beta, N|q).\end{aligned} \tag{3.10}$$

Theorem 3.15. *Let $k, m, N \in \mathbb{N}_0$, $n \in \{0, 1, 2, \dots, N\}$, $\alpha, \beta, \in \mathbb{R}$. For $0 < \alpha q < 1$ and $0 < \beta q < 1$, the sequence of q-Hahn polynomials*

(i) $\{\tilde{Q}_n(\bar{x}; \frac{\alpha}{q^k}, \beta, N|q)\}_{n=0}^N$, *with $\alpha > 1$, is quasi-orthogonal of order $k \leq n-1$ with respect to the discrete weight $w(x)$ on the interval $(1, q^{-N})$ and the polynomials have at least $(n-k)$ real, distinct zeros in $(1, q^{-N})$;*

(ii) $\{\tilde{Q}_n(\bar{x}; \alpha, \frac{\beta}{q^m}, N|q)\}_{n=0}^N$, *$\beta > 1$, is quasi-orthogonal of order $m \leq n-1$ with respect to $w(x)$ on $(1, q^{-N})$ and the polynomials have at least $(n-m)$ real, distinct zeros in the interval $(1, q^{-N})$;*

(iii) $\{\tilde{Q}_n(\bar{x}; \frac{\alpha}{q^k}, \frac{\beta}{q^m}, N|q)\}_{n=0}^N$, *$\alpha, \beta > 1$, is quasi-orthogonal of order $k+m \leq n-1$ with respect to $w(x)$ on $(1, q^{-N})$ and the polynomials have at least $n-(k+m)$ real, distinct zeros in $(1, q^{-N})$.*

Proof.

(i) Fix $\alpha > 1, \beta \in \mathbb{R}$, such that $0 < \alpha q < 1$, $0 < \beta q < 1$. From Lemma 1.4 and (3.9a), it follows that $\tilde{Q}_n(\bar{x}; \frac{\alpha}{q}, \beta, N|q)$ is quasi-orthogonal of order one on $(1, q^{-N})$. From Lemma 1.6 we know that at least $(n-1)$ zeros of $\tilde{Q}_n(\bar{x}; \frac{\alpha}{q}, \beta, N|q)$ lie in the interval $(1, q^{-N})$. By iteration, we can express $\tilde{Q}_n(\bar{x}; \frac{\alpha}{q^k}, \beta, N|q)$ as a linear combination of $\tilde{Q}_n(\bar{x}; \alpha, \beta, N|q)$, $\tilde{Q}_{n-1}(\bar{x}; \alpha, \beta, N|q), \dots, \tilde{Q}_{n-k}(\bar{x}; \alpha, \beta, N|q)$, and the results follow from Lemmas 1.4 and 1.6.

(ii) Fix $\beta > 1, \alpha \in \mathbb{R}$, such that $0 < \alpha q < 1$, $0 < \beta q < 1$. The quasi-orthogonality follows in the same way as in (i), by using (3.9b).

(iii) Fix $\alpha > 1$ and $\beta > 1$ such that $0 < \alpha q < 1$ and $0 < \beta q < 1$. From (3.10), we see that $\tilde{Q}_n(\bar{x}; \frac{\alpha}{q}, \frac{\beta}{q}, N|q)$ can be written as a linear combination of $\tilde{Q}_n(\bar{x}; \alpha, \beta, N|q)$, $\tilde{Q}_{n-1}(\bar{x}; \alpha, \beta, N|q)$ and $\tilde{Q}_{n-2}(\bar{x}; \alpha, \beta, N|q)$ and it follows from Lemma 1.4 that the sequence $\tilde{Q}_n(\bar{x}; \frac{\alpha}{q}, \frac{\beta}{q}, \gamma; q)$ is quasi-orthogonal of order two on $(1, q^{-N})$. By iteration, we can express $\tilde{Q}_n(\bar{x}; \frac{\alpha}{q^k}, \frac{\beta}{q^m}, N|q)$ as a linear combination of $\tilde{Q}_n(\bar{x}; \alpha, \beta, N|q)$, $\tilde{Q}_{n-1}(\bar{x}; \alpha, \beta, N|q)$, $\dots$, $\tilde{Q}_{n-(k+m)}(\bar{x}; \alpha, \beta, N|q)$, and the result follows directly from Lemma 1.4. It follows from Lemma 1.6 that at least $n-(k+m)$ zeros of $\tilde{Q}_n(\bar{x}; \frac{\alpha}{q^k}, \frac{\beta}{q^m}, \gamma; q)$ lie in the interval $(1, q^{-N})$.

$\square$

Theorem 3.16. *Let $N \in \mathbb{N}_0$, $n \in \{0, 1, 2, \dots, N\}$, $\alpha, \beta \in \mathbb{R}$, $0 < \alpha q, \beta q < 1$, and let $x_{n,j}, j \in \{1, 2, \dots, n\}$, denote the zeros of $\tilde{Q}_n(\bar{x}; \alpha, \beta, N|q)$, $y_{n,j}, j \in \{1, 2, \dots, n\}$, the zeros of $\tilde{Q}_n(\bar{x}; \frac{\alpha}{q}, \beta, N|q)$ and $z_{n,j}, j \in \{1, 2, \dots, n\}$, the zeros of $\tilde{Q}_n(\bar{x}; \alpha, \frac{\beta}{q}, N|q)$. Then*

(i) if $\alpha > 1$, $y_{n,1} < 1 < x_{n,1} < x_{n-1,1} < y_{n,2} < x_{n,2} < \cdots < x_{n-1,n-1} < y_{n,n} < x_{n,n} < q^{-N}$;

(ii) if $\beta > 1$, $1 < x_{n,1} < z_{n,1} < x_{n-1,1} < x_{n,2} < z_{n,2} < \cdots < x_{n-1,n-1} < x_{n,n} < q^{-N} < z_{n,n}$.

Proof.

(i) From (3.9a) we obtain the value $a_n = \frac{\alpha\,(q^n-1)(\beta\,q^n-1)\left(q^n-q^{N+1}\right)}{q^N(\alpha\beta\,q^{2\,n}-1)(\alpha\,\beta\,q^{2\,n}-q)}$, which is positive when we take into consideration the values of the parameters. The interlacing result, from which we can deduce the location of $y_{n,n}$, follows from Lemma 1.8 (ii).

In order to prove that $y_{n,1}$ does not lie in the interval of orthogonality, i.e., $y_{n,1} < 1$, we use the fact that

$$ {}_3\phi_2\left(\begin{array}{c} q^{-n}, \alpha\beta q^{n+1}, 1 \\ \alpha q, q^{-N} \end{array}\middle| q;q\right) = 1 \text{ since } (1;q)_n = \begin{cases} 1, & \text{if } n=0 \\ 0, & \text{if } n \neq 0 \end{cases} $$

and Lemma 1.7. Consider

$$ f_n(1) = \frac{\tilde{Q}_n(1;\alpha,\beta,N|q)}{\tilde{Q}_{n-1}(1;\alpha,\beta,N|q)} = -\frac{(\alpha\,q^n-1)\,(\alpha\,\beta\,q^n-1)\left(q^n-q^{N+1}\right)}{(\alpha\beta\,q^{2\,n}-1)\,(\alpha\,\beta\,q^{2\,n}-q)}, $$

which is negative for the appropriate parameter values. We thus have

$$ -a_n - f_n(1) = \frac{(\alpha-1)\left(q^n-q^{N+1}\right)}{(\alpha\,\beta\,q^{2n}-q)\,q^N} < 0, $$

i.e., $-a_n < f_n(1) < 0$, and the result follows from Lemma 1.7 (i).

(ii) From (3.9b) we obtain $a_n = -\frac{\alpha\,\beta\,\left(q^n-q^{N+1}\right)(\alpha\,q^n-1)\,(q^n-1)\,q^n}{(\alpha\beta\,q^{2\,n}-1)\,(\alpha\,\beta\,q^{2\,n}-q)\,q^N}$, which is negative. The interlacing result, from which we can deduce the location of $z_{n,1}$, follows from Lemma 1.8 (i).

The polynomial $\tilde{Q}_n(\bar{x};\alpha,\beta,N|q)$ evaluated at $\bar{x} = q^{-N}$, can be written in terms of a ${}_2\phi_1$-hypergeometric function. We apply (3.5), and simplify, to obtain

$$ f_n(q^{-N}) = \frac{\tilde{Q}_n(q^{-N};\alpha,\beta,N|q)}{\tilde{Q}_{n-1}(q^{-N};\alpha,\beta,N|q)} = -\frac{\alpha\,(\beta\,q^n-1)\,(\alpha\beta\,q^n-1)\left(-q^{N+1}+q^n\right)q^n}{(\alpha\,\beta\,q^{2\,n}-q)\,(\alpha\,\beta\,q^{2\,n}-1)\,q^N}. $$

When taking into consideration the values of the parameters,

$$ -a_n - f_n(q^{-N}) = -\frac{\alpha\,(\beta-1)\left(q^n-q^{N+1}\right)q^n}{(\alpha\,\beta\,q^{2\,n}-q)q^N} < 0 $$

and the result follows from Lemma 1.7 (ii).

□

Remark 3.17. *We cannot say anything about the location of the zeros of* $\tilde{Q}_n(\bar{x};\frac{\alpha}{q^2},\beta,N|q)$, *since the coefficient of* $\tilde{Q}_{n-2}(\bar{x};\alpha,\beta,N|q)$, *in the equation*

$$ \tilde{Q}_n(\bar{x};\frac{\alpha}{q^2},\beta,N|q) = \tilde{Q}_n(\bar{x};\alpha,\beta,N|q) + \frac{\alpha\,(q+1)\,(q^n-1)\,(\beta q^n-1)\left(q^n-q^{N+1}\right)}{(\alpha\beta q^{2\,n}-q^2)\,(\alpha\beta q^{2\,n}-1)\,q^N}\tilde{Q}_{n-1}(\bar{x};\alpha,\beta,N|q) $$
$$ +\frac{(\alpha q)^2\,(q^n-1)\,(\beta q^n-q)\left(q^n-q^{N+1}\right)(q^n-q)\,(\beta q^n-1)\left(q^n-q^{N+2}\right)}{(\alpha\beta q^{2\,n}-q^2)^2\,(\alpha\beta q^{2\,n}-q)\,(\alpha\beta q^{2\,n}-q^3)\,q^{2\,N}}\tilde{Q}_{n-2}(\bar{x};\alpha,\beta,N|q), $$

that can be obtained from (3.9a), is positive (cf. [Brezinski et al., 2004, Theorem 4]). The same is true for the location of the zeros of $\tilde{Q}_n(\bar{x};\alpha,\frac{\beta}{q^2},N|q)$ *and the equation can be found in the accompanying Maple file.*

Theorem 3.18. *Let $N \in \mathbb{N}_0$, $n \in \{0, 1, 2, \dots, N\}$, $\alpha, \beta > 1$. All the zeros of $\tilde{Q}_n(\bar{x}; \frac{\alpha}{q}, \frac{\beta}{q}, N|q)$ are real and distinct and if $z_{n,j}, j \in \{1, 2, \dots, n\}$, are the zeros of $\tilde{Q}_n(\bar{x}; \frac{\alpha}{q}, \frac{\beta}{q}, N|q)$, then $z_{n,1} < 1$ and $q^{-N} < z_{n,n}$.*

Proof. Fix $\alpha > 1$ and $\beta > 1$ such that $0 < \alpha q < 1$ and $0 < \beta q < 1$. We use (3.10) with a_n the coefficient of $\tilde{Q}_{n-1}(\bar{x}; \alpha, \beta, N|q)$ and b_n the coefficient of $\tilde{Q}_{n-2}(\bar{x}; \alpha, \beta, N|q)$. By taking into account the values of the parameters, we see that

$$b_n = -\frac{\alpha^2 \beta \left(\beta\, q^n - q\right) \left(q^n - q^{N+1}\right) \left(q^n - q^{N+2}\right) \left(\alpha\, q^n - q\right) \left(q^n - 1\right) \left(q^n - q\right) q^{n+1}}{\left(\alpha\, \beta\, q^{2\,n} - q\right) \left(\alpha\, \beta\, q^{2\,n} - q^3\right) \left(\alpha\, \beta\, q^{2\,n} - q^2\right)^2 q^{2\,N}} < 0,$$

and it follows from [Brezinski et al., 2004, Theorem 4] that $z_{n,j}, j \in \{1, 2, \dots, n\}$, are real.

In order to determine the location of $z_{n,1}$ and $z_{n,n}$, we use [Joulak, 2005, Theorem 9]. Since

$$\begin{aligned} f_n(1) f_{n-1}(1) + a_n f_{n-1}(1) + b_n &= \frac{\tilde{Q}_n(1; \alpha, \beta, N|q)}{\tilde{Q}_{n-2}(1; \alpha, \beta, N|q)} + a_n \frac{\tilde{Q}_{n-1}(1; \alpha, \beta, N|q)}{\tilde{Q}_{n-2}(1; \alpha, \beta, N|q)} + b_n \\ &= \frac{(\alpha - 1) \left(\alpha\, q^n - q\right) \left(q^n - q^{N+2}\right) \left(q^n - q^{N+1}\right) q}{\left(\alpha \beta\, q^{2n} - q^2\right) \left(\alpha\, \beta\, q^{2n} - q^3\right) q^{2N}} \\ &< 0, \end{aligned}$$

it follows that $z_{n,1} < 1$. Furthermore,

$$f_n(q^{-N}) f_{n-1}(q^{-N}) + a_n f_{n-1}(q^{-N}) + b_n = \frac{\alpha^2 \left(\beta - 1\right) \left(\beta\, q^n - q\right) \left(q^n - q^{N+1}\right) \left(q^n - q^{N+2}\right) q^n}{\left(\alpha \beta\, q^{2n} - q^2\right) \left(\alpha\, \beta\, q^{2n} - q^3\right) q^{2N}} < 0$$

and $q^{-N} < z_{n,n}$. □

Remark 3.19. *(i) When we let $\beta = 0$ in the definition of the q-Hahn polynomials, we obtain the affine q-Krawtchouk polynomials [Koekoek et al., 2010, Section 14.16] $\tilde{K}_n^{Aff}(\bar{x}; \alpha, N; q)$, orthogonal on $(1, q^{-N})$ if $0 < \alpha q < 1$. When we fix $\alpha > 1$, such that $0 < \alpha q < 1$, the quasi-orthogonality of the polynomials $\tilde{K}_n^{Aff}(\bar{x}; \frac{\alpha}{q^k}, N; q), k < n$, on $(1, q^{-N})$ follows directly from (3.10), with $\beta = 0$. If $x_{n,j}, j \in \{1, 2, \dots, n\}$, denote the zeros of $\tilde{K}_n^{Aff}(\bar{x}; \alpha, N; q)$ and $y_{n,j}, j \in \{1, 2, \dots, n\}$, the zeros of $\tilde{K}_n^{Aff}(\bar{x}; \frac{\alpha}{q}, N; q)$, the interlacing result in Theorem 3.16 (i) follows.*

(ii) Since $\lim_{\alpha \to \infty} \tilde{Q}_n(\bar{x}; \alpha, p, N|q) = \tilde{K}_n^{qtm}(\bar{x}; p, N; q)$, [Koekoek et al., 2010, Section 14.14], we obtain from (3.9b), the equation

$$\tilde{K}_n^{qtm}\left(\bar{x}; \frac{p}{q}, N; q\right) = \tilde{K}_n^{qtm}(\bar{x}; p, N; q) + \frac{\left(q^{N+1} - q^n\right)\left(q^n - 1\right)}{p q^{2\,n+N}} \tilde{K}_{n-1}^{qtm}(\bar{x}; p, N; q).$$

For $q^{-N} < p < q^{-N+1}$, the quantum q-Krawtchouk polynomials $\tilde{K}_n^{qtm}(\bar{x}; \frac{p}{q^k}, N; q)$ are quasi-orthogonal of order $k < n$ and the interlacing result in Theorem 3.16 (ii) follows, where $x_{n,j}, j \in \{1, 2, \dots, n\}$, denote the zeros of $\tilde{K}_n^{qtm}(\bar{x}; p, N; q)$ and $z_{n,j}, j \in \{1, 2, \dots, n\}$, the zeros of $\tilde{K}_n^{qtm}\left(\bar{x}; \frac{p}{q}, N; q\right)$.

3.3.3 Bounds of the extreme zeros

The q-Hahn polynomials $\tilde{P}_n(x;\alpha,\beta,\gamma;q)$ satisfy equations of the type

$$f(x)\tilde{Q}_{n-k}(x;\alpha q^{s_1},\beta q^{s_2},N|q)=H(x)\tilde{Q}_n(x;\alpha,\beta,N|q)+G_{k-1,s_1,s_2}(x)\tilde{Q}_{n-1}(x;\alpha,\beta,N|q),$$

with $s_1+s_2\in\{0,1,\ldots,2k\}$. Let $B^{(1)}_{k,s_1,s_2}$ and $B^{(2)}_{k,s_1,s_2}$ be the smallest and the largest zero of G_{k-1,s_1,s_2}, respectively. For $k=2,3,\ldots$, the best upper bound for $x_{n,1}$ is $B^{(1)}_{k,2k,0}$, the zero of $G_{k-1,2k,0}$, and the best lower bound for $x_{n,n}$ is $B^{(2)}_{k,0,2k}$, the zero of $G_{k-1,0,2k}$.

3.4 The little q-Jacobi polynomials

The little q-Jacobi polynomials $\tilde{p}_n(q^x;\alpha,\beta|q)$ are orthogonal with respect to the discrete weight $w(x;\alpha,\beta)=\frac{(\beta q;q)_x(\alpha q)^x}{(q;q)_x}$ for $0<\alpha q<1$, $\beta q<1$ on $(0,1)$.

3.4.1 Interlacing properties

Proposition 3.20.

$$\tilde{p}_n(x;\alpha,\beta|q)=\tilde{p}_n(x;\alpha q,\beta|q)+\frac{\alpha\, q^n\,(q^n-1)\,(\beta\, q^n-1)}{(\alpha\beta\, q^{2\,n+1}-1)\,(\alpha\,\beta\, q^{2\,n}-1)}\tilde{p}_{n-1}(x;\alpha q,\beta|q); \tag{3.11a}$$

$$\tilde{p}_n(x;\alpha,\beta|q)=\tilde{p}_n(x;\alpha,\beta q|q)-\frac{\alpha\,\beta\, q^{2\,n}\,(q^n-1)\,(\alpha\, q^n-1)}{(\alpha\beta\, q^{2\,n+1}-1)\,(\alpha\,\beta\, q^{2\,n}-1)}\tilde{p}_{n-1}(x;\alpha,\beta q|q); \tag{3.11b}$$

$$\begin{aligned}\tilde{p}_n(x;\alpha,\beta|q)&=\frac{(\alpha\, q-1)\,(\alpha\beta\, q^{2\,n+1}-1)\,\tilde{p}_n(x;\alpha q^2,\beta|q)}{(\alpha\, q^{n+1}-1)\,(\alpha\,\beta\, q^{n+1}-1)}\\&+\frac{\alpha\, q\,(q^n-1)\,(\beta q^n-1)\,((\alpha\,\beta q^{2\,n+2}-1)\,x+q^n\,(\alpha\, q-1))\,\tilde{p}_{n-1}(x;\alpha q^2,\beta|q)}{(\alpha\, q^{n+1}-1)\,(\alpha\,\beta q^{2\,n+2}-1)\,(\alpha\,\beta\, q^{n+1}-1)};\end{aligned} \tag{3.11c}$$

$$\begin{aligned}\tilde{p}_n(x;\alpha,\beta|q)&=-\frac{(\alpha\,\beta\, q^{n+1}(q^n-1)+1-\beta\, q^{n+1})\,(\alpha\,\beta\, q^{2\,n+1}-1)\,\tilde{p}_n(x;\alpha,\beta q|q)}{(\beta\, q^{n+1}-1)\,(\alpha\,\beta\, q^{n+1}-1)}\\&+\frac{\alpha\,\beta\, q^{2\,n}\,(\beta\, q^2x-1)\,(q^n-1)\,(\alpha\, q^n-1)\,\tilde{p}_{n-1}(x;\alpha,\beta q^2|q)}{(\beta\, q^{n+1}-1)\,(\alpha\,\beta\, q^{n+1}-1)};\end{aligned} \tag{3.11d}$$

$$\tilde{p}_n(x;\alpha,\beta q|q)=\tilde{p}_n(x;\alpha q,\beta|q)+\frac{\alpha\, q^n\,(q^n-1)}{\beta\,\alpha\, q^{2\,n+1}-1}\tilde{p}_{n-1}(x;\alpha q,\beta q|q). \tag{3.11e}$$

For equations (3.11d) and (3.11e), one can also refer to [Gochhayat et al., 2016, Eqs. (9), (10)], respectively.

Theorem 3.21. *Let $0<\alpha q<1$ and $\beta q<1$ and denote the zeros of $\tilde{p}_n(x;\alpha,\beta|q)$ by $0<x_{n,1}<x_{n,2}<\cdots<x_{n,n}<1$, the zeros of $\tilde{p}_n(x;\alpha q,\beta|q)$ by $y_{n,1}<y_{n,2}<\cdots<y_{n,n}$, the zeros of $\tilde{p}_n(x;\alpha q^2,\beta|q)$ by $Y_{n,1}<Y_{n,2}<\cdots<Y_{n,n}$, the zeros of $\tilde{p}_n(x;\alpha,\beta q|q)$ by $z_{n,1}<z_{n,2}<\cdots<z_{n,n}$, the zeros of $\tilde{p}_n(x;\alpha,\beta q^2|q)$ by $Z_{n,1}<Z_{n,2}<\cdots<Z_{n,n}$ and the zeros of $\tilde{p}_n(x;\alpha q,\beta q|q)$ by $t_{n,1}<t_{n,2}<\cdots<t_{n,n}$. Then, for $i\in\{1,2,\ldots,n-1\}$,*

(a) $x_{n,i}<y_{n,i}<y_{n-1,i}<x_{n,i+1}<y_{n,i+1}$,

(b) $z_{n,i}<x_{n,i}<z_{n-1,i}<z_{n,i+1}<x_{n,i+1}$ *if* $\beta>0$ *and* $x_{n,i}<z_{n,i}<z_{n-1,i}<x_{n,i+1}<z_{n,i+1}$ *if* $\beta<0$,

(c) $x_{n,i} < y_{n,i} < Y_{n,i} < Y_{n-1,i} < x_{n,i+1} < y_{n,i+1} < Y_{n,i+1}$,

(d) $x_{n,i} < z_{n,i} < Z_{n-1,i} < x_{n,i+1} < z_{n,i+1}$ *if* $\beta < 0$,

(e) $z_{n,i} < y_{n,i} < t_{n-1,i} < z_{n,i+1} < y_{n,i+1}$ *if* $\beta < 0$.

Proof. Let $0 < \alpha q < 1$ and $\beta q < 1$. We note that, since $0 < q < 1$, $q^n - 1 < 0$, $\alpha q^n - 1 < 0$, $\beta q^n - 1 < 0$, $\alpha\beta q^n - 1 < 0$, for all positive integers n.

The polynomials on the right-hand side of each of the equations (3.11a) and (3.11b) belong to the same orthogonal sequence, therefore their zeros interlace and satisfy the property (1.1). Each of these equations are thus in the form of (1.2), i.e.,

$$f_n(x) = a(x)p_n(x) + b(x)q_{n-1}(x),$$

with $a(x) = 1$. Furthermore,

(a) $b(x)$ in (3.11a) is a positive constant (w.r.t. x) and the result follows from Corollary 1.3 (a);

(b) taking into consideration the restrictions on the parameters, $b(x)$ in (3.11b) is a positive constant if $\beta < 0$ and $b(x)$ is negative when $\beta > 0$. The result follows from applying Corollary 1.3 to the different situations.

(c) The polynomials $\tilde{p}_n(x; \alpha q^2, \beta|q)$ and $\tilde{p}_{n-1}(x; \alpha q^2, \beta|q)$ belong to the same orthogonal sequence and their zeros satisfy (1.1). (3.11c) is in the form of (1.2) and taking into consideration the restrictions on the parameters, $a(x)$ is a positive constant.

$$\begin{aligned} b(x) &= \frac{\alpha\, q\, (q^n - 1)\, (\beta q^n - 1)}{(\alpha\, q^{n+1} - 1)\, (\alpha\, \beta q^{2\,n+2} - 1)\, (\alpha\, \beta\, q^{n+1} - 1)} \left(\left(\alpha\, \beta q^{2\,n+2} - 1\right) x + q^n\, (\alpha\, q - 1) \right) \\ &= -k^2 \left(\left(\alpha\, \beta q^{2\,n+2} - 1\right) x + q^n\, (\alpha\, q - 1) \right),\ k \in \mathbb{R}, \end{aligned}$$

represents a linear function with derivative $-k^2(\alpha\,\beta q^{2\,n+2} - 1) > 0$, intersecting the x-axis at $x = \frac{-q^n(\alpha\, q - 1)}{(\alpha q)\,(\beta q) q^{2\,n-1}} < 0$ for $\beta q < 1$. $b(x)$ is thus positive on $(0, 1)$ and from Corollary 1.3 (a) we deduce that $x_{n,i} < Y_{n,i} < Y_{n-1,i} < x_{n,i+1} < Y_{n,i+1}$ for each $i \in \{1, 2, \ldots, n-1\}$. Furthermore, by replacing α with αq in (3.11a), we obtain $y_{n,i} < Y_{n,i} < Y_{n-1,i} < y_{n,i+1} < Y_{n,i+1}$ for each $i \in \{1, 2, \ldots, n-1\}$ and by combining these two interlacing results with the fact that $x_{n,i} < y_{n,i}$ for each $i \in \{1, 2, \ldots, n\}$, the required interlacing follows.

(d) Let $\beta < 0$. By replacing β with βq in (3.11b), we obtain $z_{n,i} < Z_{n,i} < Z_{n-1,i} < z_{n,i+1} < Z_{n,i+1}$ for each $i \in \{1, 2, \ldots, n-1\}$ and equation (3.11d) is in the form of (1.2). Under the condition that $\beta < 0$, the coefficient of $\tilde{p}_n(x; \alpha, \beta q|q)$ is a positive constant. The coefficient of $\tilde{p}_{n-1}(x; \alpha, \beta q^2|q)$ is

$$b(x) = \frac{\alpha\, \beta\, q^{2\,n}\, (q^n - 1)\, (\alpha\, q^n - 1)}{(\beta\, q^{n+1} - 1)\, (\alpha\, \beta\, q^{n+1} - 1)} \left(\beta\, q^2 x - 1\right) = -k^2 \left(\beta q^2 x - 1\right),\ k \in \mathbb{R},$$

that represents a linear function with positive derivative, intersecting the negative x-axis and $b(x)$ is thus positive on $(0, 1)$. The result follows from Corollary 1.3 (a).

(e) Assume $\beta < 0$. From (b) we know that the zeros of $\tilde{p}_n(x;\alpha,\beta|q)$ and $\tilde{p}_{n-1}(x;\alpha,\beta q|q)$ interlace. By replacing α by αq, it follows that $y_{n,i} < t_{n-1,i} < y_{n,i+1}$ for each $i \in \{1,2,\cdots,n-1\}$. Equation (3.11e) is in the form of (1.2) with $a(x) = 1$ and, taking into consideration the restrictions on the parameters, $b(x)$ is a positive constant. The result follows from Corollary 1.3 (a).

□

Corollary 3.22. *For each $i \in \{1,2,\ldots,n-1\}$,*

(a) $x_{n,i} < x_{n-1,i} < y_{n-1,i} < x_{n,i+1}$*,*

(b) $x_{n,i} < z_{n-1,i} < x_{n-1,i} < x_{n,i+1}$ *if* $\beta > 0$ *and* $x_{n,i} < x_{n-1,i} < z_{n-1,i} < x_{n,i+1}$ *if* $\beta < 0$*,*

(c) $x_{n,i} < z_{n,i} < y_{n,i} < x_{n,i+1} < z_{n,i+1} < y_{n,i+1}$ *if* $\beta < 0$.

Proof.

(a) We combine the interlacing of the zeros of $\tilde{p}_n(x;\alpha,\beta|q)$ and $\tilde{p}_{n-1}(x;\alpha,\beta|q)$ with the results proved in Theorem 3.21 (a) to obtain the required interlacing.

(b) We combine the interlacing of the zeros of $\tilde{p}_n(x;\alpha,\beta|q)$ and $\tilde{p}_{n-1}(x;\alpha,\beta|q)$ with the result of Theorem 3.21 (b).

(c) Let $\beta < 0$. This result follows from the interlacing proved in Theorem 3.21 (a), (b) and (e).

□

Remark 3.23. *(i) We note that our results differ from the interlacing results for the little q-Jacobi polynomials, given in [Gochhayat et al., 2016, Section 3]. In [Gochhayat et al., 2016, Theorem 2], the values of x, given as the zeros of the polynomial $p_n(x;\alpha,\beta|q)$, are actually the zeros y of the polynomial $p_n(q^y;\alpha,\beta|q)$. The same is true for the interlacing results in [Gochhayat et al., 2016, Theorems 4,5,6 and 7];*

(ii) When $\beta = 0$ in the definition of the little q-Jacobi polynomials, we obtain the little q-Laguerre (or Wall) polynomials $\tilde{p}_n(x;\alpha|q)$, that are orthogonal on $(0,1)$ when $0 < \alpha q < 1$. The interlacing results in Theorem 3.21 (a) and (c) and Corollary 12 (a) follow from (3.11a) and (3.11c) (with $\beta = 0$), where $x_{n,i}, y_{n,i}$ and $Y_{n,i}, i \in \{1,2,\ldots,n\}$ are the zeros of $\tilde{p}_n(x;\alpha|q)$, $\tilde{p}_n(x;\alpha q|q)$ and $\tilde{p}_n(x;\alpha q^2|q)$, respectively.

3.4.2 Quasi-orthogonality

Consider the recurrence equations (cf. (3.11a) and (3.11b))

$$\tilde{p}_n\left(x;\frac{\alpha}{q},\beta|q\right) = \tilde{p}_n(x;\alpha,\beta|q) + \frac{\alpha\, q^n\,(q^n-1)\,(\beta\, q^n-1)}{(\alpha\beta\, q^{2\,n}-1)\,(\alpha\,\beta\, q^{2\,n}-q)}\tilde{p}_{n-1}(x;\alpha,\beta|q); \tag{3.12a}$$

$$\tilde{p}_n\left(x;\alpha,\frac{\beta}{q}|q\right) = \tilde{p}_n(x;\alpha,\beta|q) - \frac{\alpha\,\beta\, q^{2\,n}\,(q^n-1)\,(\alpha\, q^n-1)}{(\alpha\beta\, q^{2\,n}-1)\,(\alpha\,\beta\, q^{2\,n}-q)}\tilde{p}_{n-1}(x;\alpha,\beta|q). \tag{3.12b}$$

Corollary 3.24.

$$\tilde{p}_n\left(x;\frac{\alpha}{q},\frac{\beta}{q}|q\right)=\tilde{p}_n(x;\alpha,\beta|q)-\frac{\alpha\,q^n\,(\alpha\,\beta\,q^{2\,n}-q^{n+1}\beta-\beta\,q^n+q)\,(q^n-1)}{(\alpha\,\beta\,q^{2\,n}-q^2)\,(\alpha\,\beta\,q^{2\,n}-1)}\tilde{p}_{n-1}(x;\alpha,\beta|q)$$
$$-\frac{\alpha^2\beta q^{3\,n+1}\,(q^n-1)\,(\beta\,q^n-q)\,(\alpha\,q^n-q)\,(q^n-q)}{(\alpha\,\beta\,q^{2\,n}-q^2)^2\,(\alpha\,\beta\,q^{2\,n}-q)\,(\alpha\,\beta\,q^{2\,n}-q^3)}\tilde{p}_{n-2}(x;\alpha,\beta|q). \tag{3.13}$$

Theorem 3.25. *Let $k,m\in\mathbb{N}_0$, $\alpha,\beta\in\mathbb{R}$. For $0<\alpha q<1$ and $0<\beta q<1$, the sequence of little q-Jacobi polynomials*

(i) $\{\tilde{p}_n(x;\frac{\alpha}{q^k},\beta|q)\}_{n\geq 0}$, with $\alpha>1$, is quasi-orthogonal of order $k\leq n-1$ with respect to $w(x)=\frac{(\beta q;q)_x(\alpha q)^x}{(q;q)_x}$ on the interval $(0,1)$ and the polynomials have at least $(n-k)$ real, distinct zeros in $(0,1)$;

(ii) $\{\tilde{p}_n(x;\alpha,\frac{\beta}{q^m}|q)\}_{n\geq 0}$, $\beta>1$, is quasi-orthogonal of order $m\leq n-1$ with respect to $w(x)$ on $(0,1)$ and the polynomials have at least $(n-m)$ real, distinct zeros in $(0,1)$;

(iii) $\{\tilde{p}_n(x;\frac{\alpha}{q^k},\frac{\beta}{q^m}|q)\}_{n\geq 0}$, $\alpha,\beta>1$, is quasi-orthogonal of order $k+m\leq n-1$ with respect to $w(x)$ on $(0,1)$ and the polynomials have at least $n-(k+m)$ real, distinct zeros in $(0,1)$.

Proof.

(i) Fix $\alpha>1,\beta\in\mathbb{R}$, such that $0<\alpha q<1$, $0<\beta q<1$. From Lemma 1.4 and (3.12a), it follows that $\tilde{p}_n(x;\frac{\alpha}{q},\beta|q)$ is quasi-orthogonal of order one on $(0,1)$. By iteration, we can express $\tilde{p}_n(x;\frac{\alpha}{q^k},\beta|q)$ as a linear combination of $\tilde{p}_n(x;\alpha,\beta|q)$, $\tilde{p}_{n-1}(x;\alpha,\beta|q)$, $\ldots$, $\tilde{p}_{n-k}(x;\alpha,\beta|q)$, and the result follows from Lemma 1.4. The location of the $(n-k)$ real, distinct zeros of $\tilde{p}_n(x;\frac{\alpha}{q^k},\beta|q), k\in\{1,2,\ldots,n-1\}$, follows from Lemma 1.6.

(ii) Fix $\beta>1,\alpha\in\mathbb{R}$, such that $0<\alpha q<1$, $0<\beta q<1$. The quasi-orthogonality follows in the same way as in (i), by using (3.12b).

(iii) Fix $\alpha>1$ and $\beta>1$ such that $0<\alpha q<1$ and $0<\beta q<1$. From (3.13), we see that $\tilde{p}_n(x;\frac{\alpha}{q},\frac{\beta}{q}|q)$ can be written as a linear combination of $\tilde{p}_n(x;\alpha,\beta|q)$, $\tilde{p}_{n-1}(x;\alpha,\beta|q)$ and $\tilde{p}_{n-2}(x;\alpha,\beta|q)$, and it follows from Lemma 1.4 that the sequence $\tilde{p}_n(x;\frac{\alpha}{q},\frac{\beta}{q}|q)$ is quasi-orthogonal of order two on $(0,1)$. By iteration, we can express $\tilde{p}_n(x;\frac{\alpha}{q^k},\frac{\beta}{q^m}|q)$ as a linear combination of $\tilde{p}_n(x;\alpha,\beta|q),\tilde{p}_{n-1}(x;\alpha,\beta|q),\ldots,\tilde{p}_{n-(k+m)}(x;\alpha,\beta|q)$, and the results follow directly from Lemmas 1.4 and 1.6.

□

Theorem 3.26. *Let $\alpha,\beta,\in\mathbb{R}$, $0<\alpha q,\beta q<1$, and suppose $x_{n,j},j\in\{1,2,\ldots,n\}$, denote the zeros of $\tilde{p}_n(x;\alpha,\beta|q)$, $y_{n,j},j\in\{1,2,\ldots,n\}$, the zeros of $\tilde{p}_n(x;\frac{\alpha}{q},\beta|q)$ and $z_{n,j},j\in\{1,2,\ldots,n\}$, the zeros of $\tilde{p}_n(x;\alpha,\frac{\beta}{q}|q)$. Then*

(i) if $\alpha>1$, $y_{n,1}<0<x_{n,1}<x_{n-1,1}<y_{n,2}<\cdots<x_{n-1,n-1}<y_{n,n}<x_{n,n}<1$;

(ii) if $\beta>1$, $0<x_{n,1}<z_{n,1}<x_{n-1,1}<x_{n,2}<\cdots<x_{n-1,n-1}<x_{n,n}<1<z_{n,n}$.

Proof.

(i) From (3.12a) we obtain the value $a_n = \frac{\alpha(q^n-1)(\beta\, q^n-1)q^n}{(\alpha\,\beta\, q^{2\,n}-1)(\alpha\,\beta\, q^{2\,n}-q)} > 0$. The interlacing result, as well as the position of $y_{n,n}$, follows from Lemma 1.8 (ii).

To obtain the position of $y_{n,1}$ we use Lemma 1.7 and when we consider the given parameter values,

$$f_n(0) = \frac{\tilde{p}_n(0;\alpha,\beta|q)}{\tilde{p}_{n-1}(0;\alpha,\beta|q)} = -\frac{(\alpha\,\beta\, q^n-1)\,(\alpha\, q^n-1)\, q^n}{(\alpha\,\beta\, q^{2\,n}-1)\,(\alpha\,\beta\, q^{2\,n}-q)} < 0.$$

We thus have

$$-a_n - f_n(0) = \frac{q^n\,(\alpha-1)}{\alpha\,\beta\, q^{2\,n}-q} < 0$$

and the result follows from Lemma 1.7 (i).

(ii) From (3.12b), we obtain the value $a_n = -\frac{\alpha\beta(q^n-1)(\alpha\, q^n-1)q^{2\,n}}{(\alpha\,\beta\, q^{2\,n}-1)(\alpha\,\beta\, q^{2\,n}-q)} < 0$. The interlacing result, as well as the position of $z_{n,1}$, follows from Lemma 1.8 (i).

To obtain the position of $z_{n,n}$, we use Lemma 1.7, and when we consider the given parameter values,

$$f_n(1) = \frac{\tilde{p}_n(1;\alpha,\beta|q)}{\tilde{p}_{n-1}(1;\alpha,\beta|q)} = -\frac{\alpha\,(\beta-1)\, q^{2\,n}}{\alpha\,\beta\, q^{2\,n}-q} > 0.$$

We thus have

$$-a_n - f_n(1) = -\frac{\alpha\,(\beta-1)\, q^{2\,n}}{\alpha\,\beta\, q^{2\,n}-q} > 0$$

and it follows from Lemma 1.7 (ii) that $1 < z_{n,n}$.

□

Theorem 3.27. *Let $\alpha,\beta > 1$. All the zeros of $\tilde{p}_n(x;\frac{\alpha}{q},\frac{\beta}{q}|q)$, denoted by $z_{n,j}$, $j \in \{1,2,\ldots,n\}$, are real and distinct and $z_{n,1} < 0$ and $1 < z_{n,n}$.*

Proof. Fix $\alpha > 1$ and $\beta > 1$ such that $0 < \alpha q < 1$ and $0 < \beta q < 1$. We use (3.13), with a_n the coefficient of $\tilde{p}_{n-1}(x;\alpha,\beta|q)$ and b_n the coefficient of $\tilde{p}_{n-2}(x;\alpha,\beta|q)$. By taking into account the values of the parameters, we see that

$$b_n = -\frac{\alpha^2\beta\,(\beta\, q^n-q)\,(\alpha\, q^n-q)\,(q^n-1)\,(q^n-q)\, q^{3\,n+1}}{(\alpha\,\beta\, q^{2\,n}-q)\,(\alpha\,\beta\, q^{2\,n}-q^3)\,(\alpha\,\beta\, q^{2\,n}-q^2)^2} < 0,$$

and it follows from [Brezinski et al., 2004, Theorem 4] that $z_{n,j}$, $j \in \{1,2,\ldots,n\}$, are real and distinct.

In order to determine the location of $z_{n,1}$ and $z_{n,n}$, we use [Joulak, 2005, Theorem 9]. Since

$$f_n(0)f_{n-1}(0) + a_n f_{n-1}(0) + b_n = \frac{(\alpha-1)\,(\alpha\, q^n-q)\, q^{2n+1}}{(\alpha\,\beta\, q^{2n}-q^3)\,(\alpha\,\beta\, q^{2n}-q^2)} < 0,$$

it follows that $z_{n,1} < 1$. Furthermore,

$$f_n(1)f_{n-1}(1) + a_n f_{n-1}(1) + b_n = \frac{\alpha^2\,(\beta-1)\,(\beta\, q^n-q)\, q^{3\,n}}{(\alpha\,\beta\, q^{2\,n}-q^2)\,(\alpha\,\beta\, q^{2\,n}-q^3)} < 0,$$

and $1 < z_{n,n}$. □

Remark 3.28. *(i) From (3.12a) we obtain*

$$\tilde{p}_n\left(x;\frac{\alpha}{q^2},\beta|q\right) = \tilde{p}_n(x;\alpha,\beta|q) + \frac{\alpha\, q^n\,(q+1)\,(q^n-1)\,(\beta\, q^n-1)}{(\alpha\,\beta\, q^{2\,n}-q^2)\,(\alpha\,\beta\, q^{2\,n}-1)} \times \tilde{p}_{n-1}(x;\alpha,\beta|q) + b_n\tilde{p}_{n-2}(x;\alpha,\beta|q),$$

with

$$b_n = \frac{\alpha^2 q^{2\,n+2}\,(q^n-1)\,(\beta\, q^n-q)\,(q^n-q)\,(\beta\, q^n-1)}{(\alpha\,\beta\, q^{2\,n}-q^2)^2\,(\alpha\,\beta\, q^{2\,n}-q)\,(\alpha\,\beta\, q^{2\,n}-q^3)}$$

and

$$C_n - b_n = \frac{q^{2\,n+1}\,(\alpha-q)\,(q^n-q)\,(\beta q^n-q)\,\alpha}{(\alpha\beta q^{2\,n}-q^3)\,(\alpha\beta q^{2\,n}-q^2)^2},$$

where $-C_n$ is the coefficient of $\tilde{p}_{n-2}(x;\alpha,\beta|q)$ in the three-term recurrence equation of the little q-Jacobi polynomials [Koekoek et al., 2010, (14.12.4)]. Since $C_n < b_n$, there is an interlacing between $(n-2)$ zeros of $\tilde{p}_n(x;\frac{\alpha}{q^2},\beta|q)$ and the $(n-1)$ zeros of $\tilde{p}_{n-1}(x;\alpha,\beta|q)$ (cf. [Joulak, 2005, Theorem 15]).

(ii) When $\beta = 0$ in the definition of the little q-Jacobi polynomials, we obtain the little q-Laguerre (or Wall) polynomials $\tilde{p}_n(x;\alpha|q)$, that are orthogonal on $(0,1)$ when $0 < \alpha q < 1$. The quasi-orthogonality of $\{\tilde{p}_n(x;\frac{\alpha}{q^k}|q)\}_{n\geq 0}$, for $k < n$, when $\alpha > 1$, $0 < \alpha q < 1$, follows directly from (3.12a) (with $\beta = 0$). The location of the zeros of the order one quasi-orthogonal polynomial $\tilde{p}_n(x;\frac{\alpha}{q}|q)$ is given in Theorem 3.26 (i), where $x_{n,j}$, $j \in \{1,2,\ldots,n\}$, denote the zeros of $\tilde{p}_n(x;\alpha|q)$ and $y_{n,j}$, $j \in \{1,2,\ldots,n\}$, the zeros of $\tilde{p}_n(x;\frac{\alpha}{q}|q)$.

3.4.3 Bounds of the extreme zeros

For the little q-Jacobi polynomials $\tilde{p}_n(x;\alpha,\beta|q)$, we have equations of type

$$f(x)\tilde{p}_{n-k}(x;\alpha q^{s_1},\beta q^{s_2}|q) = H(x)\tilde{p}_n(x;\alpha,\beta|q) + G_{k-1,s_1,s_2}(x)\tilde{p}_{n-1}(x;\alpha,\beta|q),$$

with $s_1+s_2 \in \{0,1,\ldots,2k\}$. Let $B^{(1)}_{k,s_1,s_2}$ and $B^{(2)}_{k,s_1,s_2}$ be the smallest and the largest zero of G_{k-1,s_1,s_2}, respectively. For $k=2$, we obtain

$$\begin{aligned}&\alpha\, q^3 x^4\,(q^n-q)\,(\beta\, q^n-q)\,\left(\alpha\beta\, q^{2\,n}-1\right)^2\left(\alpha\beta\, q^{2\,n}-q\right)\tilde{p}_{n-2}(x;\alpha q^4,\beta|q)\\ &= a_2(x)\tilde{p}_n(x;\alpha,\beta|q) + q^{2n}(\alpha q^2-1)(\alpha q^n-1)(\alpha\beta q^n-1)G_{1,4,0}(x)\tilde{p}_{n-1}(x;\alpha,\beta|q),\end{aligned}$$

with

$$\begin{aligned}a_2(x) = \left(\alpha\beta\, q^{2\,n-1};q\right)_2 q\Big(&\alpha\, qx\,(q^n-q)\,(\beta\, q^n-q)\,\left(\left(\alpha\beta\, q^{2\,n}-1\right)q^2x + (q+1)\left(\alpha\, q^2-1\right)q^n\right)\\ &+ q^{2\,n}\,(\alpha\, q;q)_3\Big)\end{aligned}$$

and $G_{1,4,0}(x) = Ax+B$ so that

$$x_{n,1} \leq B^{(1)}_{2,4,0} = -\frac{B}{A} = \frac{(\alpha\, q^3-1)\,(\alpha\, q-1)\, q^{n-1}}{(\alpha\,\beta\, q^{2\,n+1}+1)(\alpha q^2+1) - \alpha\, q^{n+1}(\beta+1)(q+1)}.$$

For $k=3$, the best upper bound for $x_{n,1}$ is

$$x_{n,1} < B^{(1)}_{3,6,0} = \left(-b-\sqrt{b^2-4ac}\right)/(2a),$$

with

$$a = q^3 \left(\alpha^2 q^6 + \alpha\, q^3 + 1\right) \left(\alpha^2 \beta^2 q^{4\,n+2} + 1\right) - \alpha\, q^{n+4} \left(\alpha\, q^3 + 1\right) \left(\beta + 1\right) \left(q^2 + q + 1\right)$$
$$\times \left(\alpha\beta\, q^{2\,n+1} + 1\right) + \alpha\, q^{2\,n+4} \left(\alpha^2 \beta\, q^6 + q \left(\beta\, q + q^2 + q + 1\right) \left(\beta\, q^2 + \beta\, q + \beta + q\right) \alpha + \beta\right),$$
$$b = -q^{n+1} \left(\alpha\, q^2 - 1\right) \left(\alpha\, q^4 - 1\right) \left(\alpha\,\beta \left(\alpha\, q^3 + 1\right) q^{2\,n+1} - \alpha \left(q^2 + 1\right) \left(\beta + 1\right) q^{n+1} + \alpha\, q^3 + 1\right)$$
$$\times \left(q + 1\right), \quad c = q^{2\,n} \left(\alpha\, q^5 - 1\right) \left(\alpha\, q - 1\right) \left(\alpha\, q^4 - 1\right) \left(\alpha\, q^2 - 1\right).$$

The best upper bounds for $x_{n,1}$ are the smallest zeros $B^{(1)}_{k,2k,0}$, $k \geq 4$. However, we remark that we don't obtain good bounds for the largest zeros from our mixed recurrence relations.

n, q	10, 0.4	5, 0.4	15, 0.9	20, 0.98
$\alpha,\ \beta$	0.2, 0.3	2.1, 2.4	0.5, -3.4	0.1, -5
zero $x_{n,1}$	0.000229210625	0.00273372029	0.0367	0.288
bound $B^{(1)}_{3,6,0}$	0.00022921089	0.00273372034	0.0374	0.318
bound $B^{(1)}_{2,4,0}$	0.0002307	0.0027371	0.046	0.4049

Table 3.1: Bounds for the extreme zeros of the little q-Jacobi polynomials $\tilde{p}_n(x; \alpha, \beta|q)$

Remark 3.29. *In [Gupta and Muldoon, 2007, Eq. (4.2) and Eq. (4.3)], the bounds of the smallest zero $\tilde{x}_{n,1}$ of the little q-Jacobi polynomials $\tilde{p}_n((1-q)x; \alpha, \beta|q)$ (that we denote here by $GM_{4.2}$ and $GM_{4.3}$ for [Gupta and Muldoon, 2007, Eq. (4.2)] and [Gupta and Muldoon, 2007, Eq. (4.3)], respectively) were given. As shown in Table 3.2, the upper bound $\frac{1}{1-q} B^{(1)}_{3,6,0}$ is more accurate than the upper bounds obtained in [Gupta and Muldoon, 2007, Eq. (4.2) and Eq. (4.3)].*

n, q	10, 0.4	5, 0.4	15, 0.9	20, 0.98
$\alpha,\ \beta$	0.2, 0.3	2.1, 2.4	0.5, -3.4	0.1, -5
zero $x_{n,1}/(1-q)$	0.0003820177	0.004556200485	0.367	14.3998
bound $\frac{1}{1-q} B^{(1)}_{3,6,0}$	0.00038201816	0.004556200569	0.374	15.9198
bound $GM_{4.3}$	0.0004186	0.0045692	0.4698	21.382
bound $GM_{4.2}$	0.000527796	0.004898	0.709	24.2655

Table 3.2: Bounds for the extreme zeros of the little q-Jacobi polynomials $\tilde{p}_n((1-q)x; \alpha, \beta|q)$

3.5 The q-Meixner polynomials

We note that in the definition of the q-Meixner polynomials, we set $\bar{x} = q^{-x}$, i.e., $x = \frac{\ln \bar{x}}{-\ln q}$, and as x increases on $(0, \infty)$, $\bar{x}$ will increase on $(1, \infty)$. The variable x in our equations thus represents $\bar{x}$ in the definition of the polynomials and for $0 < \beta q < 1$ and $\gamma > 0$, the polynomial $\tilde{M}_n(x; \beta, \gamma; q)$ is orthogonal on $(1, \infty)$.

3.5.1 Interlacing properties

Proposition 3.30.

$$\tilde{M}_n(x;\beta,\gamma q;q) = \tilde{M}_n(x;\beta,\gamma;q) + \gamma\, q^{-2n+1}\,(q^n-1)\,(\beta q^n-1)\,\tilde{M}_{n-1}(x;\beta,\gamma;q); \tag{3.14a}$$

$$\begin{aligned}\tilde{M}_n(x;\beta,\gamma q^2;q) &= -\frac{(\beta\,\gamma\,q\,(q^n-1)-q\gamma-1)\,q^n\tilde{M}_n(x;\beta,\gamma;q)}{\gamma q+q^n}\\ &+\frac{\gamma\,q^{-n+1}\,(\beta q^n-1)\,(q^n-1)\,(\gamma\,\beta\,q+q^n x+\gamma q+1)\,\tilde{M}_{n-1}(x;\beta,\gamma;q)}{\gamma q+q^n};\end{aligned} \tag{3.14b}$$

$$\tilde{M}_n(x;\beta,\gamma;q) = \frac{(\gamma\,\beta\,q+q^n x)\,\tilde{M}_n(x;\beta q,\gamma;q)}{q^n(\gamma\,\beta\,q+x)} - \frac{\gamma\,\beta\,\left(q^n+\gamma\right)(q^n-1)}{q^{3\,n-2}(\gamma\,\beta\,q+x)}\tilde{M}_{n-1}(x;\beta q,\gamma;q); \tag{3.14c}$$

$$\tilde{M}_n(x;\beta,\gamma q;q) = \tilde{M}_n(x;\beta q,\gamma;q) - \gamma q^{-2n+1}\,(q^n-1)\,\tilde{M}_{n-1}(x;\beta q,\gamma;q). \tag{3.14d}$$

Theorem 3.31. *Let $0<\beta q<1$ and $\gamma>0$ and denote the zeros of $\tilde{M}_n(x;\beta,\gamma;q)$ by $1<x_{n,1}<x_{n,2}<\cdots<x_{n,n}<\infty$, the zeros of $\tilde{M}_n(x;\beta q,\gamma;q)$ by $y_{n,1}<y_{n,2}<\cdots<y_{n,n}$, the zeros of $\tilde{M}_n(x;\beta,\gamma q;q)$ by $z_{n,1}<z_{n,2}<\cdots<z_{n,n}$ and the zeros of $\tilde{M}_n(x;\beta,\gamma q^2;q)$ by $Z_{n,1}<Z_{n,2}<\cdots<Z_{n,n}$. Then, for $i\in\{1,2,\ldots,n-1\}$,*

(a) $z_{n,i}<x_{n,i}<x_{n-1,i}<z_{n,i+1}<x_{n,i+1}$,

(b) $Z_{n,i}<x_{n,i}<x_{n-1,i}<Z_{n,i+1}<x_{n,i+1}$,

(c) $z_{n,i}<x_{n,i}<y_{n,i}<y_{n-1,i}<z_{n,i+1}<x_{n,i+1}<y_{n,i+1}$.

Proof. Let $0<\beta q<1$ and $\gamma>0$. Since $0<q<1$, it follows that $q^n-1<0$ and $\beta q^n-1<0$.

The polynomials on the right-hand side of each of the equations (3.14a) - (3.14d) belong to the same orthogonal sequence, therefore their zeros interlace and satisfy the property (1.1). Each of these equations thus is in the form of (1.2) with

(a) $a(x)=1$ and $b(x)>0$ in (3.14a) and the required interlacing follows from Corollary 1.3 (a);

(b) $a(x)>0$ in (3.14b) and, taking in consideration the restrictions on the parameters,

$$b(x)=\frac{\gamma\,(\beta q^n-1)\,(q^n-1)}{q^{n-1}(\gamma q+q^n)}\,(q^n x+\gamma\,\beta\,q+\gamma q+1)$$

is a linear function with positive derivative and is positive on $(1,\infty)$. The interlacing follows from Corollary 1.3 (a);

(c) Taking into consideration the restrictions on the parameters, the coefficients of both polynomials on the right-hand side of (3.14c) are positive on $(1,\infty)$, and following Corollary 1.3 (a), $x_{n,i}<y_{n,i}<y_{n-1,i}<x_{n,i+1}<y_{n,i+1}$ for each $i\in\{1,2,\ldots,n-1\}$. Furthermore, the coefficients of both polynomials on the right-hand side of (3.14d) are positive constants and applying Corollary 1.3 (a) for a second time, we obtain $z_{n,i}<y_{n,i}<y_{n-1,i}<z_{n,i+1}<y_{n,i+1}$ for each $i\in\{1,2,\ldots,n-1\}$. It is known, from (a), that $z_{n,i}<x_{n,i}$ for each $i\in\{1,2,\ldots,n\}$, and the required combined interlacing follows.

$\square$

Corollary 3.32. *For $i \in \{1, 2, \ldots, n-1\}$,*

(a) $z_{n,i} < z_{n-1,i} < x_{n-1,i} < y_{n-1,i} < z_{n,i+1}$,

(b) $Z_{n,i} < Z_{n-1,i} < x_{n-1,i} < Z_{n,i+1}$,

(c) $z_{n,i} < x_{n,i} < x_{n-1,i} < y_{n-1,i} < z_{n,i+1} < x_{n,i+1}$.

Proof.

(a) The result follows from Theorem 3.31 (c) and the interlacing of the zeros of $\tilde{M}_n(x;\beta,\gamma q;q)$ and $\tilde{M}_{n-1}(x;\beta,\gamma q;q)$.

(b) The result follows from Theorem 3.31 (b) and the interlacing of the zeros of $\tilde{M}_n(x;\beta,\gamma q^2;q)$ and $\tilde{M}_{n-1}(x;\beta,\gamma q^2;q)$.

(c) We combine the interlacing of the zeros of $\tilde{M}_n(x;\beta,\gamma;q)$ and $\tilde{M}_{n-1}(x;\beta,\gamma;q)$ with the result of Theorem 3.31 (c) to obtain the required interlacing.

$\square$

Remark 3.33. *(i) In general, the zeros of $\tilde{M}_n(x;\beta,\gamma;q)$ and $\tilde{M}_{n-1}(x;\beta,\gamma q;q)$ do not interlace. These polynomials satisfy*

$$\tilde{M}_{n-1}(x;\beta,\gamma q;q) = -\frac{q^{2n-1}\tilde{M}_n(x;\beta,\gamma;q)}{\gamma q + q^n} + \frac{b(x)\tilde{M}_{n-1}(x;\beta,\gamma;q)}{q\,(\gamma q + q^n)}$$

with $b(x) = q^{2n}x + \gamma q\,(\beta q^n + q^n - 1)$, which represents a linear function that changes sign on $(1,\infty)$ for $0 < \beta q < 1$ and $\gamma > 0$. For example, when $n = 2, \beta = 1, \gamma = 5, q = 0.1$, the zeros of $\tilde{M}_n(x;\beta,\gamma;q)$ are $\{42.15, 5413.85\}$ and the zero of $\tilde{M}_{n-1}(x;\beta,\gamma q;q)$ is $\{5.50\}$;

(ii) When $\beta = 0$ in the definition of the q-Meixner polynomials, we obtain the q-Charlier polynomials $\tilde{C}_n(x;\gamma;q)$. The interlacing results in Theorem 3.31 (a) and (b) and Corollary 3.32 (b) follow from (3.14a) and (3.14b) (with $\beta = 0$), where $x_{n,i}, y_{n,i}$ and $Z_{n,i}, i \in \{1, 2, \ldots, n\}$, are the zeros of $\tilde{C}_n(x;\gamma;q)$,$\tilde{C}_n(x;\gamma q;q)$ and $\tilde{C}_n(x;\gamma q^2;q)$, respectively.

3.5.2 Quasi-orthogonality

The q-Meixner polynomials $\tilde{M}_n(\bar{x};\beta,\gamma;q)$ with $\bar{x} = q^{-x}$, are orthogonal with respect to the discrete weight $\frac{(\beta q;q)_x \gamma^x q^{\binom{x}{2}}}{(q,-\beta\gamma q;q)_x}$, when $0 \le \beta q < 1$, $\gamma > 0$, $\bar{x} \in (1,\infty)$, and satisfy

$$\tilde{M}_n\left(\bar{x};\frac{\beta}{q},\gamma;q\right) = \frac{(q^n x + \beta\,\gamma)}{(\beta\,\gamma + x)q^n}\tilde{M}_n(\bar{x};\beta,\gamma;q) - \frac{\beta\gamma\,(q^n+\gamma)\,(q^n-1)\,q}{(\beta\,\gamma + x)q^{3n}}\tilde{M}_n(\bar{x};\beta,\gamma;q).$$

The polynomial $\tilde{M}_n(\bar{x};\frac{\beta}{q^k},\gamma;q), k < n$, is not quasi-orthogonal with respect to $\frac{(\beta q;q)_x \gamma^x q^{\binom{x}{2}}}{(q,-\beta\gamma q;q)_x}$, on $(1,\infty)$, since it cannot be written as a linear combination of the polynomials $\tilde{M}_n(\bar{x};\beta,\gamma;q)$, $\tilde{M}_{n-1}(\bar{x};\beta,\gamma;q), \ldots, \tilde{M}_{n-k}(\bar{x};\beta,\gamma;q)$. Since $\gamma > 0$, we also have $\frac{\gamma}{q} > 0$ or $\gamma q > 0$ and the sequences $\tilde{M}_n(\bar{x};\beta,\frac{\gamma}{q};q)$ or $\tilde{M}_n(\bar{x};\beta,\gamma q;q)$, are orthogonal on $(1,\infty)$ for $0 \le \beta q < 1$. We therefore do not consider q-shifts of γ.

3.5.3 Bounds of the extreme zeros

The q-Meixner polynomials $\tilde{M}_n(\bar{x};\beta,\gamma;q)$ satisfy recurrence relations of type
$f(x)\tilde{M}_{n-k}(x;\beta,\gamma q^{-s};q) = H(x)\tilde{M}_n(x;\beta,\gamma;q)+G_{k-1,s}(x)\tilde{M}_{n-1}(x;\beta,\gamma;q)$, $s\in\{0,1,\ldots,2k\}$.
Let $B_{k,s}^{(1)}$ and $B_{k,s}^{(2)}$ be the smallest and the largest zero of $G_{k-1,s}(x)$, respectively. The bounds $x_{n,1} < B_{k,2k}^{(1)}$ and $x_{n,n} > B_{k,0}^{(2)}$ are more accurate for $k = 2,3,\ldots$. We have $B_{3,0}^{(2)} = (-b+\sqrt{b^2-4c})/2$ with

$$b = q^{1-2n}\left(q^n\gamma\beta - \gamma q^2 + q^n\gamma - q^n - \gamma\right)(q+1)\,,\quad c = q^{-2n+3}\Big(\beta^2\gamma^2 + \beta\gamma^2 - \beta\gamma + \gamma^2 - \gamma + 1\Big) - q^{-n+2}\gamma\beta - q^{3-4n}\gamma\left(q^2+q+1\right)\left(q^n\gamma\beta + q^n\gamma - \gamma q - q^n\right).$$

The expression for the bound $B_{3,6}^{(1)}$ is huge and will not be displayed here. However, we have in Table 3.3 some values of the bounds for some random parameters.

$n,\ q,\ \beta,\ \gamma$	5, 0.5, 0.15, 3	10, 0.9, 0.1, 0.5	20, 0.98, 0.05, 25
zero $x_{n,1}$	4.26295	1.048	12.061
bound $B_{4,8}^{(1)}$	4.263796	1.12	12.698
bound $B_{3,6}^{(1)}$	4.299	1.27	14.08
bound $B_{3,0}^{(2)}$	2415.251	10.78	100.04
bound $B_{4,0}^{(2)}$	2415.757878	11.28	109.65
zero $x_{n,n}$	2415.757968	11.35	114.32

Table 3.3: Bounds for the extreme zeros of the q-Meixner polynomials $\tilde{M}_n(\bar{x};\beta,\gamma;q)$

3.6 The q-Krawtchouk polynomials

The q-Krawtchouk polynomials $\tilde{K}_n(\bar{x};p,N;q)$ with $\bar{x} = q^{-x}$ and $n\in\{0,1,\ \ldots,\ N\}$, are orthogonal for $p>0$ with respect to the discrete weight $w(x) = \dfrac{(q^{-N};q)_x(-p)^x}{(q;q)_x}$ on $(1,q^{-N})$.

3.6.1 Interlacing properties

Proposition 3.34.

$$\tilde{K}_n(x;p,N;q) = \tilde{K}_n(x;pq,N;q) + \frac{pq^n\left(q^{N+1}-q^n\right)\left(q^n-1\right)}{q^N\left(1+pq^{2n}\right)\left(q+pq^{2n}\right)}\tilde{K}_{n-1}(x;pq,N;q); \tag{3.15a}$$

$$\tilde{K}_n(x;p,N;q) = \frac{\left(pq^{2n}+1\right)\left(pq^{N+1}+1\right)\tilde{K}_n(x;pq^2,N;q)}{\left(pq^n+1\right)\left(pq^{n+N+1}+1\right)} + \frac{p\left(q^n-1\right)\left(q^{N+1}-q^n\right)\left(q^N\left(pq^{2n+1}+1\right)x + q^n\left(pq^{1+N}+1\right)\right)\tilde{K}_{n-1}(x;pq^2,N;q)}{q^N\left(pq^n+1\right)\left(pq^{n+N+1}+1\right)\left(pq^{2n+1}+1\right)}. \tag{3.15b}$$

Theorem 3.35. *Let $p > 0$, $n \in \{0, 1, \dots, N\}$ and denote the zeros of $\tilde{K}_n(x; p, N; q)$ by $1 < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < q^{-N}$, the zeros of $\tilde{K}_n(x; pq, N; q)$ by $y_{n,1} < y_{n,2} < \cdots < y_{n,n}$ and the zeros of $\tilde{K}_n(x; pq^2, N; q)$ by $Y_{n,1} < Y_{n,2} < \cdots < Y_{n,n}$. Then, for each $i \in \{1, 2, \dots, n-1\}$,*

(a) $x_{n,i} < y_{n,i} < y_{n-1,i} < x_{n,i+1} < y_{n,i+1}$,

(b) $x_{n,i} < Y_{n,i} < Y_{n-1,i} < x_{n,i+1} < Y_{n,i+1}$.

Proof. Let $p > 0$, $n \in \{0, 1, \dots, N\}$. We note that $q^n - 1 < 0$ and since $q^m < q^n$ for $m > n$, $q^{N+1} - q^n < 0$. Since the polynomials on the right-hand side of both equations (3.15a) and (3.15b) belong to the same orthogonal sequences, their zeros interlace and both these equations are in the form of (1.2). The required interlacing follows from Corollary 1.3 (a), since

(a) both $a(x)$ and $b(x)$ in (3.15a) are positive constants;

(b) taking into account the restrictions on the parameters, it is clear that $a(x)$ is a positive constant and $b(x) > 0$ represents a linear function that does not change sign on $(1, q^{-N})$.

□

Corollary 3.36. *For $i \in \{1, 2, \dots, n-1\}$,*

(a) $x_{n,i} < x_{n-1,i} < y_{n-1,i} < x_{n,i+1}$,

(b) $x_{n,i} < y_{n,i} < Y_{n,i} < Y_{n-1,i} < x_{n,i+1} < y_{n,i+1} < Y_{n,i+1}$.

Proof.

(a) The result follows directly from Theorem 3.35 (a) and the interlacing of the zeros of $\tilde{K}_n(x; p, N; q)$ and $\tilde{K}_{n-1}(x; p, N; q)$.

(b) When we replace p by pq in (3.15a), we obtain, using the same argument as in the proof of Theorem 3.35 (a), that $y_{n,i} < Y_{n,i} < Y_{n-1,i} < y_{n,i+1} < Y_{n,i+1}$, for each $i \in \{1, 2, \dots, n-1\}$. We combine this with the interlacing results in Theorem 3.35 (a) and (b), which leads to the required result.

□

3.6.2 Quasi-orthogonality

The polynomials $\tilde{K}_n(\bar{x}; \frac{p}{q^k}, N; q)$ are orthogonal for $p > 0$ with respect to $\dfrac{(q^{-N}; q)_x (\frac{-p}{q^k})^x}{(q; q)_x}$ on $(1, q^{-N})$. By iterating the equation

$$K_n\left(\bar{x}; \frac{p}{q}, N; q\right) = K_n(\bar{x}; p, N; q) - \frac{p\,(q^n - 1)\left(q^n - q^{N+1}\right) q^{n+1}}{\left(q^{2n} p + q\right)\left(q^{2n} p + q^2\right) q^N} K_{n-1}(\bar{x}; p, N; q),$$

we can write $\tilde{K}_n(\bar{x}; \frac{p}{q^k}, N; q)$ as a linear combination of the polynomials $\tilde{K}_{n-j}(\bar{x}; p, N; q)$, $j \in \{0, 1, \dots, k\}$, and the polynomials $\tilde{K}_n(\bar{x}; \frac{p}{q^k}, N; q)$ are also quasi-orthogonal for $p > 0$ on $(1, q^{-N})$ with respect to $w(x)$.

3.6.3 Bounds of the extreme zeros

The polynomials $\tilde{K}_n(x;p,N;q)$ verify equations of type

$$f(x)\tilde{K}_{n-k}(x;pq^s,N;q) = H(x)\tilde{K}_n(x;p,N;q) + G_{k-1,s}(x)\tilde{K}_{n-1}(x;p,N;q),$$

with $s \in \{0,1,\ldots,2k\}$. Let $B^{(1)}_{k,s}$, $B^{(2)}_{k,s}$ be the smallest and the largest zeros of $G_{k-1,s}$, respectively. For $k=3$ and $s=6$, we derive the upper bound $B^{(1)}_{3,6} = -\frac{b_1+\sqrt{b_1^2-4a_1c_1}}{2a_1}$ of $x_{n,1}$ with

$$\begin{aligned}
&a_1 = q^{2N}\left(pq^{2n}+q\right)\left(q^4+pq^{2n}\right)\left(q^5+pq^{2n}\right)\left(pq^{2n}+q^2\right), \quad b_1 = -\left(q^4+q^{2n}p\right)(q+1)\\
&\times\left(q^{2n}p+q^2\right)q^{n+N+1}\left(\left(q^{2n}p-q^3\right)\left(q^Np-1\right)+p\left(q^2+1\right)\left(q^{N+1}+1\right)q^n\right),\\
&c_1 = \left(q^{4n}p^2+q^6\right)q^{2n+3}\left(q^{2N}p^2-q^Np+1\right)+p\left(q^{2n}p-q^3\right)\left(q^{N+1}+1\right)\left(q^Np-1\right)q^{3+3n}\\
&\times\left(q^2+q+1\right)-p\Big(p^2q^{2+2N}-p\left(q^{N+2}+q^{N+1}+q^N+1\right)\left(q^{N+2}+q^2+q+1\right)+q^2\Big)q^{4+4n},
\end{aligned}$$

and for $s=0$ the lower bound $B^{(2)}_{3,0} = \left(-b_2+\sqrt{b_2^2-4a_2c_2}\right)/(2a_2)$ of $x_{n,n}$ with

$$\begin{aligned}
a_2 &= \left(p^2q^{2N+6}-pq^{N+3}+1\right)\left(q^{4n}p^2+1\right)q^{2N+3}\\
&+p\left(q^{2n}p-1\right)\left(q^{N+1}+1\right)\left(pq^{N+3}-1\right)q^{n+2N+3}\left(q^2+q+1\right)\\
&-\left(1+p^2q^{2N+6}-q\left(q^{N+2}+q^2+q+1\right)\left(q^{N+2}+q^{N+1}+q^N+1\right)p\right)pq^{2n+2N+3},\\
b_2 &= -\left(pq^{N+2}+1\right)\left(pq^{N+4}+1\right)\left(\left(q^{2n}p-1\right)\left(pq^{N+3}-1\right)+p\left(q^2+1\right)\left(q^{N+1}+1\right)q^n\right)\\
&\times(q+1)q^{n+N+1}, \quad c_2 = q^{2n}\left(pq^{N+2}+1\right)\left(pq^{N+5}+1\right)\left(pq^{N+4}+1\right)\left(pq^{N+1}+1\right).
\end{aligned}$$

As shown in Table 3.4 for $k=4$, the accurate bounds for $x_{n,1}$ are the smallest zeros $B^{(1)}_{k,2k}$ of $G_{k-1,2k}$ and the accurate bounds for $x_{n,n}$ are the largest zeros $B^{(2)}_{k,0}$ of $G_{k-1,0}$.

$n,\ p,\ N,\ q$	10, 7, 15, 0.9	5, 0.5, 10, 0.98	20, 1, 20, 0.85
zero $x_{n,1}$	1.022	1.048	1.140878
bound $B^{(1)}_{4,8}$	1.084	1.051	1.1410586
bound $B^{(1)}_{3,6}$	1.1996	1.06	1.14539
bound $B^{(2)}_{3,0}$	4.069	1.18	1.56
bound $B^{(2)}_{4,0}$	4.4699	1.204	1.91
zero $x_{n,n}$	4.685	1.218	25.8

Table 3.4: Bounds for the extreme zeros of the q-Krawtchouk polynomials $\tilde{K}_n(x;p,N;q)$

3.7 The q-Laguerre polynomials

The q-Laguerre polynomials $\tilde{L}^{(\alpha)}_n(x;q)$ are orthogonal for $\alpha > -1$ on $(0,\infty)$ with respect to the weight function $w(x) = \frac{x^\alpha}{(-x;q)_\infty}$. In [Moak, 1981], relations between different sequences of q-Laguerre polynomials are provided and interlacing results between the zeros of different sequences of these polynomials are given in [Jordaan and Toókos, 2010], [Moak, 1981].

3.7.1 Interlacing properties

Proposition 3.37.

$$\tilde{L}_n^{(\alpha)}(x;q) = \tilde{L}_n^{(\alpha+1)}(x;q) - q^{-2n-\alpha}\,(q^n-1)\,\tilde{L}_{n-1}^{(\alpha+1)}(x;q) \quad \textit{(cf. [Moak, 1981, Eq. (4.12)]);} \tag{3.16a}$$

$$\tilde{L}_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1}-1)\,q^n\,\tilde{L}_n^{(\alpha+2)}(x;q)}{q^{n+\alpha+1}-1} + \frac{(q^{n+\alpha+1}x - q^{\alpha+1}+1)\,(q^n-1)\,\tilde{L}_{n-1}^{(\alpha+2)}(x;q)}{q^{n+\alpha+1}(q^{n+\alpha+1}-1)}. \tag{3.16b}$$

Theorem 3.38. *Let $\alpha > -1$. We denote the zeros of $\tilde{L}_n^{(\alpha)}(x;q)$ by $0 < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < \infty$, the zeros of $\tilde{L}_n^{(\alpha+1)}(x;q)$ by $y_{n,1} < y_{n,2} < \cdots < y_{n,n}$ and the zeros of $\tilde{L}_n^{(\alpha+2)}(x;q)$ by $Y_{n,1} < Y_{n,2} < \cdots < Y_{n,n}$. Then, for $i \in \{1,2,\ldots,n-1\}$,*

(a) $x_{n,i} < y_{n,i} < y_{n-1,i} < x_{n,i+1} < y_{n,i+1}$ (cf. [Moak, 1981, Theorem 3]),

(b) $x_{n,i} < Y_{n,i} < Y_{n-1,i} < x_{n,i+1} < Y_{n,i+1}$.

Proof. Let $\alpha > -1$. We note that $q^n - 1 < 0$ and $q^{n+\alpha} - 1 < 0$.

(a) Since $\tilde{L}_n^{(\alpha+1)}(x;q)$ and $\tilde{L}_{n-1}^{(\alpha+1)}(x;q)$ belong to the same orthogonal sequence, the interlacing property (1.1) is satisfied and (3.16a) is in the form of (1.2). Both $a(x)$ and $b(x)$ are positive constants and the result follows from Corollary 1.3 (a).

(b) The polynomials $\tilde{L}_n^{(\alpha+2)}(x;q)$ and $\tilde{L}_{n-1}^{(\alpha+2)}(x;q)$ belong to the same orthogonal sequence, which implies (1.1) is satisfied and equation (3.16b) is in the form of (1.2). For the given values of the parameters, $a(x)$ is a positive constant and

$$b(x) = \frac{q^n-1}{q^{n+\alpha+1}(q^{n+\alpha+1}-1)}\left(q^{n+\alpha+1}x - q^{\alpha+1}+1\right) > 0$$

on $(0,\infty)$ and the interlacing follows from Corollary 1.3 (a).

□

Corollary 3.39. *For $i \in \{1,2,\ldots,n-1\}$,*

(a) $x_{n,i} < x_{n-1,i} < y_{n-1,i} < Y_{n-1,i} < x_{n,i+1}$,

(b) $x_{n,i} < y_{n,i} < Y_{n,i} < Y_{n-1,i} < x_{n,i+1} < y_{n,i+1} < Y_{n,i+1}$.

Proof.

(a) See [Jordaan and Toókos, 2010, Theorem 5.1].

(b) When we replace α by $\alpha+1$ in (3.16a), we obtain, using the same argument as in the proof of Theorem 3.38 (a), that $y_{n,i} < Y_{n,i} < Y_{n-1,i} < y_{n,i+1} < Y_{n,i+1}$, for each $i \in \{1,2,\ldots,n-1\}$. We combine this with the results in Theorem 3.38 (a) and (b) to obtain the result.

□

Remark 3.40. *In [Jordaan and Toókos, 2010], the result in Corollary (3.39) (a) is extended to also include a continuous shift of the parameter α. Furthermore, examples are provided to show that, in general, interlacing breaks down between the zeros of: $\tilde{L}_n^{(\alpha)}(x;q)$ and $\tilde{L}_n^{(\alpha+3)}(x;q)$, $\tilde{L}_n^{(\alpha)}(x;q)$ and $\tilde{L}_{n-1}^{(\alpha+3)}(x;q)$ and $\tilde{L}_n^{(\alpha+1)}(x;q)$ and $\tilde{L}_{n-1}^{(\alpha)}(x;q)$.*

3.7.2 Quasi-orthogonality

Consider the equation (cf. [Moak, 1981, Eq. (4.12)] and (3.16a)

$$\tilde{L}_n^{(\alpha-1)}(x;q) = \tilde{L}_n^{(\alpha)}(x;q) - \frac{(q^n-1)\,q}{q^{2\,n+\alpha}}\,\tilde{L}_{n-1}^{(\alpha)}(x;q). \tag{3.17}$$

Theorem 3.41. *Let $k \in \mathbb{N}_0$ and $\alpha \in \mathbb{R}$. For $-1 < \alpha < 0$ and $k \in \{1, 2, \ldots, n-1\}$, the sequence of q-Laguerre polynomials $\{\tilde{L}_n^{(\alpha-k)}(x;q)\}_{n\geq 0}$ is quasi-orthogonal of order k on the interval $(0,\infty)$ with respect to $w(x)$ and the polynomials have at least $(n-k)$ real, distinct zeros in $(0,\infty)$.*

Proof. Fix $-1 < \alpha < 0$. From Lemma 1.4 and (3.17) it follows that $\tilde{L}_n^{(\alpha-1)}(x;q)$ is quasi-orthogonal of order one on $(0,\infty)$. By iteration, we can express $\tilde{L}_n^{(\alpha-k)}(x;q)$ as a linear combination of $\tilde{L}_n^{(\alpha)}(x;q), \tilde{L}_{n-1}^{(\alpha)}(x;q), \ldots, \tilde{L}_{n-k}^{(\alpha)}(x;q)$, and the result follows from Lemma 1.4. The location of the $(n-k)$ real, distinct zeros of $\tilde{L}_n^{(\alpha-k)}(x;q), k \in \{1, 2, \ldots, n-1\}$, follows from Lemma 1.6. □

Theorem 3.42. *Let $-1 < \alpha < 0$ and denote the zeros of $\tilde{L}_n^{(\alpha)}(x;q)$ by $x_{n,j}, j \in \{1, 2, \ldots, n\}$, and the zeros of $\tilde{L}_n^{(\alpha-1)}(x;q)$ by $y_{n,j}, j \in \{1, 2, \ldots, n\}$. Then*

$$y_{n,1} < 0 < x_{n,1} < x_{n-1,1} < y_{n,2} < x_{n,2} < \cdots < x_{n-1,n-1} < y_{n,n} < x_{n,n}.$$

Proof. From (3.17), we obtain the value $a_n = \frac{-(q^n-1)q}{q^{2\,n+\alpha}} > 0$. The interlacing result, as well as the position of $y_{n,n}$, follows from Lemma 1.8 (ii).

To obtain the position of $y_{n,1}$, we use Lemma 1.7, and when we consider the given parameter values,

$$f_n(0) = \frac{(q^{n+\alpha}-1)\,q}{q^{2\,n+\alpha}} < 0.$$

We thus have

$$-a_n - f_n(0) = -\frac{(q^\alpha-1)\,q}{q^{n+\alpha}} < 0$$

and since $-a_n < f_n(0) < 0$, the result follows from Lemma 1.7 (i). □

3.7.3 Bounds of the extreme zeros

The q-Laguerre polynomials $\tilde{L}_n^{(\alpha)}(x;q)$ are solution of equations of type

$$f(x)\tilde{L}_{n-k}^{(\alpha+s)}(x;q) = H(x)\tilde{L}_n^{(\alpha)}(x;q) + G_{k-1,s}(x)\tilde{L}_{n-1}^{(\alpha)}(x;q),\ s \in \{0, 1, \ldots, 2k\}. \tag{3.18}$$

Let $B_{k,s}^{(1)}$, $B_{k,s}^{(2)}$ be the smallest and the largest zeros of $G_{k-1,s}$, respectively. For $k = 3$ and $s = 0$, we derive the lower bound $B_{3,0}^{(2)}$ of $x_{n,n}$ given by

$$x_{n,n} > B_{3,0}^{(2)} = \frac{1}{2q^{2n+\alpha}}\Big(q\,(q+1)\,(q^2+1-q^{\alpha+n}-q^n) + \Big(\,(1+q)^2\,q^2\,(q^\alpha q^n - q^2 + q^n - 1)^2$$
$$- 4\,q^{2\,n+3}\,(q^{2\,\alpha}+1+q^\alpha) + 4\,(q^\alpha+1)\,q^{n+3}\,(1+q+q^2) - 4\,q^4\,(1+q+q^2)\,\Big)^{1/2}\Big),$$

and for $s = 6$ the upper bound $B_{3,6}^{(1)}$ of $x_{n,1}$ given by

$$x_{n,1} < B_{3,6}^{(1)} = -\frac{b + \sqrt{b^2 - 4ac}}{2a},$$

where

$$\begin{aligned}
a &= \left(\left(q^{6+2\,\alpha} + q^{\alpha+3} + 1\right) q^{2\,n} - \left(q^2 + q + 1\right)\left(q^n q^\alpha q^3 + q^n - q\right)\right) q^{2\,\alpha+3},\\
b &= \left(q^{\alpha+n+3} - q^2 + q^n - 1\right) q^{\alpha+1}\left(q + 1\right)\left(q^{\alpha+2} - 1\right)\left(q^{\alpha+4} - 1\right),\\
c &= \left(q^{\alpha+2} - 1\right)\left(q^{\alpha+4} - 1\right)\left(q^{\alpha+5} - 1\right)\left(q^{\alpha+1} - 1\right).
\end{aligned}$$

As shown in Table 3.5 for $k = 4$, the sharpest bounds for $x_{n,1}$ are the smallest zeros $B_{k,2k}^{(1)}$ of $G_{k-1,2k}$ and the sharpest bounds of $x_{n,n}$ are the largest zeros $B_{k,0}^{(2)}$ of $G_{k-1,0}$.

$n,\ \alpha,\ q$	$10,\ -0.9,\ 0.8$	$20,\ 65,\ 0.95$	$40,\ -0.95,\ 0.9$
zero $x_{n,1}$	0.00524429273450741	11.31	0.0005479977042140575
bound $B_{4,8}^{(1)}$	0.00524429273450819	11.54	0.0005479977042140596
bound $B_{3,6}^{(1)}$	0.0052442928	12.46	0.0005479977053
bound $B_{3,0}^{(2)}$	110.1179	461.13	9496.159
bound $B_{4,0}^{(2)}$	111.79	498.71	10012.19
zero $x_{n,n}$	111.83	509.576	10086.19

Table 3.5: Bounds for the extreme zeros of the q-Laguerre polynomials $\tilde{L}_n^{(\alpha)}(x;q)$

Remark 3.43. *Gupta and Muldoon [2007] considered the q-Laguerre polynomials*

$$L_n^{(\alpha)}((1-q)x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_1\phi_1\left(\begin{matrix} q^{-n} \\ q^{\alpha+1}\end{matrix}\,\middle|\, q; -q^{n+\alpha+1}(1-q)x\right),\ \alpha > -1,\ x \in (0,\infty).$$

As shown in Table 3.6, our upper bounds for the smallest zero $x_{n,1}/(1-q)$ of $L_n^{(\alpha)}((1-q)x;q)$ are sharper than the bounds given in [Gupta and Muldoon, 2007, Eq. (4.8) and (4.9)] (that we denote here by $GM_{4.8}$ and $GM_{4.9}$, respectively, for the bound [Gupta and Muldoon, 2007, Eq. (4.8)] and [Gupta and Muldoon, 2007, (4.9)]). Moreover, in Gupta and Muldoon [2007] lower bounds for the largest zeros $\tilde{x}_{n,n}$ were not given whereas we obtain significant lower bounds for the largest zeros $x_{n,n}/(1-q)$.

$n,\ \alpha,\ q$	$10,\ -0.9,\ 0.8$	$20,\ 65,\ 0.95$	$40,\ -0.95,\ 0.9$
zero $\tilde{x}_{n,1}$	0.026221463672537	226.22	0.005479977042140575
bound $\frac{1}{1-q}B_{4,8}^{(1)}$	0.02622146367254	230.84	0.005479977042140596
bound $\frac{1}{1-q}B_{3,6}^{(1)}$	0.0262214643551	249.19	0.005479977053
bound $GM_{4.9}$	0.0262386	317.63	0.005481029
bound $GM_{4.8}$	0.02718	431.105	0.0056003
bound $\frac{1}{1-q}B_{3,0}^{(2)}$	550.5895	9222.6	94961.59
bound $\frac{1}{1-q}B_{4,0}^{(2)}$	558.955	9974.2	100121.92
zero $\tilde{x}_{n,n}$	559.15	10191.51	10086.19

Table 3.6: Bounds for the extreme zeros of the q-Laguerre polynomials $\tilde{L}_n^{(\alpha)}((1-q)x;q)$

3.8 The alternative q-Charlier or q-Bessel polynomials

3.8.1 Interlacing properties

Proposition 3.44.

$$\tilde{y}_n(x;\alpha;q) = \tilde{y}_n(x;\alpha q;q) - \frac{\alpha\, q^{2\,n}\,(q^n-1)}{(q+\alpha\, q^{2\,n})\,(1+\alpha\, q^{2\,n})}\tilde{y}_{n-1}(x;\alpha q;q); \tag{3.19a}$$

$$\tilde{y}_n(x;\alpha;q) = \frac{(\alpha q^{2\,n}+1)\,\tilde{y}_n(x;\alpha q^2;q)}{\alpha\, q^n+1} - \frac{\alpha\, q^n\,(q^n-1)\,((\alpha\, q^{2\,n+1}+1)x+q^n)\,\tilde{y}_{n-1}(x;\alpha q^2;q)}{(\alpha\, q^{2\,n+1}+1)\,(\alpha\, q^n+1)}. \tag{3.19b}$$

Theorem 3.45. *Let $\alpha > 0$. We denote the zeros of $\tilde{y}_n(x;\alpha;q)$ by $0 < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < 1$, the zeros of $\tilde{y}_n(x;\alpha q;q)$ by $z_{n,1} < z_{n,2} < \cdots < z_{n,n}$ and the zeros of $\tilde{y}_n(x;\alpha q^2;q)$ by $Z_{n,1} < Z_{n,2} < \cdots < Z_{n,n}$. Then, for $i \in \{1,2,\ldots,n-1\}$,*

(a) $x_{n,i} < z_{n,i} < z_{n-1,i} < x_{n,i+1} < z_{n,i+1}$,

(b) $x_{n,i} < Z_{n,i} < Z_{n-1,i} < x_{n,i+1} < Z_{n,i+1}$.

Proof. Let $\alpha > 0$. The polynomials on the right-hand side of each of the equations (3.19a) and (3.19b) belong to the same orthogonal sequence and their zeros satisfy (1.1), therefore these equations are in the form of (1.2). Taking into consideration the values of the parameters,

(a) both $a(x)$ and $b(x)$ in (3.19a) are positive constants and the result follows from Corollary 1.3 (a).

(b) $a(x)$ in (3.19b) is a positive constant and $b(x)$ represents a linear function that does not change sign on $(0,1)$ and the interlacing follows from Corollary 1.3 (a).

□

Corollary 3.46. *For $i \in \{1,2,\ldots,n-1\}$,* *(a)* $x_{n,i} < x_{n-1,i} < z_{n-1,i} < Z_{n-1,i} < x_{n,i+1}$,

(b) $x_{n,i} < z_{n,i} < Z_{n,i} < Z_{n-1,i} < x_{n,i+1} < z_{n,i+1} < Z_{n,i+1}$.

Proof.

(a) The result follows from Theorem 3.45 (a) and (b) and the interlacing of the zeros of $\tilde{y}_n(x;\alpha;q)$ and $\tilde{y}_{n-1}(x;\alpha;q)$.

(b) When we replace α by αq in (3.19a), we deduce that $z_{n,i} < Z_{n,i} < Z_{n-1,i} < z_{n,i+1} < Z_{n,i+1}$ and we combine this with the interlacing results in Theorem 3.45 (a) and (b) to obtain the required result.

$\square$

3.8.2 Quasi-orthogonality

The polynomials $\tilde{y}_n(q^x;\alpha;q)$ are orthogonal for $\alpha > 0$ with respect to $w(x) = \dfrac{\alpha^x}{(q;q)_x} q^{\binom{x+1}{2}}$ on $(0,\infty)$. By iterating the equation

$$\tilde{y}_n(\bar{x};\frac{\alpha}{q};q) = \tilde{y}_n(\bar{x};\alpha;q) - \frac{\alpha q^{2n+1}(q^n-1)\,\tilde{y}_{n-1}(\bar{x};\alpha;q)}{(\alpha q^{2n}+q)(\alpha q^{2n}+q^2)},$$

where $\bar{x} = q^x \in (0,1)$, we can write $\tilde{y}_n(\bar{x};\frac{\alpha}{q^k};q)$ as a linear combination of the polynomials $\tilde{y}_{n-j}(\bar{x};\alpha;q), j \in \{0,1,\ldots,k\}$, and the polynomials $\tilde{y}_n(\bar{x};\frac{\alpha}{q^k};q)$ are also quasi-orthogonal for $\alpha > 0$ on $(0,1)$ with respect to $w(x)$.

3.8.3 Bounds of the extreme zeros

The polynomials $\tilde{y}_n(x;\alpha;q)$ are solutions of recurrence equations of type

$$f(x)\tilde{y}_{n-k}(x;\alpha q^s;q) = H(x)\tilde{y}_n(x;\alpha;q) + G_{k-1,s}(x)\tilde{y}_{n-1}(x;\alpha;q),\ s \in \{0,1,\ldots,2k\}.$$

Let $B^{(1)}_{k,s}$ be the smallest zero of $G_{k-1,s}$. For $k=3$, the best upper bound for $x_{n,1}$ is

$$x_{n,1} < B^{(1)}_{3,6} = \left(-b - \sqrt{b^2-4ac}\right)/(2a),$$

with

$$\begin{aligned}
a &= q^{4n+3}\alpha^2 - q^{3+n}\left(q^{2n}\alpha-1\right)\left(q^2+q+1\right)\alpha + q^{2n+3}\left(q^3\alpha+q^2\alpha+\alpha q-1\right)\alpha + q^3,\\
b &= q^{n+1}\left(-q^{n+2}\alpha + q^{2n}\alpha - \alpha q^n - 1\right)(q+1)\,,\ c = q^{2n}.
\end{aligned}$$

The best upper bounds for $x_{n,1}$ are the zeros $B^{(1)}_{k,2k}$, as shown for $k=4$ in Table 3.7.

3.9 The Al-Salam-Carlitz I polynomials

The Al-Salam-Carlitz I polynomials $\tilde{U}^{(\alpha)}_n(x;q)$ are orthogonal for $\alpha < 0$ on $(\alpha,1)$ with respect to the weight function $w(x) = (qx, \frac{qx}{\alpha};q)_\infty$.

$n,\ \alpha,\ q$	5, 5, 0.45	20, 50, 0.9	50, 5, 0.95
zero $x_{n,1}$	0.03536656765	0.0079156	0.0314
bound $B_{4,8}^{(1)}$	0.03536656789	0.007974	0.0323
bound $B_{3,6}^{(1)}$	0.0353685	0.0084	0.0349

Table 3.7: Bounds for the extreme zeros of the alternative q-Charlier $\tilde{y}_n(x;\alpha;q)$

3.9.1 Interlacing properties

Proposition 3.47.

$$\tilde{U}_n^{(\alpha)}(x;q) = \tilde{U}_n^{(\alpha q)}(x;q) + \alpha(q^n-1)\tilde{U}_{n-1}^{(\alpha q)}(x;q). \tag{3.20}$$

Theorem 3.48. *Let $\alpha < 0$ and denote the zeros of $\tilde{U}_n^{(\alpha)}(x;q)$ by $\alpha < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < 1$ and the zeros of $\tilde{U}_n^{(\alpha q)}(x;q)$ by $\alpha q < y_{n,1} < y_{n,2} < \cdots < y_{n,n} < 1$. Then, for $i \in \{1,2,\ldots,n-1\}$, $x_{n,i} < y_{n,i} < y_{n-1,i} < x_{n,i+1} < y_{n,i+1}$.*

Proof. Let $\alpha < 0$. Since $\tilde{U}_n^{(\alpha q)}(x;q)$ and $\tilde{U}_{n-1}^{(\alpha q)}(x;q)$ belong to the same orthogonal sequence, the interlacing property (1.1) is satisfied and (3.21) is in the form of (1.2). Taking into consideration the values of the parameters, $a(x) > 0$ and $b(x) > 0$ are constants on $(\alpha,1)$ and the result follows from Corollary 1.3 (a). ☐

Corollary 3.49. *For $i \in \{1,2,\ldots,n-1\}$, $x_{n,i} < x_{n-1,i} < y_{n-1,i} < x_{n,i+1}$.*

Proof. The result follows from Theorem 3.48 and the interlacing of the zeros of $\tilde{U}_n^{(\alpha)}(x;q)$ and $\tilde{U}_{n-1}^{(\alpha)}(x;q)$. ☐

In general, the zeros of $\tilde{U}_n^{(\alpha)}(x;q)$ do not interlace with the zeros of $\tilde{U}_n^{(\alpha q^2)}(x;q)$ or with the zeros of $\tilde{U}_{n-1}^{(\alpha q^2)}(x;q)$. For example, when $n=2, \alpha=-16$ and $q=0.9$, the zeros of $\tilde{U}_n^{(\alpha)}(x;q)$ are $\{-15.77,-12.78\}$, the zeros of $\tilde{U}_n^{(\alpha q^2)}(x;q)$ are $\{-12.64,-10.08\}$ and the zero of $\tilde{U}_{n-1}^{(\alpha q^2)}(x;q)$ is $\{-11.96\}$.

3.9.2 Quasi-orthogonality

The polynomials $\tilde{U}_n^{(\frac{\alpha}{q^k})}(x;q), k < n$, are orthogonal with respect to $w(x) = (qx, \frac{q^{k+1}x}{\alpha};q)_\infty$ on the interval $(\frac{\alpha}{q^k},1)$ and we will prove that they are quasi-orthogonal with respect to $w(x)$ on $(\alpha,1)$. Using the equation

$$\tilde{U}_n^{(\frac{\alpha}{q})}(x;q) = \tilde{U}_n^{(\alpha)}(x;q) + \alpha q^{-1}(q^n-1)\tilde{U}_{n-1}^{(\alpha)}(x;q), \tag{3.21}$$

we deduce that

$$\tilde{U}_n^{(\frac{\alpha}{q^2})}(x;q) = \tilde{U}_n^{(\alpha)}(x;q) + \frac{\alpha\,(q^n-1)\,(q+1)}{q^2}\tilde{U}_{n-1}^{(\alpha)}(x;q) + \frac{\alpha^2\,(q^n-1)\,(q^n-q)}{q^4}\tilde{U}_{n-2}^{(\alpha)}(x;q). \tag{3.22}$$

Theorem 3.50. *Let $k \in \mathbb{N}_0$ and $\alpha < 0$. The sequence of Al-Salam-Carlitz I polynomials $\{\tilde{U}_n^{(\frac{\alpha}{q^k})}(x;q)\}_{n\geq 0}$ is quasi-orthogonal with respect to $w(x)$ on $(\alpha,1)$ and the polynomials have at least $(n-k)$ real, distinct zeros in $(\alpha,1)$.*

Proof. From Lemma 1.4 and (3.21) it follows that $\tilde{U}_n^{(\frac{\alpha}{q})}(x;q)$ is quasi-orthogonal of order one on $(\alpha,1)$. By iteration, we can express $\tilde{U}_n^{(\frac{\alpha}{q^k})}(x;q)$ as a linear combination of $\tilde{U}_n^{(\alpha)}(x;q)$, $\tilde{U}_{n-1}^{(\alpha)}(x;q)$, $\ldots$, $\tilde{U}_{n-k}^{(\alpha)}(x;q)$, and the result follows from Lemma 1.4. The location of the $(n-k)$ real, distinct zeros of $\tilde{U}_n^{(\frac{\alpha}{q^k})}(x;q), k \in \{1,2,\ldots,n-1\}$, follows from Lemma 1.6. $\square$

Remark 3.51. *We can also obtain (3.21) from the generating function [Koekoek et al., 2010, Eq. (14.24.10)] of the Al-Salam-Carlitz I polynomials*

$$\frac{(t,\alpha t;q)_\infty}{(xt;q)_\infty} = \sum_{n=0}^{\infty} \frac{U_n^{(\alpha)}(x;q)}{(q;q)_n} t^n, \tag{3.23}$$

from which it follows that

$$\frac{(t,\frac{\alpha}{q^k}t;q)_\infty}{(xt;q)_\infty} = \sum_{n=0}^{\infty} \frac{U_n^{\left(\frac{\alpha}{q^k}\right)}(x;q)}{(q;q)_n} t^n, \;\; k \in \{1,2,\ldots\}. \tag{3.24}$$

From the relation

$$\frac{(a;q)_\infty}{(aq^n;q)_\infty} = (a;q)_n,$$

we obtain, when $a = \frac{\alpha}{q^k}t$ and $n = k$,

$$\left(\frac{\alpha}{q^k}t;q\right)_\infty = \left(\frac{\alpha}{q^k}t;q\right)_k (\alpha t;q)_\infty.$$

By using (3.23), (3.24) becomes

$$\left(\frac{\alpha}{q^k}t;q\right)_k \sum_{n=0}^{\infty} \frac{U_n^{(\alpha)}(x;q)}{(q;q)_n} t^n = \sum_{n=0}^{\infty} \frac{U_n^{\left(\frac{\alpha}{q^k}\right)}(x;q)}{(q;q)_n} t^n, \;\; k \in \{1,2,\ldots\}.$$

Expanding $\left(\frac{\alpha}{q^k}t;q\right)_k$ and equating powers of t yields $U_n^{\left(\frac{\alpha}{q^k}\right)}(x;q)$ as a linear combination of $U_{n-j}^{(\alpha)}(x;q)$, $j \in \{0,1,\ldots,k\}$. In particular, for $k=1$ and $k=2$, we get (3.21) and (3.22), respectively.

Theorem 3.52. *Let $\alpha < 0$ and denote the zeros of $\tilde{U}_n^{(\alpha)}(x;q)$ by $x_{n,j}, j \in \{1,2,\ldots,n\}$, and the zeros of $\tilde{U}_n^{(\frac{\alpha}{q})}(x;q)$ by $y_{n,j}, j \in \{1,2,\ldots,n\}$. Then*

(i) $\frac{\alpha}{q} < y_{n,1} < x_{n,1} < x_{n-1,1} < y_{n,2} < \cdots < x_{n-1,n-1} < y_{n,n} < x_{n,n} < 1$ *and, additionally, if $\alpha < \frac{q^n}{q^n-1}$, then $y_{n,1} < \alpha < x_{n,1}$;*

(ii) $(n-2)$ *zeros of $\tilde{U}_n^{(\frac{\alpha}{q^2})}(x;q)$ interlace with the n zeros of $\tilde{U}_n^{(\alpha)}(x;q)$ if $\alpha < \frac{q^{n+1}}{q^n-1}$.*

Proof.

(i) From (3.21) we obtain the value $a_n = \frac{\alpha(q^n-1)}{q} > 0$. The interlacing result, as well as the position of $y_{n,n}$, follows from Lemma 1.8 (ii). The position of $y_{n,1}$ cannot be determined, since

$$f_n(\alpha) = \frac{\tilde{U}_n^{(\alpha)}(\alpha;q)}{\tilde{U}_{n-1}^{(\alpha)}(\alpha;q)} = -q^{n-1} < 0$$

and the sign of

$$-a_n - f_n(\alpha) = -\frac{\alpha(q^n-1)}{q} + q^{n-1} = \frac{\alpha(1-q^n)+q^n}{q}$$

varies as the parameters vary within the allowed regions. However, if $\alpha < \frac{q^n}{q^n-1}$, then $-a_n < f_n(\alpha) < 0$ and from Lemma 1.7(i), it follows that $y_{n,1} < \alpha$.

(ii) The coefficient of $\tilde{U}_{n-2}^{(\alpha)}(x;q)$ in (3.22) is

$$b_n = \frac{(q^n-1)\,(q^n-q)\,\alpha^2}{q^4}.$$

From the three-term recurrence equation of the Al-Salam-Carlitz I polynomials [Koekoek et al., 2010, Eq. (14.24.4)]

$$\tilde{U}_n^{(\alpha)}(x;q) = \left(x - \frac{q^n\,(\alpha+1)}{q}\right)\tilde{U}_{n-1}^{(\alpha)}(x;q) - \frac{\alpha\,q^n\,(q^n-q)}{q^3}\tilde{U}_{n-2}^{(\alpha)}(x;q),$$

we obtain $C_n = \frac{\alpha\,q^n\,(q^n-q)}{q^3}$ and since

$$b_n - C_n = \frac{\alpha\,(q^n-q)\,(\alpha\,q^n - q^{n+1} - \alpha)}{q^4} > 0$$

when $\alpha < \frac{q^{n+1}}{q^n-1}$, the interlacing result follows from [Joulak, 2005, Theorem 15 (ii)].

□

3.9.3 Bounds of the extreme zeros

Finding inner bounds by using Theorem 1.11 is not possible for the Al-Salam Carlitz I polynomial system $\{\tilde{U}_n^{(\alpha)}(x;q)\}_{n\geq 0}$, since the polynomials $\tilde{U}_n^{(\alpha)}(x;q)$ are orthogonal for $\alpha < 0$ on $(\alpha,1)$ and shifting α to αq^k will result in a change of the interval of orthogonality.

3.10 The Al-Salam-Carlitz II polynomials

3.10.1 Interlacing properties

Proposition 3.53.

$$\tilde{V}_n^{(\alpha q)}(x;q) = \tilde{V}_n^{(\alpha)}(x;q) - \alpha\,q(q^n-1)q^{-n}\tilde{V}_{n-1}^{(\alpha)}(x;q); \tag{3.25a}$$

$$\tilde{V}_n^{(\alpha q^2)}(x;q) = \left(\alpha q^{n+1} + 1 - \alpha\,q\right)\tilde{V}_n^{(\alpha)}(x;q) - \alpha q^{-n+1}\,(q^n-1)\,(q^n x + 1 - \alpha\,q)\,\tilde{V}_{n-1}^{(\alpha)}(x;q). \tag{3.25b}$$

Theorem 3.54. *Let $0 < \alpha q < 1$. Denote the zeros of $\tilde{V}_n^{(\alpha)}(x;q)$ by $1 < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < \infty$, the zeros of $\tilde{V}_n^{(\alpha q)}(x;q)$ by $y_{n,1} < y_{n,2} < \cdots < y_{n,n}$ and the zeros of $\tilde{V}_n^{(\alpha q^2)}(x;q)$ by $Y_{n,1} < Y_{n,2} < \cdots < Y_{n,n}$. Then, for $i \in \{1,2,\ldots,n-1\}$,*

(a) $y_{n,i} < x_{n,i} < x_{n-1,i} < y_{n,i+1} < x_{n,i+1}$,

(b) $Y_{n,i} < x_{n,i} < x_{n-1,i} < Y_{n,i+1} < x_{n,i+1}$.

Proof. Let $0 < \alpha q < 1$. Since $\tilde{V}_n^{(\alpha)}(x;q)$ and $\tilde{V}_{n-1}^{(\alpha)}(x;q)$ belong to the same orthogonal sequence, the interlacing property (1.1) is satisfied and both (3.25a) and (3.25b) are in the form of (1.2).

(a) Taking into consideration the values of the parameters, both the coefficients of $\tilde{V}_n^{(\alpha)}(x;q)$ and $\tilde{V}_{n-1}^{(\alpha)}(x;q)$ in (3.25a) are positive constants and the result follows from Corollary 1.3 (a).

(b) Taking into consideration the restrictions on the parameters, $a(x)$ in (3.25b) is a positive constant and $b(x) = \frac{\alpha(1-q^n)}{q^{n-1}}\,(q^n x - \alpha\, q + 1)$ represents a linear function with positive values on $(1,\infty)$. The result follows from Corollary 1.3 (a).

□

Corollary 3.55. *For $i \in \{1,2,\ldots,n-1\}$,*

(a) $y_{n,i} < y_{n-1,i} < x_{n-1,i} < y_{n,i+1}$,

(b) $Y_{n,i} < y_{n,i} < x_{n,i} < x_{n-1,i} < Y_{n,i+1} < y_{n,i+1} < x_{n,i+1}$,

(c) $Y_{n,i} < Y_{n-1,i} < x_{n-1,i} < Y_{n,i+1}$.

Proof.

(a) The result follows from Theorem 3.54 (a) and the interlacing of the zeros of $\tilde{V}_n^{(\alpha q)}(x;q)$ and $\tilde{V}_{n-1}^{(\alpha q)}(x;q)$.

(b) By replacing α with αq in (3.25a), we obtain $Y_{n,i} < y_{n,i} < y_{n-1,i} < Y_{n,i+1} < y_{n,i+1}$. We combine this with the interlacing results in Theorem 3.54 (a) and (b) to obtain the required result.

(c) The result follows directly from Theorem 3.54 (b) and the interlacing of the zeros of $\tilde{V}_n^{(\alpha q^2)}(x;q)$ and $\tilde{V}_{n-1}^{(\alpha q^2)}(x;q)$.

□

Remark 3.56. *In general, the zeros of $\tilde{V}_n^{(\alpha)}(x;q)$ and $\tilde{V}_{n-1}^{(\alpha q)}(x;q)$ do not interlace. These polynomials satisfy*

$$\tilde{V}_{n-1}^{(\alpha q)}(x;q) = -q^{n-1}\tilde{V}_n^{(\alpha)}(x;q) + b(x)\tilde{V}_{n-1}^{(\alpha)}(x;q)$$

with $b(x) = q^{-1}\,(q^n x - \alpha\, q)$, a function that changes sign on $(1,\infty)$ for $0 < \alpha q < 1$. However, when we restrict α in such a way that $0 < \alpha q < q^n < 1$, the zeros interlace as follows: $x_{n,i} < y_{n-1,i} < x_{n-1,i} < x_{n,i+1}$ for each $i \in \{1,2,\ldots,n-1\}$.

3.10.2 Quasi-orthogonality

The Al-Salam-Carlitz II polynomials $\tilde{V}_n^{(\alpha)}(\bar{x};q)$ with $\bar{x}=q^{-x}$, are orthogonal with respect to the discrete weight $w(x)=\frac{q^{x^2}\alpha^x}{(q;q)_x(\alpha q;q)_x}$, when $0\leq \alpha q<1, \bar{x}\in(1,\infty)$, and satisfy

$$\tilde{V}_n^{\left(\frac{\alpha}{q}\right)}(x;q)=-\frac{(q^n x-\alpha)}{(\alpha-x)q^n}\tilde{V}_n^{(\alpha)}(x;q)-\frac{\alpha q\,(q^n-1)}{(\alpha-x)q^{2\,n}}\tilde{V}_{n-1}^{(\alpha)}(x;q).$$

The polynomial $\tilde{V}_n^{\left(\frac{\alpha}{q^k}\right)}(x;q), k<n$, is not quasi-orthogonal with respect to $w(x)$, on $(1,\infty)$, since it cannot be written as a linear combination of the polynomials $\tilde{V}_n^{(\alpha)}(x;q)$, $\tilde{V}_{n-1}^{(\alpha)}(x;q),\ldots,\tilde{V}_{n-k}^{(\alpha)}(x;q)$.

3.10.3 Bounds of the extreme zeros

The Al-Salam Carlitz II polynomials $\tilde{V}_n^{(\alpha)}(q^{-x};q)$ are orthogonal for $0<\alpha q<1$ on $(0,\infty)$ with respect to

$$w(x;\alpha)=\frac{q^{x^2}\alpha^x}{(q;q)_k(\alpha q;q)_k}\quad\text{and}\quad\frac{w(x;\alpha q^{-k})}{w(x;\alpha)}=\frac{\left(\frac{1}{\alpha}q^{-x};q\right)_k}{\left(\frac{1}{\alpha};q\right)_k}=c_k(q^{-x};\alpha)$$

is a polynomial of degree k in the variable q^{-x}. However, when we substitute α with $\alpha q^{-k}, k\in\{1,2,\ldots\}$, the condition $0<\alpha q<1$ is not satisfied. Therefore, finding inner bounds by using Theorem 1.11 is not possible for the Al-Salam Carlitz II polynomials $\tilde{V}_n^{(\alpha)}(q^{-x};q)$.

Chapter 4

Classical orthogonal polynomials on quadratic and q-quadratic lattices

The aim of this chapter is to study the quasi-orthogonality and the interlacing properties of the zeros of some families of classical orthogonal polynomials on quadratic and q-quadratic lattices. Starting from the hypergeometric representations of classical orthogonal polynomials, we get mixed recurrence equations they satisfied. In converse, if we have for example the three-term recurrence equations satisfied by a classical orthogonal polynomial family on quadratic or q-quadratic lattices, can we recover its hypergeometric representation? In this chapter, we implement an algorithm to identify classical orthogonal polynomials on a quadratic or a q-quadratic lattice from their recurrence relations (see [Tcheutia, 2019]).

4.1 Introduction

Foupouagnigni showed in [Foupouagnigni, 2008] that classical orthogonal polynomials on a quadratic or q-quadratic lattice satisfy a second-order divided difference equation of the form

$$\phi(x(s))\mathbb{D}_x^2 p_n(x(s)) + \psi(x(s))\mathbb{S}_x\mathbb{D}_x p_n(x(s)) + \lambda_n p_n(x(s)) = 0, \tag{4.1}$$

where $\phi(x) = ax^2 + bx + c$, $\psi(x) = dx + e$ $(d \neq 0)$, are polynomials of degree at most 2 and of degree one, respectively, the operators $\mathbb{D}_x$ and $\mathbb{S}_x$ are given by

$$\mathbb{D}_x f(x(s)) = \frac{f(x(s+\frac{1}{2})) - f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})}, \quad \mathbb{S}_x f(x(s)) = \frac{f(x(s+\frac{1}{2})) + f(x(s-\frac{1}{2}))}{2},$$

and $x(s)$ is a quadratic or q-quadratic lattice defined by [Magnus, 1995]

$$x(s) = \begin{cases} c_1 q^s + c_2 q^{-s} + c_3 \text{ if } 0 < q < 1, \\ c_4 s^2 + c_5 s + c_6 \quad \text{if } q = 1, \end{cases} \quad c_1, \ldots, c_6 \in \mathbb{C}.$$

Note that (4.1) is equivalent to a difference or q-difference equation of the form (see [Koekoek et al., 2010, chaps. 9, 14])

$$\lambda_n y(x(s)) = B(s)y(x(s+1)) - (B(s) + D(s))y(x(s)) + D(s)y(x(s-1)), \tag{4.2}$$

with

$$\phi(x(s)) = -\frac{1}{2}\Big(x(s+\frac{1}{2}) - x(s-\frac{1}{2})\Big)\Big((x(s+1)-x(s))B(s) + (x(s)-x(s-1))D(s)\Big),$$

$$\psi(x(s)) = (x(s)-x(s+1))B(s) + (x(s)-x(s-1))D(s).$$

Following the work by Foupouagnigni [2008], Njionou Sadjang et al. [2015b] proved that the Wilson and the continuous dual Hahn polynomials are solutions of a divided-difference equation of the form

$$\phi(x)\mathbf{D}_x^2 p_n(x) + \psi(x)\mathbf{S}_x\mathbf{D}_x p_n(x) + \lambda_n p_n(x) = 0, \tag{4.3}$$

where the operators $\mathbf{S}_x$ and the Wilson operator (see [Cooper, 2002], [Ismail and Stanton, 2012]) $\mathbf{D}_x$ are defined by

$$\mathbf{D}_x f(x) = \frac{f\Big(x+\frac{i}{2}\Big) - f\Big(x-\frac{i}{2}\Big)}{2ix}, \quad \mathbf{S}_x f(x) = \frac{f\Big(x+\frac{i}{2}\Big) + f\Big(x-\frac{i}{2}\Big)}{2}.$$

Using the same approach, Tcheutia et al. [2017] derived a divided–difference equation of type

$$\phi(x)\delta_x^2 p_n(x) + \psi(x)\mathbf{S}_x\delta_x p_n(x) + \lambda_n p_n(x) = 0, \tag{4.4}$$

satisfied by the continuous Hahn and the Meixner–Pollaczek polynomials, where the difference operator δ_x (see [Olver et al., 2010, p. 436], compare [Koekoek et al., 2010, p. 201 and 214], [Njionou Sadjang, 2013], [Njionou Sadjang et al., 2015a], [Tratnik, 1989, Eq. (1.15)]) is defined as follows:

$$\delta_x f(x) = \frac{f\Big(x+\frac{i}{2}\Big) - f\Big(x-\frac{i}{2}\Big)}{i}.$$

(4.3) and (4.4) are equivalent to the difference equation (see [Koekoek et al., 2010, chap. 9])

$$\lambda_n y(x) = B(x)y(x+i) - (B(x)+D(x))y(x) + D(x)y(x-i), \tag{4.5}$$

with

$$\phi(x) = x((2x+i)B(x) + (2x-i)D(x)),\ \psi(x) = -i((2x+i)B(x) - (2x-i)D(x)),$$

and

$$\phi(x) = \frac{1}{2}(B(x)+D(x)),\ \psi(x) = -i(B(x)-D(x)),$$

respectively.

The coefficients of the divided-difference equations given in the forms (4.1), (4.3) or (4.4) can be used for instance to compute the three–term recurrence relation or some structure formulae, from which one can derive the inversion coefficients of classical orthogonal polynomials on a quadratic and a q-quadratic lattice (see e. g. [Foupouagnigni et al., 2013], [Njionou Sadjang et al., 2015b], [Tcheutia, 2014], [Tcheutia et al., 2017] and references therein).

The hypergeometric and the basic hypergeometric representations of classical orthogonal polynomials on a quadratic or a q-quadratic lattice are given below (see [Koekoek

et al., 2010] for more details):
1. Askey-Wilson

$$p_n(x;a,b,c,d|q) = \frac{(ab,ac,ad;q)_n}{a^n} {}_4\phi_3\left(\begin{array}{c} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{array}\middle| q;q\right), \quad x=\cos\theta,$$

$$\text{with } p_n(x;a,b,c,d|q) = 2^n(abcdq^{n-1};q)_n \tilde{p}_n(x;a,b,c,d|q);$$

2. q-Racah

$$R_n(\mu(x);\alpha,\beta,\gamma,\delta|q) = {}_4\phi_3\left(\begin{array}{c} q^{-n}, \alpha\beta q^{n+1}, q^{-x}, \gamma\delta q^{x+1} \\ \alpha q, \beta\delta q, \gamma q \end{array}\middle| q;q\right), \quad n=0,1,\dots,N,$$

where

$$\mu(x) := q^{-x} + \gamma\delta q^{x+1}, \quad \alpha q = q^{-N} \text{ or } \beta\delta q = q^{-N} \text{ or } \gamma q = q^{-N},$$

with a nonnegative integer N and

$$R_n(\mu(x);\alpha,\beta,\gamma,\delta|q) = \frac{(\alpha\beta q^{n+1};q)_n}{(\alpha q,\beta\delta q,\gamma q;q)_n} \tilde{R}_n(\mu(x);\alpha,\beta,\gamma,\delta|q);$$

3. Continuous dual q-Hahn

$$p_n(x;a,b,c|q) = \frac{(ab,ac;q)_n}{a^n} {}_3\phi_2\left(\begin{array}{c} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac \end{array}\middle| q;q\right), \quad x=\cos\theta,$$

$$\text{with } p_n(x;a,b,c|q) = 2^n \tilde{p}_n(x;a,b,c|q);$$

4. Continuous q-Hahn

$$p_n(x;a,b,c,d;q) = \frac{(abe^{2i\hat{\theta}},ac,ad;q)_n}{(ae^{i\hat{\theta}})^n} {}_4\phi_3\left(\begin{array}{c} q^{-n}, abcdq^{n-1}, ae^{i(\theta+2\hat{\theta})}, ae^{-i\theta} \\ abe^{2i\hat{\theta}}, ac, ad \end{array}\middle| q;q\right),$$

$$x=\cos(\theta+\hat{\theta}), \text{ with } p_n(x;a,b,c,d;q) = 2^n(abcdq^{n-1};q)_n \tilde{p}_n(x;a,b,c,d;q);$$

5. Dual q-Hahn

$$R_n(\mu(x);\gamma,\delta,N|q) = {}_3\phi_2\left(\begin{array}{c} q^{-n}, q^{-x}, \gamma\delta q^{x+1} \\ \gamma q, q^{-N} \end{array}\middle| q;q\right), \quad n=0,1,\dots,N,$$

where $\mu(x) := q^{-x} + \gamma\delta q^{x+1}$, with $\tilde{R}_n(\mu(x);\gamma,\delta,N|q) = (\gamma q, q^{-N};q)_n R_n(\mu(x);\gamma,\delta,N|q)$;
6. Al-Salam-Chihara

$$Q_n(x;a,b|q) = \frac{(ab;q)_n}{a^n} {}_3\phi_2\left(\begin{array}{c} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0 \end{array}\middle| q;q\right), \quad x=\cos\theta,$$

with $Q_n(x;a,b|q) = 2^n\tilde{Q}_n(x;a,b|q)$;
7. q-Meixner-Pollaczek

$$P_n(x;a|q) = a^{-n}e^{-in\hat{\theta}}\frac{(a^2;q)_n}{(q;q)_n}{}_3\phi_2\left(\begin{matrix} q^{-n}, ae^{i(\theta+2\hat{\theta})}, ae^{-i\theta} \\ a^2, 0 \end{matrix}\middle| q;q\right),\ x = \cos(\theta+\hat{\theta}),$$

$$\text{with } P_n(x;a|q) = \frac{2^n}{(q;q)_n}\tilde{P}_n(x;a|q);$$

8. Continuous q-Jacobi

$$P_n^{(\alpha,\beta)}(x|q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}{}_4\phi_3\left(\begin{matrix} q^{-n}, q^{n+\alpha+\beta+1}, q^{\frac{\alpha}{2}+\frac{1}{4}}e^{i\theta}, q^{\frac{\alpha}{2}+\frac{1}{4}}e^{-i\theta} \\ q^{\alpha+1}, -q^{\frac{\alpha+\beta+1}{2}}, -q^{\frac{\alpha+\beta+2}{2}} \end{matrix}\middle| q;q\right),\ x = \cos\theta,$$

$$\text{with } P_n^{(\alpha,\beta)}(x|q) = \frac{2^n q^{(\frac{\alpha}{2}+\frac{1}{4})n}(q^{n+\alpha+\beta+1};q)_n}{(q, -q^{\frac{1}{2}(\alpha+\beta+1)}, -q^{\frac{1}{2}(\alpha+\beta+2)};q)_n}\tilde{P}_n^{(\alpha,\beta)}(x|q);$$

9. Continuous q-Ultraspherical / Rogers

$$C_n(x;\beta|q) = \frac{(\beta^2;q)_n}{(q;q)_n}\beta^{-\frac{n}{2}}{}_4\phi_3\left(\begin{matrix} q^{-n}, \beta^2q^n, \beta^{\frac{1}{2}}e^{i\theta}, \beta^{\frac{1}{2}}e^{-i\theta} \\ \beta q^{\frac{1}{2}}, -\beta, -\beta q^{\frac{1}{2}} \end{matrix}\middle| q;q\right),\ x = \cos\theta,$$

$$\text{with } C_n(x;\beta|q) = \frac{2^n(\beta;q)_n}{(q;q)_n}\tilde{C}_n(x;\beta|q);$$

10. Dual q-Krawtchouk

$$K_n(\lambda(x);c,N|q) = {}_3\phi_2\left(\begin{matrix} q^{-n}, q^{-x}, cq^{x-N} \\ q^{-N}, 0 \end{matrix}\middle| q;q\right),\ n = 0,1,\ldots,N,$$

where $\lambda(x) := q^{-x} + cq^{x-N}$, with $\tilde{K}_n(\lambda(x);c,N|q) = (q^{-N};q)_nK_n(\lambda(x);c,N|q)$;
11. Continuous big q-Hermite

$$H_n(x;a|q) = a^{-n}{}_3\phi_2\left(\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ 0, 0 \end{matrix}\middle| q;q\right),\ x = \cos\theta,\ \text{with } H_n(x;a|q) = 2^n\tilde{H}_n(x;a|q);$$

12. Continuous q-Laguerre

$$P_n^{(\alpha)}(x|q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}{}_3\phi_2\left(\begin{matrix} q^{-n}, q^{\frac{\alpha}{2}+\frac{1}{4}}e^{i\theta}, q^{\frac{\alpha}{2}+\frac{1}{4}}e^{-i\theta} \\ q^{\alpha+1}, 0 \end{matrix}\middle| q;q\right),\ x = \cos\theta,$$

$$\text{with } P_n^{(\alpha)}(x|q) = \frac{2^nq^{(\frac{\alpha}{2}+\frac{1}{4})n}}{(q;q)_n}\tilde{P}_n^{(\alpha)}(x|q);$$

13. Wilson

$$W_n(x^2;a,b,c,d)=(a+b)_n(a+c)_n(a+d)_n\,{}_4F_3\left(\left.\begin{matrix}-n,n+a+b+c+d-1,a+ix,a-ix\\ a+b,a+c,a+d\end{matrix}\right|1\right),$$

$$\text{with } W_n(x^2;a,b,c,d)=(-1)^n(n+a+b+c+d-1)_n\tilde{W}_n(x^2;a,b,c,d);$$

14. Racah

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta)={}_4F_3\left(\left.\begin{matrix}-n,n+\alpha+\beta+1,-x,x+\gamma+\delta+1\\ \alpha+1,\beta+\delta+1,\gamma+1\end{matrix}\right|1\right),\ n=0,1,\dots,N,$$

where

$$\lambda(x)=x(x+\gamma+\delta+1),\ \ \alpha+1=-N \text{ or } \beta+\delta+1=-N \text{ or } \gamma+1=-N$$

with a nonnegative integer N, and

$$R_n(\lambda(x);\alpha,\beta,\gamma,\delta)=\frac{(n+\alpha+\beta+1)_n}{(\alpha+1)_n(\beta+\delta+1)_n(\gamma+1)_n}\tilde{R}_n(\lambda(x);\alpha,\beta,\gamma,\delta);$$

15. Continuous dual Hahn

$$S_n(x^2;a,b,c)=(a+b)_n(a+c)_n\,{}_3F_2\left(\left.\begin{matrix}-n,a+ix,a-ix\\ a+b,a+c\end{matrix}\right|1\right),$$

with $S_n(x^2;a,b,c)=(-1)^n\tilde{S}_n(x^2;a,b,c)$;

16. Continuous Hahn

$$p_n(x;a,b,c,d)=i^n\frac{(a+c)_n(a+d)_n}{n!}\,{}_3F_2\left(\left.\begin{matrix}-n,n+a+b+c+d-1,a+ix\\ a+c,a+d\end{matrix}\right|1\right),$$

$$\text{with } \ p_n(x;a,b,c,d)=\frac{(n+a+b+c+d-1)_n}{n!}\tilde{p}_n(x;a,b,c,d);$$

17. Dual Hahn

$$R_n(\lambda(x);\gamma,\delta,N)={}_3F_2\left(\left.\begin{matrix}-n,-x,x+\gamma+\delta+1\\ \gamma+1,-N\end{matrix}\right|1\right),\ n=0,1,\dots,N,$$

where $\lambda(x)=x(x+\gamma+\delta+1)$, $\tilde{R}_n(\lambda(x);\gamma,\delta,N)=(\gamma+1)_n(-N)_nR_n(\lambda(x);\gamma,\delta,N)$;

18. Meixner-Pollaczek

$$P_n^{(\lambda)}(x;\theta)=\frac{(2\lambda)_n}{n!}e^{in\theta}\,{}_2F_1\left(\left.\begin{matrix}-n,\lambda+ix\\ 2\lambda\end{matrix}\right|1-e^{-2i\theta}\right),\ P_n^{(\lambda)}(x;\theta)=\frac{(2\sin\theta)^n}{n!}\tilde{P}_n^{(\lambda)}(x;\theta).$$

4.2 Interlacing properties of the zeros of some families of classical orthogonal polynomials on quadratic and q-quadratic lattices

4.2.1 The Wilson polynomials

The Wilson polynomials $\tilde{W}_n(x^2;a,b,c,d)$ are orthogonal on $(0,\infty)$ with respect to

$$w(x;a,b,c,d) = \left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right|^2, \tag{4.6}$$

for $Re(a,b,c,d) > 0$ and non-real parameters occur in conjugate pairs. Furthermore, the weight function is clearly independent of the order in which the parameters a,b,c and d occur. By shifting b to $b+1$, c to $c+1$ or d to $d+1$, the same interlacing results are obtained as by shifting a to $a+1$. We note that the polynomial $W_n(x^2;a,b,c,d)$ has n zeros in $(0,\infty)$, namely $x^2_{n,1}, x^2_{n,2}, \ldots, x^2_{n,n}$. Let $\tilde{W}_n(x^2) = \tilde{W}_n(x^2;a,b,c,d)$.

Proposition 4.1.

$$\tilde{W}_n(x^2;a,b,c,d) = \tilde{W}_n(x^2;a+1,b,c,d) \tag{4.7a}$$
$$+\frac{n(c+d+n-1)(b+d+n-1)(b+c+n-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)}\tilde{W}_{n-1}(x^2;a+1,b,c,d);$$

$$\tilde{W}_n(x^2;a,b+1,c,d) = \tilde{W}_n(x^2;a+1,b,c,d) \tag{4.7b}$$
$$+\frac{n(b-a)(c+d+n-1)}{2n+a+b+c+d-1}\tilde{W}_{n-1}(x^2;a+1,b+1,c,d).$$

Theorem 4.2. *Suppose $a,b,c,d>0$. Denote the zeros of $\tilde{W}_n(x^2;a,b,c,d)$ by $x^2_{n,1} < x^2_{n,2} < \cdots < x^2_{n,n}$, the zeros of $\tilde{W}_n(x^2;a+1,b,c,d)$ by $x^{(a)^2}_{n,1} < x^{(a)^2}_{n,2} < \cdots < x^{(a)^2}_{n,n}$, the zeros of $\tilde{W}_n(x^2;a,b+1,c,d)$ by $x^{(b)^2}_{n,1} < x^{(b)^2}_{n,2} < \cdots < x^{(b)^2}_{n,n}$, the zeros of $\tilde{W}_n(x^2;a+1,b+1,c,d)$ by $x^{(a,b)^2}_{n,1} < x^{(a,b)^2}_{n,2} < \cdots < x^{(a,b)^2}_{n,n}$. Then,*

(a) $x^2_{n,i} < x^{(a)^2}_{n,i} < x^{(a)^2}_{n-1,i} < x^2_{n,i+1} < x^{(a)^2}_{n,i+1}$;

(b) if $b-a>0$, $x^{(b)^2}_{n,i} < x^{(a)^2}_{n,i} < x^{(a,b)^2}_{n-1,i} < x^{(b)^2}_{n,i+1} < x^{(a)^2}_{n,i+1}$, *and*
if $b-a<0$, $x^{(a)^2}_{n,i} < x^{(b)^2}_{n,i} < x^{(a,b)^2}_{n-1,i} < x^{(a)^2}_{n,i+1} < x^{(b)^2}_{n,i+1}$.

Proof. Suppose a,b,c,d are positive real numbers.

(a) Since $\tilde{W}_n(x^2;a,b,c,d)$ and $\tilde{W}_{n-1}(x^2;a,b,c,d)$ belong to the same orthogonal sequence, their zeros interlace and (4.7a) is in the form of (1.2), with $a(x)=1$ and $b(x)>0$. The result follows from Corollary 1.3 (a).

(b) Equation (4.7b) is in the form of (1.2), with $a(x)=1$, $b(x)<0$ if $b-a<0$ and the result follows from Corollary 1.3 (b), $b(x)>0$ if $b-a>0$ and the result follows from Corollary 1.3 (a).

□

4.2.2 The Racah polynomials

The Racah polynomials $\tilde{R}_n(\lambda(x);\alpha,\beta,\gamma,\delta)$, $n\in\{0,1,2,\ldots,N\}$, with $\lambda(x)=x(x+\gamma+\delta+1)$, are orthogonal on $(0,N)$ with respect to the weight function

$$w(x)=\frac{(\alpha+1)_x(\beta+\delta+1)_x(\gamma+1)_x(\gamma+\delta+1)_x((\gamma+\delta+3)/2)_x}{(-\alpha+\gamma+\delta+1)_x(-b+\gamma+1)_x(\delta+1)_x((\gamma+\delta+1)/2)_x}$$

if $\alpha+1=-N$ or $\beta+\delta+1=-N$ or $\gamma+1=-N$ with N a nonnegative integer. Since shifting γ or δ will change $\lambda(x)$, we will only consider shifts in α and β.

Proposition 4.3.

$$\begin{aligned}\tilde{R}_n(\lambda(x);\alpha,\beta,\gamma,\delta)&=\tilde{R}_n(\lambda(x);\alpha+1,\beta,\gamma,\delta) &(4.8a)\\&-\frac{(\beta+n)\,(\beta+\delta+n)\,(\gamma+n)\,n}{(2\,n+\alpha+\beta+1)\,(2\,n+\alpha+\beta)}\tilde{R}_{n-1}(\lambda(x);\alpha+1,\beta,\gamma,\delta);\\ \tilde{R}_n(\lambda(x);\alpha,\beta,\gamma,\delta)&=\tilde{R}_n(\lambda(x);\alpha,\beta+1,\gamma,\delta) &(4.8b)\\&-\frac{(\alpha+n)\,(\alpha-\delta+n)\,(\gamma+n)\,n}{(2\,n+\alpha+\beta+1)\,(2\,n+\alpha+\beta)}\tilde{R}_{n-1}(\lambda(x);\alpha,\beta+1,\gamma,\delta);\\ \tilde{R}_n(\lambda(x);\alpha,\beta+1,\gamma,\delta)&=\tilde{R}_n(\lambda(x);\alpha+1,\beta,\gamma,\delta) &(4.8c)\\&+\frac{n\,(\gamma+n)\,(\alpha-\beta-\delta)}{2\,n+\alpha+\beta+1}\tilde{R}_{n-1}(\lambda(x);\alpha+1,\beta+1,\gamma,\delta).\end{aligned}$$

From the latter equations, we can make some assumptions on the parameters α,β,γ to derive interlacing properties of the zeros of the Racah polynomials.

4.2.3 The continuous Hahn polynomials

The continuous Hahn polynomials $\tilde{P}_n(x;a,b,c,d)$ are orthogonal on $\mathbb{R}$ with respect to $w(x)=\Gamma(a+ix)\Gamma(b+ix)\Gamma(c-ix)\Gamma(d-ix)$ for $Re(a,b,c,d)>0$, $c=\bar{a}$ and $d=\bar{b}$. We have for this family equations like

$$\begin{aligned}\tilde{P}_n(x;a,b,c,d)&=\tilde{P}_n(x;a+1,b,c,d)\\&+\frac{ni\,(b+c+n-1)\,(b+d+n-1)}{(2\,n+a+b+c+d-2)\,(2\,n+a+b+c+d-1)}\tilde{P}_{n-1}(x;a+1,b,c,d);\end{aligned}$$

$$\begin{aligned}\tilde{P}_n(x;a,b+1,c,d)&=\tilde{P}_n(x;a+1,b,c,d)\\&+\frac{ni\,(b-a)}{2\,n+a+b+c+d-1}\tilde{P}_{n-1}(x;a+1,b+1,c,d).\end{aligned}$$

Like in the latter equations, there is the complex i appearing in the second term of the right-hand sides of the mixed recurrence equations satisfied by the continuous Hahn polynomials. Therefore, we can not apply our method to deduce the interlacing properties of the zeros of $\tilde{P}_n(x;a,b,c,d)$.

4.2.4 The Askey-Wilson polynomials

The weight function of the Askey-Wilson polynomials

$$w(x;a,b,c,d|q)=\frac{1}{\sqrt{1-x^2}}\left|\frac{(e^{2i\theta};q)_\infty}{(ae^{i\theta},be^{i\theta},ce^{i\theta},de^{i\theta};q)_\infty}\right|^2, \tag{4.9}$$

is independent of the order of the parameters a,b,c and d and by shifting b to bq, c to cq or d to dq, we obtain the same interlacing results as by shifting a to aq.

Proposition 4.4.

$$\begin{aligned}\tilde{p}_n(x;a,b,c,d|q)=\tilde{p}_n(x;aq,b,c,d|q)&\qquad\qquad (4.10a)\\ -\frac{a(1-q^n)(1-cdq^{n-1})(1-bdq^{n-1})(1-bcq^{n-1})}{2(1-abcdq^{2n-1})(1-abcdq^{2n-2})}&\tilde{p}_{n-1}(x;aq,b,c,d|q),\\ \tilde{p}_n(x;a,bq,c,d|q)=\tilde{p}_n(x;aq,b,c,d|q)+\frac{(b-a)(1-q^n)(1-cdq^{n-1})}{2(1-abcdq^{2n-1})}&\tilde{p}_{n-1}(x;aq,bq,c,d|q).\\ &\qquad\qquad (4.10b)\end{aligned}$$

Theorem 4.5. *Suppose a,b,c,d are real and $\max(|a|,|b|,|c|,|d|)<1$. Denote the zeros of $\tilde{p}_n(x;a,b,c,d|q)$ by $-1<x_{n,1}<x_{n,2}<\cdots<x_{n,n}<1$, the zeros of $\tilde{p}_n(x;aq,b,c,d|q)$ by $-1<x^{(a)}_{n,1}<x^{(a)}_{n,2}<\cdots<x^{(a)}_{n,n}<1$, the zeros of $\tilde{p}_n(x;a,bq,c,d|q)$ by $-1<x^{(b)}_{n,1}<x^{(b)}_{n,2}<\cdots<x^{(b)}_{n,n}<1$, the zeros of $\tilde{p}_n(x;aq,bq,c,d|q)$ by $-1<x^{(a,b)}_{n,1}<x^{(a,b)}_{n,2}<\cdots<x^{(a,b)}_{n,n}<1$. Then,*

(a) if $-1<a<0$, $x_{n,i}<x^{(a)}_{n,i}<x^{(a)}_{n-1,i}<x_{n,i+1}<x^{(a)}_{n,i+1}$, and
if $0<a<1$, $x^{(a)}_{n,i}<x_{n,i}<x^{(a)}_{n-1,i}<x^{(a)}_{n,i+1}<x_{n,i+1}$;

(b) if $b-a>0$, $x^{(b)}_{n,i}<x^{(a)}_{n,i}<x^{(a,b)}_{n-1,i}<x^{(b)}_{n,i+1}<x^{(a)}_{n,i+1}$, and
if $b-a<0$, $x^{(a)}_{n,i}<x^{(b)}_{n,i}<x^{(a,b)}_{n-1,i}<x^{(a)}_{n,i+1}<x^{(b)}_{n,i+1}$.

Proof. Suppose a,b,c,d are real and $\max(|a|,|b|,|c|,|d|)<1$. Then $\max(|ac|,|ad|,|bc|,|bd|,|cd|,|abcd|)<1$ and, for $n\in\mathbb{N}$, $1-acq^n>0$, $1-bcq^n>0$, $1-bdq^n>0$, $1-cdq^n>0$ and $1-abcdq^n>0$. Since $\tilde{p}_n(x;aq,b,c,d|q)$ and $\tilde{p}_{n-1}(x;aq,b,c,d|q)$ belong to the same orthogonal sequence, their zeros interlace and (4.10a) is in the form of (1.2), with $a(x)=1$ and

(a) $b(x)>0$ if $-1<a<0$ and the result follows from Corollary 1.3 (a) and $b(x)<0$ if $0<a<1$ and the result follows from Corollary 1.3 (b).

(b) Since by shifting b to bq, we obtain the same interlacing results as by shifting a to aq and we have $x_{n,i}<x^{(b)}_{n,i}<x^{(b)}_{n-1,i}<x_{n,i+1}<x^{(b)}_{n,i+1}$ if $-1<b<0$, and $x^{(b)}_{n,i}<x_{n,i}<x^{(b)}_{n-1,i}<x^{(b)}_{n,i+1}<x_{n,i+1}$ if $0<b<1$. By replacing a by aq, it follows that $x^{(a)}_{n,i}<x^{(a,b)}_{n-1,i}<x^{(a)}_{n,i+1}$ for each $i\in\{1,2,\ldots,n-1\}$. Equation (4.10b) is in the form of (1.2), with $a(x)=1$, $b(x)<0$ if $b-a<0$ and the result follows from Corollary 1.3 (b), $b(x)>0$ if $b-a>0$ and the result follows from Corollary 1.3 (a).

∎

The following result follows directly:

Corollary 4.6. *For $i \in \{1,2,\dots,n-1\}$,*

(a) if $-1 < a < 0$ and $0 < b < 1$, $x^{(b)}_{n,i} < x_{n,i} < x^{(a)}_{n,i} < x^{(a,b)}_{n-1,i} < x^{(b)}_{n,i+1} < x_{n,i+1} < x^{(a)}_{n,i+1}$;

(b) if $-1 < b < 0$ and $0 < a < 1$, $x^{(a)}_{n,i} < x_{n,i} < x^{(b)}_{n,i} < x^{(a,b)}_{n-1,i} < x^{(a)}_{n,i+1} < x_{n,i+1} < x^{(b)}_{n,i+1}$.

Remark 4.7. *The following systems of polynomials follow from the Askey-Wilson polynomials:*

(i) By setting $d = 0$, we obtain the monic continuous dual q-Hahn polynomials $\tilde{p}_n(x;a,b,c|q)$, $x = \cos\theta$, orthogonal on $(-1,1)$ with respect to $w(x;a,b,c,0|q)$ in (4.9) where a,b,c are real and $\max(|a|,|b|,|c|) < 1$;

(ii) By setting $c = d = 0$, we obtain the monic Al-Salam Chihara polynomials $\tilde{Q}_n(x;a,b|q)$, $x = \cos\theta$, orthogonal on $(-1,1)$ with respect to $w(x;a,b,0,0|q)$ in (4.9) where a,b are real and $\max(|a|,|b|) < 1$;

(iii) By setting $b = c = d = 0$, we obtain the monic continuous big q-Hermite polynomials $\tilde{H}_n(x;a|q)$, $x = \cos\theta$, orthogonal on $(-1,1)$ with respect to $w(x;a,0,0,0|q)$ in (4.9) where a is real and $|a| < 1$.

(iv) By the substitutions $\theta \to \theta + \phi$, $a \to ae^{i\phi}$, $b \to be^{i\phi}$, $c \to ce^{-i\phi}$ and $d \to de^{-i\phi}$ we obtain the monic continuous q-Hahn polynomials $\tilde{p}_n(x;a,b,c,d;q)$, $x = \cos(\theta+\phi)$, orthogonal on $(-\pi,\pi)$ with respect to

$$w(\cos(\theta+\phi);a,b,c,d|q) = \left|\frac{(e^{2i(\theta+\phi)};q)_\infty}{(ae^{i(\theta+\phi)}, be^{i(\theta+\phi)}, ce^{i\theta}, de^{i\theta};q)_\infty}\right|^2,$$

if $c = a$ and $d = b$ and, if a and b are real and $\max(|a|,|b|) < 1$, or if $b = \overline{a}$ and $|a| < 1$. Using the above substitution in (4.10a), we obtain

$$\begin{aligned}&\tilde{p}_n(x;a,b,c,d;q) = \tilde{p}_n(x;aq,b,c,d;q)\\ &- \frac{a(1-q^n)(e^{i\phi} - cdq^{n-1}e^{-i\phi})(1-bdq^{n-1})(1-bcq^{n-1})}{2(1-abcdq^{2n-1})(1-abcdq^{2n-2})}\tilde{p}_{n-1}(x;aq,b,c,d;q)\end{aligned}$$

and we can not apply our method to deduce the interlacing properties of the zeros of $\tilde{p}_n(x;a,b,c,d;q)$, since it is not possible to determine if $e^{i\phi} - cdq^{n-1}e^{-i\phi}$ is positive or negative.

Corollary 4.8. *Suppose a,b,c,d are real and $\max(|a|,|b|,|c|,|d|) < 1$. Then for each of the systems in (i) - (iii) above, we have, for $i \in \{1,2,\dots,n-1\}$,*

(a) $x_{n,i} < y_{n,i} < y_{n-1,i} < x_{n,i+1} < y_{n,i+1}$ if $-1 < a < 0$;

(b) $y_{n,i} < x_{n,i} < y_{n-1,i} < y_{n,i+1} < x_{n,i+1}$ if $0 < a < 1$,

where $-1 < x_{n,1} < x_{n,2} < \cdots < x_{n,n} < 1$ are the zeros of the polynomial $\tilde{p}_n(x;a,b,c|q)$ in (i) ($\tilde{Q}_n(x;a,b|q)$, $\tilde{H}_n(x;a|q)$), and $-1 < y_{n,1} < y_{n,2} < \cdots < y_{n,n} < 1$ are the zeros of the polynomial with a shifted to aq, i.e., $\tilde{p}_n(x;aq,b,c|q)$ ($\tilde{Q}_n(x;aq,b|q)$, $\tilde{H}_n(x;aq|q)$).

4.2.5 The q-Racah polynomials

The q-Racah polynomials are orthogonal on $(0, N)$ if $\alpha q = q^{-N}$ or $\beta\delta q = q^{-N}$ or $\gamma q = q^{-N}$, and N a nonnegative integer. In order to prove interlacing results, we make some assumptions on the parameters.

Proposition 4.9.

$$\begin{aligned}\tilde{R}_n(\mu(x);\alpha,\beta,\gamma,\delta|q) = {} & \tilde{R}_n(\mu(x);\alpha q,\beta,\gamma,\delta|q) \\ & - \frac{\alpha q(1-q^n)(1-\beta q^n)(1-\gamma q^n)(1-\beta\delta q^n)}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+1})}\tilde{R}_{n-1}(\mu(x);\alpha q,\beta,\gamma,\delta|q);\end{aligned} \tag{4.11a}$$

$$\begin{aligned}\tilde{R}_n(\mu(x);\alpha,\beta,\gamma,\delta|q) = {} & \tilde{R}_n(\mu(x);\alpha,\beta q,\gamma,\delta|q) \\ & + \frac{\beta q(1-q^n)(1-\alpha q^n)(1-\gamma q^n)(\alpha q^n-\delta)}{(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+1})}\tilde{R}_{n-1}(\mu(x);\alpha,\beta q,\gamma,\delta|q).\end{aligned} \tag{4.11b}$$

Theorem 4.10. *We denote the zeros of $\tilde{R}_n(\mu(x);\alpha,\beta,\gamma,\delta|q)$ by $\mu(0) < \mu_{n,1} < \mu_{n,2} < \cdots < \mu_{n,n} < \mu(N)$, the zeros of $\tilde{R}_n(\mu(x);\alpha q,\beta,\gamma,\delta|q)$ by $\mu(0) < \mu^{(\alpha)}_{n,1} < \mu^{(\alpha)}_{n,2} < \cdots < \mu^{(\alpha)}_{n,n} < \mu(N)$ and the zeros of $\tilde{R}_n(\mu(x);\alpha,\beta q,\gamma,\delta|q)$ by $\mu(0) < \mu^{(\beta)}_{n,1} < \mu^{(\beta)}_{n,2} < \cdots < \mu^{(\beta)}_{n,n} < \mu(N)$ and we assume that $(1-\alpha\beta q^{2n})(1-\alpha\beta q^{2n+1}) > 0$, $\gamma q < 1$ and $0 < \delta q < 1$.*

(a) Let $\alpha q = q^{-N} > 1$. If $\beta q < 1$ and $\beta\delta q < 1$, then, for $i \in \{1,2,\ldots,n-1\}$, $\mu^{(\alpha)}_{n,i} < \mu_{n,i} < \mu^{(\alpha)}_{n-1,i} < \mu^{(\alpha)}_{n,i+1} < \mu_{n,i+1}$;

(b) Let $\beta\delta q = q^{-N} > 1$. If $\alpha q < 1$ and $\alpha q^n < \delta$, then, for $i \in \{1,2,\ldots,n-1\}$, $\mu^{(\beta)}_{n,i} < \mu_{n,i} < \mu^{(\beta)}_{n-1,i} < \mu^{(\beta)}_{n,i+1} < \mu_{n,i+1}$.

Proof. The polynomials on the right-hand side of both equations (4.11a) and (4.11b) belong to the same orthogonal sequences, their zeros interlace and these equations are both in the form of (1.2), with $a(x) = 1$.

(a) Let $\alpha q = q^{-N} > 1$ and assume that $\beta q < 1$ and $\beta\delta q < 1$. Then $1 - \beta\delta q^n > 0$, i.e., $b(x) < 0$ and the result follows from Corollary 1.3 (b);

(b) Let $\beta\delta q = q^{-N} > 1$ and assume that $\alpha q < 1$ and $\alpha q^n < \delta$. Then $b(x)$ is a negative constant and the result follows from Corollary 1.3 (b).

□

Remark 4.11. *When we take $\beta = 0, \gamma q = q^{-N}$ and $\delta \to \alpha\delta q^{N+1}$ in the definition of the q-Racah polynomials, we obtain the monic dual q-Hahn polynomials, i.e.,*

$$\tilde{R}_n(\mu(x);\alpha,0,q^{-N-1},\alpha\delta q^{N+1}|q) = \tilde{R}_n(\mu(x);\alpha,\delta,N|q), n \in \{0,1,\ldots,N\},$$

with $\mu(x) = q^{-x} + \alpha\delta q^{x+1}$, and (4.11a) becomes

$$\tilde{R}_n(\mu(x);\alpha,\delta,N|q) = \tilde{R}_n(\mu(x);\alpha q,\delta,N|q) - \alpha q(1-q^n)(1-q^{n-N-1})\tilde{R}_{n-1}(\mu(x);\alpha q,\delta,N|q),$$

with $0 < \alpha q < 1$ and $0 < \delta q < 1$, and, since $-\alpha q(1-q^n)(1-q^{n-N-1}) > 0$, the zeros interlace as follows:

$$\mu_{n,i} < \mu^{(\alpha)}_{n,i} < \mu^{(\alpha)}_{n-1,i} < \mu_{n,i+1} < \mu^{(\alpha)}_{n,i+1}, i \in \{1,2,\ldots,n-1\},$$

where $\mu_{n,i}$ are the zeros of $\tilde{R}_n(\mu(x);\alpha,\delta,N|q)$ and $\mu^{(\alpha)}_{n,i}, i \in \{1,2,\ldots,n\}$, the zeros of $\tilde{R}_n(\mu(x);\alpha q,\delta,N|q)$.

For the monic dual q-Hahn polynomials we also get the following results.

Proposition 4.12.

$$\tilde{R}_n(\mu(x);\alpha,\delta q,N|q) = \tilde{R}_n(\mu(x);\alpha q,\delta,N|q) + \alpha q^{-N}(1-q^n)(q^n-q^{N+1})\tilde{R}_{n-1}(\mu(x);\alpha q,\delta,N|q); \tag{4.12a}$$

$$\tilde{R}_n(\mu(x);\alpha,\delta q,N|q) = \frac{1-\alpha\delta\, q^{x+N+3-n}}{1-\alpha\delta q^{x+N+3}}\tilde{R}_n(\mu(x);\alpha q,\delta q,N|q) + \frac{\alpha\,(1-q^n)\left(q^n-q^{N+1}\right)\left(1-\delta\, q^{N+2-n}\right)}{q^N(1-\alpha\,\delta q^{x+N+3})}\tilde{R}_{n-1}(\mu(x);\alpha q,\delta q,N|q). \tag{4.12b}$$

Theorem 4.13. *Let $n\in\{0,1,\ldots,N\}$, $0<\alpha q<1$ and $0<\delta q<1$, and denote the zeros of $\tilde{R}_n(\mu(x);\alpha,\delta q,N|q)$ by $\mu(0)<\mu_{n,1}^{(\delta)}<\mu_{n,2}^{(\delta)}<\cdots<\mu_{n,n}^{(\delta)}<\mu(N)$, the zeros of $\tilde{R}_n(\mu(x);\alpha q,\delta,N|q)$ by $\mu(0)<\mu_{n,1}^{(\alpha)}<\mu_{n,2}^{(\alpha)}<\cdots<\mu_{n,n}^{(\alpha)}<\mu(N)$ and the zeros of $\tilde{R}_n(\mu(x);\alpha q,\delta q,N|q)$ by $\mu(0)<\mu_{n,1}^{(\alpha,\delta)}<\mu_{n,2}^{(\alpha,\delta)}<\cdots<\mu_{n,n}^{(\alpha,\delta)}<\mu(N)$. Then, for $i\in\{1,2,\cdots,n-1\}$,*

(a) $\mu_{n,i}^{(\delta)}<\mu_{n,i}^{(\alpha)}<\mu_{n-1,i}^{(\alpha)}<\mu_{n,i+1}^{(\delta)}<\mu_{n,i+1}^{(\alpha)}$,

(b) $\mu_{n,i}^{(\delta)}<\mu_{n,i}^{(\alpha,\delta)}<\mu_{n-1,i}^{(\alpha,\delta)}<\mu_{n,i+1}^{(\delta)}<\mu_{n,i+1}^{(\alpha,\delta)}$.

Proof. Let $0<\alpha q<1$ and $0<\delta q<1$. Then $1-\alpha\delta q^j>0$, $1-\alpha q^j>0$, $1-\delta q^j>0$ if $j>0$ and equations (4.12a)–(4.12b) are in the form of (1.2) and under the given assumptions, the results follows from Corollary 1.3 (a). □

4.3 Quasi-orthogonal polynomials on quadratic and q-quadratic lattices

4.3.1 The Wilson polynomials

Proposition 4.14.

$$\tilde{W}_n(x^2;a-1,b,c,d) = \tilde{W}_n(x^2)+\frac{n(c+d+n-1)(b+d+n-1)(b+c+n-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-3)}\tilde{W}_{n-1}(x^2); \tag{4.13a}$$

$$\tilde{W}_n(x^2;a,b-1,c,d) = \tilde{W}_n(x^2)+\frac{n(c+d+n-1)(a+d+n-1)(a+c+n-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-3)}\tilde{W}_{n-1}(x^2); \tag{4.13b}$$

$$\tilde{W}_n(x^2;a,b,c-1,d) = \tilde{W}_n(x^2)+\frac{n(a+d+n-1)(b+d+n-1)(a+b+n-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-3)}\tilde{W}_{n-1}(x^2); \tag{4.13c}$$

$$\tilde{W}_n(x^2;a,b,c,d-1) = \tilde{W}_n(x^2)+\frac{n(a+c+n-1)(b+c+n-1)(a+b+n-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-3)}\tilde{W}_{n-1}(x^2). \tag{4.13d}$$

Theorem 4.15. *Let a,b,c and d be such that $Re(a,b,c,d)>0$. Consider $k_1,k_2,k_3,k_4\in\{0,1,\ldots,n-1\}$, such that $k_1+k_2+k_3+k_4\le n-1$. The sequence $\{\tilde{W}_n(x^2;a-k_1,b-k_2,c-k_3,d-k_4)\}_{n\ge0}$, with $0<Re(a)<1$ (if $k_1\neq0$), $0<Re(b)<1$ (if $k_2\neq0$),*

$0 < Re(c) < 1$ (if $k_3 \neq 0$) and $0 < Re(d) < 1$ (if $k_4 \neq 0$), is quasi-orthogonal of order $k_1 + k_2 + k_3 + k_4 \leq n-1$ with respect to the weight $w(x)$ on $(0, \infty)$ and the polynomials have at least $n - (k_1 + k_2 + k_3 + k_4)$ real, distinct zeros in $(0, \infty)$.

Proof. Fix a such that $0 < Re(a) < 1$. From Lemma 1.4 and (4.13a), it follows that $\tilde{W}_n(x^2; a-1, b, c, d)$ is quasi-orthogonal of order one on $(0, \infty)$. By iteration, we can express $\tilde{W}_n(x^2; a-k, b, c, d)$ as a linear combination of $\tilde{W}_n(x^2; a, b, c, d)$, $\tilde{W}_{n-1}(x^2; a, b, c, d)$, $\ldots, \tilde{W}_{n-k}(x^2; a, b, c, d)$ and from Lemma 1.4 it follows that $\tilde{W}_n(x^2; a-k, b, c, d), 0 < Re(a) < 1$, is quasi-orthogonal of order $k \leq n-1$ on $(0, \infty)$. Furthermore, from Lemma 1.6 we know that at least $(n-k)$ real, distinct zeros of $\tilde{W}_n(x^2; a-k, b, c, d)$, $k \in \{1, 2, \ldots, n-1\}$, lie in $(0, \infty)$, i.e., at least $(n-k)$ of the zeros $(x_{n,1})^2, (x_{n,2})^2, \ldots, (x_{n,n})^2$, lie in $(0, \infty)$.

When we fix the parameter b, (or c, d) such that $0 < Re(b) < 1$ (or $0 < Re(c) < 1, 0 < Re(d) < 1$), we can prove in the same way, using (4.13b) (or (4.13c), (4.13d)), that the polynomial $\tilde{W}_n(x^2; a, b-k, c, d)$ (alternatively $\tilde{W}_n(x^2; a, b, c-k, d)$, or $\tilde{W}_n(x^2; a, b, c, d-k)$) is quasi-orthogonal of order k on $(0, \infty)$. Using an iteration process, we can write $\tilde{W}_n(x^2; a-k_1, b-k_2, c-k_3, d-k_4)$ with $0 < Re(a) < 1$ (if $k_1 \neq 0$), $0 < Re(b) < 1$ (if $k_2 \neq 0$), $0 < Re(c) < 1$ (if $k_3 \neq 0$) and $0 < Re(d) < 1$ (if $k_4 \neq 0$), in the form of (1.5) and the results follow from Lemmas 1.4 and 1.6. □

Theorem 4.16. *Consider a, b, c, d, such that $Re(b, c, d) > 0$, $0 < Re(a) < 1$ and non-real parameters occur in conjugate pairs. Let $x^2_{n,i}, i \in \{1, 2, \ldots, n\}$, denote the zeros of $\tilde{W}_n(x^2; a, b, c, d)$ and $y^2_{n,i}, i \in \{1, 2, \ldots, n\}$, the zeros of $\tilde{W}_n(x^2; a-1, b, c, d)$. Then*

$$y^2_{n,1} < x^2_{n,1} < x^2_{n-1,1} < y^2_{n,2} < x^2_{n,2} < \cdots < x^2_{n-1,n-1} < y^2_{n,n} < x^2_{n,n}.$$

Proof. From (4.13a), we obtain $a_n = \frac{n(c+d+n-1)(b+d+n-1)(b+c+n-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-3)}$, which is positive and the interlacing result, as well as the position of $y^2_{n,n}$, follows from Lemma 1.8 (ii). □

4.3.2 The Racah polynomials

Proposition 4.17.

$$\tilde{R}_n(\lambda(x); \alpha-1, \beta, \gamma, \delta) = \tilde{R}_n(\lambda(x); \alpha, \beta, \gamma, \delta) - \frac{(\beta+n)(\beta+\delta+n)(\gamma+n)\,n}{(2n+\alpha+\beta)(2n+\alpha+\beta-1)} \tilde{R}_{n-1}(\lambda(x); \alpha, \beta, \gamma, \delta); \tag{4.14a}$$

$$\tilde{R}_n(\lambda(x); \alpha, \beta-1, \gamma, \delta) = \tilde{R}_n(\lambda(x); \alpha, \beta, \gamma, \delta) - \frac{(\alpha+n)(\alpha-\delta+n)(\gamma+n)\,n}{(2n+\alpha+\beta)(2n+\alpha+\beta-1)} \tilde{R}_{n-1}(\lambda(x); \alpha, \beta, \gamma, \delta). \tag{4.14b}$$

Theorem 4.18. *Let $k \in \{1, 2, \ldots, n-1\}$. The sequence of Racah polynomials*

(i) $\{\tilde{R}_n(\lambda(x); \alpha-k, \beta, \gamma, \delta)\}_{n=0}^N$, with $\alpha = -N-1$, is quasi-orthogonal of order k with respect to the weight $w(x)$ on $(0, \lambda(N))$ and the polynomials have at least $(n-k)$ real, distinct zeros in $(0, \lambda(N))$;

(ii) $\{\tilde{R}_n(\lambda(x); \alpha, \beta-k, \gamma, \delta)\}_{n=0}^N$, with $\beta = -N-\delta-1$, is quasi-orthogonal of order k with respect to the weight $w(x)$ on $(0, \lambda(N))$ and the polynomials have at least $(n-k)$ real, distinct zeros in $(0, \lambda(N))$.

Proof.

(i) Let $\alpha = -N-1$. From Lemma 1.4 and (4.14a), it follows that $\tilde{R}_n(\lambda(x);\alpha-1,\beta,\gamma,\delta)$ is quasi-orthogonal of order one on $(0,\lambda(N))$. By iteration, we can express $\tilde{R}_n(\lambda(x);\alpha-k,\beta,\gamma,\delta)$ as a linear combination of $\tilde{R}_n(\lambda(x);\alpha,\beta,\gamma,\delta)$, $\tilde{R}_{n-1}(\lambda(x);\alpha,\beta,\gamma,\delta)$, $\ldots$, $\tilde{R}_{n-k}(\lambda(x);\alpha,\beta,\gamma,\delta)$ and the result follows from Lemma 1.4. Furthermore, from Lemma 1.6 we know that at least $(n-k)$ real, distinct zeros of $\tilde{R}_n(\lambda(x);\alpha-k,\beta,\gamma,\delta), k\in\{1,2,\ldots,n-1\}$, lie in $(0,\lambda(N))$.

(ii) Let $\beta=-N-\delta-1$. The result follows in the same way from (4.14b) and Lemmas 1.4 and 1.6.

□

As in the case of the q-Racah polynomials, we obtain different interlacing results for values of n larger than $\frac{N}{2}+1$, that we show in the next theorem.

Theorem 4.19. *Consider $n\le N$ and let $x_{n,i}, i\in\{1,2,\ldots,n\}$ denote the zeros of $\tilde{R}_n(\lambda(x);\alpha,\beta,\gamma,\delta)$, $y_{n,i}, i\in\{1,2,\ldots,n\}$, the zeros of $\tilde{R}_n(\lambda(x);\alpha-1,\beta,\gamma,\delta)$ and $z_{n,i}, i\in\{1,2,\ldots,n\}$, the zeros of $\tilde{R}_n(\lambda(x);\alpha,\beta-1,\gamma,\delta)$. Then, for $n>\frac{N}{2}+1$,*

(i) if $\alpha=-N-1$ and $\beta>0,\ \delta>0,\ \gamma>0$, we have

$$0<x_{n,1}<y_{n,1}<x_{n-1,1}<x_{n,2}<y_{n,2}<\cdots<x_{n-1,n-1}<x_{n,n}<y_{n,n};$$

(ii) if $\beta=-N-\delta-1$ and $\alpha>0,\ \gamma>0,\ \alpha-\delta>0$, we have

$$0<x_{n,1}<z_{n,1}<x_{n-1,1}<x_{n,2}<z_{n,2}<\cdots<x_{n-1,n-1}<x_{n,n}<z_{n,n}.$$

Proof. Under the above hypotheses, the coefficients of $\tilde{R}_n(\lambda(x);\alpha,\beta,\gamma,\delta)$ in (4.14a) and (4.14b) are negative and the interlacing results follow from Lemma 1.8 (i). □

4.3.3 The continuous Hahn polynomials

Proposition 4.20.

$$\begin{aligned}\tilde{P}_n(x;a-1,b,c,d) &= \tilde{P}_n(x;a,b,c,d) &&(4.15a)\\ &\quad+\frac{i\,(b+c+n-1)\,(b+d+n-1)\,n}{(2\,n+a+b+c+d-3)\,(2\,n+a+b+c+d-2)}\tilde{P}_{n-1}(x;a,b,c,d);\\ \tilde{P}_n(x;a,b-1,c,d) &= \tilde{P}_n(x;a,b,c,d) &&(4.15b)\\ &\quad+\frac{i\,(a-1+d+n)\,(a-1+c+n)\,n}{(2\,n+a+b+c+d-3)\,(2\,n+a+b+c+d-2)}\tilde{P}_{n-1}(x;a,b,c,d);\\ \tilde{P}_n(x;a,b,c-1,d) &= \tilde{P}_n(x;a,b,c,d) &&(4.15c)\\ &\quad-\frac{i\,(b+d+n-1)\,(a-1+d+n)\,n}{(2\,n+a-3+b+c+d)\,(2\,n+a-2+b+c+d)}\tilde{P}_{n-1}(x;a,b,c,d);\end{aligned}$$

$$\begin{aligned}\tilde{P}_n(x;a,b,c,d-1) &= \tilde{P}_n(x;a,b,c,d) &&(4.15d)\\ &\quad-\frac{i\,(b+c+n-1)\,(a-1+c+n)\,n}{(2\,n+a-3+b+c+d)\,(2\,n+a-2+b+c+d)}\tilde{P}_{n-1}(x;a,b,c,d).\end{aligned}$$

Corollary 4.21.

$$\tilde{P}_n(x; a-1, b, c-1, d) = \tilde{P}_n(x) - \frac{i\,(a+d-b-c)\,(b+d+n-1)\,n}{(2n+a-4+b+c+d)\,(2n+a-2+b+c+d)}\tilde{P}_{n-1}(x)$$
$$+ \frac{(b+d+n-2)\,(a-2+d+n)\,(n-1)\,(b+c-2+n)\,(b+d+n-1)\,n}{(2n-5+a+b+c+d)\,(2n+a-4+b+c+d)^2\,(2n+a-3+b+c+d)}\tilde{P}_{n-2}(x); \quad (4.16a)$$
$$\tilde{P}_n(x; a, b-1, c, d-1) = \tilde{P}_n(x) + \frac{i\,(a+d-b-c)\,(a-1+c+n)\,n}{(2n-4+a+b+c+d)\,(2n+a-2+b+c+d)}\tilde{P}_{n-1}(x)$$
$$+ \frac{n\,(a+d+n-2)\,(a-1+c+n)\,(b+c+n-2)\,(a+c+n-2)\,(n-1)}{(2n-5+a+b+c+d)\,(2n-4+a+b+c+d)^2\,(2n+a-3+b+c+d)}\tilde{P}_{n-2}(x). \quad (4.16b)$$

Theorem 4.22. *Consider a, b, c, d such that $Re(a, b, c, d) > 0$, $c = \bar{a}$ and $d = \bar{b}$. Consider $k_1, k_2, k_3, k_4 \in \{0, 1, \ldots, n-1\}$, such that $k_1 + k_2 + k_3 + k_4 \leq n-1$. The sequence of continuous Hahn polynomials $\{\tilde{P}_n(x; a-k_1, b-k_2, c-k_3, d-k_4)\}_{n\geq 0}$, with $0 < Re(a) = Re(c) < 1$ (if $k_1 \neq 0$), $0 < Re(b) = Re(d) < 1$ (if $k_2 \neq 0$), $0 < Re(a) = Re(c) < 1$ (if $k_3 \neq 0$) and $0 < Re(b) = Re(d) < 1$ (if $k_4 \neq 0$), is quasi-orthogonal of order $k_1 + k_2 + k_3 + k_4 \leq n-1$ with respect to the weight $w(x)$ on $\mathbb{R}$ and the polynomials have at least $n - (k_1 + k_2 + k_3 + k_4)$ real, distinct zeros.*

Proof. Fix a and c such that $0 < Re(a) = Re(c) < 1$. From Lemma 1.4 and (4.15a), it follows that $\tilde{P}_n(x; a-1, b, c, d)$ is quasi-orthogonal of order one on $\mathbb{R}$. By iteration, we can express $\tilde{P}_n(x; a-k, b, c, d)$ as a linear combination of $\tilde{P}_n(x; a, b, c, d)$, $\tilde{P}_{n-1}(x; a, b, c, d)$, $\ldots, \tilde{P}_{n-k}(x; a, b, c, d)$ and it follows from Lemma 1.4 that $\tilde{P}_n(x; a-1, b, c, d)$ is quasi-orthogonal of order one on $\mathbb{R}$. By using an iteration process, we can write $\tilde{P}_n(x; a-k, b, c, d)$ as a linear combination of orthogonal continuous Hahn polynomials and it is quasi-orthogonal of order $k \leq n-1$. Furthermore, from Lemma 1.6 we know that at least $(n-k)$ zeros of $\tilde{P}_n(x; a-k, b, c, d), k \in \{1, 2, \ldots, n-1\}$, are real and distinct. In the same way, using (4.15c), we can prove that $\tilde{P}_n(x; a, b, c-k, d), 0 < Re(a) = Re(c) < 1$ is quasi-orthogonal of order $k \leq n-1$ on $\mathbb{R}$. By fixing b and d such that $0 < Re(b) = Re(d) < 1$, we can prove the quasi-orthogonality of $\tilde{P}_n(x; a, b-k, c, d)$ and $\tilde{P}_n(x; a, b-k, c, d)$, using (4.15b), (4.15d) and Lemma 1.4.

Using an iteration process, we can write $\tilde{P}_n(x; a-k_1, b-k_2, c-k_3, d-k_4)$, with $0 < Re(a) = Re(c) < 1$ (if $k_1 \neq 0$), $0 < Re(b) = Re(d) < 1$ (if $k_2 \neq 0$), $0 < Re(a) = Re(c) < 1$ (if $k_3 \neq 0$) and $0 < Re(b) = Re(d) < 1$ (if $k_4 \neq 0$), as a linear combination of orthogonal continuous Hahn polynomials and the results follow from Lemmas 1.4 and 1.6. □

Theorem 4.23. *Consider a, b, c, d such that $Re(a, b, c, d) > 0$, $c = \bar{a}$ and $d = \bar{b}$.*

(i) Let $0 < Re(a) = Re(c) < 1$. Then $n-2$ zeros of $\tilde{P}_n(x; a-1, b, c-1, d)$ interlace with the zeros of $\tilde{P}_{n-1}(x; a, b, c, d)$;

(ii) Let $0 < Re(b) = Re(d) < 1$. Then $n-2$ zeros of $\tilde{P}_n(x; a, b-1, c, d-1)$ interlace with the zeros of $\tilde{P}_{n-1}(x; a, b, c, d)$.

Proof. In this proof $-C_n$ refers to the coefficient of $\tilde{P}_{n-2}(x; a, b, c, d)$ in the three-term recurrence equation of the continuous Hahn polynomials (cf. [Koekoek et al., 2010, (9.4.3)]), involving the polynomials $\tilde{P}_n(x; a, b, c, d)$, $\tilde{P}_{n-1}(x; a, b, c, d)$ and $\tilde{P}_{n-2}(x; a, b, c, d)$.

(i) Let $0 < Re(a), Re(c) < 1$. We consider the coefficient b_n of $\tilde{P}_{n-2}(x; a, b, c, d)$ in (4.16a). Then

$$C_n - b_n = \frac{(a+c-2)\,(n-2+b+d)\,(a+d+n-2)\,(n-1)\,(b+c+n-2)}{(2\,n-5+a+b+c+d)\,(2\,n-4+a+b+c+d)^2},$$

and when we take into consideration the specific restrictions on the parameters, we observe that $C_n < b_n$ and the result follows from [Joulak, 2005, Theorem 15 (ii)].

(ii) Let $0 < Re(b), Re(d) < 1$ and let b_n be the coefficient of $\tilde{P}_{n-2}(x; a, b, c, d)$ in (4.16b). Then

$$C_n - b_n = \frac{(b+d-2)(n-1)(b+c+n-2)(a+d+n-2)(a+c+n-2)}{(2n-5+a+b+c+d)(2n-4+a+b+c+d)^2} < 0,$$

when we take into consideration the specific restrictions on the parameters and the result follows from [Joulak, 2005, Theorem 15 (ii)].

□

4.3.4 The Askey-Wilson polynomials

We will now fix $a > 0$ such that $q < |a| < 1$ and for these values of a, the polynomial $\tilde{p}_n(x; a, b, c, d|q)$ is orthogonal on $(-1, 1)$ with respect to $w(x; a, b, c, d|q)$ defined in (4.9). In what follows, we assume that $|a| = \max(|a|, |b|, |c|, |d|) < 1$. Should this not be the case, the order in which the parameters occur, can be changed.

We will thus only consider the equations in which a is shifted to $\frac{a}{q^k} > 1$ (or $\frac{a}{q^k} < -1$ should $a < 0$), and we will prove that the polynomials $\tilde{p}_n(x; \frac{a}{q^k}, b, c, d|q), k \in \{1, 2, \dots, n-1\}$, are quasi-orthogonal of order k on $(-1, 1)$. We use the equation

$$\begin{aligned}\tilde{p}_n(x; \frac{a}{q}, b, c, d|q) = {} & \tilde{p}_n(x; a, b, c, d|q) \\ & - \frac{aq(q^n-1)(cdq^n-q)(bdq^n-q)(bcq^n-q)}{2(abcdq^{2n}-q^3)(abcdq^{2n}-q^2)}\tilde{p}_{n-1}(x; a, b, c, d|q).\end{aligned} \tag{4.17}$$

Theorem 4.24. *Let a, b, c, d be real, or they occur in complex conjugate pairs if complex, and $\max(|a|, |b|, |c|, |d|) < 1$, and let $w(x; a, b, c, d|q)$ be as defined in (4.9). For a such that $q < |a| < 1$, the sequence of Askey-Wilson polynomials $\{\tilde{p}_n(x; \frac{a}{q^k}, b, c, d|q)\}_{n\geq 0}$ is quasi-orthogonal of order $k < n$ with respect to the weight $w(x; a, b, c, d|q)$ on the interval $(-1, 1)$ and the polynomials have at least $(n-k)$ real, distinct zeros in $(-1, 1)$.*

Proof. Suppose $q < |a| < 1$. From Lemma 1.4 and (4.17), it follows that $\tilde{p}_n(x; \frac{a}{q}, b, c, d|q)$ is quasi-orthogonal of order one on $(-1, 1)$. By iteration, we can express $\tilde{p}_n(x; \frac{a}{q^k}, b, c, d|q)$ as a linear combination of $\tilde{p}_n(x; a, b, c, d|q), \tilde{p}_{n-1}(x; a, b, c, d|q), \dots, \tilde{p}_{n-k}(x; a, b, c, d|q)$ and the result follows from Lemma 1.4. The location of the $(n-k)$ real, distinct zeros of $\tilde{p}_n(x; \frac{a}{q^k}, b, c, d|q), k \in \{1, 2, \dots, n-1\}$, follows from Lemma 1.6. □

Theorem 4.25. *Let a, b, c, d be real, or they occur in complex conjugate pairs if complex. Suppose $|a| = \max(|a|, |b|, |c|, |d|) < 1$, $q < |a| < 1$ and let $x_{n,i}, i \in \{1, 2, \dots, n\}$, denote the zeros of $\tilde{p}_n(x; a, b, c, d|q)$ and $y_{n,i}, i \in \{1, 2, \dots, n\}$, the zeros of $\tilde{p}_n(x; \frac{a}{q}, b, c, d|q)$. Then*

(i) if $a > 0$, $-1 < x_{n,1} < y_{n,1} < x_{n-1,1} < x_{n,2} < y_{n,2} < \cdots < x_{n-1,n-1} < x_{n,n} < y_{n,n}$;

(ii) if $a < 0$, $y_{n,1} < x_{n,1} < x_{n-1,1} < y_{n,2} < x_{n,2} < \cdots < x_{n-1,n-1} < y_{n,n} < x_{n,n} < 1$.

Proof. Suppose $|a| = \max(|a|, |b|, |c|, |d|) < 1$. The coefficient of $\tilde{p}_{n-1}(x; a, b, c, d|q)$ in (4.17) is

$$a_n = -\frac{aq(q^n-1)(cdq^n-q)(bdq^n-q)(bcq^n-q)}{2(abcdq^{2n}-q^3)(abcdq^{2n}-q^2)}.$$

(i) Consider the case $a > 0$ and fix a such that $q < a < 1$. Then $a_n < 0$ for the given parameter values and the interlacing result, as well as the position of $y_{n,1}$, follows from Lemma 1.8 (i).

(ii) Now we consider the case $a < 0$ and fix a such that $-1 < a < -q$. Then $a_n > 0$ and the interlacing result, as well as the position of $y_{n,n}$, follows from Lemma 1.8 (ii).

□

4.3.5 The q-Racah polynomials

The q-Racah polynomials $\tilde{R}_n(\mu(x);\alpha,\beta,\gamma,\delta|q)$ with $\mu(x) = q^{-x} + \gamma\delta q^{x+1}$, are orthogonal for $n \in \{0, 1, \ldots, N\}$, with respect to the discrete weight function

$$w(x) = \frac{(\alpha q, \beta\delta q, \gamma q, \gamma\delta q; q)_x (1 - \gamma\delta q^{2x+1})}{(q, \frac{\gamma\delta q}{\alpha}, \frac{\gamma q}{\beta}, \delta q; q)_x (\alpha\beta q)^x (1 - \gamma\delta q)}$$

for $\alpha q = q^{-N}$ or $\beta\delta q = q^{-N}$ or $\gamma q = q^{-N}$, where N is a nonnegative integer. Shifting the parameter γ or δ will change $\mu(x)$ and we will only consider shifts of α and β. From (4.11a) and (4.11b) we obtain

$$\begin{aligned}\tilde{R}_n(\mu(x);\tfrac{\alpha}{q},\beta,\gamma,\delta|q) = {} & \tilde{R}_n(\mu(x);\alpha,\beta,\gamma,\delta|q) \\ & - \frac{\alpha q(1-q^n)(1-\beta q^n)(1-\gamma q^n)(1-\beta\delta q^n)}{(1-\alpha\beta q^{2n})(q-\alpha\beta q^{2n})}\tilde{R}_{n-1}(\mu(x);\alpha,\beta,\gamma,\delta|q);\end{aligned} \tag{4.18a}$$

$$\begin{aligned}\tilde{R}_n(\mu(x);\alpha,\tfrac{\beta}{q},\gamma,\delta|q) = {} & \tilde{R}_n(\mu(x);\alpha,\beta,\gamma,\delta|q) \\ & + \frac{\beta q(1-q^n)(1-\alpha q^n)(1-\gamma q^n)(\alpha q^n-\delta)}{(1-\alpha\beta q^{2n})(q-\alpha\beta q^{2n})}\tilde{R}_{n-1}(\mu(x);\alpha,\beta,\gamma,\delta|q).\end{aligned} \tag{4.18b}$$

Theorem 4.26. *Let $k \in \{1, 2, \ldots, n-1\}$. The sequence of q-Racah polynomials*

(i) $\{\tilde{R}_n(\mu(x);\frac{\alpha}{q^k},\beta,\gamma,\delta|q)\}_{n=0}^N$, with $\alpha = q^{-N-1}$, is quasi-orthogonal of order k with respect to the weight $w(x)$ on $(\mu(0),\mu(N))$ and the polynomials have at least $(n-k)$ real, distinct zeros on $(\mu(0),\mu(N))$;

(ii) $\{\tilde{R}_n(\mu(x);\alpha,\frac{\beta}{q},\gamma,\delta|q)\}_{n=0}^N$, with $\beta = \frac{q^{-N-1}}{\delta}$, is quasi-orthogonal of order k with respect to the weight $w(x)$ on $(\mu(0),\mu(N))$ and the polynomials have at least $(n-k)$ real, distinct zeros on $(\mu(0),\mu(N))$.

Proof.

(i) Let $\alpha = q^{-N-1}$. From Lemma 1.4 and (4.18a), it follows that $\tilde{R}_n(\mu(x);\frac{\alpha}{q},\beta,\gamma,\delta|q)$ is quasi-orthogonal of order one on $(\mu(0),\mu(N))$. By iteration, we can express $\tilde{R}_n(\mu(x);\frac{\alpha}{q^k},\beta,\gamma,\delta|q)$ as a linear combination of $\tilde{R}_n(\mu(x);\alpha,\beta,\gamma,\delta|q)$, $\tilde{R}_{n-1}(\mu(x);\alpha,\beta,\gamma,\delta|q)$, …, $\tilde{R}_{n-k}(\mu(x);\alpha,\beta,\gamma,\delta|q)$ and the result follows from Lemma 1.4. Furthermore, from Lemma 1.6 we know that at least $(n-k)$ real, distinct zeros of $\tilde{R}_n(\mu(x);\frac{\alpha}{q^k},\beta,\gamma,\delta)$, $k \in \{1, 2, \ldots, n-1\}$, lie in $(\mu(0),\mu(N))$.

(ii) Let $\beta = \frac{q^{-N-1}}{\delta}$. The result follows in the same way from (4.18b) and Lemmas 1.4 and 1.6.

□

For values of n larger than $\frac{N}{2}+1$, we obtain the following interlacing results.

Theorem 4.27. *Consider $n \leq N$ and let $x_{n,i}, i \in \{1,2,\dots,n\}$, denote the zeros of $\tilde{R}_n(\mu(x);\alpha,\beta,\gamma,\delta|q)$, $y_{n,i}, i \in \{1,2,\dots,n\}$, the zeros of $\tilde{R}_n(\mu(x);\frac{\alpha}{q},\beta,\gamma,\delta|q)$ and $z_{n,i}$, $i \in \{1,2,\dots,n\}$, the zeros of $\tilde{R}_n(\mu(x);\alpha,\frac{\beta}{q},\gamma,\delta|q)$. Then, for $n > \frac{N}{2}+1$,*

(i) if $\alpha = q^{-N-1}$ and $\beta q < 1,\ \gamma q < 1,\ \beta\delta q < 1$,

$$\mu(0) < x_{n,1} < y_{n,1} < x_{n-1,1} < x_{n,2} < y_{n,2} < \cdots < x_{n-1,n-1} < x_{n,n} < y_{n,n};$$

(ii) if $\beta = \frac{q^{-N-1}}{\delta}$ and $\alpha q < 1,\ \gamma q < 1,\ \frac{\alpha}{\delta} q < 1$, we have

$$\mu(0) < x_{n,1} < z_{n,1} < x_{n-1,1} < x_{n,2} < z_{n,2} < \cdots < x_{n-1,n-1} < x_{n,n} < z_{n,n}.$$

Proof. Under the above hypotheses, the coefficients of $\tilde{R}_{n-1}(\mu(x);\alpha,\beta,\gamma,\delta|q)$ in (4.18a) and (4.18b) are negative and the interlacing results follow from Lemma 1.8 (i). □

4.4 Three-term recurrence equations satisfied by classical orthogonal polynomials on a quadratic or a q-quadratic lattice

Every orthogonal polynomial system $\{p_n(x)\}_{n\geq 0}$ satisfies a three-term recurrence relation of the type

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x) \ (n = 0,1,2,\dots,p_{-1} \equiv 0), \tag{4.19}$$

with $C_n A_n A_{n-1} > 0$. Moreover, Favard's theorem [Chihara, 1978, Section 4] states that the converse is true. Starting from a difference equation of type (4.2) or (4.5) given in [Koekoek et al., 2010], we deduce the divided-difference equation of type (4.1), (4.3) or (4.4) satisfied by each classical orthogonal polynomial on a quadratic or q-quadratic lattice. Some of them can be found in [Foupouagnigni, 2008], [Njionou Sadjang et al., 2015b], [Tcheutia et al., 2017]. They will also be recovered using the algorithms implemented in the next section with the three-term recurrence relation given as input. Considering the divided-difference equations of type (4.1), (4.3) or (4.4) as input, we recall in this section a general method to derive three–term recurrence relations (4.19) for classical orthogonal polynomials on a quadratic or a q-quadratic lattice in terms of the coefficients a,b,c,d,e of the given polynomials $\phi(x) = ax^2+bx+c$ and $\psi(x) = dx+e$.

The polynomial basis $(x^n)_{n\geq 0}$ is not appropriate for the operators $\mathbb{D}_x$, $\mathbf{D}_x$ and δ_x (see e. g. [Foupouagnigni, 2008], [Witte, 2015]). From the definitions of the classical orthogonal polynomials on a quadratic or a q-quadratic lattice, the natural polynomial bases appropriate for the operator $\mathbb{D}_x$ are the bases $\{B_n(a,x)\}_{n\geq 0}$, $\{\xi_n(\gamma,\delta,\mu(x))\}_{n\geq 0}$ and $\{\chi_n(\gamma,\delta,\lambda(x))\}_{n\geq 0}$ defined by

$$B_n(a,x) = (aq^s;q)_n(aq^{-s};q)_n = \prod_{k=0}^{n-1}(1-2axq^k+a^2q^{2k}),\ n \geq 1,\ B_0(a,x) \equiv 1, \tag{4.20}$$

where $x = x(s) = \cos\theta = \dfrac{q^s + q^{-s}}{2}$, $q^s = e^{i\theta}$;

$$\begin{cases} \xi_n(\gamma,\delta,\mu(x)) = (q^{-x};q)_n(\gamma\delta q^{x+1};q)_n = \prod_{k=0}^{n-1}(1+\gamma\delta q^{2k+1} - \mu(x)q^k),\ n\geq 1, \\ \xi_0(\gamma,\delta,\mu(x)) \equiv 1, \end{cases} \tag{4.21}$$

with $\mu(x) = q^{-x} + \gamma\delta q^{x+1}$;

$$\begin{cases} \chi_n(\gamma,\delta,\lambda(x)) = (-x)_n(x+\gamma+\delta+1)_n = \prod_{k=0}^{n-1}\Big(k(\gamma+\delta+k+1) - \lambda(x)\Big),\ n\geq 1, \\ \chi_0(\gamma,\delta,\lambda(x)) \equiv 1, \end{cases} \tag{4.22}$$

for $\lambda(x) = x(x+\gamma+\delta+1)$. The basis $\{\vartheta_n(a,x)\}_{n\geq 0}$ defined by

$$\vartheta_n(a,x) = (a+ix)_n(a-ix)_n; \tag{4.23}$$

is appropriate for the operator $\mathbf{D}_x$ whereas the corresponding basis for δ_x is $\{(a+ix)_n\}_{n\geq 0}$. The elements of the basis $\{B_n(a,x)\}_{n\geq 0}$, $\{(a+ix)_n\}_{n\geq 0}$, $\{\xi_n(\gamma,\delta,\mu(x))\}_{n\geq 0}$ or $\{\chi_n(\gamma,\delta,\lambda(x))\}_{n\geq 0}$ are polynomials of degree n in the variables x, x, $\mu(x)$ or $\lambda(x)$, respectively, and the elements of the basis $\{\vartheta_n(a,x)\}_{n\geq 0}$ are polynomials of degree n in the variable x^2.

4.4.1 Polynomials expanded in the basis $\{\vartheta_n(\alpha,x)\}_{n\geq 0}$

In this basis are expanded the Wilson polynomials $W_n(x^2;a,b,c,d)$ and the continuous dual Hahn polynomials $S_n(x^2;a,b,c)$. The procedure to find the coefficients of the recurrence equation (4.19) (with x substituted by x^2) in terms of the coefficients a, b, c, d, e of $\phi(x)$ and $\psi(x)$ is as follows (cf. [Foupouagnigni et al., 2013], [Koepf and Schmersau, 1998], [Koepf and Schmersau, 2002], [Njionou Sadjang et al., 2015b], [Tcheutia et al., 2017]):

1. Substitute

$$p_n(x) := p_n(x^2) = k_n\vartheta_n(\alpha,x) + k_n'\vartheta_n(\alpha,x) + k_n''\vartheta_{n-2}(\alpha,x) + \ldots \tag{4.24}$$

in the divided-difference equation (4.3) (with x substituted by x^2). Next multiply this equation by $\vartheta_1(\alpha,x)$ and use the relations [Njionou Sadjang et al., 2015b]

$$\vartheta_1(\alpha,x)\mathbf{D}_x^2\vartheta_n(\alpha,x) = \eta(n)\eta(n-1)\vartheta_{n-1}(\alpha,x),$$
$$\vartheta_1(\alpha,x)\mathbf{S}_x\mathbf{D}_x\vartheta_n(\alpha,x) = \eta(n)\left(\beta(\alpha+\frac{1}{2},n-1)\vartheta_{n-1}(\alpha,x) + \vartheta_n(\alpha,x)\right),$$
$$\vartheta_1(\alpha,x)\vartheta_n(\alpha,x) = \nu(\alpha,n)\vartheta_n(\alpha,x) + \vartheta_{n+1}(\alpha,x),$$

with

$$\eta(n) = n,\ \beta(\alpha,n) = -n(n+\alpha-\frac{1}{2}),\ \nu(\alpha,n) = -(n^2+2\alpha n).$$

2. To eliminate the terms $x^2\vartheta_n(\alpha,x)$ and $x^4\vartheta_n(\alpha,x)$, use twice the relation [Njionou Sadjang et al., 2015b]

$$x^2\vartheta_n(\alpha,x) = -(n+\alpha)^2\vartheta_n(\alpha,x) + \vartheta_{n+1}(\alpha,x). \tag{4.25}$$

3. Equate the coefficients of $\vartheta_{n+1}(\alpha, x)$ to get $\lambda_n = -n((n-1)a + d)$ in (4.3), and the coefficients of $\vartheta_n(\alpha, x)$ and $\vartheta_{n-1}(\alpha, x)$ to get $k'_n/k_n,\ k''_n/k_n \in \mathbb{Q}(n)$, respectively.

4. Substitute the expression of p_n given by (4.24) in the recurrence relation (4.19) (with x substituted by x^2) and use (4.25). By equating the coefficients of $\vartheta_{n+1}(\alpha, x)$, $\vartheta_n(\alpha, x)$, $\vartheta_{n-1}(\alpha, x)$, we get A_n, B_n and C_n, respectively, in terms of k_n, k'_n and k''_n.

5. Substituting the values of k'_n and k''_n given in step 3 in these equations yields the three unknowns in terms of a, b, c, d, e, n, k_{n-1}, k_n, k_{n+1}.

Proposition 4.28. *Let $p_n(x) := p_n(x^2) = k_n\vartheta_n(\alpha, x) + k'_n\vartheta_n(\alpha, x) + k''_n\vartheta_{n-2}(\alpha, x) + \dots$ $(n \in \mathbb{N}_0)$ be a family of polynomial solutions of the divided-difference equation* (4.3). *Then the recurrence equation* (4.19) *(with x substituted by x^2) holds with*

$$\frac{k_n}{k_{n+1}}A_n = 1,$$
$$\frac{k_n}{k_{n+1}}B_n = -\frac{n\,(n-1)\,a\,(2\,an^2 - 2\,an + 4\,nd - 2\,b - d) - nd\,(2\,b + d - 2\,nd) + e\,(2\,a - d)}{((2\,n-2)\,a + d)\,(2\,an + d)},$$
$$\frac{k_{n-1}}{k_{n+1}}C_n = \frac{n\,(an - 2\,a + d)}{(2\,an - a + d)\,(2\,an - 3\,a + d)\,(2\,an - 2\,a + d)^2} \times \Big\{ (n-1)^6\,a^3 + (n-1)\,db^2$$
$$+ \left(-2\,(n-1)^4\,b + 3\,(n-1)^5\,d - 4\,c\,(n-1)^2\right)a^2 + \left(-2\,(n-1)^2\,d^2 + de\right)b + (-en - c + e)\,d^2$$
$$+ \left((n-1)^2\,b^2 - 4\,(n-1)^3\,db + 3\,(n-1)^4\,d^2 - (n-1)\,(en + 4\,c - e)\,d - e^2\right)a + (n-1)^3\,d^3 \Big\}.$$

4.4.2 Polynomials expanded in the basis $\{(\alpha + ix)_n\}_{n\geq 0}$

The polynomials expanded in this basis are the continuous Hahn polynomials $p_n(x; a, b, c, d)$ and the Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x; \theta)$. The action of the operators δ_x and $\mathbf{S}_x$ on the basis $(\alpha + ix)_n$ is given by [Tcheutia et al., 2017]

$$\begin{aligned}
(\alpha + ix)\delta_x^2(\alpha + ix)_n &= -n(n-1)(\alpha + ix)_{n-1};\\
(\alpha + ix)\mathbf{S}_x\delta_x(\alpha + ix)_n &= ni(\alpha + ix)_n - \frac{n(n-1)}{2}i(\alpha + ix)_{n-1};\\
(\alpha + ix)(\alpha + ix)_n &= (\alpha + ix)_{n+1} - n(\alpha + ix)_n;\\
x(\alpha + ix)_n &= -i(\alpha + ix)_{n+1} + i(n + \alpha)(\alpha + ix)_n.
\end{aligned}$$

We suppose that

$$p_n(x) = k_n(\alpha + ix)_n + k'_n(\alpha + ix)_{n-1} + k''_n(\alpha + ix)_{n-2} + \dots .$$

Using the same approach as in Section 4.4.1, it follows that $\lambda_n = -n((n-1)a + d)$ in (4.4) and the following result holds.

Proposition 4.29. *Let $p_n(x) = k_n(\alpha + ix)_n + k'_n(\alpha + ix)_{n-1} + k''_n(\alpha + ix)_{n-2} + \dots$ $(n \in \mathbb{N}_0)$ be a family of polynomial solutions of the divided-difference equation* (4.4). *Then the* *recurrence equation* (4.19) *is valid with*

$$\begin{aligned}
&\frac{k_n}{k_{n+1}}A_n = i,\\
&\frac{k_n}{k_{n+1}}B_n = i\frac{(2\,bn^2-2\,bn-2\,e)\,a+d\,(2\,bn+e)}{(2\,an-2\,a+d)\,(2\,an+d)},\\
&\frac{k_{n-1}}{k_{n+1}}C_n = n\Big\{-8\,n\,(n-2)\,(n-1)^4\,a^5+\Big(-4\,\left(7\,n^2-13\,n+2\right)(n-1)^3\,d\\
&+32\,cn\,(n-2)\,(n-1)^2\Big)a^4+\Big(-8\,n\,(n-2)\,(n-1)^2\,b^2-2\,\left(19\,n^2-34\,n+10\right)\\
&\times(n-1)^2\,d^2+16\,c\,(n-1)\left(5\,n^2-9\,n+2\right)d+8\,e^2n\,(n-2)\Big)a^3\\
&+\Big(-4\,(n-1)\left(5\,n^2-9\,n+2\right)db^2-8\,en\,(n-2)\,db+\left(72\,n^2-128\,n+48\right)d^2c\\
&-(n-1)\,(5\,n-3)\,(5\,n-6)\,d^3+(12\,n-8)\,e^2d\Big)a^2+\Big(-4\,(4\,n-3)\,(n-1)\,d^2b^2\\
&+(-12\,n+8)\,ed^2b+(28\,n-24)\,d^3c-(8\,n-7)\,(n-1)\,d^4+4\,e^2d^2\Big)a\\
&+(-4\,n+4)\,d^3b^2-4\,bd^3e+(1-n)\,d^5+4\,cd^4\Big\}\\
&\Big/\Big\{4\,(2\,an-2\,a+d)^2\,(2\,an-3\,a+d)\,(2\,an+d)\,(2\,an-a+d)\Big\}.
\end{aligned}$$

4.4.3 Polynomials expanded in the basis $\{\chi_n(\gamma,\delta,\lambda(x))\}_{n\geq 0}$

In this basis, we have the Racah polynomials $R_n(\lambda(x);\alpha,\beta,\gamma,\delta)$, the Dual Hahn polynomials $R_n(\lambda(x);\gamma,\delta,N)$. We get by direct computations the action of the operators $\mathbb{D}_x$ and $\mathbb{S}_x$ on $\chi_n(\gamma,\delta,\lambda(x))$ given by

$$\begin{aligned}
&\chi_1(\gamma,\delta,\lambda(x))\;\mathbb{D}_x^2\;\chi_n(\gamma,\delta,\lambda(x)) = \eta(n)\eta(n-1)\;\chi_{n-1}(\gamma,\delta,\lambda(x)),\\
&\chi_1(\gamma,\delta,\lambda(x))\;\mathbb{S}_x\mathbb{D}_x\;\chi_n(\gamma,\delta,\lambda(x)) = \eta(n)\left(\beta(\frac{1}{2},\gamma,\delta,n-1)\;\chi_{n-1}(\gamma,\delta,\lambda(x))+\chi_n(\gamma,\delta,\lambda(x))\right),\\
&\chi_1(\gamma,\delta,\lambda(x))\;\chi_n(\gamma,\delta,\lambda(x)) = \nu(\gamma,\delta,n)\;\chi_n(\gamma,\delta,\lambda(x))+\chi_{n+1}(\gamma,\delta,\lambda(x)),\\
&\lambda(x)\;\chi_n(\gamma,\delta,\lambda(x)) = \mu(\gamma,\delta,n)\;\chi_n(\gamma,\delta,\lambda(x))-\chi_{n+1}(\gamma,\delta,\lambda(x)),
\end{aligned}$$

where

$$\begin{aligned}
&\mu(\gamma,\delta,n) = n(n+\gamma+\delta+1),\;\;\nu(\gamma,\delta,n) = -n(n+\gamma+\delta+1),\\
&\beta(a,\gamma,\delta,n) = \frac{-n(2n+\gamma+\delta+2a)}{2},\quad \eta(n) = -n.
\end{aligned}$$

We set

$$p_n(x) := p_n(\lambda(x)) = k_n\;\chi_n(\gamma,\delta,\lambda(x))+k_n'\;\chi_{n-1}(\gamma,\delta,\lambda(x))+k_n''\;\chi_{n-2}(\gamma,\delta,\lambda(x))+\dots,$$

use the latter structure relations satisfied by $\chi_n(\gamma,\delta,\lambda(x))$, and proceed as in Section 4.4.1 to get $\lambda_n = -n((n-1)a+d)$ in (4.1) and the following result.

Proposition 4.30. *Let $p_n(x) := p_n(\lambda(x)) = k_n\ \chi_n(\gamma,\delta,\lambda(x)) + k_n'\ \chi_{n-1}(\gamma,\delta,\lambda(x)) + k_n''\ \chi_{n-2}(\gamma,\delta,\lambda(x)) + \dots$ $(n \in \mathbb{N}_0)$ be a family of polynomial solutions of the divided-difference equation (4.1) (where $\phi(x) \leftarrow \phi(\lambda(x))$ and $\psi(x) \leftarrow \psi(\lambda(x))$). Then the recurrence equation (4.19) (with $x \leftarrow \lambda(x)$) is valid with*

$$\frac{k_n}{k_{n+1}}A_n = -1,$$
$$\frac{k_n}{k_{n+1}}B_n = -\frac{an\,(n-1)\,(2\,an^2-2\,an+4\,dn+2\,b-d)+2\,bdn+n\,(2\,n-1)\,d^2+de-2\,ae}{(2\,an-2\,a+d)\,(2\,an+d)},$$
$$\begin{aligned}\frac{k_{n-1}}{k_{n+1}}C_n = &-\frac{n\,(an-2\,a+d)}{4\,(2\,an-a+d)\,(2\,an-3\,a+d)\,(2\,an-2\,a+d)^2} \times \Big\{ (4\,c+(-4\,n+4)\,e)\,d^2\\
&+ \Big(-8\,(n-1)^4\,b+4\,(n-1)^3\,\big(2\,\delta^2+4\,\delta\,\gamma+2\,\gamma^2-3\,n^2+4\,\delta+4\,\gamma+6\,n-1\big)\,d\\
&+16\,c\,(n-1)^2\Big)a^2+\Big((n-1)^2\,\big(5\,\delta^2+10\,\delta\,\gamma+5\,\gamma^2-12\,n^2+10\,\delta+10\,\gamma+24\,n-7\big)\,d^2\\
&-4\,(n-1)^2\,b^2-16\,(n-1)^3\,db+\big((16\,n-16)\,c-4\,e\,(n-1)^2\big)\,d+4\,e^2\Big)a\\
&+4\,(n-1)^4\,(n+\delta+\gamma)\,(-n+2+\delta+\gamma)\,a^3+(-4\,n+4)\,db^2\\
&+\big(-8\,(n-1)^2\,d^2-4\,de\big)\,b+(n-1)\,(-2\,n+3+\delta+\gamma)\,(2\,n-1+\delta+\gamma)\,d^3\Big\}.\end{aligned}$$

4.4.4 Polynomials expanded in the basis $\{B_n(\alpha,x)\}_{n\geq 0}$

The following polynomial families are expanded in the basis $\{B_n(\alpha,x)\}$: the Askey-Wilson polynomials $p_n(x;a,b,c,d|q)$, the continuous dual q-Hahn polynomials $p_n(x;a,b,c|q)$, the continuous q-Hahn polynomials $p_n(x;a,b,c,d;q)$, the Al-Salam-Chihara polynomials $Q_n(x;a,b|q)$, the q-Meixner-Pollaczek polynomials $P_n(x;a|q)$, the continuous q-Jacobi polynomials $P_n^{(\alpha,\beta)}(x|q)$, the continuous q-ultraspherical/Rogers polynomials $C_n(x;\beta|q)$, the continuous big q-Hermite polynomials $H_n(x;a|q)$, the continuous q-Laguerre polynomials $P_n^{(\alpha)}(x|q)$. The procedure to find the coefficients of the recurrence equation (4.19) in terms of the coefficients $a,\ b,\ c,\ d,\ e$ of $\phi(x)$ and $\psi(x)$ is as in Section 4.4.1:

1. Substitute

$$p_n(x) = k_nB_n(\alpha,x)+k_n'B_{n-1}(\alpha,x)+k_n''B_{n-2}(\alpha,x)+\dots \tag{4.26}$$

in the divided-difference equation (4.1). Next multiply this equation by $B_1(\alpha,x)$ and use the relations [Foupouagnigni et al., 2013]

$$B_1(\alpha,x)\mathbb{D}_x^2B_n(\alpha,x) = \eta(\alpha,n)\eta(\alpha\sqrt{q},n-1)B_{n-1}(\alpha,x), \tag{4.27}$$
$$B_1(\alpha,x)\mathbb{S}_x\mathbb{D}_xB_n(\alpha,x) = \eta(\alpha,n)\left(\beta_1(\alpha\sqrt{q},n-1)B_{n-1}(\alpha,x)+\beta_2(n-1)B_n(\alpha,x)\right), \tag{4.28}$$
$$B_1(\alpha,x)B_n(\alpha,x) = \nu_1(\alpha,n)B_n(\alpha,x)+\nu_2(n)B_{n+1}(\alpha,x), \tag{4.29}$$

with

$$\eta(\alpha,n) = \frac{2\alpha(1-q^n)}{q-1},\quad \beta_1(\alpha,n) = \frac{1}{2}(1-\alpha^2q^{2n-1})(1-q^{-n}),$$
$$\beta_2(n) = \frac{1}{2}+\frac{1}{2q^n},\quad \nu_1(\alpha,n) = (1-q^{-n})(1-\alpha^2q^n),\quad \nu_2(n) = q^{-n}.$$

2. To eliminate the terms $xB_n(\alpha,x)$ and $x^2B_n(\alpha,x)$, use the relations [Foupouagnigni et al., 2013]

$$\begin{aligned} xB_n(\alpha,x) &= \mu_1(\alpha,n)B_n(\alpha,x)+\mu_2(\alpha,n)B_{n+1}(\alpha,x), && (4.30)\\ x^2B_n(\alpha,x) &= \mu_1^2(\alpha,n)B_n(\alpha,x)+\mu_2(\alpha,n)(\mu_1(\alpha,n)+\mu_1(\alpha,n+1))B_{n+1}(\alpha,x)\\ &\quad+\mu_2(\alpha,n)\mu_2(\alpha,n+1)B_{n+2}(\alpha,x), && (4.31)\end{aligned}$$

with

$$\mu_1(\alpha,n)=\frac{1+\alpha^2q^{2n}}{2\alpha q^n},\quad \mu_2(\alpha,n)=\frac{-1}{2\alpha q^n}.$$

3. Equating the coefficients of $B_{n+1}(\alpha,x)$ gives

$$\lambda_n=-\frac{1}{2}\,\frac{(q^n-1)\left(2\sqrt{q}\,(q^n-q)\,a+(q-1)\,(q^n+q)\,d\right)}{q^n\,(q-1)^2}. \tag{4.32}$$

Equating the coefficients of $B_n(\alpha,x)$ and $B_{n-1}(\alpha,x)$ gives $k_n'/k_n,\ k_n''/k_n\in\mathbb{Q}(q^n,\sqrt{q})$.

4. Substitute the expression of p_n given by (4.26) in the recurrence relation (4.19) and use (4.30). By equating the coefficients of $B_{n+1}(\alpha,x)$, $B_n(\alpha,x)$, $B_{n-1}(\alpha,x)$, we get A_n, B_n and C_n, respectively, given as

$$A_n\frac{k_n}{k_{n+1}}=\frac{1}{\mu_2(\alpha,n)},\quad B_n\frac{k_n}{k_{n+1}}=-\frac{\mu_2(\alpha,n-1)k_n'}{\mu_2(\alpha,n)k_n}+\frac{k_{n+1}'}{k_{n+1}}-\frac{\mu_1(\alpha,n)}{\mu_2(\alpha,n)}, \tag{4.33}$$

$$\begin{aligned} C_n\frac{k_{n-1}}{k_{n+1}} &= -\frac{\mu_2(\alpha,n-1)(k_n')^2}{\mu_2(\alpha,n)k_n^2}+\frac{\mu_2(\alpha,n-2)k_n''}{\mu_2(\alpha,n)k_n}-\frac{k_{n+1}''}{k_{n+1}}+\frac{k_n'k_{n+1}'}{k_nk_{n+1}} && (4.34)\\ &\quad-\frac{(\mu_1(\alpha,n)-\mu_1(\alpha,n-1))k_n'}{\mu_2(\alpha,n)k_n}.\end{aligned}$$

5. Substituting the values of k_n' and k_n'' given in step 3 in these equations yields the three unknowns given in terms of α, a, b, c, d, e, n, k_{n-1}, k_n, k_{n+1}, for $N=q^n$, $Q=\sqrt{q}$, by:

$$
\begin{aligned}
&\frac{k_n}{k_{n+1}}A_n = -2\alpha N,\\
&\frac{k_n}{k_{n+1}}B_n = 2\alpha\, N^2\Big\{\Big(\Big(\left(-2\,Q^3-2\,Q\right)b+\left(Q^4-2\,Q^2+1\right)e\Big)\left(Q^2d+2\,Qa-d\right)N^2\\
&-\left(Q^2+1\right)\Big(\left(\left(-2\,Q^5+4\,Q^3-2\,Q\right)e-4\,Q^4b-4\,Q^2b\right)a+\left(Q^6-Q^4-Q^2+1\right)de\\
&+\left(2\,Q^5b-4\,Q^3b+2\,Qb\right)d\Big)N-Q^2\left(\left(-2\,Q^3-2\,Q\right)b+\left(Q^4-2\,Q^2+1\right)e\right)\left(Q^2d-2\,Qa-d\right)\Big)\Big\}\\
&\Big/\Big\{\left(\left(Q^2d+2\,Qa-d\right)N^2+Q^6d-2\,Q^5a-Q^4d\right)\left(\left(Q^2d+2\,Qa-d\right)N^2+Q^2d-2\,Qa-d\right)\Big\},\\
&\frac{k_{n-1}}{k_{n+1}}C_n = \alpha^2N^2\left(N-1\right)\Big(\left(Q^2d+2\,Qa-d\right)N+Q^6d-2\,Q^5a-Q^4d\Big)\Big\{\Big(\left(Q^2d+2\,Qa-d\right)^3N^6\\
&-4\,Q^3\left(Q^2d+2\,Qa-d\right)^2\left(a+2\,c\right)N^5+Q^4\left(Q^2d+2\,Qa-d\right)\Big(Q^4d^2-4\,Q^4e^2-4\,Q^2a^2+16\,Q^2b^2\\
&-2\,Q^2d^2+8\,Q^2e^2+d^2-4\,e^2\Big)N^4-8\,Q^7\Big(Q^4ad^2+2\,Q^4ae^2-4\,Q^4bde+2\,Q^4cd^2-4\,Q^2a^3\\
&-8\,Q^2a^2c+8\,Q^2ab^2-2\,Q^2ad^2-4\,Q^2ae^2+8\,Q^2bde-4\,Q^2cd^2+ad^2+2\,ae^2-4\,bde+2\,cd^2\Big)N^3\\
&-Q^8\left(Q^2d-2\,Qa-d\right)\left(Q^4d^2-4\,Q^4e^2-4\,Q^2a^2+16\,Q^2b^2-2\,Q^2d^2+8\,Q^2e^2+d^2-4\,e^2\right)N^2\\
&-4\,Q^{11}\left(Q^2d-2\,Qa-d\right)^2\left(a+2\,c\right)N-Q^{12}\left(Q^2d-2\,Qa-d\right)^3\Big)\Big\}\\
&\Big/\Big\{Q^2\left(\left(Q^2d+2\,Qa-d\right)N^2+Q^6d-2\,Q^5a-Q^4d\right)^2\Big(\left(Q^2d+2\,Qa-d\right)N^2\\
&+Q^4d-2\,Q^3a-Q^2d\Big)\left(\left(Q^2d+2\,Qa-d\right)N^2+Q^8d-2\,Q^7a-Q^6d\right)\Big\}.
\end{aligned}
$$

4.4.5 Polynomials expanded in the basis $\{\xi_n(\gamma,\delta,\mu(x))\}_{n\geq 0}$

The polynomials represented in this basis are: the q–Racah polynomials $R_n(\mu(x);\alpha,\beta,\gamma,\delta|q)$, the dual q–Hahn polynomials $R_n(\mu(x);\gamma,\delta,N|q)$, the dual q–Krawtchouk polynomials $K_n(\mu(x);c,N|q)$. By direct computations, we have the structure relations

$$
\begin{aligned}
\xi_1(\gamma,\delta,\mu(x))\mathbb{D}_x^2\xi_n(\gamma,\delta,\mu(x)) &= \eta(1,n)\eta(\sqrt{q},n-1)\xi_{n-1}(\gamma,\delta,\mu(x));\\
\xi_1(\gamma,\delta,\mu(x))\mathbb{S}_x\mathbb{D}_x\xi_n(\gamma,\delta,\mu(x)) &= \eta(1,n)\Big(\beta_1(\sqrt{q},\gamma,\delta,n-1)\xi_{n-1}(\gamma,\delta,\mu(x))\\
&\quad+\beta_2(n-1)\xi_n(\gamma,\delta,\mu(x))\Big);\\
\xi_1(\gamma,\delta,\mu(x))\xi_n(\gamma,\delta,\mu(x)) &= \nu_1(\gamma,\delta,n)\xi_n(\gamma,\delta,\mu(x))+\nu_2(n)\xi_{n+1}(\gamma,\delta,\mu(x));\\
\mu(x)\xi_n(\gamma,\delta,\mu(x)) &= \mu_1(\gamma,\delta,n)\xi_n(\gamma,\delta,\mu(x))+\mu_2(n)\xi_{n+1}(\gamma,\delta,\mu(x));
\end{aligned}
$$

where

$$
\mu_1(\gamma,\delta,n)=\frac{1+\gamma\delta q^{2n+1}}{q^n},\ \mu_2(n)=\frac{-1}{q^n},\ \nu_1(\gamma,\delta,n)=(1-q^{-n})(1-\gamma\delta q^{n+1}),\ \nu_2(n)=q^{-n},
$$
$$
\beta_1(a,\gamma,\delta,n)=\frac{1}{2}(1-a^2\gamma\delta q^{2n})(1-q^{-n}),\quad \beta_2(n)=\frac{1+q^n}{2q^n},\quad \eta(a,n)=\frac{a(1-q^n)}{q-1}.
$$

We suppose now that

$$
p_n(x) := p_n(\mu(x)) = k_n\xi_n(\gamma,\delta,\mu(x))+k'_n\xi_{n-1}(\gamma,\delta,\mu(x))+k''_n\xi_{n-2}(\gamma,\delta,\mu(x))+\dots.
$$

To get the coefficients A_n, B_n and C_n of (4.19), we proceed as in Section 4.4.4 and obtain λ_n given by (4.32) and for $N = q^n$, $Q = \sqrt{q}$,

$$\frac{k_n}{k_{n+1}} A_n = -N,$$
$$\frac{k_n}{k_{n+1}} B_n = N^2 \Big\{ \Big(\left(\left(-2\,Q^3 - 2\,Q\right) b + \left(Q^4 - 2\,Q^2 + 1\right) e\right) \left(Q^2 d + 2\,Qa - d\right) N^2$$
$$- \left(Q^2 + 1\right) \left(-2\,\left(Q^4 e + 2\,Q^3 b - 2\,Q^2 e + 2\,Qb + e\right) Qa + d \left(Q^2 - 1\right)^2 \left(Q^2 e + 2\,Qb + e\right)\right) N$$
$$- Q^2 \left(\left(-2\,Q^3 - 2\,Q\right) b + \left(Q^4 - 2\,Q^2 + 1\right) e\right) \left(Q^2 d - 2\,Qa - d\right) \Big) \Big\}$$
$$\Big/ \Big\{ \left(\left(Q^2 d + 2\,Qa - d\right) N^2 + Q^6 d - 2\,Q^5 a - Q^4 d\right) \left(\left(Q^2 d + 2\,Qa - d\right) N^2 + Q^2 d - 2\,Qa - d\right) \Big\},$$
$$\frac{k_{n-1}}{k_{n+1}} C_n = (N-1)\, N^2 \Big(\left(-2\,Q^5 + 2\,NQ\right) a + \left(Q^6 - Q^4 + NQ^2 - N\right) d\Big)$$
$$\times \Big\{ \Big(\delta\,\gamma\, \left(Q^2 d + 2\,Qa - d\right)^3 N^6 - 2\,Q \left(Q^2 d + 2\,Qa - d\right)^2 \left(2\,Q^2 a \delta\,\gamma + c\right) N^5$$
$$+ Q^2 \left(Q^2 d + 2\,Qa - d\right) \Big(\gamma\,Q^2 \Big(\left(Q^2 d - d\right)^2 - 4\,Q^2 a^2\Big) \delta - \left(Q^2 e - e\right)^2 + 4\,Q^2 b^2\Big) N^4$$
$$- 4\,Q^5 \Big(\Big(\left(Q^2 d - d\right)^2 - 4\,Q^2 a^2\Big) c + 2\,\gamma\,Q^2 a \Big(\left(Q^2 d - d\right)^2 - 4\,Q^2 a^2\Big) \delta + \Big(Q^4 e^2 + 4\,Q^2 b^2 - 2\,Q^2 e^2$$
$$+ e^2\Big) a - 2\,bde \left(Q^2 - 1\right)^2 \Big) N^3 - Q^6 \left(Q^2 d - 2\,Qa - d\right) \Big(\gamma\,Q^2 \Big(\left(Q^2 d - d\right)^2 - 4\,Q^2 a^2\Big) \delta + 4\,Q^2 b^2$$
$$- \left(Q^2 e - e\right)^2 \Big) N^2 - 2\,Q^9 \left(Q^2 d - 2\,Qa - d\right)^2 \left(2\,Q^2 a \delta\,\gamma + c\right) N - \gamma\,Q^{12} \delta\, \left(Q^2 d - 2\,Qa - d\right)^3 \Big) \Big\}$$
$$\Big/ \Big\{ \left(\left(Q^2 d + 2\,Qa - d\right) N^2 + Q^8 d - 2\,Q^7 a - Q^6 d\right) \Big(\left(Q^2 d + 2\,Qa - d\right) N^2$$
$$+ Q^4 d - 2\,Q^3 a - Q^2 d\Big) \left(\left(Q^2 d + 2\,Qa - d\right) N^2 + Q^6 d - 2\,Q^5 a - Q^4 d\right)^2 \Big\}.$$

4.5 Extension of the algorithms implemented in the Maple package `retode`

As shown in the last section 4.4, the classical orthogonal polynomials on a quadratic or a q-quadratic lattice satisfy a recurrence equation

$$p_n(x) = (A_n x + B_n) p_n(x) - C_n p_{n-1}(x),$$

with A_n, B_n, C_n given explicitly. If a holonomic recurrence equation (i. e. linear, homogeneous with polynomial coefficients)

$$q_n(x) p_{n+2}(x) + r_n(x) p_{n+1}(x) + s_n(x) p_n(x) = 0 \ (q_n(x),\ r_n(x),\ s_n(x) \in \mathbb{Q}[n,x]), \quad (4.35)$$

is given as input, the Maple implementations `rec2ortho` of Koorwinder and Swarttouw or `retode` of Koepf and Schmersau can identify its solution which is a (linear transformation of a) classical orthogonal polynomial system of a continuous, a discrete or a q-discrete variable, if applicable. The two implementations `rec2ortho` and `retode` do not handle classical orthogonal polynomials on a quadratic or q-quadratic lattice. Alhaidari [2017] (see also [Alhaidari, 2019]) submitted (as open problem during the 14th International Symposium on Orthogonal Polynomials, Special Functions and Applications) two polynomials defined by their three-term recurrence relations and initial values. He was interested by the derivation of their weight functions, generating functions, orthogonality relations, etc.. Motivated by this open problem and as suggested in the comments by Van Assche [2019], our aim in this section is to implement, using the same approach as Koepf and Schmersau [2002], algorithms to test whether a given holonomic recurrence equation has classical orthogonal polynomial solutions on a quadratic or a q-quadratic lattice. The algorithms were explicitly given and explained in [Koepf and Schmersau, 2002] for classical orthogonal polynomials of a continuous, a discrete or a q-discrete variable and we will adapt them here for classical orthogonal polynomials on a quadratic or a q-quadratic lattice according to the basis in which the polynomials are expanded.

4.5.1 Polynomials expanded in the basis $\{\vartheta_n(\alpha, x)\}_{n\geq 0}$

Algorithm 1 (cf. [Koepf and Schmersau, 2002, Algorithms 1 and 2]). This algorithm takes as input a holonomic three-term recurrence equation of type (4.35) and decides if it has (a linear transformation of) classical orthogonal polynomial solutions expanded in the basis $\{\vartheta_n(\alpha, x)\}_{n\geq 0}$, and returns its divided-difference equation if applicable.

1. `Input`: A holonomic three-term recurrence equation

$$q_n(x)p_{n+2}(x) + r_n(x)p_{n+1}(x) + s_n(x)p_n(x) = 0 \ (q_n(x),\ r_n(x),\ s_n(x) \in \mathbb{Q}[n, x]).$$

2. `Shift`: Shift by $\max\{n \in \mathbb{N}_0 \mid n$ is a zero of either $q_{n-1}(x)$ or $s_n(x)\}+1$ if necessary.

3. `Rewriting`: Rewrite the recurrence equation in the form

$$p_{n+1}(x) = t_n(x)p_n(x) + u_n(x)p_{n-1}(x) \quad (t_n(x), u_n(x) \in \mathbb{Q}(n, x)).$$

   If either $t_n(x)$ is not a polynomial of degree one in x or $u_n(x)$ is not a constant with respect to x, return "`no classical orthogonal polynomial solution exists`"; exit.

4. `Linear transformation`: Use the linear transformation $x \mapsto (x - g)/f$ with unknowns f and g to rewrite the recurrence equation.

5. `Standardization`: Rewrite the latter recurrence equation as

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x) \quad (A_n,\ B_n,\ C_n\ \in \mathbb{Q}(n),\ A_n \neq 0). \quad (4.36)$$

   Since $\frac{k_n}{k_{n+1}} A_n = 1$ in Proposition 4.28, define

$$\frac{k_{n+1}}{k_n} := A_n = \frac{v_n}{w_n} \quad (v_n,\ w_n\ \in \mathbb{Q}[n]).$$

6. `Make monic`: Since $p_n(x) = k_n\tilde{p}_n(x)$ where $\tilde{p}_n(x)$ is the monic family, we can rewrite (4.36) as

$$\tilde{p}_{n+1}(x) = (x + \tilde{B}_n)\tilde{p}_n(x) - \tilde{C}_n\tilde{p}_{n-1}(x),$$

$$\text{with} \quad \tilde{B}_n := \frac{B_n}{A_n} \in \mathbb{Q}(n) \quad \text{and} \quad \tilde{C}_n := \frac{C_n}{A_n A_{n-1}} \in \mathbb{Q}(n),$$

and bring these rational functions in lowest terms. According to Proposition 4.28, if the degree of the numerator of $\tilde{B}_n$ is larger than 4, if the degree of the denominator of $\tilde{B}_n$ is larger than 2, if the degree of the numerator of $\tilde{C}_n$ is larger than 8, or if the degree of the denominator of $\tilde{C}_n$ is larger than 4, then return "`no classical orthogonal polynomial solution exists`".

7. `Polynomial identities`: Set

$$\tilde{B}_n = \frac{k_n}{k_{n+1}} B_n, \ \tilde{C}_n = \frac{k_{n-1}}{k_{n+1}} C_n,$$

with the right-hand sides given in Proposition 4.28, in terms of the unknowns a, b, c, d, e. Multiply these identities by their common denominators, and bring them therefore in polynomial form.

8. `Equating coefficients`: Equate the coefficients of the powers of n in the two resulting equations. This results in a nonlinear system in the unknowns a, b, c, d, e, f and g. Solve this system by Gröbner bases methods. If the system has no solution or only one with $a = d = 0$, then return '`no classical orthogonal polynomial solution exists`"; exit.

9. `Output`: Return the solution vector (a, b, c, d, f, g) of the last step, the divided-difference equation (4.3) together with the information $\frac{k_{n+1}}{k_n}$ and $y = fx + g$.

Example 4.31. *For the first example, we consider the three-term recurrence equation satisfied by the Wilson polynomials*

$$\tilde{W}_n(x^2) := \tilde{W}_n(x^2; a, b, c, d) = \frac{W_n(x^2; a, b, c, d)}{(a+b)_n (a+c)_n (a+d)_n}$$

given by [Koekoek et al., 2010, Eq. (9.1.4)]

$$-(a^2 + x^2)\tilde{W}_n(x^2) = A_n \tilde{W}_{n+1}(x^2) - (A_n + C_n)\tilde{W}_n(x^2) + C_n \tilde{W}_{n-1}(x^2),$$

where

$$A_n = \frac{(n+a+b+c+d-1)(n+a+b)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(2n+a+b+c+d)},$$
$$C_n = \frac{n(n+b+c-1)(n+b+d-1)(n+c+d-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)}.$$

Using our implementation, the result is obtained by

```
> A[n]:=(n+a+b+c+d-1)*(n+a+b)*(n+a+c)*(n+a+d)/((2*n+a+b+c+d-1)
> *(2*n+a+b+c+d))
```

$$A_n := \frac{(n+a+b+c+d-1)(n+a+b)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)\quad(2n+a+b+c+d)}$$

```
> C[n]:=n*(n+b+c-1)*(n+b+d-1)*(n+c+d-1)/((2*n+a+b+c+d-2)
> *(2*n+a+b+c+d-1))
```

$$C_n := \frac{n(n+b+c-1)(n+b+d-1)(n+c+d-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)}$$

```
> RecWilson:= -(a^2+x)*p(n)=A[n]*p(n+1)-(A[n]+C[n])*p(n)+C[n]*p(n-1):
```

```
> strict:=true:
> REtoWilsonDE(subs(n=n+1, RecWilson),p(n),x)
```

$$\text{`Warning: parameters have the values`}, \left\{a = a, b = -a\left(ab + ac + ad + cb + bd + cd\right),\right.$$
$$\left. c = a^2bcd, d = a^2 + ab + ac + ad, e = -a^2bc - a^2bd - a^2cd - abcd\right\}$$

$$\Big[\left(abcd - abx - acx - adx - cbx - bdx - cdx + x^2\right) DD\left(DD\left(p\left(n,x\right),x\right),x\right)$$
$$-\left(abc + abd + acd + bcd - ax - bx - cx - dx\right) SS\left(DD\left(p\left(n,x\right),x\right),x\right)$$
$$-n\left(n + a + b + c + d - 1\right) p\left(n,x\right) = 0,$$
$$\frac{k_{n+1}}{k_n} = -\frac{\left(2\,n + a + b + c + d - 1\right)\left(2\,n + a + b + c + d\right)}{\left(n + a + b + c + d - 1\right)\left(n + a + b\right)\left(n + a + c\right)\left(n + a + d\right)}\Big],$$

which gives the divided-difference equation of the Wilson polynomials (see [Njionou Sadjang et al., 2015b, Thm. 2.5]), as well as the term ratio k_{n+1}/k_n. Here SS and DD stand for $\mathbf{S}_x$ and $\mathbf{D}_x$, respectively.

Example 4.32. *Alhaidari [2017] encountered two families of orthogonal polynomials on the real line defined by their three-term recurrence relations and initial values. The first system is given by*

$$\begin{aligned}\cos\theta H_n^{(\mu,\nu)}(z;\alpha,\theta) &= \Bigg(z\sin\theta\left[\left(n + \frac{\mu+\nu+1}{2}\right)^2 + \alpha\right] \\ &+ \frac{\nu^2-\mu^2}{(2n+\mu+\nu)(2n+\mu+\nu+2)}\Bigg) H_n^{(\mu,\nu)}(z;\alpha,\theta) + \frac{2(n+\mu)(n+\nu)}{(2n+\mu+\nu)(2n+\mu+\nu+1)} \\ &\times H_{n-1}^{(\mu,\nu)}(z;\alpha,\theta) + \frac{2(n+1)(n+\mu+\nu+1)}{(2n+\mu+\nu+1)(2n+\mu+\nu+2)} H_{n+1}^{(\mu,\nu)}(z;\alpha,\theta),\end{aligned} \tag{4.37}$$

with $0 \le \theta \le \pi$, $\mu,\nu > -1$, $\alpha \in \mathbb{R}$ and initial values $H_0^{(\mu,\nu)}(z;\alpha,\theta) = 1$, $H_{-1}^{(\mu,\nu)}(z;\alpha,\theta) = 0$. The second system is

$$\begin{aligned}zG_n^{(\mu,\nu)}(z;\sigma) &= \Bigg((\sigma + B_n^2)\left[\frac{\mu^2-\nu^2}{(2n+\mu+\nu)(2n+\mu+\nu+2)} + 1\right] - \frac{2n(n+\nu)}{2n+\mu+\nu} \\ &- \frac{(\mu+1)^2}{2}\Bigg) G_n^{(\mu,\nu)}(z;\sigma) - (\sigma + B_{n-1}^2)\frac{2(n+\mu)(n+\nu)}{(2n+\mu+\nu)(2n+\mu+\nu+1)} G_{n-1}^{(\mu,\nu)}(z;\sigma) \\ &- (\sigma + B_n^2)\frac{2(n+1)(n+\mu+\nu+1)}{(2n+\mu+\nu+1)(2n+\mu+\nu+2)} G_{n+1}^{(\mu,\nu)}(z;\sigma),\end{aligned} \tag{4.38}$$

with $B_n = n + 1 + \frac{\mu+\nu}{2}$, $\mu,\nu > -1$ and $\sigma \in \mathbb{R}$ and initial values $G_0^{(\mu,\nu)}(z;\sigma) = 1$, $G_{-1}^{(\mu,\nu)}(z;\sigma) = 0$.

Our implementation with

```
> BB:=n->n+1+(mu+nu)/2:
> recOpen2:=z*p(n)=((sigma+BB(n)^2)*((mu^2-nu^2)/((2*n+mu+nu)
> *(2*n+mu+nu+2))+1)-2*n*(n+nu)/(2*n+mu+nu)-(mu+1)^2/2)*p(n)
> -(sigma+BB(n-1)^2)*2*(n+mu)*(n+nu)/((2*n+mu+nu)*(2*n+mu+nu+1))*p(n-1)
> -(sigma+BB(n)^2)*2*(n+1)*(n+mu+nu+1)/((2*n+mu+nu+1)*(2*n+mu+nu+2))*p(n+1):
> strict:=false:
```

```
> REtoWilsonDE(subs(n=n+1, recOpen2),p(n),z)
```

returns six divided-difference equations (see the Maple file associated to this work) for the second recurrence equation (4.38):

$$\left(z^2+\left(\mu^2-2\,\sigma-3\right)z+\frac{1}{4}\,(\mu-1)^2\left(\mu^2+2\,\mu+4\,\sigma+1\right)\right)\mathbf{D}_z^2P_n(z/2)$$
$$+\,(4\,z+2\,(\mu-1)\,(\mu+2\,\sigma+1))\mathbf{S}_z\mathbf{D}_zP_n(z/2)-4n(n+1)P_n(z/2)=0\ (\nu=-\mu), \tag{4.39}$$

$$\left(z^2+\left(\mu^2+2\,\mu-2\,\sigma+1\right)z+\frac{1}{4}\,(\mu+1)^2\left(\mu^2+2\,\mu+4\,\sigma+1\right)\right)\mathbf{D}_z^2P_n(z/2)$$
$$-\,4\,\sigma\,(\mu+1)\,\mathbf{S}_z\mathbf{D}_zP_n(z/2)-4n(n-1)P_n(z/2)=0\ (\nu=-\mu-2), \tag{4.40}$$

$$\left(a^2z^2+\left(\left(\mu^2+2\,\mu-2\,\sigma+1\right)a^2-d\,(\mu+1)\,a-\frac{1}{2}\,d^2\right)z\right.$$
$$\left.+\,\frac{1}{4}\,(\mu+1)^2\left(\left(\mu^2+2\,\mu+4\,\sigma+1\right)a^2-2\,d\,(\mu+1)\,a+d^2\right)\right)\mathbf{D}_z^2P_n(z/2)$$
$$+\left(2\,adz-(\mu+1)\left(4\,a^2\sigma-d\,(\mu+1)\,a+d^2\right)\right)\mathbf{S}_z\mathbf{D}_zP_n(z/2)$$
$$-\,4na((n-1)a+d)P_n(z/2)=0\ (a\neq 0,\ d\neq 0)\ (\nu=-\mu-2+d/a), \tag{4.41}$$

$$((4\,\nu+1)\,z-\frac{1}{2}\,\nu^2)\mathbf{D}_z^2P_n(2z)+(\nu^2+\nu-2z)\mathbf{S}_z\mathbf{D}_zP_n(2z)+nP_n(2z)=0, \tag{4.42}$$

$\nu\neq-\frac{1}{4}$ with $\sigma=0$, $\mu=\nu-1$,

$$\left((4\,\nu+3)\,z-\frac{1}{2}\,\nu\,(\nu+1)\right)\mathbf{D}_z^2P_n(2z)+(\nu^2+2\nu-2z+\frac{1}{2})\mathbf{S}_z\mathbf{D}_zP_n(2z)+nP_n(2z)=0, \tag{4.43}$$

$\nu\neq-\frac{3}{4}$ with $\sigma=-\frac{1}{4}$, $\mu=\nu$,

$$\left((4\,\nu+5)\,z-\frac{1}{2}\,(\nu+1)^2\right)\mathbf{D}_z^2P_n(2z)+(\nu^2+3\nu-2z+2)\mathbf{S}_z\mathbf{D}_zP_n(2z)+nP_n(2z)=0, \tag{4.44}$$

$\nu\neq-\frac{5}{4}$ with $\sigma=0$, $\mu=\nu+1$. By comparison with the Wilson divided-difference equation, we deduce from the first three divided-difference equations (4.39)–(4.41) that

$$G_n^{(\mu,\nu)}(z;\sigma)=constant\times W_n(z/2;a,b,c,d)$$

where a,b,c,d are permutations of elements of the set

$$\{\frac{1}{2}(-\mu+1),\frac{1}{2}(-\mu+1),\frac{1}{2}(\mu+1)+\sqrt{-\sigma},\frac{1}{2}(\mu+1)-\sqrt{-\sigma}\},$$
$$\{\frac{1}{2}(\mu+1),\frac{1}{2}(\mu+1),-\frac{1}{2}(\mu+1)+\sqrt{-\sigma},-\frac{1}{2}(\mu+1)-\sqrt{-\sigma}\},$$
$$\{\frac{1}{2}(\mu+1),\frac{1}{2}(\mu+1),\frac{1}{2}(\delta-\mu-1)+\sqrt{-\sigma},\frac{1}{2}(\delta-\mu-1)-\sqrt{-\sigma}\},$$

for the first equation (4.39), the second equation (4.40), and the third equation (4.41) in which the parameters $a = 1$, $d = \delta$, respectively. This brings therefore a new parameter δ in the definition of the polynomial $G_n^{(\mu,\nu)}(z;\sigma)$ and we also remark that for $d = 0$ and $a = 1$ in the third equation (4.41), we recover the second equation (4.40). For the value $d = \delta = 2\mu+2$, we recover (from the identification of (4.41) with the Wilson polynomials) the solution

$$G_n^{(\mu,\mu)}(z;\sigma) = \frac{W_n(z/2;a,b,c,d)}{n!(a+b)_n(a+d)_n}$$

given in [Van Assche, 2019] where $a = c = \frac{\mu+1}{2}$, $b = \frac{\mu+1}{2} + \sqrt{-\sigma}$ and $d = \frac{\mu+1}{2} - \sqrt{-\sigma}$. Comparing the last three equations (4.42)–(4.44) with the divided-difference equation of the continuous dual Hahn polynomials $S_n(2z;a,b,c)$, we deduce that

$$G_n^{(\nu-1,\nu)}(z;0) = constant \times S_n(2z;a,b,c),$$

with $a = c = \nu$, $b = \frac{1}{2}$, or $a = b = \nu, c = \frac{1}{2}$ or $b = c = \nu, a = \frac{1}{2}$, i. e., a,b,c are permutations of elements of the set $\{\nu,\nu,\frac{1}{2}\}$;

$$G_n^{(\nu,\nu)}(z;-1/4) = constant \times S_n(2z;a,b,c),$$

where a,b,c are permutations of elements of the set $\{\nu+1,\ \nu,\ \frac{1}{2}\}$;

$$G_n^{(\nu+1,\nu)}(z;0) = constant \times S_n(2z;a,b,c),$$

where a,b,c are permutations of the elements of the set $\{\nu+1,\nu+1,\frac{1}{2}\}$.

4.5.2 Polynomials expanded in the basis $\{(\alpha+ix)_n\}_{n\geq 0}$

The steps of the algorithm in this case agree with those given in Section 4.5.1. In steps 5 and 7, we use Proposition 4.29 whereas in step 6, the algorithm will return "`no classical orthogonal polynomial solution exists`" if the degree of the numerator or the denominator of $\tilde{B}_n$ is larger than 2, if the degree of the numerator of $\tilde{C}_n$ is larger than 7, or if the degree of the denominator of $\tilde{C}_n$ is larger than 5.

Example 4.33. *As example here, starting from the three-term recurrence equations (*`RE`*) [Koekoek et al., 2010, Eq. (9.4.3)] and [Koekoek et al., 2010, Eq. (9.7.3)] satisfied by the continuous Hahn and the Meixner-Pollaczek polynomials, respectively, and using our implementation with* `REtoContHahnDE(subs(n=n+1, RE), p(n), x)`*, we recover the divided-difference equations of type (4.4) satisfied by both families (see [Tcheutia et al., 2017, prop. 4]). In the output in this case, SS and DD stand for* $\mathbf{S}_x$ *and* δ_x*, respectively.*

4.5.3 Polynomials expanded in the basis $\{\chi_n(\gamma,\delta,\lambda(x))\}_{n\geq 0}$

We proceed as in the algorithm of Section 4.5.1. Here, in steps 5 and 7, we use Proposition 4.30 whereas in step 6, the algorithm will return "`no classical orthogonal polynomial solution exists`" if the degree of the numerator of $\tilde{B}_n$ is larger than 4, if the degree of the denominator of $\tilde{B}_n$ is larger than 2, if the degree of the numerator of $\tilde{C}_n$ is larger than 8, or if the degree of the denominator of $\tilde{C}_n$ is larger than 4.

Example 4.34. *If we call* `RE` *the three-term recurrence equation of the Racah or the dual Hahn polynomials given, respectively, by [Koekoek et al., 2010, Eq. (9.2.3)] and [Koekoek et al., 2010, Eq. (9.6.3)], then with our implementation* `REtoRacahDE(subs(n=n+1, RE), p(n),x)`*, we get the divided-difference equation satisfied by both families (see the associated Maple file).*

Remark 4.35. *From our implementations of sections 3.1., 3.2 and 3.3, we get for the recurrence equation (4.37) the following:*

```
> RE:=1/2*(y+y^(-1))*p(n)=(z/(2*I)*(y-y^(-1))*((n+(mu+nu+1)/2)^2+alpha)+
(nu^2-mu^2)/((2*n+mu+nu)*(2*n+mu+nu+2)))*p(n)
> +2*(n+mu)*(n+nu)/((2*n+mu+nu)*(2*n+mu+nu+1))*p(n-1)
> +2*(n+1)*(n+mu+nu+1)/((2*n+mu+nu+1)*(2*n+mu+nu+2))*p(n+1):
> strict:=false:
> REtoWilsonDE(subs(n=n+1, RE), p(n), z)
```

`'Warning: parameters have the values'`, $\{a = a, \alpha = \alpha, b = b, c = c, d = d,\}$

$$\{e = e, f = 0, g = g, \mu = \mu, \nu = \nu, y = 1\}$$

`'Warning: parameters have the values'`, $\{a = a, \alpha = \alpha, b = b, c = c, d = d,\}$

$$\{e = e, f = 0, g = g, \mu = \mu, \nu = \nu, y = 1\}$$

$$[[DD\left(DD\left(p\left(n,g\right),z\right),z\right) + \frac{\left(dg+e\right)SS\left(DD\left(p\left(n,g\right),z\right),z\right)}{ag^2+bg+c} - \frac{n\left(an-a+d\right)p\left(n,g\right)}{ag^2+bg+c} = 0],]$$

We get the same answer using

```
> REtoRacahDE(subs(n=n+1, RE),p(n),z)
```

and

```
> REtoContHahnDE(subs(n=n+1, RE),p(n),z)
```

We deduce from our implementations (which return the solution $H_n^{(\mu,\nu)}(z;\alpha,\theta) \equiv 0$) that the polynomial family $H_n^{(\mu,\nu)}(z;\alpha,\theta)$ satisfying the recurrence equation (4.37) is not related (by a linear transformation) to a known classical orthogonal polynomial sequence on a quadratic lattice expanded in the basis $\{\vartheta_n(\alpha,x)\}$, $\{(\alpha+ix)_n\}$ or $\{\chi_n(\gamma,\delta,\lambda(x))\}$. This recurrence equation may lead to a new family of orthogonal polynomials. This problem by Alhaidari remains then open.

4.5.4 Polynomials expanded in the basis $\{B_n(\alpha,x)\}_{n\geq 0}$

Algorithm 2 (see [Koepf and Schmersau, 2002, Algorithm 3]). This algorithm takes as input a holonomic three-term recurrence equation of type (4.35) and decides if it has (a linear transformation of) classical orthogonal polynomial solutions expanded in the basis $\{B_n(\alpha,x)\}_{n\geq 0}$, and returns its divided-difference equation if applicable.

1. `Input`: A holonomic three-term recurrence equation

$$q_n(x)p_{n+2}(x) + r_n(x)p_{n+1}(x) + s_n(x)p_n(x) = 0 \;\; (q_n(x),\; r_n(x),\; s_n(x) \in \mathbb{Q}[q^n, \sqrt{q}, x])\,.$$

2. `Shift`: Shift by $\max\{n \in \mathbb{N}_0 \mid n$ is a zero of either $q_{n-1}(x)$ or $s_n(x)\}+1$ if necessary.

3. `Rewriting`: Rewrite the recurrence equation in the form

$$p_{n+1}(x) = t_n(x)p_n(x) + u_n(x)p_{n-1}(x) \quad (t_n(x), u_n(x) \in \mathbb{Q}(q^n, \sqrt{q}, x))\,.$$

If either $t_n(x)$ is not a polynomial of degree one in x or $u_n(x)$ is not a constant with respect to x, return "`no classical orthogonal polynomial solution exists`"; exit.

4. `Linear transformation`: Rewrite the recurrence equation by the linear transformation $x \mapsto (x-g)/f$ with unknowns f and g.

5. `Standardization`: Rewrite the latter recurrence equation as

$$p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x) \ \ (A_n,\ B_n,\ C_n \ \in \mathbb{Q}(q^n, \sqrt{q}),\ A_n \neq 0)\,.$$

In Section 4.4.4, we get $\frac{k_n}{k_{n+1}} A_n = -2\alpha q^n$ from which we define

$$\frac{k_{n+1}}{k_n} := -\frac{1}{2\alpha q^n} A_n = \frac{v_n}{w_n} \ \ (v_n,\ w_n \ \in \mathbb{Q}[q^n, \sqrt{q}])\,.$$

6. `Make monic`: Since $p_n(x) = k_n \tilde{p}_n(x)$, the last recurrence equation becomes

$$\tilde{p}_{n+1}(x) = ((-2\alpha q^n)x + \tilde{B}_n)\tilde{p}_n(x) - \tilde{C}_n \tilde{p}_{n-1}(x),$$

$$\text{with } \ \tilde{B}_n := \frac{B_n}{A_n} \in \mathbb{Q}(q^n, \sqrt{q}) \ \text{ and } \ \tilde{C}_n := \frac{C_n}{A_n A_{n-1}} \in \mathbb{Q}(q^n, \sqrt{q}).$$

If the degree (w. r. t. $N := q^n$) of the numerator of $\tilde{B}_n$ is larger than 3, if the degree of the denominator of $\tilde{B}_n$ is larger than 4, or if the degree of the numerator or the denominator of $\tilde{C}_n$ is larger than 8, then return "`no classical orthogonal polynomial solution exists`".

7. `Polynomial identities`: Set

$$\tilde{B}_n = \frac{k_n}{k_{n+1}} B_n,\ \ \tilde{C}_n = \frac{k_{n-1}}{k_{n+1}} C_n,$$

with the right-hand sides given in Section 4.4.4, in terms of the unknowns $a,\ b,\ c,\ d,\ e$. Multiply these identities by their common denominators, and bring them therefore in polynomial form.

8. `Equating coefficients`: Equate the coefficients of the powers of $N = q^n$ in the two resulting equations. This results in a nonlinear system in the unknowns $a,\ b,\ c,\ d,\ e,\ f$ and g. Solve this system by Gröbner bases methods. If the system has no solution or only one with $a = d = 0$, then return '`no classical orthogonal polynomial solution exists`"; exit.

9. `Output`: Return the solution vector (a, b, c, d, f, g) of the last step, the divided-difference equation (4.1) together with the information $\frac{k_{n+1}}{k_n}$ and $y = fx + g$.

Example 4.36. *As illustrative example, we use our implementation to find the divided-difference equation of type* (4.1) *satisfied by the continuous Hermite polynomials.*

```
> recContinuousqHermite:=2*x*p(n)=p(n+1)+(1-q^n)*p(n-1)
```

$$recContinuousqHermite := 2\,xp\,(n) = p\,(n+1) + (1-q^n)\,p\,(n-1)$$

```
> strict:=true:
```

> *REtoAskeyWilsonDE(subs(n=n+1, recContinuousqHermite), p(n),q, x)*

$$[1/2\,\left(2\,x^2-1\right) DD\left(DD\left(p\left(n,x\right),x\right),x\right)-2\,{\frac{xSS\left(DD\left(p\left(n,x\right),x\right),x\right)\sqrt{q}}{q-1}}$$
$$+2\,{\frac{q^{3/2}\left(-1+q^n\right)p\left(n,x\right)}{q^n\left(q-1\right)^2}}=0, \frac{k_{n+1}}{k_n}=2]$$

In the result, SS and DD stand for $\mathbb{S}_x$ and $\mathbb{D}_x$, respectively. The results for the other families can be found in the accompanying Maple file.

4.5.5 Polynomials expanded in the basis $\{\xi_n(\gamma,\delta,\mu(x))\}_{n\geq 0}$

The steps of the algorithm in this case agree with those given in Section 4.5.4. In steps 5 and 7, we use the results from Section 4.4.5 whereas in step 6, the algorithm will return "`no classical orthogonal polynomial solution exists`" if the degree of the numerator of $\tilde{B}_n$ is larger than 3, if the degree of the denominator of $\tilde{B}_n$ is larger than 4, if the degree of the numerator or the denominator of $\tilde{C}_n$ is larger than 8.

Example 4.37. *If we consider for example the recurrence equation* `RE` *for the q-Racah, dual q-Hahn, dual q-Krawtchouk polynomials given, respectively, by [Koekoek et al., 2010, Eq. (14.2.3)], [Koekoek et al., 2010, Eq. (14.7.3)], [Koekoek et al., 2010, Eq. (14.17.3)], we use our implementation* `REtoqRacahDE(subs(n=n+1,RE), p(n), x, q)` *to get the divided-difference equations satisfied by the three families of polynomials and the product $\gamma\delta$.*

`Note`: The Maple implementation `retode` by Koepf and Schmersau has been updated with our extension to classical orthogonal polynomials on a quadratic or a q-quadratic lattice. The package `retode.mpl` and a worksheet `retodedemo.mw` containing the three-term recurrence equations of Section 4.4 and the examples for all the classical orthogonal polynomials on a quadratic or a q-quadratic lattice can be obtained from `http://www.mathematik.uni-kassel.de/~tcheutia/`.

Bibliography

A. D. Alhaidari. Open problem in orthogonal polynomials. *arXiv:math-ph/1709.06081v1*, 2017. v, 4, 95, 98

A. D. Alhaidari. Special orthogonal polynomials in quantum mechanics. *Symmetry*, 11, 2019. 95

I. Area, D. K. Dimitrov, E. Godoy, and A. Ronveaux. Zeros of Gegenbauer and Hermite polynomials and connection coefficients. *Math. Comp.*, 73:1937–1951, 2004. 12

I. Area, D. K. Dimitrov, E. Godoy, and F. R. Rafaeli. Inequalities for zeros of Jacobi polynomials via Obrechkoff's theorem. *Math. Comp.*, 81:991–1912, 2012. 12, 22

I. Area, D. K. Dimitrov, E. Godoy, and V. G. Paschoa. Zeros of classical orthogonal polynomials of a discrete variable. *Math. Comp.*, 82:1069–1095, 2013. 12

R. Askey. Graphs as an aid to understanding special functions. In *Asymptotic and Computational Analysis, Winnipeg 1989*, volume 124, pages 3–33. Lecture Notes in Pure and Applied Mathematics, 1990. 6

A. F. Beardon. The theorems of Stieltjes and Favard. *Comput. Methods Funct. Theory*, 11:247–262, 2011. 10

C. M. Bender, S. Boettcher, and V. M. Savage. Conjecture on the interlacing of zeros in complex Sturn-Liouville problems. *J. Math. Phys.*, 41:6381–6387, 2000. 3

O. Bottema. Die Nullstellen gewisser durch Rekursionsformeln definierter Polynome. *Proc. Amsterdam*, 34:681–691, 1931. 12, 28

C. F. Bracciali, F. Marcellán, and S. Varma. Orthogonality of quasi-orthogonal polynomials. *Filomat*, 32:6953–6977, 2018. 10

C. Brezinski, K. A. Driver, and M. Redivo-Zaglia. Quasi-orthogonality with applications to some families of classical orthogonal polynomials. *Appl. Numer. Math.*, 48:157–168, 2004. 7, 8, 9, 10, 21, 26, 43, 48, 49, 54

A. Bultheel, R. Cruz-Barroso, and M. Van Barel. On Gauss-type quadrature formulas with prescribed nodes anywhere on the real line. *Calcolo.*, 47:21–48, 2010. 9

W. Y. C. Chen, Q.-H. Hou, and Y.-P. Mu. The extended Zeilberger algorithm with parameters. *J. Symbolic Comput.*, 47:643–654, 2012. 16

L. Chihara and D. Stanton. Zeros of generalized Krawtchouk polynomials. *J. Approx. Theory*, 60:43–57, 1990. 6, 34

T. S. Chihara. On quasi-orthogonal polynomials. *Proc. Amer. Math. Soc.*, 8:765–767, 1957. 9, 10

T. S. Chihara. *An Introduction to Orthogonal Polynomials*. Gordon and Breach, New York, 1978. 2, 4, 88

S. Cooper. The Askey-Wilson operator and the ${}_6\phi_5$ summation formula. *South East Asian J. Math. Sci.*, 1:71–82, 2002. 73

G. Criscuolo, G. Mastroianni, and D. Occorsio. Convergence of extended Lagrange interpolation. *Math. Comp.*, 55:197–212, 1990. 3

D. Dickinson. On quasi-orthogonal polynomials. *Proc. Amer. Math. Soc.*, 12:185–194, 1961. 9

D. K. Dimitrov and G. P. Nikolov. Sharp bounds for the extreme zeros of classical orthogonal polynomials. *J. Approx. Theory*, 162:1793–1804, 2010. 12

D. K. Dimitrov and F. R. Rafaeli. Monotonocity of zeros of Laguerre polynomials. *J. Comput. Appl. Math.*, 233:699–702, 2009. 12

A. Draux. On quasi-orthogonal polynomials. *J. Approx. Theory*, 62:1–14, 1990. 9

V. G. Drinfeld. Quantum groups. In *Proceedings of the International Congress of Math., Berkeley, 1986*, pages 798–820. Amer. Math. Soc., 1987. 37

K. Driver and A. Jooste. Quasi-orthogonal Meixner polynomials. *Quaest. Math.*, 40: 477–490, 2017. 9

K. Driver and K. Jordaan. Interlacing of zeros of shifted sequences of one-parameter orthogonal polynomials. *Numer. Math.*, 107:615–624, 2007. 25

K. Driver and K. Jordaan. Stieltjes interlacing of zeros of Laguerre polynomials from different sequences. *Indag. Math.*, 21:204–211, 2011. 25, 26

K. Driver and K. Jordaan. Bounds for extreme zeros of some classical orthogonal polynomials. *J. Approx. Theory*, 164:1200–1204, 2012. 11, 12, 22, 27, 28

K. Driver and K. Jordaan. Zeros of quasi-orthogonal Jacobi polynomials. *SIGMA*, 12:13 pages, 2016. 9

K. Driver and K. Jordaan. Zeros of Jacobi polynomials $p_n^{(\alpha,\beta)}$, $-2 < \alpha,\ \beta < -1$. *Numer. Algorithms*, 79:1075–1085, 2018. 3, 21

K. Driver and M. E. Muldoon. Interlacing properties and bounds for zeros of some quasi-orthogonal Laguerre polynomials. *Comput. Methods Funct. Theory*, 15:645–654, 2015. 9, 26

K. Driver and M. E. Muldoon. Bounds for extreme zeros of quasi-orthogonal ultraspherical polynomials. *J. Classical Anal.*, 9:69–78, 2016. 3, 9

K. Driver, K. Jordaan, and N. Mbuyi. Interlacing of the zeros of Jacobi polynomials with different parameters. *Numer. Algorithms*, 49:143–152, 2008. 6, 20, 21

L. Fejér. Mechanische Quadraturen mit positiven Cotesschen Zahlen. *Math. Z.*, 37:287–309, 1933. 9

M. Foupouagnigni. On difference equations for orthogonal polynomials on non-uniform lattices. *J. Difference Equ. Appl.*, 14:127–174, 2008. 2, 72, 73, 88

M. Foupouagnigni, W. Koepf, D. D. Tcheutia, and P. Njionou Sadjang. Representations of q-orthogonal polynomials. *J. Symbolic Comput.*, 47:1347–1371, 2012. 13, 14, 15

M. Foupouagnigni, W. Koepf, and D. D. Tcheutia. Connection and linearization coefficients of the Askey-Wilson polynomials. *J. Symbolic Comput.*, 53:96–118, 2013. 73, 89, 92, 93

G. Gasper and M. Rahman. *Basic Hypergeometric Series*, volume 35. Encyclopedia Math. Appl., Cambridge University Press, Cambridge, 1990. 3

G. Gasper and M. Rahman. *Basic Hypergeometric Series*, volume 96. Encyclopedia Math. Appl., Cambridge University Press, Cambridge, 2nd ed. edition, 2004. 37, 42

P. Gochhayat, K. Jordaan, K. Raghavendar, and A. Swaminathan. Interlacing properties and bounds for zeros of ${}_2\phi_1$ hypergeometric and little q-Jacobi polynomials. *Ramanujan J.*, 40:45–62, 2016. 6, 7, 8, 12, 15, 50, 52

Z. Grinshpun. Special linear combinations of orthogonal polynomials. *J. Math. Anal. Appl.*, 299:1–18, 2004. 4

D. P. Gupta and M. E. Muldoon. Inequalities for the smallest zeros of Laguerre polynomials and their q-analogues. *J. Inequal. Pure Appl. Math.*, 8, 2007. 12, 26, 28, 56, 64

D. P. Gupta, M. E. H. Ismail, and D. R. Masson. Contiguous relations, basic hypergeometric functions, and orthogonal polynomials. II. Associated big q-Jacobi polynomials. *J. Math. Anal. Appl.*, 171:477–497, 1992. 6

W. Hahn. Bericht über die Nullstellen der Laguerreschen und der Hermiteschen Polynome. *Jahresbericht der Deustchen Mathematiker-Vereinigung*, 44:215–236, 1933. 27

W. Hahn. Über Orthogonalpolynome, die q-Differenzengleichungen genügen. *Math. Nachr.*, 2:4–34, 1949. 37

E. Heine. Untersuchungen über die Reihe $1+\frac{(1-q^\alpha)(1-q^\beta)}{(1-q)(1-q^\gamma)}x+\frac{(1-q^\alpha)(1-q^{\alpha+1})(1-q^\beta)(1-q^{\beta+1})}{(1-q)(1-q^2)(1-q^\gamma)(1-q^{\gamma+1})}x^2+\cdots$. *Journal für die reine und angewandte Mathematik*, 34:285–328, 1847. 6, 15, 37

M. E. H. Ismail. *Classical and Quantum Orthogonal Polynomials in One Variable*, volume 98. Encyclopedia Math. Appl., Cambridge University Press, Cambridge, 2005. 3, 6

M. E. H. Ismail and X. Li. Bound on the extreme zeros of orthogonal polynomials. *Proc. Amer. Math. Soc.*, 115, 1992. 12

M. E. H. Ismail and D. Stanton. Some combinatorial and analytical identities. *Ann. Comb.*, 16:755–771, 2012. 73

M. E. H. Ismail and X.-S. Wang. On quasi-orthogonal polynomials: their differential equations, discriminants and electrostatics. *J. Math. Anal. Appl.*, 474:1178–1197, 2019. 9

S. J. Johnston, A. Jooste, and K. Jordaan. Quasi-orthogonality of some hypergeometric polynomials. *Integral Transforms Spec. Funct.*, 27:111–125, 2016. 9, 32

A. Jooste and K. Jordaan. Bounds for zeros of Meixner and Kravchuk polynomials. *LMS J. Comput. Math.*, 17:47–57, 2014. 11, 12, 36

A. Jooste, K. Jordaan, and F. Toókos. On the zeros of Meixner polynomials. *Numer. Math.*, 124:57–71, 2013. 9, 35

A. Jooste, P. Njionou Sadjang, and W. Koepf. Inner bounds for the extreme zeros of ${}_3F_2$ hypergeometric polynomials. *Integral Transforms Spec. Funct.*, 28:361–373, 2017. 12, 19, 33

A. S. Jooste, D. D. Tcheutia, and W. Koepf. Recurrence equations involving different orthogonal polynomial sequences and applications. *submitted.* 11, 12, 19, 37

K. Jordaan and F. Toókos. Mixed recurrence relations and interlacing of the zeros of some q-orthogonal polynomials from different sequences. *Acta. Math. Hungar.*, 128:150–164, 2010. 6, 7, 8, 61, 62

K. Jordaan and K. F. Toókos. Interlacing theorems for the zeros of some orthogonal polynomials from different sequences. *Appl. Numer. Math.*, 59:2015–2022, 2009. 6, 15, 31, 34, 35

H. Joulak. A contribution to quasi-orthogonal polynomials and associated polynomials. *Appl. Numer. Math.*, 54:65–78, 2005. 8, 9, 10, 43, 49, 54, 55, 69, 86

S. Karlin and J. L. McGregor. The differential equations of birth-and-death processes and the Stieltjes moment problem. *Trans. Amer. Math. Soc.*, 85:489–546, 1957. 6

R. Koekoek, P. A. Lesky, and R. F. Swarttouw. *Hypergeometric Orthogonal Polynomials and their q-Analogues*. Springer, Berlin, 2010. 2, 3, 13, 37, 40, 44, 46, 49, 55, 68, 69, 72, 73, 85, 88, 97, 100, 101, 103

H. T. Koelink and T. H. Koornwinder. The Clebsch-Gordan coefficients for the quantum group $s_\mu u(2)$ and q-Hahn polynomials. *Proceedings A*, pages 443–456, 1989. 37

W. Koepf. *Hypergeometric Summation – An algorithmic approach to summation and special function identities*. Springer, London, second edition, 2014. 4, 16, 17

W. Koepf and D. Schmersau. Representations of orthogonal polynomials. *J. Comput. Appl. Math.*, 90:57–94, 1998. 13, 14, 15, 89

W. Koepf and D. Schmersau. Recurrence equations and their classical orthogonal polynomials solutions. *Appl. Math. Comput.*, 128:303–327, 2002. v, 4, 89, 96, 101

T. H. Koornwinder. Orthogonal polynomials in connection with quantum groups. In *P. Nevai (Ed.), Orthogonal Polynomials, Theory and Practice*, volume 294, pages 257–292, Dordrecht, 1990. Kluwer Academic Publishers. 37

T. H. Koornwinder. Compact quantum groups and q-special functions. In *V. Baldoni, M.A. Picardello (Eds.), Representations of Lie Groups and Quantum Groups*, pages 46–128, Longman Scientific & Technical, New York, 1994. Pitman Research Notes in Mathematics series. 37

T. H. Koornwinder and R. Swarttouw. rec2ortho: an algorithm for identifying orthogonal polynomials given by their three-term recurrence relation as special functions. *http://turing.wins.uva.nl/ thk/recentpapers/rec2ortho.html*, 1996–1998. v, 4

I. Krasikov. Bounds for zeros of the Charlier polynomials. *Methods Appl. Anal.*, 9:599–610, 2002. 12

I. Krasikov. Turán inequalities and zeros of orthogonal polynomials. *Methods Appl. Anal.*, 12:75–88, 2005. 12

I. Krasikov. On extreme zeros of classical orthogonal polynomials. *J. Comput. Appl. Math.*, 193:168–182, 2006. 12, 23, 28

I. Krasikov and A. Zarkh. On zeros of discrete orthogonal polynomials. *J. Approx. Theory*, 156:121–141, 2009. 12, 33

R. J. Levit. The zeros of the Hahn polynomials. *SIAM Review*, 9:191–203, 1967. 6, 31

F. Locher. Stability tests for linear difference forms. *Internat. Ser. Numer. Math.*, 112: 215–223, 1993. 3

A. P. Magnus. Special nonuniform lattice (snul) orthogonal polynomials on discrete dense sets of points. *J. Comput. Appl. Math.*, 65:253–265, 1995. 72

F. Marcellán and J. Petronilho. On the solution of some distributional differential equations: existence and characterizations of the classical moment functionals. *Integral Transforms Spec. Funct.*, 2:185–218, 1994. 13

P. Maroni. Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques. In *Brezinski C, Gori L, Ronveaux A, editors. Orthogonal polynomials and their applications: Proceedings of the Third International Symposium; 1990 June 1-8; Erice.*, volume 9, pages 95–130, Basel: JC Baltzer AG, 1991. IMACS Ann. Comput. Appl. Math. 9

J. C. Mason and D. C. Handscomb. *Chebyshev Polynomials*. Chapman & Hall/CRC, New York, 2002. 3

G. Mastroianni and D. Occorsio. Interlacing properties of the zeros of the orthogonal polynomials and approximation of the Hilbert transform. *Computers Math. Applic.*, 30:155–168, 1995. 3

J. C. Medem, R. Álvarez Nodarse, and F. Marcellán. On the q-polynomials: A distributional study. *J. Comput. Appl. Math.*, 135:157–196, 2001. 13, 14

D. S. Moak. The q-analogue of the Laguerre polynomials. *J. Math. Anal. Appl.*, 81:20–47, 1981. 6, 61, 62, 63

E. R. Neumann. Beiträge zur Kenntnis der Laguerreschen Polynome. *Jahresber. Deutsch. Math.-Verein.*, 30:15–35, 1921. 12, 28

A. F. Nikiforov and V. B. Uvarov. *Special Functions of Mathematical Physics*. Birkhäuser Verlag, Basel, 1988. 2, 3

A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov. Classical Orthogonal Polynomials of a Discrete Variable. *Springer Ser. Comput. Phys., Springer-Verlag, Berlin*, 1991. 2

P. Njionou Sadjang. *Moments of classical orthogonal polynomials*. Phd dissertation, Universität Kassel, 2013. 73

P. Njionou Sadjang, W. Koepf, and M. Foupouagnigni. On moments of classical orthogonal polynomials. *J. Math. Anal. Appl.*, 424:122–151, 2015a. 73

P. Njionou Sadjang, W. Koepf, and M. Foupouagnigni. On structure formulas for Wilson polynomials. *Integral Transforms Spec. Funct.*, 26:1000–1014, 2015b. 73, 88, 89, 98

F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark. *NIST Handbook of Mathematical Functions*. National Institute of Standards and Technology U.S. Department of Commerce and Cambridge University Press, 2010. 73

F. Peherstorfer. Linear combinations of orthogonal polynomials generating positive quadrature formulas. *Math. Comp.*, 55:231–241, 1990. 3

M. Petkovšek, H. Wilf, and D. Zeilberger. $A = B$. A. K. Peters, Wellesley, 1996. 16

A. P. Prudnikov, Y. A. Brychkov, and O. I. Marichev. *Integrals and Series*, volume 3. Gordon and Breach Science Publishers, New York, 1990. 6

E. D. Rainville. *Special Functions*. The Macmillan Company, New York, 1960. 6

M. Riesz. Sur le problème des moments III. *Ark. f. Mat., Astr. och Fys.*, 17:1–52, 1923. 9

J. Segura. Interlacing of the zeros of contiguous hypergeometric functions. *Numer. Algorithms*, 49:387–407, 2008. 11

J. A. Shohat. On mechanical quadratures, in particular, with positive coefficients. *Trans. Amer. Math. Soc.*, 42:461–496, 1937. 9, 10

R. F. Swarttouw. The contiguous function relations for the basic hypergeometric series. *J. Math. Anal. Appl.*, 149:151–159, 1990. 6, 15

G. Szegő. *Orthogonal Polynomials*, volume XXIII. American Mathematical Society Colloquium Publications, Providence, RI, fourth edition, 1975. 3, 6, 12, 15, 22, 27

D. D. Tcheutia. *On connection, linearization and duplication coefficients of classical orthogonal polynomials*. Ph.d. dissertation, Universität Kassel, 2014. 15, 73

D. D. Tcheutia. Recurrence equations and their classical orthogonal polynomial solutions on a quadratic or q-quadratic lattice. *J. Difference Equ. Appl.*, DOI: 10.1080/10236198.2019.1627346, 2019. 72

D. D. Tcheutia, P. Njionou Sadjang, W. Koepf, and M. Foupouagnigni. Divided–difference equation, inversion, connection, multiplication and linearization formulae of the continuous Hahn and the Meixner–Pollaczek polynomials. *Ramanujan J.*, 45:33–56, 2017. 73, 88, 89, 90, 100

D. D. Tcheutia, A. S. Jooste, and W. Koepf. Mixed recurrence equations and interlacing properties for zeros of sequences of classical q-orthogonal polynomials. *Appl. Numer. Math.*, 125:86–102, 2018a. 37

D. D. Tcheutia, A. S. Jooste, and W. Koepf. Quasi-orthogonality of some hypergeometric and q-hypergeometric polynomials. *SIGMA*, 14:26 pages, 2018b. 15, 37

M. V. Tratnik. Multivariable Meixner, Krawtchouk, and Meixner-Pollaczek polynomials. *J. Math. Phys.*, 30:2740–2749, 1989. 73

G. Valent and W. Van Assche. The impact of stieltjes' work on continued fractions and orthogonal polynomials: additional material. In *Proceedings of the International Conference on Orthogonality, Moment Problems and Continued Fractions (Delft, 1994), J. Comput. Appl. Math. 65*, pages 419–447, 1995. 3

W. Van Assche. Solution of an open problem about two families of orthogonal polynomials. *SIGMA*, 15, 2019. 4, 96, 100

R. Vidunas and T. H. Koornwinder. Algorithmic methods for special functions by computer algebra. *Webpage of the NWO project*, 2000. 6

N. S. Witte. Semi-classical orthogonal polynomial systems on non-uniform lattices, deformations of the askey table and analogs of isomonodromy. *Nagoya Math. J.*, 219: 127–234, 2015. 88

S. L. Woronowicz. Compact matrix pseudogroups. *Comm. Math. Phys.*, 111:613–665, 1987. 37