KLAUSUR

Spezielle Funktionen für Ingenieure

15.2.1999

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Bitte lassen Sie genügend Platz zwischen den Aufgaben und beschreiben Sie nur die Vorderseite der Blätter!

Zum Bestehen der Klausur sollten 10 Punkte erreicht werden.

1)	2)	3a)	3b)

Punkte:	Note:

1. Show that the following identity holds:

$$\sum_{\nu=0}^{\infty} \frac{t^{\nu+1}}{\nu+1} P_\nu(t) = \frac{1}{2} \ln\left(\frac{1+t}{1-t}\right) \; , |t| < 1 \, .$$

Hint: Use the generating function of Legendre's polynomials:

$$\sum_{\nu=0}^{\infty} t^\nu P_\nu(\xi) = \frac{1}{\sqrt{1 - 2t\,\xi + t^2}} \, , |t| < 1 \, , |\xi| \leq 1$$

and the integral:

$$\int_0^t \frac{dt'}{\sqrt{1 - 2t'\,\xi + t'^{\,2}}} = \ln\left(t - \xi + \sqrt{1 - 2t\,\xi + t^2}\right) - \ln(1 - \xi) \, .$$

(4P)

2. Let spherical coordinates be given through:

$$x = r\cos(\phi)\sin(\theta) \, , y = r\sin(\phi)\sin(\theta) \, , z = r\cos(\theta) \, ,$$

$$0 < , \quad 0 \leq \phi < 2\pi \, , \quad 0 \leq \theta \leq \pi \, ,$$

The potential equation for functions $f(r,\theta)$ being independent of ϕ reads as:

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta}\left(\sin(\theta) \frac{\partial f}{\partial \theta}\right) = 0 \, .$$

Show that any function

$$f_\nu(r,\theta) = r^\nu \, P_\nu(\cos(\theta)) \, , \nu = 0, 1, 2, \ldots$$

solves the potential equation.

Find a solution of the form

$$f(r,\theta) = \sum_{\nu=0}^{\infty} c_\nu f_\nu(r,\theta)$$

which satis£es the condition:

$$f(r,0) = \frac{1}{1+r} \, , \quad 0 < r \leq r_0 < 1 \, .$$

(6P)

3. Consider the differential equation

$$y'' + \frac{3}{x}\, y' + x\, y = 0\,.$$

(a) Show that the equation possesses a solution of the form:

$$y_1(x) = c_0 \left(1 - \frac{1}{15} x^3 + \ldots\right)\,.$$

With the ansatz

$$y_2(x) = u(x) y_1(x)$$

a second solution can be obtained. Give a differential equation for $u(x)$.

(b) Through the transformation of the independent variable

$$\xi = \frac{1}{x}$$

a new equation for $\tilde{y}(\xi) = y\left(\frac{1}{\xi}\right)$ is obtained.

Is the point $\xi = 0$ a regular singular point of the new equation?
(10P)

Solutions

1.) Legendre's polynomials are generated through

$$\sum_{\nu=0}^{\infty} t^{\nu} P_{\nu}(\xi) = \frac{1}{\sqrt{1-2t\,\xi+t^2}}\,,\, |t|<1\,,\, |\xi|\leq 1\,.$$

Integration on both sides gives:

$$\int_0^t \sum_{\nu=0}^{\infty} t'^{\,\nu} P_{\nu}(\xi)\, dt' = \int_0^t \frac{1}{\sqrt{1-2t'\,\xi+t'^{\,2}}}\, dt'\,.$$

On the left-hand side we integrate termwise and on the right-hand side we use the hint:

$$\sum_{\nu=0}^{\infty} \frac{t^{\nu+1}}{\nu+1} P_{\nu}(\xi) = \ln\left(t-\xi+\sqrt{1-2\,t\,\xi+t^2}\right) - \ln(1-\xi)\,.$$

By setting $\xi = t$ we obtain:

$$\begin{aligned}
\sum_{\nu=0}^{\infty} \frac{t^{\nu+1}}{\nu+1} P_{\nu}(t) &= \ln\left(\sqrt{1-2\,t^2+t^2}\right)) - \ln(1-t) \\
&= \frac{1}{2}\,\ln(1-t^2) - \ln(1-t) \\
&= \frac{1}{2}\,\ln(1-t^2) - \frac{1}{2}\,\ln((1-t)^2) \\
&= \frac{1}{2}\,\ln\left(\frac{1-t^2}{(1-t)^2}\right) \\
&= \frac{1}{2}\,\ln\left(\frac{1+t}{1-t}\right)\,.
\end{aligned}$$

2.) An easy calculation shows that:

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial r^{\nu}}{\partial r}\right) = \begin{cases} 0 & , \quad \text{if} \quad \nu = 0\,, \\ \frac{\partial}{\partial r} \nu\, r^{\nu+1} = \nu\,(\nu+1)\, r^{\nu}\,, & \text{if} \quad \nu > 0\,. \end{cases}$$

We have $P_0(t) = 1$. Hence it follows that

$$\frac{\partial}{\partial \theta}\left(\sin(\theta) \frac{\partial P_{\nu}(\cos(\theta))}{\partial \theta}\right) = 0$$

for $\nu = 0$. In the case $\nu > 0$ we obtain by differentiation:

$$\begin{aligned}
&\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial P_\nu(\cos(\theta))}{\partial\theta}\right)\\
&\quad= -\frac{\partial}{\partial\theta}\left(((\sin(\theta))^2\, P'_\nu(\cos(\theta))\right)\\
&\quad= (\sin(\theta))^3\, P''_\nu(\cos(\theta)) - 2\,\sin(\theta)\,\cos(\theta)\, P'_\nu(\cos(\theta))\,.
\end{aligned}$$

Multiplying by $\frac{1}{\sin(\theta)}$ yields

$$\begin{aligned}
&\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial f}{\partial\theta}\right)\\
&= \begin{cases} 0 & ,\text{if}\,\nu = 0\,,\\ (\sin(\theta))^2 P''_\nu(\cos(\theta)) - 2\cos(\theta) P'_\nu(\cos(\theta)) & ,\text{if}\,\nu > 0\,,\end{cases}
\end{aligned}$$

where

$$P'_\nu(\xi) = \frac{dP_\nu(\xi)}{d\xi}\,, \quad P''_\nu(\xi) = \frac{d^2 P_\nu(\xi)}{d^2\xi}\,.$$

Now inserting $f_\nu(r,\theta) = r^\nu\, P_\nu(\cos(\theta))$ into the potential equation gives

$$\begin{aligned}
&\nu\,(\nu+1)\, r^\nu P_\nu(\cos(\theta)) + \left((\sin(\theta))^2 P''_\nu(\cos(\theta)) - 2\cos(\theta) P'_\nu(\cos(\theta))\right) r^\nu\\
&= -\, r^\nu\,\left(\xi^2 - 1) P''_\nu(\xi) + 2\xi P'_\nu(\xi) - \nu\,(\nu+1)\, r^\nu P_\nu(\xi)\right) = 0\,,
\end{aligned}$$

where we use Legendre's equation and $\xi = \cos(\theta))$.

Inserting $\theta = 0$ into the series expansion we obtain:

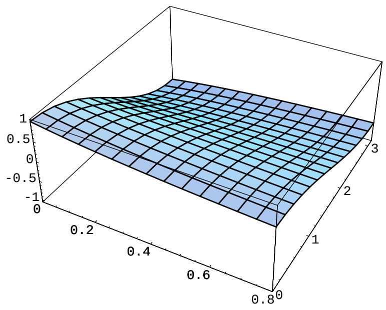
$$f(r,0) = \sum_{\nu=0}^{\infty} c_\nu\, r^\nu\, P_\nu(\cos(0)) = \sum_{\nu=0}^{\infty} c_\nu\, r^\nu\,.$$

Next we expand $f(r,0)$ with the help of the geometric series:

$$f(r,0) = \frac{1}{1+r} = \frac{1}{1-(-r)} = \sum_{\nu=0}^{\infty}(-1)^\nu\, r^\nu\,.$$

This expansion even holds for $|r| \leq r_0$. Therefore

$$c_\nu = (-1)^\nu\,.$$



Die Funktion
$f(r,\theta) =$
$\sum_{\nu=0}^{\infty}(-1)^{\nu}\, r^{\nu}\, P_{\nu}(\cos(\theta))$

3.a) The given differential equation is of the type

$$y'' + p_1(x)\, y' + p_2(x)\, y = 0\,,$$

where

$$p_1(x) = \frac{P_1(x)}{x}\,, \quad p_2(x) = \frac{P_2(x)}{x^2}$$

and

$$P_1(x) = 3 + 0\,x + 0\,x^2 + \cdots\,,$$
$$P_2(x) = 0 + 0\,x + 0\,x^2 + 1\cdot x^3 + 0\,x^4 + \cdots\,.$$

Following the Fuchsian theory we consider the determining equation

$$\rho\,(\rho - 1) + 3\,\rho = 0\,.$$

Since the solutions of this equation become

$$\rho_1 = 0\,, \quad \rho_2 = -2\,,$$

the Fuchsian theory assures a solution of the following type:

$$y_1(x) = x^0 \sum_{\nu=0}^{\infty} c_\nu\, x^{nu} = \sum_{\nu=0}^{\infty} c_\nu\, x^{nu}\,.$$

Differentiating twice:

$$\begin{aligned} y_1'(x) &= \sum_{\nu=1}^{\infty} \nu\, c_\nu\, x^{nu-1}\,, \\ y_1''(x) &= \sum_{\nu=2}^{\infty} \nu\,(\nu-1)\, c_\nu\, x^{nu-2}\,, \end{aligned}$$

and inserting into the differential equation gives:

$$\begin{aligned}
&2 \cdot 1\, c_2\, x^0 + 3 \cdot 2\, c_3\, x^1 + 3 \cdot 2\, c_4\, x^2 + \cdots \\
&\frac{3}{x} \left(1 \cdot c_1\, x^0 + 2 \cdot c_2\, x^1 + 3 \cdot c_3\, x^2 + \cdots\right) \\
&x\ \left(c_0\, x^0 + c_1\, x^1 + c_2\, x^2 + \cdots\right) \\
&= 0\,.
\end{aligned}$$

Comparing coef£cients yields the following conditions:

$$\begin{aligned}
x^{-1}: &\quad 3\, c_1 = 0\,, \\
x^0: &\quad 2\, c_2 = 0\,, \\
x^1: &\quad c_0 + 9\, c_3 + 6\, c_3 = 0 \\
x^2: &\quad c_1 + 12\, c_4 + 8\, c_4 = 0 \\
&\vdots
\end{aligned}$$

The coef£cient c_0 can be arbitrarily choosen and we obtain an expansion beginning with

$$y_1(x) = c_0 \left(1 - \frac{1}{15} x^3 + \ldots\right)\,.$$

We can as well use the Fuchsian theory for obtaining the expansion. Writing

$$\begin{aligned}
f_0(\rho) &= \rho\,(\rho - 1) + 3\,\rho\,, \\
f_3(\rho) &= 1\,, \\
f_\nu(\rho) &= 0\,, \quad \nu > 0\,, \nu \neq 3\,,
\end{aligned}$$

we have to determine the coef£cients from the system

$$\begin{aligned}
&c_0\, f_0(0) = 0\,, \\
&c_1\, f_0(0+1) + c_0\, f_1(0) = 0\,, \\
&c_2\, f_0(0+2) + c_1\, f_1(0+1) + c_0\, f_2(0) = 0 \\
&c_3\, f_0(0+3) + c_2\, f_1(0+2) + c_1\, f_2(0+1) + c_0\, f_3(0) = 0 \\
&\vdots
\end{aligned}$$

which obviously gives the same result.

Differentiation of the product $y_2(x) = y_1(x)\,u(x)$ yields:

$$\begin{aligned} y_2'(x) &= y_1'(x)\,u(x) + y_1(x)\,u'(x)\,, \\ y_2''(x) &= y_1''(x)\,u(x) + 2\,y_1'(x)\,u'(x) + y_1(x)\,u''(x)\,. \end{aligned}$$

Inserting y_2 into

$$y'' + \frac{3}{x}\,y' + x\,y = 0$$

gives the condition

$$\begin{aligned} & y_1''(x)\,u(x) + 2\,y_1'(x)\,u'(x) + y_1(x)\,u''(x) \\ & + \frac{3}{x}\,(y_1'(x)\,u(x) + y_1(x)\,u'(x)) \\ & + x\,y_1(x)\,u(x) = 0\,. \end{aligned}$$

Through the fact that y_1 solves the equation the above condition is simpli£ed to

$$y_1(x)\,u''(x) + 2\,y_1'(x)\,u'(x) + \frac{3}{x}\,y_1(x)\,u'(x) = 0\,.$$

Finally we obtain the following £rst order equation for u':

$$u''(x) + \left(2\,\frac{y_1'(x)}{y_1(x)} + \frac{3}{x}\right)\,u'(x) = 0\,.$$

3.b) Using $x = \dfrac{1}{\xi}$ and $y(x) = \tilde{y}\left(\dfrac{1}{x}\right)$ we obtain

$$\begin{aligned} y'(x) &= \frac{d\tilde{y}}{d\xi}\left(\frac{1}{x}\right)\,\left(-\frac{1}{x^2}\right) \\ &= -\xi^2\,\frac{d\tilde{y}}{d\xi}(\xi)\,, \\ y''(x) &= -\xi^2\,\frac{d^2\tilde{y}}{d\xi^2}\left(\frac{1}{x}\right)\,\left(-\frac{1}{x^2}\right) - 2\,\xi\,\left(-\frac{1}{x^2}\right)\,\frac{d\tilde{y}}{d\xi}\left(\frac{1}{\xi}\right) \\ &= \xi^4\,\frac{d^2\tilde{y}}{d\xi^2}(\xi) + 2\,\xi^3\,\frac{d\tilde{y}}{d\xi}(\xi)\,. \end{aligned}$$

Inserting these results into

$$y'' + \frac{3}{x}\,y' + x\,y = 0$$

gives

$$\xi^4 \, \frac{d^2\tilde{y}}{d\xi^2}(\xi) + 2\,\xi^3 \, \frac{d\tilde{y}}{d\xi}(\xi) - 3\,\xi\,\xi^2 \, \frac{d\tilde{y}}{d\xi}(\xi) + \frac{1}{\xi}\,\tilde{y}(\xi) = 0\,.$$

i.e.

$$\frac{d^2\tilde{y}}{d\xi^2}(\xi) - \frac{1}{\xi}\,\frac{d\tilde{y}}{d\xi}(\xi) + \frac{1}{\xi^5}\,\tilde{y}(\xi) = 0\,.$$

According to the Fuchsian theorem the equation

$$\frac{d^2\tilde{y}}{d\xi^2}(\xi) + p_1(\xi)\,\frac{d\tilde{y}}{d\xi}(\xi) + p_2(\xi)\,\tilde{y}(\xi) = 0$$

posseses a regular singular point at $\xi = 0$ if and only if near $\xi = 0$ there exist analytical functions P_1 and P_2 with the property

$$p_1(\xi) = \frac{P_1(\xi)}{\xi}\,, \quad p_2(\xi) = \frac{P_2(\xi)}{\xi^2}\,.$$

Since we have $P_2 = \dfrac{1}{\xi^4}$ the point $\xi = 0$ is not a regular singular point.