Satz

In[1]:= `Abs[Cross[a, b]]^2 == Abs[a]^2 Abs[b]^2 - (a.b)^2`

Out[1]= $|a\times b|^2 == |a|^2\,|b|^2 - (a.b)^2$

Beweis

In[2]:= `Länge[a_] := Sqrt[Sum[a[[k]]^2, {k, 1, Length[a]}]]`

$$\text{Länge}[a\_] := \sqrt{\sum_{k=1}^{\text{Length}[a]} a[[k]]^{\wedge}2}$$

In[3]:= `a = {xa, ya, za}`

Out[3]= {xa, ya, za}

In[4]:= `Länge[a]`

Out[4]= $\sqrt{xa^2 + ya^2 + za^2}$

In[5]:= `b = {xb, yb, zb}`

Out[5]= {xb, yb, zb}

In[6]:= `Cross[a, b]`

Out[6]= {ya zb – yb za, xb za – xa zb, xa yb – xb ya}

linke Seite

In[7]:= `Länge[Cross[a, b]]^2`

Out[7]= $(xa\,yb - xb\,ya)^2 + (xb\,za - xa\,zb)^2 + (ya\,zb - yb\,za)^2$

In[8]:= `links = Expand[%]`

Out[8]= $xa^2\,yb^2 + xa^2\,zb^2 - 2\,xa\,xb\,ya\,yb - 2\,xa\,xb\,za\,zb + xb^2\,ya^2 + xb^2\,za^2 + ya^2\,zb^2 - 2\,ya\,yb\,za\,zb + yb^2\,za^2$

rechte Seite

In[9]:= `Länge[a]^2 * Länge[b]^2 - (a.b)^2`

Out[9]= $(xa^2 + ya^2 + za^2)(xb^2 + yb^2 + zb^2) - (xa\,xb + ya\,yb + za\,zb)^2$

In[10]:= `rechts = Expand[%]`

Out[10]= $xa^2\,yb^2 + xa^2\,zb^2 - 2\,xa\,xb\,ya\,yb - 2\,xa\,xb\,za\,zb + xb^2\,ya^2 + xb^2\,za^2 + ya^2\,zb^2 - 2\,ya\,yb\,za\,zb + yb^2\,za^2$

In[11]:= `links - rechts`

Out[11]= 0

Spatprodukt

In[12]:= `c = {xc, yc, zc}`

Out[12]= {xc, yc, zc}

In[13]:= `Spat[a_, b_, c_] := Expand[Cross[a, b].c]`

In[14]:=
```
Spat[a, b, c]
```
Out[14]= $xa\, yb\, zc - xa\, yc\, zb - xb\, ya\, zc + xb\, yc\, za + xc\, ya\, zb - xc\, yb\, za$

In[15]:=
```
Spat[a, b, c] + Spat[b, a, c]
```
Out[15]= 0

■ Entwicklungssatz: Satz 3.10

In[16]:=
```
links = Cross[a, Cross[b, c]]
```
Out[16]= $\{xb\, ya\, yc + xb\, za\, zc - xc\, ya\, yb - xc\, za\, zb,\ -xa\, xb\, yc + xa\, xc\, yb + yb\, za\, zc - yc\, za\, zb,\ -xa\, xb\, zc + xa\, xc\, zb - ya\, yb\, zc + ya\, yc\, zb\}$

In[17]:=
```
rechts = (a.c) * b - (a.b) * c
```
Out[17]= $\{xb\,(xa\, xc + ya\, yc + za\, zc) - xc\,(xa\, xb + ya\, yb + za\, zb),\ yb\,(xa\, xc + ya\, yc + za\, zc) - yc\,(xa\, xb + ya\, yb + za\, zb),\ zb\,(xa\, xc + ya\, yc + za\, zc) - zc\,(xa\, xb + ya\, yb + za\, zb)\}$

In[18]:=
```
Expand[links - rechts]
```
Out[18]= $\{0, 0, 0\}$

■ Satz von Lagrange

In[19]:=
```
d = {xd, yd, zd}
```
Out[19]= $\{xd, yd, zd\}$

In[20]:=
```
links = Cross[a, b].Cross[c, d]
```
Out[20]= $(xa\, yb - xb\, ya)\,(xc\, yd - xd\, yc) + (xb\, za - xa\, zb)\,(xd\, zc - xc\, zd) + (ya\, zb - yb\, za)\,(yc\, zd - yd\, zc)$

In[21]:=
```
rechts = (a.c) * (b.d) - (a.d) * (b.c)
```
Out[21]= $(xa\, xc + ya\, yc + za\, zc)\,(xb\, xd + yb\, yd + zb\, zd) - (xa\, xd + ya\, yd + za\, zd)\,(xb\, xc + yb\, yc + zb\, zc)$

In[22]:=
```
Expand[links - rechts]
```
Out[22]= 0

■ Parallelogrammidentität

In[23]:=
```
L = 2 (Länge[a]^2 + Länge[b]^2)
```
Out[23]= $2\left(xa^2 + xb^2 + ya^2 + yb^2 + za^2 + zb^2\right)$

In[24]:=
```
R = Länge[a + b]^2 + Länge[a - b]^2
```
Out[24]= $(xa - xb)^2 + (xa + xb)^2 + (ya - yb)^2 + (ya + yb)^2 + (za - zb)^2 + (za + zb)^2$

In[25]:=
```
Expand[R - L]
```
Out[25]= 0