■ Gram-Schmidt-Orthogonalisierung

In[1]:=
```
Länge[v_] := Sqrt[ Sum[v[[k]] Conjugate[v[[k]]], {k, 1, Length[v]}] ]
```

In[2]:=
```
Skalarprodukt[u_, v_] := Sum[u[[k]] Conjugate[v[[k]]], {k, 1, Length[v]}]
```

In[3]:=
```
{1, i}.{1 + i, i}
```

Out[3]=
$$i$$

In[4]:=
```
Skalarprodukt[{1, i}, {1 + i, i}]
```

Out[4]=
$$2 - i$$

In[5]:=
```
Clear[GramSchmidt]
GramSchmidt[A__] := Module[{a, b, e, k, l},
  a = {A};
  Do[
   b[l] = a[[l]] - Sum[(Skalarprodukt[a[[l]], e[k]]) * e[k], {k, 1, l-1}];
   e[l] = b[l] / Länge[b[l]], {l, 1, Length[a]}];
  Transpose[Table[e[k], {k, 1, Length[a]}]]
 ]
```

■ Beispiel 5.27

In[7]:=
```
gs = GramSchmidt[{1, 3}, {2, 7}]
```

Out[7]=
$$\begin{pmatrix} \frac{1}{\sqrt{10}} & \frac{2-\frac{\sqrt{\frac{2}{5}}+\frac{21}{\sqrt{10}}}{\sqrt{10}}}{\sqrt{\left(7-\frac{3\left(\sqrt{\frac{2}{5}}+\frac{21}{\sqrt{10}}\right)}{\sqrt{10}}\right)^2+\left(2-\frac{\sqrt{\frac{2}{5}}+\frac{21}{\sqrt{10}}}{\sqrt{10}}\right)^2}} \\ \frac{3}{\sqrt{10}} & \frac{7-\frac{3\left(\sqrt{\frac{2}{5}}+\frac{21}{\sqrt{10}}\right)}{\sqrt{10}}}{\sqrt{\left(7-\frac{3\left(\sqrt{\frac{2}{5}}+\frac{21}{\sqrt{10}}\right)}{\sqrt{10}}\right)^2+\left(2-\frac{\sqrt{\frac{2}{5}}+\frac{21}{\sqrt{10}}}{\sqrt{10}}\right)^2}} \end{pmatrix}$$

In[8]:=
```
Simplify[gs]
```

Out[8]=
$$\begin{pmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{pmatrix}$$

In[9]:=
```
Clear[GramSchmidtOhneNormierung]
GramSchmidtOhneNormierung[A__] := Module[{a, b, k, l},
  a = {A};
  Do[
   b[l] = a[[l]] - Sum[Skalarprodukt[a[[l]], b[k]]/Länge[b[k]]^2 * b[k], {k, 1, l-1}],
   {l, 1, Length[a]}];
  Transpose[Table[b[k], {k, 1, Length[a]}]]
 ]
```

In[11]:=
```
GramSchmidtOhneNormierung[{1, 3}, {2, 7}]
```

Out[11]=
$$\begin{pmatrix} 1 & -\frac{3}{10} \\ 3 & \frac{1}{10} \end{pmatrix}$$

■ Beispiel 5.28

In[12]:=
```
GramSchmidt[{1, i}, {1 + i, i}]
```

Out[12]=
$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

In[13]:=
```
GramSchmidtOhneNormierung[{1, i}, {1 + i, i}]
```

Out[13]=
$$\begin{pmatrix} 1 & \frac{i}{2} \\ i & \frac{1}{2} \end{pmatrix}$$

■ Beispiel 5.29

In[14]:=
```
gs = GramSchmidt[{1, 1, 2}, {1, 3, -4}]
```

Out[14]=
$$\begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1-\frac{-4\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{6}}}{\sqrt{6}}}{\sqrt{\left(-4-\sqrt{\frac{2}{3}}\left(-4\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{6}}\right)\right)^2+\left(1-\frac{-4\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{6}}}{\sqrt{6}}\right)^2+\left(3-\frac{-4\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{6}}}{\sqrt{6}}\right)^2}} \\ \frac{1}{\sqrt{6}} & \frac{3-\frac{-4\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{6}}}{\sqrt{6}}}{\sqrt{\left(-4-\sqrt{\frac{2}{3}}\left(-4\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{6}}\right)\right)^2+\left(1-\frac{-4\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{6}}}{\sqrt{6}}\right)^2+\left(3-\frac{-4\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{6}}}{\sqrt{6}}\right)^2}} \\ \sqrt{\frac{2}{3}} & \frac{-4-\sqrt{\frac{2}{3}}\left(-4\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{6}}\right)}{\sqrt{\left(-4-\sqrt{\frac{2}{3}}\left(-4\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{6}}\right)\right)^2+\left(1-\frac{-4\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{6}}}{\sqrt{6}}\right)^2+\left(3-\frac{-4\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{6}}}{\sqrt{6}}\right)^2}} \end{pmatrix}$$

In[15]:=
```
Simplify[gs]
```

Out[15]=
$$\begin{pmatrix} \frac{1}{\sqrt{6}} & \sqrt{\frac{5}{42}} \\ \frac{1}{\sqrt{6}} & \frac{11}{\sqrt{210}} \\ \sqrt{\frac{2}{3}} & -4\sqrt{\frac{2}{105}} \end{pmatrix}$$

In[16]:=
```
GramSchmidtOhneNormierung[{1, 1, 2}, {1, 3, -4}]
```

Out[16]=
$$\begin{pmatrix} 1 & \frac{5}{3} \\ 1 & \frac{11}{3} \\ 2 & -\frac{8}{3} \end{pmatrix}$$

■ Weitere Beispiele

In[17]:=
```
GramSchmidt[{1, 0, 0}, {1, 1, 0}, {1, 1, 1}]
```

Out[17]=
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[18]:=
```
GramSchmidt[{1, 1, 1}, {1, 1, 0}, {1, 0, 0}]
```

Out[18]=
$$\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\sqrt{\frac{2}{3}} & 0 \end{pmatrix}$$

In[19]:=
```
GramSchmidt[{1, 2, 3, 4, 5}, {1, 0, 0, 0, 0},
 {0, 1, 0, 0, 0}, {0, 0, 1, 0, 0}, {0, 0, 0, 1, 0}]
```

Out[19]=
$$\begin{pmatrix} \frac{1}{\sqrt{55}} & 3\sqrt{\frac{6}{55}} & 0 & 0 & 0 \\ \frac{2}{\sqrt{55}} & -\frac{\sqrt{\frac{2}{165}}}{3} & \frac{5}{3\sqrt{3}} & 0 & 0 \\ \frac{3}{\sqrt{55}} & -\frac{1}{\sqrt{330}} & -\frac{1}{5\sqrt{3}} & \frac{\sqrt{\frac{41}{2}}}{5} & 0 \\ \frac{4}{\sqrt{55}} & -\frac{2\sqrt{\frac{2}{165}}}{3} & -\frac{4}{15\sqrt{3}} & -\frac{6\sqrt{\frac{2}{41}}}{5} & \frac{5}{\sqrt{41}} \\ \sqrt{\frac{5}{11}} & -\frac{\sqrt{\frac{5}{66}}}{3} & -\frac{1}{3\sqrt{3}} & -\frac{3}{\sqrt{82}} & -\frac{4}{\sqrt{41}} \end{pmatrix}$$

In[20]:=
```
GramSchmidtOhneNormierung[{1, 0, 0}, {1, 1, 0}, {1, 1, 1}]
```

Out[20]=
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[21]:=
```
GramSchmidtOhneNormierung[{1, 1, 1}, {1, 1, 0}, {1, 0, 0}]
```

Out[21]=
$$\begin{pmatrix} 1 & \frac{1}{3} & \frac{1}{2} \\ 1 & \frac{1}{3} & -\frac{1}{2} \\ 1 & -\frac{2}{3} & 0 \end{pmatrix}$$

In[22]:=
```
GramSchmidtOhneNormierung[{1, 2, 3, 4, 5},
 {1, 0, 0, 0, 0}, {0, 1, 0, 0, 0}, {0, 0, 1, 0, 0}, {0, 0, 0, 1, 0}]
```

Out[22]=
$$\begin{pmatrix} 1 & \frac{54}{55} & 0 & 0 & 0 \\ 2 & -\frac{2}{55} & \frac{25}{27} & 0 & 0 \\ 3 & -\frac{3}{55} & -\frac{1}{9} & \frac{41}{50} & 0 \\ 4 & -\frac{4}{55} & -\frac{4}{27} & -\frac{6}{25} & \frac{25}{41} \\ 5 & -\frac{1}{11} & -\frac{5}{27} & -\frac{3}{10} & -\frac{20}{41} \end{pmatrix}$$