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Fourth order q-difference equation for the first associated of the q-classical orthogonal polynomials

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Abstract

We derive the fourth-order q-difference equation satisfied by the first associated of the q-classical orthogonal polynomials. The coefficients of this equation are given in terms of the polynomials σ and τ which appear in the q-Pearson difference equation $D_q(\sigma\,\rho)=\tau\rho$ defining the weight ρ of the q-classical orthogonal polynomials inside the q-Hahn tableau. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

The fourth-order difference equation for the associated polynomials of all classical discrete polynomials were given for all integers r (order of association) in [5], using the properties of the Stieltjes functions of the associated linear forms.

On the other hand, the equation for the first associated ($r=1$) of all classical discrete polynomials was obtained in [13] using a useful relation proved in [2]. In this work, mimicking the approach used in [13] we give a single fourth-order q-difference equation which is valid for the first associated of all q-classical orthogonal polynomials. This equation is important for some connection coefficient problems [10], and also in order to represent finite modifications inside the Jacobi matrices of the q-classical starting family [14]. q-classical orthogonal polynomials involved in this work belong to

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the q-Hahn class as introduced by Hahn [8]. They are represented by the basic hypergeometric series appearing at the level ${}_3\phi_2$ and not at the level ${}_4\phi_3$ of the Askey–Wilson orthogonal polynomials.

The orthogonality weight ρ (defined in the interval I) for q-classical orthogonal polynomials is defined by a Pearson-type q-difference equation

$$D_q(\sigma\rho) = \tau\rho, \tag{1}$$

where the q-difference operator D_q is defined [8] by

$$D_q\, f(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0,\ 0 < q < 1, \tag{2}$$

and $D_q f(0) := f'(0)$ by continuity, provided that $f'(0)$ exists. σ is a polynomial of degree at most two and τ is polynomial of degree one.

The monic polynomials $P_n(x;q)$, orthogonal with respect to ρ, satisfy the second-order q-difference equation

$$\mathscr{Q}_{2,n}[y(x)] \equiv [\sigma(x) D_q\, D_{1/q} + \tau(x) D_q + \lambda_{q,n}\mathscr{I}_d] y(x) = 0, \tag{3}$$

an equation which can be written in the q-shifted form

$$[(\sigma_1 + \tau_1 t_1)\mathscr{T}_q^{\,2} - ((1+q)\sigma_1 + \tau_1 t_1 - \lambda_{q,n}\, t_1^2)\mathscr{T}_q + q\sigma_1 \mathscr{I}_d] y(x) = 0, \tag{4}$$

with

$$\begin{aligned} \lambda_{q,n} &= -[n]_q \left\{ \tau' + [n-1]_{\frac{1}{q}} \frac{\sigma''}{2q} \right\}, \quad [n]_q = \frac{1-q^n}{1-q}, \\ \sigma_i &\equiv \sigma(q^i x), \quad \tau_i \equiv \tau(q^i x), \quad t_i \equiv t(q^i x), \quad t(x) = (q-1)x \end{aligned} \tag{5}$$

and the geometric shift $\mathscr{T}_q$ defined by

$$\mathscr{T}_q^{\,i} f(x) = f(q^i x), \quad \mathscr{T}_q^{\,0} \equiv \mathscr{I}_d \ (\equiv \text{identity operator}). \tag{6}$$

2. Fourth-order q-difference equation for the first associated $P_{n-1}^{(1)}(x;q)$ of the q-classical orthogonal polynomial

The first associated of $P_{n-1}(x;q)$ is a monic polynomial of degree $n-1$, denoted by $P_{n-1}^{(1)}(x;q)$, and defined by

$$P_{n-1}^{(1)}(x;q) = \frac{1}{\gamma_0} \int_I \frac{P_n(s;q) - P_n(x;q)}{s - x} \rho(s)\, \mathrm{d}_q s, \tag{7}$$

where γ_0 is given by $\gamma_0 = \int_I \rho(s)\, \mathrm{d}_q s$ and the q-integral is defined in [7].

The polynomials $P_n(x;q) \equiv P_n^{(0)}(x;q)$ and $P_n^{(1)}(x;q)$ satisfy also the following three-term recurrence relation [4] for $r = 0$ and $r = 1$, respectively,

$$\begin{aligned} P_{n+1}^{(r)}(x;q) &= (x - \beta_{n+r}) P_n^{(r)}(x;q) - \gamma_{n+r}\, P_{n-1}^{(r)}(x;q), \quad n \geqslant 1, \\ P_0^{(r)}(x;q) &= 1, \quad P_1^{(r)}(x;q) = x - \beta_r. \end{aligned} \tag{8}$$

Relation (7) can be written as

$$P^{(1)}_{n-1}(x;q) = \rho(x)Q_n(x;q) - P_n(x;q)\rho(x)Q_0(x;q), \tag{9}$$

where

$$Q_n(x;q) = \frac{1}{\gamma_0\,\rho(x)}\int_I \frac{P_n(s;q)}{s-x}\rho(s)\,\mathrm{d}_q s.$$

It is well-known [15] that $Q_n(x;q)$ also satisfies Eq. (3); hence by (9)

$$\mathscr{Q}_{2,n}\left[\frac{P^{(1)}_{n-1}(x;q)}{\rho(x)} + P_n(x;q)Q_0(x;q)\right] = 0. \tag{10}$$

In a first step, we eliminate $\rho(x)$ and $Q_0(x;q)$ in Eq. (10) using Eqs. (1) and (3) for $P_n(x;q)$. This can be easily carried out using a computer algebra system — we used Maple V Release 4 [3] — and gives the relation

$$(\sigma_1+\tau_1 t_1)\mathscr{Q}^*_{2,n-1}\left[P^{(1)}_{n-1}(x;q)\right] = \left[e\mathscr{T}_q + f\mathscr{I}_d\right]P_n(x;q), \tag{11}$$

with

$$\begin{aligned}
\mathscr{Q}^*_{2,n-1} &= \sigma_2\mathscr{T}_q^2 - ((1+q)\sigma_1+\tau_1 t_1 - \lambda_{q,n}t_1^2)\mathscr{T}_q + q(\sigma+\tau t)\mathscr{I}_d,\\
e &= \left(\frac{\sigma''}{2}-\tau'\right)((1+q)\sigma_1+\tau_1 t_1-\lambda_{q,n}t_1^2)t_1,\\
f &= -\left(\frac{\sigma''}{2}-\tau'\right)((q+1)\sigma_1+\tau_1 t_1)t_1.
\end{aligned} \tag{12}$$

In a second step, we use Eqs. (11), (12) and the fact that the polynomials $P_n(x;q)$ satisfy Eq. (3), again. This gives — after some computations with Maple V.4 — the operator $\mathscr{Q}^{**}_{2,n-1}$ annihilating the right-hand side of Eq. (11),

$$\begin{aligned}
\mathscr{Q}^{**}_{2,n-1} &= (\sigma_3+\tau_3 t_3)[q^2A_1+(1+q)\sigma_2+\tau_2 t_2]\mathscr{T}_q^2 - [q^3A_1(\sigma_2+\tau_2 t_2)+A_3(\sigma_2+qA_1)]\mathscr{T}_q\\
&\quad + q\sigma_1[q^2A_2+(1+q)\sigma_3+\tau_3 t_3)]\,\mathscr{I}_d,
\end{aligned} \tag{13}$$

where $A(x) = (1+q)\sigma(x)+\tau(x)t(x)-\lambda_{q,n}\,t(x)^2$ and $A_j \equiv A_j(x) \equiv A(q^j x),\ j=1,2,3.$

We therefore obtain the factorized form of the fourth-order q-difference equation satisfied by each $P^{(1)}_{n-1}(x;q)$,

$$\mathscr{Q}^{**}_{2,n-1}\frac{\mathscr{Q}^*_{2,n-1}}{q^2(q-1)^2x^2}[P^{(1)}_{n-1}(x;q)] = 0. \tag{14}$$

3. Limiting situations, comments and example

(1) Since $\lim_{q\to 1} D_q = \mathrm{d}/\mathrm{d}x$, from Eqs. (12) and (13), we recover by a limit process the factorized form of the fourth-order differential equation satisfied by the first associated $P^{(1)}_{n-1}(x)$ of the (continuous) classical orthogonal polynomials P_{n-1} [12],

$$\mathscr{Q}^{**c}_{2,n-1}\,\mathscr{Q}^{*c}_{2,n-1}[P^{(1)}_{n-1}(x)] = 0, \tag{15}$$

with

$$\mathcal{Q}^{*c}_{2,n-1} = \lim_{q\to 1} \frac{\mathcal{Q}^{*}_{2,n-1}}{q^2(q-1)^2x^2} = \sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (2\sigma' - \tau)\frac{\mathrm{d}}{\mathrm{d}x} + (\sigma'' - \tau' + \lambda_n)\mathcal{I}_d,$$

$$\mathcal{Q}^{**c}_{2,n-1} = \frac{1}{4\sigma(x)}\lim_{q\to 1} \frac{\mathcal{Q}^{**}_{2,n-1}}{q^2(q-1)^2x^2} = \sigma\frac{\mathrm{d}^2}{\mathrm{d}x^2} + (\sigma' + \tau)\frac{\mathrm{d}}{\mathrm{d}x} + (\tau' + \lambda_n)\mathcal{I}_d,$$

where $\lambda_n \equiv \lim_{q\to 1} \lambda_{q,n} = -n\,[(n-1)\frac{\sigma''}{2} + \tau']$.

(2) If the polynomials σ and τ are such that $\sigma'' = 2\tau'$ [12–14], then the right-hand side of Eq. (11) is equal to zero, and the first associated $P^{(1)}_{n-1}$ satisfies the second (instead of fourth)-order difference equation

$$\mathcal{Q}^{*}_{2,n-1}[P^{(1)}_{n-1}(x;q)] = 0.$$

For the little q-Jacobi polynomials $p_n(x;a,b|q)$ [1, 9]

$$\sigma(x) = \frac{x(x-1)}{q}, \quad \tau(x) = \frac{1 - aq + (abq^2 - 1)x}{q(q-1)},$$

and for the big q-Jacobi polynomials $P_n(x;a,b,c;q)$ [1, 9]

$$\sigma(x) = acq - (a+c)x + \frac{x^2}{q}, \quad \tau(x) = \frac{cq + aq(1-(b+c)q) + (abq^2-1)x}{q(q-1)},$$

the constant $\sigma'' - 2\tau'$ is equal to $2(1-abq)/(q-1)$. Therefore, the first associated of the little q-Jacobi polynomials (resp. big q-Jacobi polynomials) is still in the little q-Jacobi (resp. big q-Jacobi) family when $abq = 1$.

Computations involving the coefficients β_n and γ_n (see Eq. (8)) given in [1, 6, 11] and use of Maple V.4 generate the following relations between the monic little q-Jacobi (resp. monic big q-Jacobi) polynomials and their respective first associated

$$p^{(1)}_n\left(x;a,\frac{1}{qa}\,|q\right) = (aq)^n\, p_n\left(\frac{x}{aq};\frac{1}{a},aq\,|q\right), \tag{16}$$

$$P^{(1)}_n\left(x;a,\frac{1}{qa,},c;q\right) = (a)^n\, P_n\left(\frac{x}{a};\frac{1}{a},aq,c\,q;q\right). \tag{17}$$

(3) The results given in this paper (see Eqs. (11) and (13)), which agree with the ones obtained using the Stieltjes properties of the associated linear form [6], can be used for connection problems, expanding the first associated $P^{(1)}_{n-1}$ in terms of P_n, in the same spirit as in [10]. We have also computed the coefficients of the fourth-order q-difference equation satisfied by the first associated of the q-classical orthogonal polynomials appearing in the q-Hahn tableau. In particular, from the big q-Jacobi polynomials, we derive by limit processes [9] the fourth-order differential (resp. q-difference) equation satisfied by the first associated of the classical (resp. q-classical) orthogonal polynomials.

(4) For the little q-Jacobi polynomials for example, the operators $\mathcal{Q}^{*}_{2,n-1}$ and $\mathcal{Q}^{**}_{2,n-1}$ are given below, with the notation: $v = q^n$.

$$\mathcal{Q}^{*}_{2,n-1} = qx[(q^2x-1)\mathcal{T}_q^2 - v^{-1}(-v - av + q^2xabv^2 + qx)\mathcal{T}_q + a(-1+bqx)\mathcal{I}_d],$$

$$\begin{aligned}\mathcal{Q}^{**}_{2,n-1} = {} & v^{-1}\, q^4x^2[qa(-1+bq^4x)(q^3xabv + q^3xabv^2 + q^2xv + q^2x - qv - qav - v - av)\mathcal{T}_q^2 \\ & - v^{-1}(q^5x^2 + av^2 + qv^2 - q^2xv^2 - q^3xabv^3 + q^7x^2a^2b^2v^3 \\ & - q^3xa^2bv^3 - q^5xabv^3 + q^2a^2v^2 - q^5xabv^2 - q^5xa^2bv^2 + q^2av^2 \\ & - q^5xa^2bv^3 - q^2xav - q^4xav - q^2xv - q^4xv - q^3xav + q^5x^2v \\ & - q^3xv + q^7x^2a^2b^2v^4 + q^6x^2abv - q^4xa^2bv^3 + qa^2v^2 - q^2xav^2 \\ & + 2q^6x^2abv^2 + q^6x^2abv^3 + 2qav^2 + v^2 - q^4xabv^3)\mathcal{T}_q \\ & + (-1+qx)(q^4xabv + q^4xabv^2 + q^3xv + q^3x - qv - qav - v - av)\mathcal{I}_d].\end{aligned}$$

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