

# Properties of some finite families of classical orthogonal polynomials

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This paper looks at various groups of new families of orthogonal polynomials that were defined in the last two decades. We show that they are intimately related to known systems so that they are not really new and some of their properties can be determined from known results. In particular, we derive a new hypergeometric representation of a large polynomial family defined by M. Masjed-Jamei (2004) and we show that these polynomials can be written in terms of the Jacobi polynomials with complex parameters. We derive new properties of these polynomials including their three-term recurrence relation, structure relations, moments, connection coefficients and linearization coefficients that were not given in the original paper although the method used to derive them are not new.

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## 1. Introduction

The generalized hypergeometric series  ${}_pF_q$  (see e.g. [15]) is defined by

$${}_pF_q \left( \begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| x \right) := \sum_{m=0}^{\infty} A_m x^m = \sum_{m=0}^{\infty} \frac{(a_1)_m \cdots (a_p)_m}{(b_1)_m \cdots (b_q)_m} \frac{x^m}{m!},$$

where  $(a_i)_m$  denotes the Pochhammer symbol (or shifted factorial) defined by

$$(a_i)_m = \begin{cases} 1 & \text{if } m = 0 \\ a_i(a_i + 1)(a_i + 2) \cdots (a_i + m - 1) & \text{if } m \in \mathbb{N}. \end{cases}$$

We say that a term  $A_m$  is a hypergeometric term with respect to  $m$  if  $\frac{A_{m+1}}{A_m} \in \mathbb{Q}(m)$ , i.e. is a rational function in the variable  $m$ .

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The summand  $\alpha_m = A_m x^m$  of a generalized hypergeometric series is a hypergeometric term since

$$\frac{\alpha_{m+1}}{\alpha_m} = \frac{(m+a_1)\cdots(m+a_p)}{(m+b_1)\cdots(m+b_q)} \frac{x}{m+1}.$$

We note that no denominator parameter  $b_j$  is allowed to be zero or a negative integer. If any numerator parameter  $a_i$  is zero or a negative integer, the series terminates.

The classical continuous orthogonal polynomials are defined in terms of hypergeometric series and crucial due to their numerous applications (see e.g. [2] and references therein) in the approximation theory, Gaussian quadrature processes, differential and difference equations, various physics problems like quantum mechanics, etc.. For example (see e.g. [5]), the Laguerre polynomials made their appearance in the quantum mechanics of the hydrogen atom; a connection was found between the Laguerre polynomials and classic derangements, i.e., fixed-point-free permutations, thus giving a combinatorial interpretation of Laguerre polynomials. The electrostatic energy for a system of  $n+2$  charges,  $n$  of which are free unit charges on  $[-1, 1]$ , where at  $+1$  there is a charge  $p > 0$  and at  $-1$  a charge  $q > 0$ , attains a minimum at the zeros of the Jacobi polynomials  $P_n^{(2p-1, 2q-1)}(x)$  (see e.g. [10], [28]).

Let us present definitions of some families of orthogonal polynomials that were published in the last two decades and were called new.

The Chebyshev polynomials of fifth kind  $C_j(x)$  are defined in [18, Section 3.7] and considered in [1]. Their power series representations are given by

$$\begin{aligned} C_{2j}(x) &= (2j+1) \sum_{r=0}^j \frac{(-1)^r (2j-r)!}{r! 2^{2r} (2j-2r+1)!} x^{2j-2r} \\ &= x^{2j} {}_2F_1 \left( \begin{matrix} -j, -j - \frac{1}{2} \\ -2j \end{matrix} \middle| \frac{1}{x^2} \right), \\ C_{2j+1}(x) &= \frac{\Gamma(j + \frac{5}{2})}{(2j+1)!} \sum_{r=0}^j \frac{(-1)^r \binom{j}{j-r} (2j-r+1)!}{\Gamma(j-r + \frac{5}{2})} x^{2j-2r+1} \\ &= x^{2j+1} {}_2F_1 \left( \begin{matrix} -j, -j - \frac{3}{2} \\ -2j-1 \end{matrix} \middle| \frac{1}{x^2} \right). \end{aligned}$$

We would like to show that these polynomials are intimately related to some known polynomials, namely Chebyshev polynomials of the first kind and Gegenbauer polynomials defined as follows.

The Jacobi polynomials  $P_n^{(\alpha, \beta)}(x)$  are given by ([12, (9.8.1)])

$$(1) \quad P_n^{(\alpha, \beta)}(x) = \frac{(\alpha + 1)_n}{n!} {}_2F_1 \left( \begin{matrix} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{matrix} \middle| \frac{1-x}{2} \right).$$

The Gegenbauer (or ultraspherical) polynomials are the Jacobi polynomials with  $\alpha = \beta = \lambda - 1/2$ ,  $\lambda \neq 0$  ([12, (9.8.19)] and [22, (15.9.3)])

$$(2) \quad \begin{aligned} C_n^{(\lambda)}(x) &= \frac{(2\lambda)_n}{(\lambda + 1/2)_n} P_n^{(\lambda-1/2, \lambda-1/2)}(x) \\ &= \frac{(2\lambda)_n}{n!} {}_2F_1 \left( \begin{matrix} -n, n + 2\lambda \\ \lambda + \frac{1}{2} \end{matrix} \middle| \frac{1-x}{2} \right) \\ &= (2x)^n \frac{(\lambda)_n}{n!} {}_2F_1 \left( \begin{matrix} -\frac{1}{2}n, \frac{1}{2}(1-n) \\ 1 - \lambda - n \end{matrix} \middle| \frac{1}{x^2} \right). \end{aligned}$$

The Chebyshev polynomials of the first kind can be obtained from the Jacobi polynomials by taking  $\alpha = \beta = -1/2$  ([12, (9.8.35)])

$$T_n(x) = \frac{P_n^{(-1/2, -1/2)}(x)}{P_n^{(-1/2, -1/2)}(1)} = {}_2F_1 \left( \begin{matrix} -n, n \\ 1/2 \end{matrix} \middle| \frac{1-x}{2} \right).$$

**Proposition 1.** *We show that:*

$$(3) \quad C_{2j}(x) = \frac{1}{4^j x} T_{2j+1}(x),$$

$$(4) \quad C_{2j+1}(x) = \frac{(-1)^j}{(2j+1)4^j} x C_{2j}^{(2)}(\sqrt{1-x^2}).$$

*Proof.* The left and right hand sides of Equations (3) and (4) are, respectively, solutions of the following recurrence relations with the same initial conditions:

$$p(j) + (-16x^2 + 8)p(j+1) + 16p(j+2) = 0, \quad p(0) = 1, \quad p(1) = x^2 - \frac{3}{4},$$

and

$$\begin{aligned} &16(2j+5)(2j+3)^2 p(j+2) - 8(2j+3)(8j^2x^2 + 32jx^2 - 4j^2 + 30x^2 \\ &- 16j - 17)p(j+1) + (2j+1)(2j+5)^2 p(j) = 0, \\ &p(0) = x, \quad p(1) = \frac{1}{6}x(6x^2 - 5); \end{aligned}$$

from which the equalities (3) and (4) follow. The latter recurrence relations are computed using Zeilberger's algorithm ([15], [24]) which generates a holonomic recurrence relation for a series with a summand which is a hypergeometric term.  $\square$

In [1], the authors studied the moments, inversion and linearization problems for the Chebyshev polynomials of fifth kind. Motivated by this work in [1], we consider the following remaining families of classical orthogonal polynomials defined by Masjed-Jamei:

$$(5) \quad \begin{aligned} J_n^{(p,q)}(x; a, b, c, d) &= (-1)^n ((ab + cd) + i(ad - bc))^n (n + 1 - 2p)_n \\ &\times \sum_{k=0}^n \binom{n}{k} \left( \frac{a^2 + c^2}{(ab + cd) + i(ad - bc)} \right)^k \\ &\times {}_2F_1 \left( \begin{matrix} k - n, p - n - iq/2 \\ 2p - 2n \end{matrix} \middle| \frac{2(ad - bc)}{(ad - bc) - i(ab + cd)} \right) x^k \quad (\text{see [17]}), \end{aligned}$$

solutions of the differential equation

$$(6) \quad \sigma(x)y_n''(x) + \tau(x)y_n'(x) - n(n + 1 - 2p)(a^2 + c^2)y_n(x) = 0,$$

where  $\sigma(x) = (ax + b)^2 + (cx + d)^2$ ,  $\tau(x) = (2(1 - p)(a^2 + c^2)x + q(ad - bc) + 2(1 - p)(ab + cd))$ ;

$$(7) \quad \begin{aligned} M_n^{(p,q)}(x) &= (-1)^n n! \sum_{k=0}^n \binom{p - n - 1}{k} \binom{q + n}{n - k} (-x)^k \quad (\text{see [16], [19, p. 15]}) \\ &= (-1)^n n! \binom{n + q}{n} {}_2F_1 \left( \begin{matrix} -n, -p + n + 1 \\ q + 1 \end{matrix} \middle| -x \right), \end{aligned}$$

$$(8) \quad \begin{aligned} N_n^{(p)}(x) &= (-1)^n \sum_{k=0}^n k! \binom{p - n - 1}{k} \binom{n}{n - k} (-x)^k \\ &= (-1)^n {}_2F_0 \left( \begin{matrix} -n, -p + n + 1 \\ \phantom{q + 1} \end{matrix} \middle| -x \right) \quad (\text{see [16], [19, p. 18]}), \end{aligned}$$

$$\begin{aligned}
 I_n^{(p)}(x) &= n! \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{p-1}{n-k} \binom{n-k}{k} (2x)^{n-2k} \quad (\text{see [16], [19, p. 20]}) \\
 (9) \quad &= n! \binom{p-1}{n} (2x)^n {}_2F_1 \left( \begin{array}{c} -\frac{1}{2}n, -\frac{1}{2}(n-1) \\ p-n \end{array} \middle| -\frac{1}{x^2} \right).
 \end{aligned}$$

We introduce the Bessel polynomials  $y_n(x; \alpha)$  defined for  $n = 0, 1, 2, \dots, N$ ,  $\alpha < -2N - 1$  by ([12, (9.13.1)])

$$(10) \quad y_n(x; \alpha) = {}_2F_0 \left( \begin{array}{c} -n, n + \alpha + 1 \\ \end{array} \middle| -\frac{x}{2} \right).$$

Some of the families defined by Masjed-Jamei are intimately related with well-known families. In fact, the following relations are valid [16]:

$$\begin{aligned}
 M_n^{(-\alpha-\beta, \alpha)} \left( \frac{1}{2}(x-1) \right) &= (-1)^n n! P_n^{(\alpha, \beta)}(x); \quad N_n^{(-\alpha)} \left( \frac{x}{2} \right) = (-1)^n y_n(x; \alpha); \\
 I_n^{(p)}(x) &= \frac{n! \Gamma(p)}{(1-p)_n (i)^n \Gamma(p-n)} C_n^{(1-p)}(ix),
 \end{aligned}$$

where  $\Gamma$  is the Gamma function [12, P. 3]. Given that the inversion, connection, linearization and the moment formulas of the first kind Chebyshev polynomials, the Gegenbauer polynomials, the Jacobi and Bessel polynomials are well-known in the literature (see e.g. [13], [21], [25] and references therein, see also [2] and [3] for some recent results) we could deduce the same results for some families defined by Masjed-Jamei. In fact, if for example

$$\begin{aligned}
 x^n &= \sum_{m=0}^n I_m(n, \alpha) y_m(x, \alpha), \\
 y_n(x, \alpha) &= \sum_{m=0}^n C_m(n, \alpha, \beta) y_m(x; \beta), \\
 y_n(x, \alpha) y_m(x, \alpha) &= \sum_{k=0}^{n+m} L_k(m, n, \alpha) y_k(x, \alpha),
 \end{aligned}$$

it follows from the relation  $N_n^{(\alpha)}(x) = (-1)^n y_n(2x; -\alpha)$  that

$$\begin{aligned} x^n &= \sum_{m=0}^n 2^{-n} (-1)^m I_m(n, -\alpha) N_m^{(\alpha)}(x), \\ N_n^{(\alpha)}(x) &= \sum_{m=0}^n (-1)^{n+m} C_m(n, -\alpha, -\beta) N_m^{(\beta)}(x), \\ N_n^{(\alpha)}(x) N_m^{(\alpha)}(x) &= \sum_{k=0}^{n+m} (-1)^{n+m+k} L_k(m, n, -\alpha) N_k^{(\alpha)}(x). \end{aligned}$$

Given that the characterizations of the Jacobi, Gegenbauer and Bessel polynomials are well-known, the ones of the families  $M_n^{(p,q)}(x)$ ,  $N_n^{(p)}(x)$ ,  $I_n^p(x)$  can be deduced using the latter relations. However, for the family  $J_n^{(p,q)}(x; a, b, c, d)$ , to the best of our knowledge, currently only very few characterizations that were given in [17] are known. In addition, there was no established relation between the family  $J_n^{(p,q)}(x; a, b, c, d)$  and a known classical orthogonal polynomial family.

The main aims of the manuscript are to

- show that the Chebyshev polynomials of fifth kind are related to the Chebyshev polynomials of the first kind and the Gegenbauer polynomials,
- establish the relation between the polynomials  $M_n^{(p,q)}(x)$ ,  $N_n^{(p)}(x)$ ,  $I_n^p(x)$  and the Jacobi, Gegenbauer, Bessel polynomials, respectively,
- derive the hypergeometric representation of  $J_n^{(p,q)}(x; a, b, c, d)$ , the relation between  $J_n^{(p,q)}(x; a, b, c, d)$  and the Jacobi polynomials as well as the inversion, connection, linearization and moment formulae of the family  $J_n^{(p,q)}(x; a, b, c, d)$ .

As novelty in this manuscript,

- we derive in Section 2 some structure relations satisfied by the family  $J_n^{(p,q)}(x; a, b, c, d)$ ;
- new hypergeometric representations of  $J_n^{(p,q)}(x; a, b, c, d)$  are given in Theorem 3 and a new identity connecting this polynomial family directly with the Jacobi polynomials with complex parameters is established in Theorem 4;
- Section 3 is devoted to the moment, inversion, connection and linearization formulae of the family  $J_n^{(p,q)}(x; a, b, c, d)$ . The interlacing of the zeros of  $J_n^{(p,q)}(x; a, b, c, d)$  and  $J_{n-1}^{(p-1,q)}(x; a, b, c, d)$  is discussed in Section 4.

All these results are new although the methods used to derive them are not.

## 2. Structure relations satisfied by the family

$$\mathbf{J}_n^{(p,q)}(x; a, b, c, d)$$

As any families of classical orthogonal polynomials, the family  $\left(J_n^{(p,q)}(x; a, b, c, d)\right)_{n \geq 0}$  satisfies a three-term recurrence relation as well as some additional structure relations.

**Proposition 2.** *For the family  $y_n(x) := J_n^{(p,q)}(x; a, b, c, d)$ , the following relations are valid:*

$$\begin{aligned} xy_n(x) &= -\frac{n+1-2p}{2(2n+1-2p)(n+1-p)(a^2+c^2)}y_{n+1}(x) \\ &\quad - \frac{2(n-p)^2(ab+cd) - pq(ad-bc) + 2ab(n-p) + 2cd(n-p)}{2(a^2+c^2)(n-p)(n+1-p)}y_n(x) \\ (11) \quad &\quad + \frac{n(4n^2 - 8np + 4p^2 + q^2)(ad-bc)^2}{2(a^2+c^2)(2n+1-2p)(n-p)}y_{n-1}(x); \end{aligned}$$

$$\begin{aligned} (12) \quad y_n(x) &= -\frac{n+1-2p}{2(2n+1-2p)(n+1-p)(a^2+c^2)(n+1)}y'_{n+1}(x) \\ &\quad + \frac{q(ad-bc)}{2(a^2+c^2)(n-p)(n+1-p)}y'_n(x) \\ &\quad - \frac{n(4(n-p)^2 + q^2)(ad-bc)^2}{2(a^2+c^2)(2n+1-2p)(n-p)(n-2p)}y'_{n-1}(x); \end{aligned}$$

$$\begin{aligned} xy'_n(x) &= -\frac{n(n+1-2p)}{2(2n+1-2p)(n+1-p)(a^2+c^2)(n+1)}y'_{n+1}(x) \\ &\quad - \frac{2(n-p)(ab+cd)(n+1-p) - pq(ad-bc) + adq - bcq}{2(a^2+c^2)(n-p)(n+1-p)}y'_n(x) \\ (13) \quad &\quad + \frac{n(n+1-2p)(4(n-p)^2 + q^2)(ad-bc)^2}{(a^2+c^2)(2n+1-2p)(n-p)(n-2p)}y'_{n-1}(x); \end{aligned}$$

$$\begin{aligned}
\sigma(x)y'_n(x) &= -\frac{n(n+1-2p)}{2(2n+1-2p)(n+1-p)}y_{n+1}(x) \\
&\quad -\frac{(n+1-2p)nq(ad-bc)}{2(n-p)(n+1-p)}y_n(x) \\
&\quad -\frac{n(n+1-2p)\left(4(n-p)^2+q^2\right)(ad-bc)^2}{2(2n+1-2p)(n-p)}y_{n-1}(x).
\end{aligned}
\tag{14}$$

*Proof.* We imitate the method that was introduced in [13] and also used in [9]. First write  $J_n^{(p,q)}(x; a, b, c, d) = k_n x^n + k'_n x^{n-1} + k''_n x^{n-2} + \dots$ , substitute it in the differential equation (6), and equate the first three highest coefficients to get  $k'_n$  and  $k''_n$  in terms of  $k_n$ .

To get the coefficients of the three-term recurrence relation

$$xy_n(x) = a_n y_{n+1}(x) + b_n y_n(x) + c_n y_{n-1}(x) \tag{15}$$

and the structure relations

$$y_n(x) = \hat{a}_n y'_{n+1}(x) + \hat{b}_n y'_n(x) + \hat{c}_n y'_{n-1}(x) \tag{16}$$

$$xy'_n(x) = \alpha_n^* y'_{n+1}(x) + \beta_n^* y'_n(x) + \gamma_n^* y'_{n-1}(x) \tag{17}$$

$$\sigma(x)y'_n(x) = \alpha_n y_{n+1}(x) + \beta_n y_n(x) + \gamma_n y_{n-1}(x), \tag{18}$$

we substitute  $J_n^{(p,q)}(x; a, b, c, d) = k_n x^n + k'_n x^{n-1} + k''_n x^{n-2} + \dots$  in each relation, equate again the first three highest coefficients and the results follow by linear algebra.  $\square$

**Remark.** The structure relations in Proposition 2 could also be derived using [8, Theorem 8.1, pp. 104-105].

The representation (5) of the polynomial  $J_n^{(p,q)}(x; a, b, c, d)$  is in terms of a double sum and therefore may not be easily used. For this reason, we find new hypergeometric representations of the polynomials  $J_n^{(p,q)}(x; a, b, c, d)$ .

**Theorem 3.** *The polynomials  $J_n^{(p,q)}(x; a, b, c, d)$  have the following hypergeo-*

metric representations:

$$\begin{aligned}
 J_n^{(p,q)}(x; a, b, c, d) &= (-1)^n (n+1-2p)_n (a^2 + c^2)^n \left( x + \frac{ab+cd}{a^2+c^2} \right)^n \\
 &\text{if } ad - bc = 0; \text{ and for } ad - bc \neq 0, \\
 J_n^{(p,q)}(x; a, b, c, d) &= \frac{2^n (ad - bc)^n (-p+1-iq/2)_n}{i^n} \\
 &\times {}_2F_1 \left( \begin{matrix} -n, n+1-2p \\ -p+1-iq/2 \end{matrix} \middle| \frac{i(a^2+c^2)}{2(ad-bc)} \left( x - \frac{i(ad-bc)-(ab+cd)}{a^2+c^2} \right) \right) \\
 &= \frac{2^n (ad - bc)^n (-p+1+iq/2)_n}{(-i)^n} \\
 &\times {}_2F_1 \left( \begin{matrix} -n, n+1-2p \\ -p+1+iq/2 \end{matrix} \middle| -\frac{i(a^2+c^2)}{2(ad-bc)} \left( x + \frac{i(ad-bc)+(ab+cd)}{a^2+c^2} \right) \right).
 \end{aligned}$$

*Proof.* When  $ad - bc = 0$ , (5) becomes

$$J_n^{(p,q)}(x; a, b, c, d) = (-1)^n (ab+cd)^n (n+1-2p)_n \sum_{k=0}^n \binom{n}{k} \left( \frac{a^2+c^2}{ab+cd} \right)^k x^k$$

and the result follows from the binomial theorem.

Suppose  $ad - bc \neq 0$  and let  $x_0$  be any of the zeros  $\frac{i(ad-bc)-ab-cd}{a^2+c^2}$  or  $-\frac{i(ad-bc)+ab+cd}{a^2+c^2}$  of  $\sigma(x) = (ax+b)^2 + (cx+d)^2$ . We substitute

$$(19) \quad J_n^{(p,q)}(x; a, b, c, d) = \sum_{m=0}^n A_m(n) (x - x_0)^m$$

in the differential equation (6), rewrite  $x$  as  $(x - x_0) + x_0$  in  $\sigma(x)$  and  $\tau(x)$  and use the fact that  $\left( (x - x_0)^n \right)_{n \geq 0}$  is a basis to get a recurrence equation satisfied by the coefficients  $A_m(n)$ .

Now, by using the Petkovšek-van-Hoeij algorithm [15, 23, 29], we solve the obtained recurrence relation and get the solution up to a multiplicative constant. The result follows by equating the leading coefficients in order to get the constant.  $\square$

We would like to note that using any of the two hypergeometric representations given in Theorem 3, a variant of Zeilberger's algorithm generates both the differential equation (6) as well as the recurrence equation (11) [15, Chapters 10 and 7]. Also by some special algorithms [15, Chapter 13], these

two equations can be easily determined from the Rodrigues representation [17, Eq. (4)]

$$J_n^{(p,q)}(x; a, b, c, d) = (-1)^n \left( (ax + b)^2 + (cx + d)^2 \right)^p \exp \left( -q \arctan \frac{ax + b}{cx + d} \right) \\ \times \frac{d^n}{dx^n} \left( \left( (ax + b)^2 + (cx + d)^2 \right)^{n-p} \exp \left( q \arctan \frac{ax + b}{cx + d} \right) \right)$$

by the commands

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g:=(-1)^n*((a*x+b)^2+(c*x+d)^2)^p*exp(-q*arctan((a*x+b)/(c*x+d)));
h:=((a*x+b)^2+(c*x+d)^2)^(n-p)*exp(q*arctan((a*x+b)/(c*x+d)));
DE:=rodriguesdiffEq(g,h,n,J(x));
RE:=rodriguesrecursion(g,h,x,J(n));
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respectively.

It is proved in [6] (see also [8, Theorem 8.2, p. 116]) through an equivalence relation that, up to constant factors and affine changes of variables, the four families of polynomials Hermite, Laguerre, Jacobi, and Bessel are the only families of classical orthogonal polynomials. A more general form of this result, applicable to arbitrary families on quadratic and  $q$ -quadratic lattices is discussed in [7]. It is therefore important to find the relation between  $J_n^{(p,q)}(x; a, b, c, d)$  and the above mentioned families.

**Theorem 4.** *The polynomial  $J_n^{(p,q)}(x; a, b, c, d)$  are connected directly to the Jacobi polynomials with complex parameters through the relation*

$$(20) \quad J_n^{(p,q)}(x; a, b, c, d) = \frac{2^n (ad - bc)^n n!}{(-i)^n} P_n^{(-p+iq/2, -p-iq/2)} \left( \frac{i((a^2 + c^2)x + ab + cd)}{ad - bc} \right).$$

*Proof.* We assume  $ad \neq bc$  and set

$$u = \frac{i(ad - bc) - ab - cd}{a^2 + c^2}; \quad v = -\frac{iad - ibc + ab + cd}{a^2 + c^2}.$$

In view of the change of variable

$$x = \frac{(v - u)y + u + v}{2} = -\frac{(ad - bc)iy + ab + cd}{a^2 + c^2} := Ay + B,$$

and

$$J_n^{(p,q)}(x; a, b, c, d) = J_n^{(p,q)}(Ay + B; a, b, c, d) = G_n(y),$$

(6) becomes

$$(y^2 - 1)G_n''(y) + (iq + 2(1 - p)y)G_n'(y) - n(n + 1 - 2p)G_n(y) = 0.$$

Comparing the latter differential equation with the differential equation

$$(1 - y^2)P_n''(y) + [\beta - \alpha - (\alpha + \beta + 2)y]P_n'(y) + n(n + \alpha + \beta + 1)P_n(y) = 0$$

satisfied by the Jacobi polynomials (1), it follows that

$$J_n^{(p,q)}(x; a, b, c, d) = a_n P_n^{(-p+iq/2, -p-iq/2)}\left(\frac{i((a^2 + c^2)x + ab + cd)}{ad - bc}\right),$$

where  $a_n = \frac{2^n(ad-bc)^n n!}{(-i)^n}$  is derived by equating the leading coefficients.  $\square$

Theorems 3 and 4 are part of the main results of this paper. Using the identity between  $J_n^{(p,q)}(x; a, b, c, d)$  and the Jacobi polynomials, many of the previous results can also be detected.

### 3. Moment, inversion, connection and linearization formulae of the family $J_n^{(p,q)}(x; a, b, c, d)$

#### 3.1. Connection formulae

In [17, Eq. (28)], an approximation of type

$$f(x) \cong \sum_{n=0}^N C_n J_n^{(p,q)}(x; a, b, c, d); \quad N < p - \frac{1}{2}$$

is derived with the coefficients  $C_n$  given in a generic form for any arbitrary function  $f(x)$ . Here we give an explicit expression of these coefficients when  $f(x)$  is a member of the polynomial family  $J_n^{(p,q)}(x; a, b, c, d)$ , that is,

$$(21) \quad J_n^{(p,q)}(x; a, b, c, d) = \sum_{m=0}^n C_m(n) J_m^{(p_1, q_1)}(x; a_1, b_1, c_1, d_1).$$

**Proposition 5.** *The connection coefficients  $C_m(n)$  in (21) for  $a = a_1, b = b_1, c = c_1, d = d_1$ , that is*

$$J_n^{(p,q)}(x; a, b, c, d) = \sum_{m=0}^n C_m(n) J_m^{(p_1, q_1)}(x; a, b, c, d),$$

are solution of the following recurrence equation:

$$\begin{aligned}
& - (n + 2 - 2p)(n - 2p + 1)(n + 1 - p)(-n - 2 + m)(m + n - 2p_1 + 3) \\
& \times S(n + 2) + (n + 2)(-2p + 3 + 2n)(n - 2p + 1) \left( m^2q - 2mp_1q + n^2q \right. \\
& - 2n^2q_1 - 2npq + 4npq_1 - 2p^2q_1 + 2p_1pq + qm + 3nq - 6nq_1 - 4pq \\
& \left. + 6pq_1 + 2q - 4q_1 \right) (ad - bc) S(n + 1) \\
& - (n + 2)(n + 1)(-p + 2 + n) \left( 4(n - p)^2 + q^2 + 8n - 8p + 4 \right) (ad - bc)^2 \\
& \times (n - 2p + 1 + m)(m - n + 2p - 2p_1) S(n) = 0.
\end{aligned}$$

The method of proof is well-known and can be found in [13, 9, 25]. We repeat the steps here to make the manuscript self-contained.

*Proof.* (compare [13], [9], [25])  $p_n := J_n^{(p,q)}(x; a, b, c, d)$  and  $q_m := J_m^{(p_1, q_1)}(x; a, b, c, d)$  are solution of the differential equation (6) with the same value of  $\sigma(x) = ax^2 + bx + c$ , and  $p = p_1, q = q_1$  in  $\tau(x)$  for  $J_m^{(p_1, q_1)}(x; a, b, c, d)$ . Note that we will denote all coefficients connected with  $q_m := J_m^{(p_1, q_1)}(x; a, b, c, d)$  by dashes. Hence we have

$$\begin{aligned}
xp_n(x) &= a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x) \\
xq_m(x) &= \bar{a}_m q_{m+1}(x) + \bar{b}_m q_m(x) + \bar{c}_m q_{m-1}(x),
\end{aligned}$$

with  $a_n, b_n, c_n, \bar{a}_m, \bar{b}_m, \bar{c}_m$  given explicitly in (11).

First, substituting  $p_n(x) = \sum_{m=0}^n C_m(n) q_m(x)$  in the three-term recurrence equation  $xp_n(x) = a_n p_{n+1}(x) + b_n p_n(x) + c_n p_{n-1}(x)$ , we get (since  $C_{n+s}(n) = 0, s = 1, 2, \dots$ )

$$\begin{aligned}
& \sum_{m=0}^n C_m(n) xq_m(x) \\
& = \sum_{m=0}^{n+1} \left( a_n C_m(n+1) q_m(x) + b_n C_m(n) q_m(x) + c_n C_m(n-1) q_m(x) \right).
\end{aligned}$$

Then we substitute  $xq_m(x)$  in the latter relation using the three-term recur-

rence equation for  $q_m(x)$ . This yields

$$\begin{aligned} & \sum_{m=0}^n C_m(n) \left( \bar{a}_m q_{m+1}(x) + \bar{b}_m q_m(x) + \bar{c}_m q_{m-1}(x) \right) \\ &= \sum_{m=0}^{n+1} \left( a_n C_m(n+1) + b_n C_m(n) + c_n C_m(n-1) \right) q_m(x). \end{aligned}$$

By appropriate index shifts, we equate the coefficient of  $q_m(x)$  to get the “cross rule”

$$(22) \quad \begin{aligned} & a_n C_m(n+1) + b_n C_m(n) + c_n C_m(n-1) \\ &= \bar{a}_{m-1} C_{m-1}(n) + \bar{b}_m C_m(n) + \bar{c}_{m+1} C_{m+1}(n). \end{aligned}$$

To deduce a second cross rule in terms of the same variables  $C_m(n+1)$ ,  $C_m(n)$ ,  $C_m(n-1)$ ,  $C_{m-1}(n)$  and  $C_{m+1}(n)$ , we examine the term  $x p'_n(x)$ . Using both three-term recurrence equations for the derivatives (see (13))

$$\begin{aligned} x p'_n(x) &= \alpha_n^* p'_{n+1}(x) + \beta_n^* p'_n(x) + \gamma_n^* p'_{n-1}(x) \\ x q'_m(x) &= \bar{\alpha}_m^* q'_{m+1}(x) + \bar{\beta}_m^* q'_m(x) + \bar{\gamma}_m^* q'_{m-1}(x), \end{aligned}$$

we get

$$\begin{aligned} x p'_n(x) &= \alpha_n^* p'_{n+1}(x) + \beta_n^* p'_n(x) + \gamma_n^* p'_{n-1}(x) \\ &\Downarrow \\ \sum_{m=0}^n C_m(n) x q'_m(x) &= \sum_{m=0}^{n+1} \left( \alpha_n^* C_m(n+1) + \beta_n^* C_m(n) + \gamma_n^* C_m(n-1) \right) q'_m(x) \\ &\Downarrow \\ \sum_{m=0}^n C_m(n) \left( \bar{\alpha}_m^* q'_{m+1}(x) + \bar{\beta}_m^* q'_m(x) + \bar{\gamma}_m^* q'_{m-1}(x) \right) &= \\ \sum_{m=0}^{n+1} \left( \alpha_n^* C_m(n+1) + \beta_n^* C_m(n) + \gamma_n^* C_m(n-1) \right) q'_m(x). \end{aligned}$$

Again, by appropriate index shifts, we can equate the coefficient of  $q'_m(x)$  to get the cross rule

$$(23) \quad \begin{aligned} & \alpha_n^* C_m(n+1) + \beta_n^* C_m(n) + \gamma_n^* C_m(n-1) \\ &= \bar{\alpha}_{m-1}^* C_{m-1}(n) + \bar{\beta}_m^* C_m(n) + \bar{\gamma}_{m+1}^* C_{m+1}(n). \end{aligned}$$

Using both derivatives rules (see (14))

$$\begin{aligned}\sigma(x)p'_n(x) &= \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x), \\ \sigma(x)q'_m(x) &= \bar{\alpha}_m q_{m+1}(x) + \bar{\beta}_m q_m(x) + \bar{\gamma}_m q_{m-1}(x),\end{aligned}$$

we get

$$\begin{aligned}\sigma(x)p'_n(x) &= \alpha_n p_{n+1}(x) + \beta_n p_n(x) + \gamma_n p_{n-1}(x) \\ \Updownarrow \\ \sum_{m=0}^n C_m(n) \sigma(x)q'_m(x) &= \sum_{m=0}^{n+1} \left( \alpha_n C_m(n+1) + \beta_n C_m(n) + \gamma_n C_m(n-1) \right) q_m(x) \\ \Updownarrow \\ \sum_{m=0}^n C_m(n) \left( \bar{\alpha}_m q_{m+1}(x) + \bar{\beta}_m q_m(x) + \bar{\gamma}_m q_{m-1}(x) \right) &= \\ \sum_{m=0}^{n+1} \left( \alpha_n C_m(n+1) + \beta_n C_m(n) + \gamma_n C_m(n-1) \right) q_m(x).\end{aligned}$$

Again, by appropriate index shifts, this results in the cross rule

$$(24) \quad \begin{aligned} & \alpha_n C_m(n+1) + \beta_n C_m(n) + \gamma_n C_m(n-1) \\ &= \bar{\alpha}_{m-1} C_{m-1}(n) + \bar{\beta}_m C_m(n) + \bar{\gamma}_{m-1} C_{m-1}(n). \end{aligned}$$

To obtain a pure recurrence equation with respect to  $n$ , we eliminate the variables  $C_{m-1}(n)$  and  $C_{m+1}(n)$ . This yields a second-order recurrence equation satisfied by the connection coefficients  $C_m(n)$ .  $\square$

Using the Petkovšek-van-Hoeij algorithm [15, 23, 29], it happens that this recurrence equation doesn't not have any hypergeometric term solution even when  $p_1 = p$  or  $q_1 = q$ . Therefore, we look for another representation of these connection coefficients.

**Proposition 6.** *The following connection formula is valid:*

$$(25) \quad J_n^{(p,q)}(x; a, b, c, d) = \sum_{m=0}^n C_m(n) J_m^{(p_1, q_1)}(x; a_1, b_1, c_1, d_1),$$

with

$$\begin{aligned}
 C_m(n) &= \sum_{k=0}^{n-m} \frac{(-2p_1 + 2m + 1)(a^2 + c^2)^{m+k} (-p + 1 - i/2q)_n (-n)_{k+m}}{(-1)^m (ad - bc)^{m+k-n} (i/2)^{n-m-k} m!k! (a_1^2 + c_1^2)^m} \\
 &\times \left( \frac{i(a_1 d_1 - b_1 c_1) - (a_1 b_1 + c_1 d_1)}{a_1^2 + c_1^2} - \frac{i(ad - bc) - (ab + cd)}{a^2 + c^2} \right)^k \\
 &\times \frac{\Gamma(m + 1 - 2p_1) \Gamma(n - 2p + 1 + k + m) \Gamma(-p + 1 - i/2q)}{\Gamma(n - 2p + 1) \Gamma(-i/2q + k + m - p + 1) \Gamma(2m + 2 - 2p_1)} \\
 &\times {}_3F_2 \left( \begin{matrix} -n + k + m, n - 2p + 1 + k + m, -i/2q_1 + m + 1 - p_1 \\ 2m + 2 - 2p_1, -i/2q + k + m - p + 1 \end{matrix} \middle| \frac{(a_1 d_1 - b_1 c_1)(a^2 + c^2)}{(a_1^2 + c_1^2)(ad - bc)} \right).
 \end{aligned}$$

In particular, if  $a = a_1$ ,  $b = b_1$ ,  $c = c_1$  and  $d = d_1$ , it follows that

$$\begin{aligned}
 J_n^{(p,q)}(x; a, b, c, d) &= \sum_{m=0}^n \frac{(-1)^m \left(\frac{i}{2}\right)^{m-n} (-n)_m \Gamma(m + n + 1 - 2p)}{m!(ad - bc)^{m-n} \Gamma(n - 2p + 1)} \\
 (26) \quad &\times \frac{\Gamma(m + 1 - 2p_1) \Gamma(n + 1 - p - \frac{1}{2}iq)}{\Gamma(m + 1 - p - \frac{1}{2}iq) \Gamma(2m - 2p_1 + 1)} \\
 &\times {}_3F_2 \left( \begin{matrix} m - n, m - p_1 + 1 - \frac{1}{2}iq_1, m + n + 1 - 2p \\ m - p + 1 - \frac{1}{2}iq, 2m + 2 - 2p_1 \end{matrix} \middle| 1 \right) J_m^{(p_1, q_1)}(x; a, b, c, d).
 \end{aligned}$$

In order to prove these results, we need the so-called inversion formulae.

**Proposition 7.** *The inversion formulae satisfied by the polynomial family  $J_n^{(p,q)}(x; a, b, c, d)$  is given as follows:*

$$\begin{aligned}
 (27) \quad &\left( x - \frac{i(ad - bc) - (ab + cd)}{a^2 + c^2} \right)^n = \sum_{m=0}^n \left( \frac{i}{2(ad - bc)} \right)^{m-n} \\
 &\times \frac{\Gamma(m + 1 - 2p) \Gamma(n + 1 - p - \frac{1}{2}iq) (-n)_m (2m - 2p + 1)}{m!(a^2 + c^2)^n \Gamma(m + 1 - p - \frac{1}{2}iq) \Gamma(m + n + 2 - 2p)} J_m^{(p,q)}(x; a, b, c, d);
 \end{aligned}$$

$$\begin{aligned}
 (28) \quad &\left( x + \frac{i(ad - bc) + (ab + cd)}{a^2 + c^2} \right)^n = \sum_{m=0}^n \left( -\frac{i}{2(ad - bc)} \right)^{m-n} \\
 &\times \frac{\Gamma(m + 1 - 2p) \Gamma(n + 1 - p + \frac{1}{2}iq) (-n)_m (2m - 2p + 1)}{m!(a^2 + c^2)^n \Gamma(m + 1 - p + \frac{1}{2}iq) \Gamma(m + n + 2 - 2p)} J_m^{(p,q)}(x; a, b, c, d).
 \end{aligned}$$

*Proof.* Let  $x_0$  be equal to  $\frac{i(ad-bc)-ab-cd}{a^2+c^2}$  or  $-\frac{i(ad-bc)+ab+cd}{a^2+c^2}$ . The polynomial

family  $p_n(x) = (x - x_0)^n$  is solution of the following relations:

$$\begin{aligned}\sigma(x)p'_n(x) &= ((ax + b)^2 + (cx + d)^2)n(x - x_0)^{n-1} \\ &= (Ax^2 + Bx + C)n(x - x_0)^{n-1} \\ &= (A_1(x - x_0)^2 + B_1(x - x_0) + C_1)n(x - x_0)^{n-1} \\ &= nA_1(x - x_0)^{n+1} + nB_1(x - x_0)^n + nC_1(x - x_0)^{n-1},\end{aligned}$$

with  $A = a^2 + c^2$ ,  $B = 2ab + 2cd$ ,  $C = b^2 + d^2$ ,  $A_1 = A$ ,  $B_1 = 2x_0A + B$ ,  $C_1 = Ax_0^2 + Bx_0 + C$ , that is,  $\sigma(x)p'_n(x) = nA_1p_{n+1}(x) + nB_1p_n(x) + nC_1p_{n-1}(x)$ ,

$$xp_n(x) = p_{n+1}(x) + x_0p_n(x), \quad xp'_n(x) = \frac{n}{n+1}p'_{n+1}(x) + x_0p'_n(x).$$

It follows from these three structure relations that the coefficients  $I_m(n)$  of the inversion formula

$$(29) \quad (x - x_0)^n = \sum_{m=0}^n I_m(n) J_m^{(p,q)}(x; a, b, c, d)$$

satisfy the cross rules

$$\begin{aligned}I_m(n+1) + x_0I_m(n) &= a_{m-1}I_{m-1}(n) + b_mI_m(n) + c_{m+1}I_{m+1}(n), \\ \frac{n}{n+1}I_m(n+1) + x_0I_m(n) &= \alpha_{m-1}^*I_{m-1}(n) + \beta_m^*I_m(n) + \gamma_{m+1}^*I_{m+1}(n).\end{aligned}$$

We solve the first cross rule with respect to  $I_m(n+1)$  and substitute it in the second one to get the following recurrence equation satisfied by  $I_m(n)$ :

$$\begin{aligned}&(2m+3-2p)(m+1-2p)(m-2p)(m+1-p)(m-n-1)I_{m-1}(n) \\ &- i(ad-bc)(inq-ipq+iq-2m^2+4pm-2p^2-2m+2p) \\ &\times (2m+3-2p)(m+1-2p)(2m-1-2p)mI_m(n) \\ &- m(m+1)(m-p)(2m-1-2p)(n+m+2-2p)(ad-bc)^2 \\ &\times (4m^2-8pm+4p^2+q^2+8m-8p+4)I_{m+1}(n),\end{aligned}$$

for  $x_0 = \frac{iad-ibc-ab-cd}{a^2+c^2}$ ; and

$$\begin{aligned} & (2m+3-2p)(m+1-2p)(m-2p)(m+1-p)(m-n-1)I_{m-1}(n) \\ & - i(ad-bc)(inq-ipq+iq+2m^2-4pm+2p^2+2m-2p) \\ & \times (2m+3-2p)(m+1-2p)(2m-1-2p)mI_m(n) \\ & - m(m+1)(m-p)(2m-1-2p)(ad-bc)^2(n+m+2-2p) \\ & \times (4m^2-8pm+4p^2+q^2+8m-8p+4)I_{m+1}(n) \end{aligned}$$

for  $x_0 = -\frac{iad-ibc+ab+cd}{a^2+c^2}$ . The Petkovšek-van-Hoeij algorithm [15, 23, 29] yields the solution of each of the latter recurrence relations up to a multiplicative constant and the results follow by equating the highest coefficients in both sides of (29).  $\square$

*Proof of Proposition 6.* Combining the representations

$$J_n^{(p,q)}(x; a, b, c, d) = \sum_{j=0}^n A_j(n)(x-x_0)^j$$

and

$$(x-x_0)^j = \sum_{m=0}^j I_m(j)J_m^{(p_1,q_1)}(x; a, b, c, d),$$

yields the connection formula

$$J_n^{(p,q)}(x; a, b, c, d) = \sum_{m=0}^n C_m(n)J_m^{(p_1,q_1)}(x; a, b, c, d),$$

with

$$C_m(n) = \sum_{j=0}^{n-m} A_{j+m}(n)I_m(j+m).$$

We use the Algorithm 2.8, p. 22 of [15] to convert the hypergeometric sums into hypergeometric notation, this algorithm is implemented in Maple in the package `hsum.mpl` by the procedure `sumtohyper`. This yields (26).

The connection formula (25) is obtained using the combination

$$J_n^{(p,q)}(x; a, b, c, d) = \sum_{k=0}^n A_k(n)(x-x_0)^k, \quad (x-x_0)^k = \sum_{j=0}^k \binom{k}{j} (x_1-x_0)^{k-j}(x-x_1)^j$$

and

$$(x - x_1)^j = \sum_{m=0}^j I_m(j) J_m^{(p_1, q_1)}(x; a_1, b_1, c_1, d_1),$$

with  $x_0 = \frac{i(ad-bc)-ab-cd}{a^2+c^2}$  and  $x_1 = \frac{i(a_1d_1-b_1c_1)-a_1b_1-c_1d_1}{a_1^2+c_1^2}$ .  $\square$

**Remark 8.** 1. *If we combine*

$$(30) \quad x^n = (x - x_0 + x_0)^n = \sum_{j=0}^n \binom{n}{j} x_0^{n-j} (x - x_0)^j$$

and the inversion formula

$$(x - x_0)^j = \sum_{m=0}^j I_m(j) J_m^{(p, q)}(x; a, b, c, d),$$

we get the inversion formula

$$x^n = \sum_{m=0}^n \left( \sum_{j=0}^{n-m} \binom{n}{j+m} x_0^{n-j-m} I_m(j+m) \right) J_m^{(p, q)}(x; a, b, c, d).$$

2. *If we apply the substitutions  $x \rightarrow \frac{i((a^2+c^2)x+ab+cd)}{ad-bc}$ ,  $\alpha \rightarrow -p + iq/2$  and  $\beta \rightarrow -p - iq/2$  in the inversion formulae*

$$(1 - x)^n = 2^n \Gamma(\alpha + n + 1) \times \sum_{m=0}^n \frac{(\alpha + \beta + 2m + 1) \Gamma(\alpha + \beta + m + 1)}{\Gamma(\alpha + m + 1) \Gamma(\alpha + \beta + n + m + 2)} (-n)_m P_m^{(\alpha, \beta)}(x)$$

(see e.g. [13], [11]),

$$(1 + x)^n = 2^n \Gamma(\beta + n + 1) \times \sum_{m=0}^n (-1)^m (-n)_m \frac{(\alpha + \beta + 2m + 1) \Gamma(\alpha + \beta + m + 1)}{\Gamma(\beta + m + 1) \Gamma(\alpha + \beta + n + m + 2)} P_m^{(\alpha, \beta)}(x)$$

(see e.g. [25] and references therein), and use the limit relation (20), (27) and (28) follows immediately.

3. Substituting  $x \rightarrow \frac{i((a^2+c^2)x+ab+cd)}{ad-bc}$ ,  $\alpha \rightarrow -p + iq/2$ ,  $\beta \rightarrow -p - iq/2$ ,  $\gamma \rightarrow -p_1 + iq_1/2$ , and  $\delta \rightarrow -p_1 - iq_1/2$  in the connection formula

$$P_n^{(\alpha,\beta)}(x) = \sum_{m=0}^n \frac{(m+\alpha+1)_{n-m}(n+\alpha+\beta+1)_m}{(n-m)!(m+\gamma+\delta+1)_m} \\ \times {}_3F_2 \left( \begin{matrix} m-n, n+m+\alpha+\beta+1, m+\gamma+1 \\ m+\alpha+1, 2m+\gamma+\delta+2 \end{matrix} \middle| 1 \right) P_m^{(\gamma,\delta)}(x)$$

(see e.g. [11, p. 257], [25] and references therein) yields the connection formula (26) in view of (20). However, (25) is completely new as could not be derived from an existing connection formula.

### 3.2. Moments

Let  $W^{(p,q)}(x; a, b, c, d) = ((ax+b)^2 + (cx+d)^2)^{-p} \exp(q \arctan(ax+b)/(cx+d))$ , the polynomial family  $(J_n^{(p,q)}(x; a, b, c, d))_n$  is orthogonal with respect to  $W^{(p,q)}(x; a, b, c, d)$  on  $(-\infty, \infty)$  and satisfies the orthogonality relation [17]

$$\int_{-\infty}^{\infty} W^{(p,q)}(x; a, b, c, d) J_n^{(p,q)}(x; a, b, c, d) J_m^{(p,q)}(x; a, b, c, d) dx \\ = \left( \frac{n!(a^2+c^2)^n \Gamma(2p-n)}{(ad-bc)^{2p-2n-1} \Gamma(2p-2n)} \int_{-\pi/2}^{\pi/2} (a \cos \theta - c \sin \theta)^{2p-2n-2} \exp(q\theta) d\theta \right) \delta_{n,m},$$

if and only if  $m, n = 0, 1, 2, \dots, N < p - 1/2$  and  $ad - bc > 0$ ,  $q \in \mathbb{R}$ .

As stated in [20], to compute the moments of a continuous or discrete distribution, different bases are considered. For example, the canonical basis  $\{x^j\}_{j \geq 0}$  is used in the continuous normal distribution, while for the Jacobi weight function  $(1-x)^\alpha(1+x)^\beta$  as the shifted beta distribution [11] on  $[-1, 1]$ , using one of the two basis  $\{(1-x)^j\}_{j \geq 0}$  or  $\{(1+x)^j\}_{j \geq 0}$  is appropriate for this purpose. Njionou in [21] showed that for all  $n \in \mathbb{N}$ , the generalized moments

$$\mu_n(\theta_n(x)) = \int_{-\infty}^{\infty} W^{(p,q)}(x; a, b, c, d) \theta_n(x) dx$$

of the polynomial family  $(J_n^{(p,q)}(x; a, b, c, d))_n$  expanded in a basis  $\theta_n(x)$  can be computed by the formula

$$\mu_n(\theta_n(x)) = I_0(n) J_0^{(p,q)}(x; a, b, c, d) \mu_0,$$

where  $\theta_n(x) = \theta_n^{(1)}(x) = \left(x - \frac{i(ad-bc)-(ab+cd)}{a^2+c^2}\right)^n$  or  $\theta_n(x) = \theta_n^{(2)}(x) = \left(x + \frac{i(ad-bc)+(ab+cd)}{a^2+c^2}\right)^n$ . This yields the following results:

**Proposition 9.** *For the polynomial family  $(J_n^{(p,q)}(x; a, b, c, d))_n$ , the following moments are valid:*

$$(31) \quad \begin{aligned} \mu_n(\theta_n^{(1)}(x)) &= \left(\frac{i(a^2+c^2)}{2(ad-bc)}\right)^{-n} \frac{\Gamma(n+1-p-1/2iq)}{(ad-bc)^{2p-1}(2-2p)_n\Gamma(1-p-1/2iq)} \\ &\times \int_{-\pi/2}^{\pi/2} (a \cos \theta - c \sin \theta)^{2p-2} \exp(q\theta) d\theta, \end{aligned}$$

$$(32) \quad \begin{aligned} \mu_n(\theta_n^{(2)}(x)) &= \left(-\frac{i(a^2+c^2)}{2(ad-bc)}\right)^{-n} \frac{\Gamma(n+1-p+1/2iq)}{(ad-bc)^{2p-1}(2-2p)_n\Gamma(1-p+1/2iq)} \\ &\times \int_{-\pi/2}^{\pi/2} (a \cos \theta - c \sin \theta)^{2p-2} \exp(q\theta) d\theta. \end{aligned}$$

**Proposition 10.** *The moments  $\mu_n(\theta_n^{(1)}(x))$  and  $\mu_n(\theta_n^{(2)}(x))$  are, respectively, solutions of the following recurrence relation:*

$$(33) \quad (a^2+c^2)(n+2-2p)\mu_{n+1} - i(ad-bc)(iq-2n+2p-2)\mu_n = 0,$$

$$(34) \quad (a^2+c^2)(n+2-2p)\mu_{n+1} - i(ad-bc)(iq+2n-2p+2)\mu_n = 0.$$

*Proof.* Integrate the Pearson equation

$$\left(\sigma(x)W^{(p,q)}(x; a, b, c, d)\right)' \theta_n(x) = \tau(x)W^{(p,q)}(x; a, b, c, d)\theta_n(x)$$

from  $-\infty$  to  $\infty$  and use, respectively, integration by parts, the boundary conditions

$$\sigma(x)w(x)x^k \Big|_{x=-\infty, \infty} = 0, \quad \forall k \geq 0$$

and rewrite  $x$  as  $x - x_0 + x_0$  in  $\sigma(x)$  and  $\tau(x)$  to get the result.  $\square$

Using the formula (see (30))

$$x^n = \sum_{m=0}^n C_m(n)\theta_m(x),$$

we deduce that

$$\mu_n := \int_{-\infty}^{\infty} W^{(p,q)}(x; a, b, c, d)x^n dx = \sum_{m=0}^n C_m(n)\mu_m(\theta_m(x)).$$

### 3.3. Linearization formula

**Proposition 11.** *The polynomial family  $J_n^{(p,q)}(x; a, b, c, d)$  satisfies the linearization formula*

$$(35) \quad J_n^{(p,q)}(x; a, b, c, d) J_m^{(p,q)}(x; a, b, c, d) = \sum_{k=0}^{n+m} L_k(m, n) J_k^{(p,q)}(x; a, b, c, d),$$

where

$$\begin{aligned} L_k(m, n) &= \sum_{l=0}^{n+m-k} \frac{(-k-l)_k (-p+1-i/2q)_n \Gamma(-i/2q+k+l+1-p) (-p+1-i/2q)_m}{(2(ad-bc))^{-n-m} i^{n+m} (a^2+c^2)^{k+l} \Gamma(2k+l+2-2p) k!} \\ &\times \frac{(-2p+2k+1) (-m)_{k+l} \Gamma(m+1-2p+k+l) \Gamma(-p+1-i/2q)}{\Gamma(m+1-2p) \Gamma(-i/2q+k+l+1-p) \Gamma(k+1+l)} \left( \frac{i/2}{ad-bc} \right)^k \\ &\times \frac{\Gamma(1-2p+k)}{\Gamma(-p+1-i/2q+k)} {}_4F_3 \left( \begin{matrix} n-2p+1, i/2q-k-l+p, -n, -k-l \\ m-k-l+1, -m+2p-k-l, -p+1-i/2q \end{matrix} \middle| 1 \right). \end{aligned}$$

*Proof.* Since

$$\begin{aligned} J_n^{(p,q)}(x; a, b, c, d) &= \sum_{i=0}^n A_i(n) (x-x_0)^i \quad \text{and} \\ J_m^{(p,q)}(x; a, b, c, d) &= \sum_{j=0}^m A_j(m) (x-x_0)^j, \end{aligned}$$

then by the Cauchy product

$$J_n^{(p,q)}(x; a, b, c, d) J_m^{(p,q)}(x; a, b, c, d) = \sum_{l=0}^{n+m} G_l(m, n) (x-x_0)^l,$$

with

$$G_l(m, n) = \sum_{i=0}^l A_i(n) A_{l-i}(m).$$

Combining the preceding result with the inversion formula

$$(x-x_0)^l = \sum_{k=0}^l I_k(l) J_k^{(p,q)}(x; a, b, c, d),$$

we get (35)

$$\begin{aligned} L_k(m, n) &= \sum_{l=0}^{n+m-k} G_{l+k}(m, n) I_k(l+k) \\ &= \sum_{l=0}^{n+m-k} \sum_{i=0}^{l+k} I_k(l+k) A_i(n) A_{l+k-i}(m). \end{aligned}$$

We use again Algorithm 2.8, p. 22 of [15] to convert the hypergeometric sum into hypergeometric notation.  $\square$

#### 4. Interlacing of the zeros the family $J_n^{(p,q)}(x; a, b, c, d)$

Using a continuous version of the algorithm implemented in [26] which was successfully implemented in [27], the following structural relation can be derived:

$$\begin{aligned} (36) \quad J_n^{(p,q)}(x; a, b, c, d) &= \frac{(n+2-2p)}{2(n-p+1)} J_n^{(p-1,q)}(x; a, b, c, d) + \frac{n}{2(n-p+1)} \\ &\times \left( 2(a^2+c^2)(n-p+1)x + 2(n-p+1)(ab+cd) \right. \\ &\left. - q(ad-bc) \right) J_{n-1}^{(p-1,q)}(x; a, b, c, d). \end{aligned}$$

The relation (36) helps us to deduce from [26, Lemma 1] that  $J_n^{(p,q)}(x; a, b, c, d)$  has an odd number of zeros between any two zeros of  $J_{n-1}^{(p-1,q)}(x; a, b, c, d)$ .

### Conclusion

In this manuscript, new hypergeometric representations of the polynomial family  $J_n^{(p,q)}(x; a, b, c, d)$  has been derived as well as a relation between  $J_n^{(p,q)}(x; a, b, c, d)$  and the Jacobi polynomials. New connection, inversion, linearization formulae for the family  $J_n^{(p,q)}(x; a, b, c, d)$  have been established using well known methods. The algorithm implemented in Maple by Koepf and Schmersau (2002) failed to detect the relation (20) between  $J_n^{(p,q)}(x; a, b, c, d)$  and the Jacobi polynomials. As future work, we are going to improve the above-mentioned algorithm to be able to detect this relation.

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