Two Classes of Special Functions Using Fourier Transforms of Generalized Ultraspherical and Generalized Hermite Polynomials

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Abstract. Some orthogonal polynomial systems are mapped onto each other by the Fourier transform. Motivated by the paper [W. Koepf, M. Masjed-Jamei, Two classes of special functions using Fourier transforms of some finite classes of classical orthogonal polynomials, *Proc. Amer. Math. Soc.,* 135 (2007) 3599-3606], in this paper we introduce two new classes of orthogonal functions, which are respectively Fourier transforms of the generalized ultraspherical polynomials and generalized Hermite polynomials, and then obtain their orthogonality relations using Parseval identity.

Keywords. generalized ultraspherical polynomials, generalized Hermite polynomials, Fourier transform, Parseval identity, hypergeometric functions

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1. Introduction. Let us begin with the differential equation [9]:

$$x^2(px^2+q)\,\Phi''_n(x)+x(rx^2+s)\,\Phi'_n(x)-\Big(n(r+(n-1)p)x^2+(1-(-1)^n)\,s/2\Big)\Phi_n(x)=0\,, \tag{1}$$

where p,q,r,s are real parameters and n is a nonnegative integer.

According to [9], one of the basis solutions of this equation is a specific class of symmetric orthogonal polynomials (with four free parameters) in the explicit form

$$\Phi_n(x)=S_n\left(\begin{matrix} r & s\\ p & q\end{matrix}\middle|\, x\right)=\sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\left(\prod_{i=0}^{[n/2]-(k+1)}\frac{\big(2i+(-1)^{n+1}+2[n/2]\big)p+r}{\big(2i+(-1)^{n+1}+2\big)\,q+s}\right)x^{n-2k}\,, \tag{2}$$

and the hypergeometric form [9]

$$\bar{S}_n\left(\begin{matrix} r & s\\ p & q\end{matrix}\middle|\, x\right)=x^n\,{}_2F_1\left(\begin{matrix}-[n/2],\quad (q-s)/2q-[(n+1)/2]\\ -(r+(2n-3)p)/2p\end{matrix}\middle|\,-\frac{q}{px^2}\right), \tag{3}$$

such that $\bar{S}_n(p,q,r,s;x)$ is the monic type of polynomials (2) satisfying the three term recurrence relation:

$$\bar{S}_{n+1}(x)=x\,\bar{S}_n(x)+C_n\begin{pmatrix} r & s\\ p & q\end{pmatrix}\bar{S}_{n-1}(x)\quad;\quad \bar{S}_0(x)=1\,,\ \bar{S}_1(x)=x\,,\ n\in\mathbf{N}\,, \tag{4}$$

in which

$$C_n\begin{pmatrix} r & s \\ p & q \end{pmatrix} = \frac{pq\,n^2 + \big((r-2p)q - (-1)^n ps\big)n + (r-2p)\,s(1-(-1)^n)/2}{(2pn+r-p)(2pn+r-3p)}, \qquad (4.1)$$

and

$$\bar{S}_n(x) = \bar{S}_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) = \prod_{i=0}^{[n/2]-1} \frac{\big(2i+(-1)^{n+1}+2\big)\,q+s}{\big(2i+(-1)^{n+1}+2[n/2]\big)p+r}\; S_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right). \qquad (4.2)$$

Since (4) is explicitly known, the norm square value of the polynomials can be obtained by using Favard theorem [1]. In other words, the generic form of the orthogonality relation of the polynomials (2) is given by [9]

$$\int_{-\alpha}^{\alpha} W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) \bar{S}_n\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) \bar{S}_m\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) dx = \left((-1)^n \prod_{i=1}^{n} C_i\begin{pmatrix} r & s \\ p & q \end{pmatrix} \int_{-\alpha}^{\alpha} W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) dx\right)\delta_{n,m}, \qquad (5)$$

in which $\delta_{n,m} = \begin{cases} 0 & (n \neq m) \\ 1 & (n = m) \end{cases}$, the weight function is defined by [9]

$$W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) = \exp\left(\int \frac{(r-2p)x^2 + s}{x(px^2+q)}\,dx\right), \qquad (6)$$

and finally α takes the standard values $1, \infty$. In this sense, note according to [9], the function $(px^2+q)W(p,q,r,s;x)$ must vanish at $x = \alpha$ in order to be valid the orthogonality property (5).

In general, four classes of symmetric orthogonal polynomials can be extracted from the differential equation (1). Two of them are infinitely orthogonal (namely the generalized ultraspherical polynomials and generalized Hermite polynomials) and two other ones, which are less known [9], are finitely orthogonal. See table 1 in this regard.

Table 1: Four special subclasses of $S_n(p,q,r,s;x)$

Definition	Weight function	Interval and Kind
$S_n\left(\begin{matrix} -2a-2b-2, & 2a \\ -1, & 1 \end{matrix}\middle| x\right)$	$W\left(\begin{matrix} -2a-2b-2, & 2a \\ -1, & 1 \end{matrix}\middle| x\right) = x^{2a}(1-x^2)^b$	$[-1,1]$, Infinite
$S_n\left(\begin{matrix} -2, & 2a \\ 0, & 1 \end{matrix}\middle| x\right)$	$W\left(\begin{matrix} -2, & 2a \\ 0, & 1 \end{matrix}\middle| x\right) = x^{2a}\exp(-x^2)$	$(-\infty,\infty)$, Infinite
$S_n\left(\begin{matrix} -2a-2b+2, & -2a \\ 1, & 1 \end{matrix}\middle| x\right)$	$W\left(\begin{matrix} -2a-2b+2, & -2a \\ 1, & 1 \end{matrix}\middle| x\right) = \frac{x^{-2a}}{(1+x^2)^b}$	$(-\infty,\infty)$, Finite
$S_n\left(\begin{matrix} -2a+2, & 2 \\ 1, & 0 \end{matrix}\middle| x\right)$	$W\left(\begin{matrix} -2a+2, & 2 \\ 1, & 0 \end{matrix}\middle| x\right) = x^{-2a}\exp(-1/x^2)$	$(-\infty,\infty)$, Finite

Note in table 1 that all four weight functions must be even and positive, i.e. the condition $(-1)^{2a}=1$ must be satisfied for any four cases.
In this paper, the general properties of the two first cases of table 1 are required. So, we restate them here in summary.

1.1. Generalized ultraspherical polynomials

If $(p,q,r,s)=(-1,1,-2a-2b-2,2a)$ is substituted in (1), then the equation

$$x^2(-x^2+1)\Phi''_n(x)-2x((a+b+1)x^2-a)\Phi'_n(x)+\big(n(2a+2b+n+1)x^2+((-1)^n-1)a\big)\Phi_n(x)=0, \tag{7}$$

appears that has the explicit monic polynomial solution [9]

$$U_n^{(a,b)}(x)=\bar{S}_n\left(\begin{matrix}-2a-2b-2, & 2a\\ -1, & 1\end{matrix}\middle|\, x\right)=\prod_{i=0}^{[n/2]-1}\frac{2i+2a+2-(-1)^n}{-2i-(2b+2a+2-(-1)^n+2[n/2])}$$
$$\times\sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\left(\prod_{i=0}^{[n/2]-(k+1)}\frac{-2i-(2b+2a+2-(-1)^n+2[n/2])}{2i+2a+2-(-1)^n}\right)x^{n-2k}, \tag{7.1}$$

which is equivalent to the hypergeometric representation

$$U_n^{(a,b)}(x)=x^n\,{}_2F_1\left(\begin{matrix}-[n/2], & -a+1/2-[(n+1)/2]\\ & -a-b-n+1/2\end{matrix}\middle|\,\frac{1}{x^2}\right). \tag{8}$$

In this sense, ${}_2F_1(.)$ is a special case of the generalized hypergeometric function [2,6] of order (p,q) as

$${}_pF_q\left(\begin{matrix}a_1 & a_2 & \dots & a_p\\ b_1 & b_2 & \dots & b_q\end{matrix}\middle|\, x\right)=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\dots(a_p)_k}{(b_1)_k(b_2)_k\dots(b_q)_k}\frac{x^k}{k!}, \tag{8.1}$$

in which $(r)_k=\prod_{i=0}^{k-1}(r+i)$ denotes the Pochhammer symbol.
The orthogonality relation corresponding to polynomials (8) is given by [9]

$$\int_{-1}^{1}x^{2a}(1-x^2)^b U_n^{(a,b)}(x)U_m^{(a,b)}(x)\,dx=\left((-1)^n\prod_{j=1}^{n}C_j\left(\begin{matrix}-2a-2b-2, & 2a\\ -1, & 1\end{matrix}\right)\int_{-1}^{1}x^{2a}(1-x^2)^b\,dx\right)\delta_{n,m}=$$
$$\frac{\Gamma\left(a+\frac{1}{2}\right)\Gamma\left(a+b-\frac{1}{2}\right)\Gamma\left([\frac{n}{2}]+1\right)\Gamma\left(b+\frac{1}{2}+[\frac{n}{2}]\right)\Gamma\left(a+1+[\frac{n}{2}]+\frac{1-(-1)^n}{2}\right)\Gamma\left(a+b+\frac{1}{2}+[\frac{n}{2}]+\frac{1-(-1)^n}{2}\right)}{\Gamma(a+1)\Gamma\left(a+b+\frac{3}{2}\right)\Gamma\left(a+b+n-\frac{1}{2}\right)\Gamma\left(a+b+n+\frac{1}{2}\right)}\delta_{n,m}, \tag{9}$$

where

$$C_j\left(\begin{matrix}-2a-2b-2, & 2a\\ -1, & 1\end{matrix}\right)=\frac{-\big(j+(1-(-1)^j)a\big)\big(j+(1-(-1)^j)a+2b\big)}{(2j+2a+2b-1)(2j+2a+2b+1)}, \tag{9.1}$$

and

$$\int_{-1}^{1} x^{2a}(1-x^2)^b\,dx = B(a+\frac{1}{2}, b+1) = \frac{\Gamma(a+1/2)\Gamma(b+1)}{\Gamma(a+b+3/2)}. \tag{9.2}$$

Relation (9) shows that constraints on the parameters a and b are respectively $a+1/2>0$, $(-1)^{2a}=1$ and $a+b-1/2>0$. Moreover, $B(\lambda_1,\lambda_2)$ in (9.2) is in general the Beta integral [2,3] having various definitions as

$$\begin{aligned} B(\lambda_1;\lambda_2) &= \int_0^1 x^{\lambda_1-1}(1-x)^{\lambda_2-1}dx = \int_{-1}^{1} x^{2\lambda_1-1}(1-x^2)^{\lambda_2-1}dx = \int_0^\infty \frac{x^{\lambda_1-1}}{(1+x)^{\lambda_1+\lambda_2}}dx \\ &= 2\int_0^{\pi/2} \sin^{(2\lambda_1-1)} x\,\cos^{(2\lambda_2-1)} x\,dx = \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda_1+\lambda_2)} = B(\lambda_2;\lambda_1), \end{aligned} \tag{10}$$

in which

$$\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx \qquad \mathrm{Re}(z)>0, \tag{10.1}$$

denotes the well-known Gamma function satisfying the fundamental recurrence relation $\Gamma(z+1)=z\Gamma(z)$.

1.2. Generalized Hermite polynomials

Similarly, if $(p,q,r,s)=(0,1,-2,2a)$ is considered in (1), then the equation

$$x^2\Phi''_n(x) - 2x(x^2-a)\Phi'_n(x) + \left(2n\,x^2 + ((-1)^n-1)a\right)\Phi_n(x) = 0\ . \tag{11}$$

has the monic polynomial solution [9]

$$H_n^{(a)}(x) = \bar{S}_n\left(\begin{matrix}-2 & 2a \\ 0 & 1\end{matrix}\,\middle|\, x\right) = (-1)^{[\frac{n}{2}]}\left(a+1-\frac{(-1)^n}{2}\right)_{[\frac{n}{2}]} \sum_{k=0}^{[n/2]} \binom{[n/2]}{k} \left(\prod_{i=0}^{[n/2]-(k+1)} \frac{-2}{2i+(-1)^{n+1}+2+2a}\right) x^{n-2k}, \tag{12}$$

which is equivalent to the hypergeometric representation

$$H_n^{(a)}(x) = x^n\,{}_2F_0\left(\begin{matrix}-[n/2], & -[n/2]-a+(-1)^n/2 \\ - & \end{matrix}\,\middle|\, -\frac{1}{x^2}\right). \tag{12.1}$$

Also, the corresponding orthogonality relation takes the form [9]

$$\begin{aligned} \int_{-\infty}^{\infty} x^{2a}e^{-x^2} H_n^{(a)}(x) H_m^{(a)}(x)\,dx &= \left(\frac{1}{2^n}\prod_{i=1}^{n}(1-(-1)^i)a+i\right)\Gamma(a+\frac{1}{2})\ \delta_{n,m} \\ &= \Gamma\left(a+\frac{1}{2}+[\frac{n+1}{2}]\right)\Gamma\left([\frac{n}{2}]+1\right)\ \delta_{n,m} \quad (a>-\frac{1}{2}), \end{aligned} \tag{13}$$

where the following identity is used

$$[\frac{n+1}{2}] - [\frac{n}{2}] = \frac{1-(-1)^n}{2}. \tag{13.1}$$

But it is known that some orthogonal polynomial systems are mapped onto each other by the Fourier transform or other integral transforms such as the Mellin and Hankel transforms. Some illustrative examples in this regard are found in e.g. [5,7,8]. In this paper, we apply this viewpoint to introduce two new classes of orthogonal functions by using Fourier transforms and Parseval identity.

2. Fourier transforms of generalized ultraspherical polynomials and generalized Hermite polynomials and their orthogonality relations

The Fourier transform of a function, say $g(x)$, is defined as [4]

$$\mathbf{F}(s) = \mathbf{F}(g(x)) = \int_{-\infty}^{\infty} e^{-isx} g(x)\, dx, \tag{14}$$

and for the inverse transform one has the formula

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{isx} \mathbf{F}(s)\, ds. \tag{14.1}$$

For $g, h \in L^2(\mathbf{R})$, the Parseval identity related to Fourier theory is given by [4]

$$\int_{-\infty}^{\infty} g(x)\, h(x)\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{F}(g(x))\, \overline{\mathbf{F}(h(x))}\, ds. \tag{15}$$

By noting (9) and (15) we now define the following specific functions

$$\begin{cases} g(x) = (\tanh x)^{2\alpha} (1-\tanh^2 x)^{\beta}\, U_n^{(c,d)}(\tanh x) \quad \text{s.t.} \ \ (-1)^{2\alpha} = 1, \\ h(x) = (\tanh x)^{2l} (1-\tanh^2 x)^{u}\, U_m^{(v,w)}(\tanh x) \quad \text{s.t.} \ \ (-1)^{2l} = 1, \end{cases} \tag{16}$$

in terms of the generalized ultraspherical polynomials to which we will apply the Fourier transform. Clearly for both above functions the Fourier transform exists. For instance, for the function $g(x)$ defined in (16) we get

$$\begin{aligned} \mathbf{F}(g(x)) &= \int_{-\infty}^{\infty} e^{-isx} (\tanh x)^{2\alpha} (1-\tanh^2 x)^{\beta}\, U_n^{(c,d)}(\tanh x)\, dx \\ &= \int_{-1}^{1} (1+t)^{-\frac{is}{2}} (1-t)^{\frac{is}{2}}\, t^{2\alpha} (1-t^2)^{\beta-1}\, U_n^{(c,d)}(t)\, dt \\ &= 2^{2\beta-1}\int_{0}^{1} (1-z)^{-\frac{is}{2}} z^{\frac{is}{2}} (1-2z)^{2\alpha} z^{\beta-1} (1-z)^{\beta-1}\, U_n^{(c,d)}(1-2z)\, dz \\ &= 2^{2\beta-1}\int_{0}^{1} (1-z)^{\beta-1-\frac{is}{2}} z^{\beta-1+\frac{is}{2}} (1-2z)^{2\alpha+n} \left(\sum_{k=0}^{[n/2]} \frac{(-[\frac{n}{2}])_k (\frac{1}{2}-c-[\frac{n+1}{2}])_k}{(-c-d-n+1/2)_k\, k!} \frac{1}{(1-2z)^{2k}} \right) dz \\ &= 2^{2\beta-1}\sum_{k=0}^{[n/2]} \frac{(-[\frac{n}{2}])_k (\frac{1}{2}-c-[\frac{n+1}{2}])_k}{(-c-d-n+1/2)_k\, k!} \left(\int_{0}^{1} (1-z)^{\beta-1-\frac{is}{2}} z^{\beta-1+\frac{is}{2}} (1-2z)^{2\alpha+n-2k}\, dz \right). \end{aligned} \tag{17}$$

On the other hand, since

$$ {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| x\right) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 (1-z)^{c-b-1} z^{b-1}(1-xz)^{-a}\, dz\,, \tag{18} $$

is the integral representation of the Gauss hypergeometric function [2,6] for $\operatorname{Re} c > \operatorname{Re} b > 0$ and $|x| \le 1$, the last integral in relation (17) can be shown in terms of (18) and we finally obtain

$$ \mathbf{F}(g(x)) = 2^{2\beta-1} B\left(\beta+\frac{is}{2}, \beta-\frac{is}{2}\right)\sum_{k=0}^{[n/2]} \frac{(-[\frac{n}{2}])_k (\frac{1}{2}-c-[\frac{n+1}{2}])_k}{(-c-d-n+1/2)_k\, k!}\, {}_2F_1\left(\begin{matrix} 2k-n-2\alpha & \beta+\frac{is}{2} \\ & 2\beta \end{matrix}\middle| 2\right). \tag{19} $$

For simplicity if we here define

$$ K_n(x; p_1, p_2, p_3, p_4) = \sum_{k=0}^{[n/2]} \frac{(-[\frac{n}{2}])_k (\frac{1}{2}-p_3-[\frac{n+1}{2}])_k}{(-p_3-p_4-n+1/2)_k\, k!}\, {}_2F_1\left(\begin{matrix} 2k-n-2p_1 & p_2+\frac{x}{2} \\ & 2p_2 \end{matrix}\middle| 2\right), \tag{20} $$

then it is clear in (19) that

$$ \mathbf{F}(g(x)) = \frac{2^{2\beta-1}}{\Gamma(2\beta)}\Gamma(\beta+\frac{is}{2})\Gamma(\beta-\frac{is}{2})\, K_n(is;\alpha,\beta,c,d)\,. \tag{21} $$

Now, by substituting (21) in Parseval’s identity (15) and noting (16) we get

$$ \begin{aligned} &2\pi\int_{-\infty}^{\infty} (\tanh x)^{2(\alpha+l)} (1-\tanh^2 x)^{\beta+u}\, U_n^{(c,d)}(\tanh x) U_m^{(v,w)}(\tanh x)dx \\ &\qquad = 2\pi\int_{-1}^{1} t^{2(\alpha+l)} (1-t^2)^{\beta+u-1}\, U_n^{(c,d)}(t) U_m^{(v,w)}(t)\, dt \\ &= \frac{2^{2\beta+2u-2}}{\Gamma(2\beta)\Gamma(2u)}\int_{-\infty}^{\infty} \Gamma(\beta+\frac{is}{2})\Gamma(\beta-\frac{is}{2})\overline{\Gamma(u+\frac{is}{2})\Gamma(u-\frac{is}{2})}\, K_n(is;a,b,c,d)\,\overline{K_m(is;l,u,v,w)}\, ds\,. \end{aligned} \tag{22} $$

On the other hand, if in the left hand side of (22) we take

$$ c = v = \alpha + l \qquad \text{and} \qquad d = w = \beta + u - 1\,, \tag{23} $$

then according to orthogonality relation (9), equation (22) finally reads as

$$ \begin{aligned} &\frac{1}{2\pi}\int_{-\infty}^{\infty} \Gamma(\beta+\frac{is}{2})\Gamma(\beta-\frac{is}{2})\Gamma(u+\frac{is}{2})\Gamma(u-\frac{is}{2})\, K_n\big(is;\alpha,\beta,\alpha+l,\beta+u-1\big) K_m\big(-is;l,u,\alpha+l,\beta+u-1\big) ds \\ &= \frac{\Gamma(2\beta)\Gamma(2u)\Gamma(\alpha+l+1/2)\Gamma(\beta+u)}{2^{2\beta+2u+2n-2}\,\Gamma(\alpha+\beta+l+u+1/2)} \prod_{j=1}^{n} \frac{\big(j+(1-(-1)^j)(\alpha+l)\big)\big(j+(1-(-1)^j)(\alpha+l)+2\beta+2u-2\big)}{\big(j+\alpha+\beta+l+u-3/2\big)\big(j+\alpha+\beta+l+u-1/2\big)}\, \delta_{n,m}\,, \end{aligned} \tag{24} $$

where $\prod_{j=1}^{0}(.) = 1$, $\beta, u > 0$ and $\alpha + l > -1/2$.

Remark 1. As (20) shows, $K_n(x;p_1,p_2,p_3,p_4)$ is not a polynomial function in the general case although it can be reduced to a polynomial when $p_1=0$. For instance, for $n=0,1,2,3$ we respectively have

$$K_0(x;p_1,p_2,p_3,p_4)={}_2F_1\left(\left.\begin{matrix}-2p_1 & p_2+\frac{x}{2}\\ 2p_2 & \end{matrix}\right|2\right),$$

$$K_1(x;p_1,p_2,p_3,p_4)={}_2F_1\left(\left.\begin{matrix}-1-2p_1 & p_2+\frac{x}{2}\\ 2p_2 & \end{matrix}\right|2\right),$$

$$K_2(x;p_1,p_2,p_3,p_4)={}_2F_1\left(\left.\begin{matrix}-2-2p_1 & p_2+\frac{x}{2}\\ 2p_2 & \end{matrix}\right|2\right)+\frac{2p_3+1}{2p_3-2p_4-3}\,{}_2F_1\left(\left.\begin{matrix}-2p_1 & p_2+\frac{x}{2}\\ 2p_2 & \end{matrix}\right|2\right),$$

$$K_3(x;p_1,p_2,p_3,p_4)={}_2F_1\left(\left.\begin{matrix}-2p_1 & p_2+\frac{x}{2}\\ 2p_2 & \end{matrix}\right|2\right)+\frac{2p_3+3}{2p_3-2p_4-5}\,{}_2F_1\left(\left.\begin{matrix}-1-2p_1 & p_2+\frac{x}{2}\\ 2p_2 & \end{matrix}\right|2\right). \tag{25}$$

Here we emphasize that for $p_1=0$, the above functions are reduced to a polynomial of exactly degree n.

Remark 2. Replacing $n=m=0$ and $\alpha=l=0$ in (24) gives a special case of the Barnes beta integral [10]:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma(a+ix)\Gamma(b+ix)\Gamma(c-ix)\Gamma(d-ix)\,dx=\frac{\Gamma(a+d)\Gamma(a+c)\Gamma(b+d)\Gamma(b+c)}{\Gamma(a+b+c+d)}, \tag{26}$$

if one applies the duplication Legendre formula [3]

$$\Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+\frac{1}{2}). \tag{27}$$

Moreover, the weight function of the orthogonality relation (24) can be simplified by using Cauchy beta integral [1,3]

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{dt}{(a+it)^c(b-it)^d}=\frac{\Gamma(c+d-1)}{\Gamma(c)\Gamma(d)}(a+b)^{1-(c+d)}, \tag{28}$$

and one of its consequences, i.e.

$$\Gamma(p+iq)\Gamma(p-iq)=\frac{2^{2-2p}\pi\,\Gamma(2p-1)}{\int_{-\pi/2}^{\pi/2}e^{2qx}\cos^{2p-2}x\,dx}, \tag{28.1}$$

which results in

$$\Gamma(\beta+\frac{is}{2})\Gamma(\beta-\frac{is}{2})\Gamma(u+\frac{is}{2})\Gamma(u-\frac{is}{2})=\frac{2^{4-2\beta-2u}\,\pi^2\,\Gamma(2\beta-1)\Gamma(2u-1)}{\int_{-\pi/2}^{\pi/2} e^{sx}\cos^{2\beta-2}x\,dx\int_{-\pi/2}^{\pi/2} e^{sx}\cos^{2u-2}x\,dx}. \quad (28.2)$$

Therefore, if we define

$$W^*(x;\lambda)=\int_{-\pi/2}^{\pi/2} e^{x\theta}\cos^{2\lambda-2}\theta\,d\theta, \quad (29)$$

the following theorem will be derived.

Theorem 1. The special function $K_n(x;p_1,p_2,p_3,p_4)$ defined in (20) satisfies an orthogonality relation as

$$\int_{-\infty}^{\infty}\frac{K_n\big(ix;\alpha,\beta,\alpha+l,\beta+u-1\big)K_m\big(-ix;l,u,\alpha+l,\beta+u-1\big)}{W^*(x;\beta)W^*(x;u)}dx=\frac{(2u-1)(2\beta-1)}{\pi\,2^{2n+1}}\times$$
$$\frac{\Gamma(\alpha+l+1/2)\Gamma(\beta+u)}{\Gamma(\alpha+\beta+l+u+1/2)}\prod_{j=1}^{n}\frac{\big(j+(1-(-1)^j)(\alpha+l)\big)\big(j+(1-(-1)^j)(\alpha+l)+2\beta+2u-2\big)}{\big(j+\alpha+\beta+l+u-3/2\big)\big(j+\alpha+\beta+l+u-1/2\big)}\,\delta_{n,m}, \quad (30)$$

where $\beta,u>1/2$, $\alpha+l>-1/2$ and $W^*(x;.)$ is defined as (29).
The mentioned approach can similarly be applied to the generalized Hermite polynomials. For this purpose, we first define the following specific functions

$$u(x)=x^{2a}e^{-\frac{1}{2}x^2}H_n^{(b)}(x)\quad\text{and}\quad v(x)=x^{2c}e^{-\frac{1}{2}x^2}H_m^{(d)}(x)\quad\text{for}\quad(-1)^{2a}=(-1)^{2c}=1. \quad (31)$$

If we take the Fourier transform for e.g. $u(x)$, we get

$$\mathbf{F}(u(x))=\int_{-\infty}^{\infty}e^{-isx}x^{2a}e^{-\frac{1}{2}x^2}H_n^{(b)}(x)\,dx$$
$$=\int_{-\infty}^{\infty}e^{-isx}e^{-\frac{1}{2}x^2}x^{2a+n}\left(\sum_{k=0}^{[n/2]}\frac{(-[n/2])_k(-[n/2]-b+(-1)^n/2)_k}{k!}(-x^{-2})^k\right)dx \quad (32)$$
$$=\sum_{k=0}^{[n/2]}\frac{(-[n/2])_k(-[n/2]-b+(-1)^n/2)_k}{k!}(-1)^k\left(\int_{-\infty}^{\infty}e^{-isx}e^{-\frac{1}{2}x^2}x^{2a+n-2k}dx\right).$$

Now, it remains to evaluate the definite integral:

$$I_{n,k}(s;a)=\int_{-\infty}^{\infty}e^{-isx}e^{-\frac{1}{2}x^2}x^{2a+n-2k}dx. \quad (32.1)$$

There are two ways to compute the above integral. First, by noting that $(-1)^{2a}=1$ in (32.1), we can directly compute $I_{n,k}(s;a)$ for $n=2m$ as follows

$$I_{2m,k}(s;a)=\int_{-\infty}^{\infty}\left(\sum_{j=0}^{\infty}\frac{(-isx)^j}{j!}\right)x^{2a+2m-2k}e^{-\frac{1}{2}x^2}dx=\sum_{j=0}^{\infty}\frac{(-1)^j i^j s^j}{j!}\left(\int_{-\infty}^{\infty}x^{j+2a+2m-2k}e^{-\frac{1}{2}x^2}dx\right)$$
$$=\sum_{r=0}^{\infty}\frac{(-1)^r s^{2r}}{(2r)!}\left(2\int_0^{\infty}x^{2r+2a+2m-2k}e^{-\frac{1}{2}x^2}dx\right)=\sum_{r=0}^{\infty}\frac{(-1)^r s^{2r}}{(2r)!}2^{r+a+m-k+\frac{1}{2}}\Gamma(r+a+m-k+\frac{1}{2}). \tag{33}$$

To simplify the last summation of (33) in terms of a hypergeometric form we need to use (27) and relations:

$$\Gamma(a+k)=(a)_k\Gamma(a),\quad \Gamma(a-k)=\frac{(-1)^k\Gamma(a)}{(1-a)_k}\quad \text{and}\quad (2n)!=(1/2)_n\,2^{2n}\,n!. \tag{33.1}$$

Therefore, (33) becomes

$$I_{2m,k}(s;a)=\sum_{r=0}^{\infty}\frac{(-1)^r s^{2r}\,2^{r+a+m-k+\frac{1}{2}}\Gamma(a+m-k+\frac{1}{2})(a+m-k+\frac{1}{2})_r}{(1/2)_r\,2^{2r}\,r!}$$
$$=2^{a+m-k+1/2}\Gamma(a+m-k+1/2)\;{}_1F_1\left(\left.\begin{matrix}a+m-k+1/2\\1/2\end{matrix}\right|-\frac{s^2}{2}\right). \tag{34}$$

Similarly this computational method can be applied for $I_{2m+1,k}(s;a)$. Just we should note in this turn that $\int_{-\infty}^{\infty}x^{j+2a+2m+1-2k}e^{-x^2/2}dx=0$ for any $j=0,2,4,\dots$. After some computations we obtain

$$I_{2m+1,k}(s;a)=(-is)\,2^{a+m-k+3/2}\Gamma(a+m-k+3/2)\;{}_1F_1\left(\left.\begin{matrix}a+m-k+3/2\\3/2\end{matrix}\right|-\frac{s^2}{2}\right). \tag{35}$$

Thus, by combining both relations (34) and (35) and using (13.1) we finally have

$$I_{n,k}(s;a)=2^{a-k+\frac{1}{2}+[\frac{n+1}{2}]}\Gamma(a-k+\frac{1}{2}+[\frac{n+1}{2}])(-is)^{\frac{1-(-1)^n}{2}}\;{}_1F_1\left(\left.\begin{matrix}\frac{1}{2}+a-k+[\frac{n+1}{2}]\\1-(-1)^n/2\end{matrix}\right|-\frac{s^2}{2}\right). \tag{36}$$

The second way to compute $I_{n,k}(s;a)$ is that we respectively consider $n=2m$ and $n=2m+1$ and directly apply the cosine and sine Fourier transforms of the function $e^{-x^2/2}x^{2a+n-2k}$, see [4, section 1.4, formula 14] and [4, section 2.4, formula 24]. In other words, by noting that $(-1)^{2a}=1$ we have

$$I_{2m,k}(s;a)=\int_{-\infty}^{\infty}\cos(sx)\,x^{2a+2m-2k}e^{-\frac{1}{2}x^2}dx-i\int_{-\infty}^{\infty}\sin(sx)\,x^{2a+2m-2k}e^{-\frac{1}{2}x^2}dx$$
$$=2\int_0^{\infty}\cos(sx)x^{2a+2m-2k}e^{-\frac{1}{2}x^2}dx=2^{a+m-k+\frac{1}{2}}\Gamma(a+m-k+\frac{1}{2})\;{}_1F_1\left(\left.\begin{matrix}a+m-k+\frac{1}{2}\\1/2\end{matrix}\right|-\frac{s^2}{2}\right), \tag{37}$$

and (38)

$$I_{2m+1,k}(s;a)=\int_{-\infty}^{\infty}\cos(sx)\,x^{2a+2m+1-2k}e^{-\frac{1}{2}x^2}dx-i\int_{-\infty}^{\infty}\sin(sx)x^{2a+2m+1-2k}e^{-\frac{1}{2}x^2}dx$$

$$=-2i\int_{0}^{\infty}\sin(sx)x^{2a+2m+1-2k}e^{-\frac{1}{2}x^2}dx=(-is)\,2^{a+m-k+\frac{3}{2}}\Gamma(a+m-k+\frac{3}{2})\;{}_1F_1\left(\begin{matrix}a+m-k+\frac{3}{2}\\3/2\end{matrix}\middle|-\frac{s^2}{2}\right).$$

Consequently, the result (36) simplifies (32) as

$$\mathbf{F}(u(x))=\Gamma(a+\frac{1}{2}+[\frac{n+1}{2}])\,2^{a+\frac{1}{2}+[\frac{n+1}{2}]}\times$$

$$(-is)^{\frac{1-(-1)^n}{2}}\sum_{k=0}^{[n/2]}\frac{(-[\frac{n}{2}])_k(\frac{1}{2}-b-[\frac{n+1}{2}])_k}{(\frac{1}{2}-a-[\frac{n+1}{2}])_k\,2^k\,k!}\,{}_1F_1\left(\begin{matrix}\frac{1}{2}+a-k+[\frac{n+1}{2}]\\1-(-1)^n/2\end{matrix}\middle|-\frac{s^2}{2}\right). \tag{39}$$

On the other hand, according to Kummer's formula [6]

$${}_1F_1\left(\begin{matrix}a\\c\end{matrix}\middle|x\right)=e^x\,{}_1F_1\left(\begin{matrix}c-a\\c\end{matrix}\middle|-x\right), \tag{40}$$

relation (39) can be transformed to

$$\mathbf{F}(u(x))=\Gamma(a+\frac{1}{2}+[\frac{n+1}{2}])\,2^{a+\frac{1}{2}+[\frac{n+1}{2}]}e^{-\frac{1}{2}s^2}\times$$

$$(-is)^{\frac{1-(-1)^n}{2}}\sum_{k=0}^{[n/2]}\frac{(-[\frac{n}{2}])_k(\frac{1}{2}-b-[\frac{n+1}{2}])_k}{(\frac{1}{2}-a-[\frac{n+1}{2}])_k\,2^k\,k!}\,{}_1F_1\left(\begin{matrix}-a+k-[\frac{n}{2}]\\1-(-1)^n/2\end{matrix}\middle|\frac{1}{2}s^2\right). \tag{41}$$

Now, by noting (41) if for simplicity we define

$$J_n(x;q_1,q_2)=x^{\frac{1-(-1)^n}{2}}\sum_{k=0}^{[n/2]}\frac{(-[\frac{n}{2}])_k(\frac{1}{2}-q_2-[\frac{n+1}{2}])_k}{(\frac{1}{2}-q_1-[\frac{n+1}{2}])_k\,2^k\,k!}\,{}_1F_1\left(\begin{matrix}-q_1+k-[\frac{n}{2}]\\1-(-1)^n/2\end{matrix}\middle|\frac{1}{2}x^2\right), \tag{42}$$

then by referring to definitions (31) and applying Parseval's identity we get

$$2\pi\int_{-\infty}^{\infty}x^{2(a+c)}e^{-x^2}H_n^{(b)}(x)\,H_m^{(d)}(x)dx$$

$$=i^{\frac{(-1)^n-(-1)^m}{2}}\frac{\Gamma(a+\frac{1}{2}+[\frac{n+1}{2}])\,\Gamma(c+\frac{1}{2}+[\frac{m+1}{2}])}{2^{-(a+\frac{1}{2}+[\frac{n+1}{2}])}2^{-(c+\frac{1}{2}+[\frac{m+1}{2}])}}\int_{-\infty}^{\infty}e^{-s^2}J_n(s;a,b)\,J_m(s;c,d)\,ds. \tag{43}$$

Finally it is sufficient to assume in (43) that $b = d = a + c$ and then refer to the orthogonality relation (13) to reach the following theorem.

Theorem 2. The special function $J_n(x; q_1, q_2)$ defined in (42) satisfies the orthogonality relation

$$\int_{-\infty}^{\infty} e^{-x^2} J_n(x;a,b)\, J_m(x;b-a,b)\, dx = \frac{\pi 2^{-b-2[\frac{n+1}{2}]}\,\Gamma([\frac{n}{2}]+1)\,\Gamma(b+\frac{1}{2}+[\frac{n+1}{2}])}{\Gamma(a+\frac{1}{2}+[\frac{n+1}{2}])\,\Gamma(b-a+\frac{1}{2}+[\frac{n+1}{2}])}\,\delta_{n,m}\,, \qquad (44)$$

where $a, b > -1/2$ and $(-1)^{2b} = 1$.

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