Connection and Linearization Coefficients of the Askey-Wilson Polynomials

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Abstract

The *linearization problem* is the problem of finding the coefficients $C_k(m,n)$ in the expansion of the product $P_n(x)Q_m(x)$ of two polynomial systems in terms of a third sequence of polynomials $R_k(x)$,

$$P_n(x)Q_m(x)=\sum_{k=0}^{n+m}C_k(m,n)R_k(x).$$

The polynomials P_n, Q_m and R_k may belong to three different polynomial families. In the case $P=Q=R$, we get the (*standard*) *linearization* or Clebsch-Gordan-type problem. If $Q_m(x)\equiv 1$, we are faced with the so-called *connection problem*.

In this paper, we compute explicitly, in a more general setting and using an algorithmic approach, the connection and linearization coefficients of the Askey-Wilson orthogonal polynomial families.

We find our results by an application of computer algebra. The major algorithmic tool for our development is a refined version of q-Petkovšek's algorithm published by Horn [14, 15] and implemented in Maple by Sprenger [26, 27] (in his package `qFPS.mpl`) which finds the q-hypergeometric term solutions of q-holonomic recurrence equations).

The major ingredients which makes this application non-trivial are

- the use of appropriate operators $\mathbb{D}_x$ and $\mathbb{S}_x$;
- the use of an appropriate basis $B_n(a,x)$ for these operators;
- and a suitable characterization of the classical orthogonal polynomials on a non-uniform lattice which was developed very recently [9].

Without this preprocessing the relevant recurrence equations are not accessible, and without the mentioned algorithm the solutions of these recurrence equations are out of reach. Furthermore, we present an algorithm to deduce the inversion coefficients for the basis $B_n(a,x)$ in terms of the Askey-Wilson polynomials.

Our results generalize and extend known results, and they can be used to deduce the connection and linearization coefficients for any family of classical orthogonal polynomial (including very classical orthogonal polynomials and classical orthogonal polynomials on non-uniform lattices), using the fact that from the Askey-Wilson polynomials, one can deduce, by specialization and/or by limiting processes, any other family of classical orthogonal polynomials. As illustration, we give explicitly the connection coefficients of the continuous q-Hahn, q-Racah and Wilson polynomials.

Keywords: Askey-Wilson polynomials, Continuous q-Hahn polynomials, q-Racah polynomials , Non-uniform lattices, Divided-difference equations, Hypergeometric representation, Inversion coefficients, Connection coefficients, Linearization coefficients
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1. Introduction

The connection and linearization coefficients have been computed explicitly for very classical orthogonal polynomials:

1. for classical orthogonal polynomials of a continuous variable in [2, 3, 18, 23];
2. for classical orthogonal polynomials of a discrete variable in [12, 18];
3. for classical orthogonal polynomials of a q-discrete variable in [10].

The very classical orthogonal polynomials are known to satisfy, respectively, the following second-order holonomic differential, difference or q-difference equations (see e.g. [16]):

$$\sigma(x)\,\frac{d^2}{dx^2}\,y(x)+\tau(x)\,\frac{d}{dx}\,y(x)+\lambda_n\,y(x)=0;$$
$$\sigma(x)\,\Delta\,\nabla\,y(x)+\tau(x)\,\nabla\,y(x)+\lambda_n\,y(x)=0;$$
$$\sigma(x)\,D_q\,D_{\frac{1}{q}}\,y(x)+\tau(x)\,D_q\,y(x)+\lambda_{n,q}\,y(x)=0,$$

where Δ, ∇ and D_q are, respectively, the forward, the backward and the Hahn operators defined by

$$\Delta f(x)=f(x+1)-f(x),\ \nabla f(x)=f(x)-f(x-1),\ D_q f(x)=\frac{f(qx)-f(x)}{(q-1)x},\ q\neq 1,\ x\neq 0,$$

with $D_q f(0)=\lim\limits_{x\to 0} D_q f(x)=f'(0)$, provided that $f'(0)$ exists.

The Askey-Wilson polynomials $p_n(x;a,b,c,d|q)$ are defined as ([4], [16])

$$p_n(x;a,b,c,d|q)=\frac{(ab,ac,ad;q)_n}{a^n}\,{}_4\phi_3\left(\begin{array}{c} q^{-n},abcdq^{n-1},aq^s,aq^{-s}\\ ab,ac,ad\end{array}\middle|\,q;q\right), \tag{1}$$

with $x:=x(s)=\cos\theta=\frac{q^s+q^{-s}}{2}$, $q^s=e^{i\theta}$, where the q-hypergeometric function ${}_r\phi_s$, and the Pochhammer symbol $(a_1,a_2,\ldots,a_k;q)_n$, are defined by

$${}_r\phi_s\left(\begin{array}{c} a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{array}\middle|\,q;z\right):=\sum_{k=0}^{\infty}A_k=\sum_{k=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_k}{(b_1,\cdots,b_s;q)_k}\left((-1)^k q^{\binom{k}{2}}\right)^{1+s-r}\frac{z^k}{(q;q)_k},$$

and

$$(a_1,\cdots,a_r;q)_k:=(a_1;q)_k\cdots(a_r;q)_k,\ \text{ with }\ (a_i;q)_k=\begin{cases}\prod\limits_{j=0}^{k-1}(1-a_i q^j) & \text{if } k=1,2,3.\cdots\\ 1 & \text{if } k=0\end{cases}.$$

It should be noticed that:

1. the summand A_k of the q-hypergeometric series is a q-hypergeometric term, i. e. $\frac{A_{k+1}}{A_k}\in\mathbb{Q}(q,q^k)$ is a rational function in the variables q and q^k,
2. a linear homogeneous recurrence equation

$$\sum_{k=0}^{K}\alpha_k(q;q^m)C_{m+k}=0$$

is called q-holonomic if the coefficients $\alpha_k(q;q^m)$ are rational w. r. t. q and polynomial functions w. r. t. the variable q^m, i. e. $\alpha_k\in\mathbb{Q}(q)[q^m]$.

The Askey-Wilson polynomials do not belong to any family of very classical orthogonal polynomials. They belong, instead, to the so-called classical orthogonal polynomials on a non-uniform lattice [11, 16, 21], which are known to satisfy a divided-difference equation of the type [5, 8, 22, 28]

$$\left\{\phi(x(s))\frac{\Delta}{\nabla x_1(s)}\frac{\nabla}{\nabla x(s)}+\frac{\psi(x(s))}{2}\left[\frac{\Delta}{\Delta x(s)}+\frac{\nabla}{\nabla x(s)}\right]+\lambda_n\right\}P_n(x(s))=0,\ n\geq 0, \tag{2}$$

where $\phi(x) = \phi_2 x^2 + \phi_1 x + \phi_0$ and $\psi(x) = \psi_1 x + \psi_0$ are polynomials of maximal degree two and one respectively, λ_n is a constant depending on the integer n and the leading coefficients ϕ_2 and ψ_1 of ϕ and ψ respectively, and $x(s)$ is a non-uniform lattice defined by [21]

$$x(s) = \begin{cases} c_1 q^s + c_2 q^{-s} + c_3 & \text{if} \quad q \neq 1, \\ c_4 s^2 + c_5 s + c_6 & \text{if} \quad q = 1. \end{cases}$$

with

$$x_\mu(s) = x\left(s + \frac{\mu}{2}\right), \quad \mu,\ c_1,\ \ldots,\ c_6 \in \mathbb{C},$$

where $\mathbb{C}$ is the set of complex numbers.

Foupouagnigni [8] showed that, by means of the companion operators $\mathbb{D}_x$ and $\mathbb{S}_x$, given by

$$\mathbb{D}_x f(x(s)) = \frac{f(x(s+\frac{1}{2})) - f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2}) - x(s-\frac{1}{2})}, \quad \mathbb{S}_x f(x(s)) = \frac{f(x(s+\frac{1}{2})) + f(x(s-\frac{1}{2}))}{2},$$

Equation (2), which characterizes classical orthogonal polynomials on a non-uniform lattice [9], can be rewritten with the same polynomial coefficients ϕ and ψ as

$$\phi(x(s))\mathbb{D}_x^2 P_n(x(s)) + \psi(x(s))\mathbb{S}_x\mathbb{D}_x P_n(x(s)) + \lambda_n P_n(x(s)) = 0. \tag{3}$$

The Askey-Wilson orthogonal polynomial family satisfies the divided-difference equation of type (3) with [8]

$$\begin{aligned} \phi(x(s)) &= 2(abcd+1)x^2(s) - (a+b+c+d+abc+abd+acd+bcd)x(s) + ab+ac+ad+bc+bd+cd-abcd-1, \\ \psi(x(s)) &= \frac{4(abcd-1)q^{\frac{1}{2}}x(s)}{q-1} + \frac{2(a+b+c+d-abc-abd-acd-bcd)q^{\frac{1}{2}}}{q-1}, \end{aligned}$$

where

$$x(s) = \frac{q^s + q^{-s}}{2} = \cos\theta.$$

In [2] Area *et al.* used the formula

$${}_r\phi_s\left(\begin{matrix} a_r \\ b_s \end{matrix}\,\middle|\, q; yx\right) = \sum_{j=0}^{\infty} \frac{[(-1)^j q^{\binom{j}{2}}]^2 y^j}{(q;q)_j}\, {}_1\phi_1\left(\begin{matrix} 0 \\ 0 \end{matrix}\,\middle|\, q; q^j y\right) {}_{r+1}\phi_s\left(\begin{matrix} q^{-j}, a_r \\ b_s \end{matrix}\,\middle|\, q; qx\right), \tag{4}$$

obtained from Verma's q-extension [29] of the Fields and Wimp [7] expansion of

$${}_{r+t}\phi_{s+u}\left(\begin{matrix} a_r, c_t \\ b_s, d_u \end{matrix}\,\middle|\, q; yw\right)$$

in powers of yw as given in ([13], (3.7.9)), to derive the inversion formula of the Askey-Wilson polynomials from which the Askey-Wilson connection formula with the same first parameter follows:

$$\begin{aligned} p_n(x;a,b,c,d|q) &= \sum_{m=0}^{n} \frac{a^{m-n} q^{m(m-n)}}{(q;q)_{n-m}} \frac{(q,ab,ac,ad;q)_n (abcdq^{n-1};q)_m}{(q,ab,ac,ad;q)_m (\alpha\beta\gamma\delta q^{m-1};q)_m} \times \\ & \quad {}_5\phi_4\left(\begin{matrix} q^{m-n}, a\beta q^m, a\gamma q^m, a\delta q^m, abcdq^{m+n-1} \\ abq^m, acq^m, adq^m, a\beta\gamma\delta q^{2m} \end{matrix}\,\middle|\, q;q\right) p_m(x;a,\beta,\gamma,\delta|q). \end{aligned} \tag{5}$$

Our method, which produces a more general result, is based on an algorithmic approach using the appropriate basis $B_n(a,x)$ for the operators $\mathbb{D}_x$ and $\mathbb{S}_x$, combined with various properties of this basis, developed in [8, 9, 11].

The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ transform a polynomial of degree n in the variable $x(s)$ into a polynomial in the variable $x(s)$ of degree $n-1$ and n, respectively (see e.g. [8], [21]). However, the monomial basis $(x^n(s))_n$ is NOT appropriate for these operators since the application of the operators $\mathbb{D}_x$ and $\mathbb{S}_x$ to the monomial $x^n(s)$ produces a linear combination (with complicated coefficients) of all monomials of degree less than or equal to $n-1$ and n respectively [8]. To solve this problem, Foupouagnigni *et al.* [11] provided an appropriate basis $(B_n(a,x))_n$, namely

$$B_n(a,x) = (aq^s;q)_n(aq^{-s};q)_n,\ n\geq 1,\ B_0(a,x)\equiv 1,\ \text{ where }\ x = x(s) = \frac{q^s+q^{-s}}{2},$$

which is a polynomial of degree n in $x(s)$, and which fulfils the following properties [9, 11]:

Proposition 1. *The q-quadratic lattice $x(s)=\frac{q^s+q^{-s}}{2}$ and the corresponding polynomial basis*

$$B_n(a,x) = (aq^s;q)_n(aq^{-s};q)_n,\ n\geq 1,\ B_0(a,x)\equiv 1,$$

fulfil the relations

$$\begin{aligned}
\mathbb{D}_x B_n(a,x) &= \eta(a,n)B_{n-1}(a\sqrt{q},x); && (6)\\
\mathbb{S}_x B_n(a,x) &= \beta_1(a,n)B_{n-1}(a\sqrt{q},x)+\beta_2(n)B_n(a\sqrt{q},x); &&\\
B_1(a,x)\mathbb{D}_x^2 B_n(a,x) &= \eta(a,n)\eta(a\sqrt{q},n-1)B_{n-1}(a,x); && (7)\\
B_1(a,x)\mathbb{S}_x\mathbb{D}_x B_n(a,x) &= \eta(a,n)\left(\beta_1(a\sqrt{q},n-1)B_{n-1}(a,x)+\beta_2(n-1)B_n(a,x)\right); && (8)\\
xB_n(a,x) &= \mu_1(a,n)B_n(a,x)+\mu_2(a,n)B_{n+1}(a,x); && (9)\\
B_1(a,x)B_n(a,x) &= \nu_1(a,n)B_n(a,x)+\nu_2(n)B_{n+1}(a,x); && (10)\\
B_1(a,x)B_n(aq,x) &= B_{n+1}(a,x), &&
\end{aligned}$$

where

$$\eta(a,n)=\tfrac{2a(1-q^n)}{q-1},\ \ \beta_1(a,n)=\tfrac{1}{2}(1-a^2q^{2n-1})(1-q^{-n}),\ \ \beta_2(n)=\tfrac{1}{2}+\tfrac{1}{2q^n},$$
$$\mu_1(a,n)=\tfrac{1+a^2q^{2n}}{2aq^n},\ \ \mu_2(a,n)=\tfrac{-1}{2aq^n},\ \nu_1(a,n)=(1-q^{-n})(1-a^2q^n),\ \ \nu_2(n)=q^{-n}.$$

In this paper, we use an algorithmic approach (see e. g. [10], [18]) to derive the connection and linearization coefficients $C_m(n)$ and $L_r(m,n)$ for the Askey-Wilson orthogonal polynomials in a more general setting (with no need for the first parameters to be identical), that is:

$$p_n(x;a,b,c,d|q) = \sum_{m=0}^{n} C_m(n)p_m(x;\alpha,\beta,\gamma,\delta|q), \tag{11}$$

$$p_n(x;a_1,b_1,c_1,d_1|q)p_m(x;a_2,b_2,c_2,d_2|q) = \sum_{r=0}^{n+m} L_r(m,n)p_r(x;\alpha,\beta,\gamma,\delta|q). \tag{12}$$

We note that by taking suitable limits and a specialization process [16], we can obtain from the Askey-Wilson connection and linearization formulas, the connection and linearization formulas for the orthogonal polynomials of the Askey and q-Askey scheme (continuous q-Hahn, q-Racah, continuous dual q-Hahn, Wilson, Al-Salam-Chihara, q-Meixner-Pollaczek, Continuous q-Jacobi, continuous big q-Hermite, continuous q-Laguerre, continuous q-Hermite, the very classical orthogonal polynomials, *etc.*), which of course can also be derived directly using the algorithm developed in this paper.

The content of this paper is organized as follows:

1. In Section 2, we use our algorithmic approach, combined with the divided-difference equation (3), to recover the known coefficients of the three-term recurrence equations, as well as the known q-hypergeometric representation of the Askey-Wilson polynomials;
2. In Section 3, we solve the inversion problem, which is then used to derive the connection formula (5) between two Askey-Wilson polynomials with identical first parameter;
3. Section 4 deals with the derivation of the connection coefficients between the bases $(B_n(a,x))_n$ and $(B_n(\alpha,x))_n$ as well as the derivation of the general connection formula (11) for the Askey-Wilson polynomials;
4. In Section 5, the linearization problem (12) of the Askey-Wilson polynomials is solved;
5. Section 6 illustrates the use of limiting and specialization processes to deduce the connection coefficients of the continuous q-Hahn, q-Racah and Wilson polynomials.

2. Three-Term Recurrence Equations and q-Hypergeometric Representation

In this section, we derive, starting from Equation (3), using the basis $(B_n(a,x))_n$ as well as the properties of the operators $\mathbb{D}_x$ and $\mathbb{S}_x$ (see Proposition 1), the coefficients of the three-term recurrence relation for the Askey-Wilson polynomials $p_n(x;a,b,c,d|q)$, the coefficients of the three-term recurrence relation for the second divided-derivative of the Askey-Wilson polynomials $\mathbb{D}_x^2 p_n(x;a,b,c,d|q)$, as well as the hypergeometric representation of the Askey-Wilson polynomials. The results obtained using our method are of course identical to the known ones [16]; however, it was useful for consistency, to get these results using our method since these methods are needed for the computation of the connection coefficients.

2.1. Three-Term Recurrence Equation of the Family $(p_n(x;a,b,c,d|q))_n$

Proposition 2. *(See* e.g. *[16], p. 417). The Askey-Wilson polynomial family satisfies a three-term recurrence equation of the form*

$$x p_n(x;a,b,c,d|q) = \alpha_n p_{n+1}(x;a,b,c,d|q) + \beta_n p_n(x;a,b,c,d|q) + \gamma_n p_{n-1}(x;a,b,c,d|q), \tag{13}$$

with

$$\alpha_n = -\frac{1}{2aq^n}\frac{k_n}{k_{n+1}}, \tag{14}$$

$$\begin{aligned}
\beta_n \;=\; & \frac{1}{2}\frac{q^n}{\left(-q^2+dcba\left(q^n\right)^2\right)\left(-1+dcba\left(q^n\right)^2\right)}\times \\
& \Big(dq^2+b^2a\left(q^n\right)^2qdc-qaq^nb^2dc-cba^2q^ndq+dcbqa^2\left(q^n\right)^2+cbaq+badq-q^2q^ndcb \\
& -cdaq^nq^2-d^2cbaq^n-cdaq^nq+d^2c^2ba^2\left(q^n\right)^2-q^ndcbq-aq^nc^2db+d^2c^2b^2a\left(q^n\right)^2+dcbq+cdaq \\
& +cq^2+q^2a+bq^2+a^2\left(q^n\right)^2c^2b^2d+\left(q^n\right)^2qad^2cb+c^2a\left(q^n\right)^2qdb+a^2\left(q^n\right)^2b^2d^2c-cbaq^nq^2 \\
& -badq^nq^2-cba^2q^nd-badq^nq-bqcaq^n-d^2cbaq^nq-dc^2bqaq^n-aq^nb^2dc\Big),
\end{aligned} \tag{15}$$

$$\gamma_n = -\frac{k_n}{2k_{n-1}}\,\frac{\left(q^ndc-q\right)\left(dbq^n-q\right)\left(q^nbc-q\right)\left(dcbaq^n-q^2\right)q^na\left(baq^n-q\right)\left(caq^n-q\right)\left(aq^nd-q\right)\left(-1+q^n\right)}{q\left(-q^2+dcba\left(q^n\right)^2\right)^2\left(-q^3+dcba\left(q^n\right)^2\right)\left(-q+dcba\left(q^n\right)^2\right)},$$

where k_n is the leading coefficient of the polynomial $p_n(x;a,b,c,d|q)$ represented in the basis $(B_n(a,x))_n$

$$p_n(x;a,b,c,d|q) = k_nB_n(a,x)+k_n'B_{n-1}(a,x)+k_n''B_{n-2}(a,x)+\ldots \tag{16}$$

and is given explicitly by $k_n = \dfrac{(abcdq^{n-1};q)_n}{(-a)^nq^{\binom{n}{2}}}$.

Proof. To prove this proposition, we mimic the method given in [18]. We will first need to compute, in terms of the leading coefficient k_n, the coefficients k_n' and k_n'' of the expansion (16) of $p_n(x;a,b,c,d|q)$ in the appropriate basis $(B_n(a,x))_n$. In order to compute these coefficients, first we substitute (16) in the divided-difference equation (3). Next we multiply this equation by $B_1(a,x)$ and use relations (7), (8) and (10). To eliminate the terms $x(s)B_k(a,x)$ and $x^2(s)B_k(a,x)$, we use relations (9) and

$$x^2(s)B_n(a,x) = \mu_1^2(a,n)B_n(a,x)+\mu_2(a,n)(\mu_1(a,n)+\mu_1(a,n+1))B_{n+1}(a,x)+\mu_2(a,n)\mu_2(a,n+1)B_{n+2}(a,x). \tag{17}$$

Equating the coefficients of $B_{n+1}(a,x)$ gives (compare [8], p. 158, (104))

$$\lambda_n = -4\,\frac{\left(-1+q^n\right)\sqrt{q}\left(abcdq^n-q\right)}{\left(q-1\right)^2q^n}. \tag{18}$$

Equating the coefficients of $B_n(a,x)$, we deduce using (18) that

$$k_n' = -\frac{\left(-1+q^n\right)\left(-q+aq^n d\right)\left(aq^n c-q\right)\left(aq^n b-q\right)}{q\left(q-1\right)\left(-q^2+abcd\left(q^n\right)^2\right)} k_n. \tag{19}$$

By equating the coefficients of $B_{n-1}(a,x)$ and using (18)-(19), one gets

$$k_n'' = \frac{\left(-1+q^n\right)\left(-q+aq^n d\right)\left(aq^n c-q\right)\left(aq^n b-q\right)\left(-q+q^n\right)\left(aq^n d-q^2\right)\left(aq^n c-q^2\right)\left(aq^n b-q^2\right)}{\left(1+q\right)\left(q-1\right)^2\left(-q^3+abcd\left(q^n\right)^2\right)\left(-q^2+abcd\left(q^n\right)^2\right)q^4} k_n. \tag{20}$$

To get the coefficients α_n, β_n and γ_n of Equation (13), we substitute the expression of p_n given by (16) in the recurrence equation (13) and use Equations (9), (18)-(20). By equating the coefficients of $B_{n+1}(a,x)$, one gets α_n. Equating the coefficients of $B_n(a,x)$ and using (14) yields β_n. Similarly, equating the coefficients of $B_{n-1}(a,x)$ and using (14)-(15) gives γ_n. □

Remark 3. *The recurrence equation* (13) *was given in ([16], p. 417) in the form*

$$2x\tilde{p}_n(x) = A_n\tilde{p}_{n+1}(x) + [a+a^{-1}-(A_n+C_n)]\tilde{p}_n(x) + C_n\tilde{p}_{n-1}(x), \tag{21}$$

where

$$\tilde{p}_n(x) := \frac{a^n p_n(x;a,b,c,d|q)}{(ab,ac,ad:q)_n}$$

and

$$\begin{aligned} A_n &= \frac{(1-abq^n)(1-acq^n)(1-adq^n)(1-abcdq^{n-1})}{a(1-abcdq^{2n-1})(1-abcdq^{2n})} \\ C_n &= \frac{a(1-q^n)(1-bcq^{n-1})(1-bdq^{n-1})(1-cdq^{n-1})}{(1-abcdq^{2n-2})(1-abcdq^{2n-1})}. \end{aligned}$$

It can be proved by direct computation that Equation (21) *is equivalent to Equation* (13) *with*

$$\alpha_n = A_n\frac{a(ab,ac,ad;q)_n}{2(ab,ac,ad;q)_{n+1}},\ \beta_n = \frac{[a+a^{-1}-(A_n+C_n)]}{2},\ \gamma_n = C_n\frac{(ab,ac,ad;q)_n}{2a(ab,ac,ad;q)_{n-1}}.$$

2.2. Three-Term Recurrence Equation of the Family $(\mathbb{D}_x^2 p_n(x;a,b,c,d|q))_n$

Foupouagnigni *et al.* proved in [9] that if $(p_n)_n$ is an orthogonal polynomial family satisfying (3), then the families $(\mathbb{D}_x^m p_n)_n$ are also orthogonal, for all $m\in\mathbb{N}$. Consequently, they also satisfy a three-term recurrence equation.

Proposition 4. *The second-order divided-difference* $\mathbb{D}_x^2 p_n$ *of the Askey-Wilson orthogonal polynomials defined by* (16) *satisfies the following recurrence equation*

$$x\mathbb{D}_x^2 p_n(x;a,b,c,d|q) = \alpha_n^\star\mathbb{D}_x^2 p_{n+1}(x;a,b,c,d|q) + \beta_n^\star\mathbb{D}_x^2 p_n(x;a,b,c,d|q) + \gamma_n^\star\mathbb{D}_x^2 p_{n-1}(x;a,b,c,d|q), \tag{22}$$

with

$$\alpha_n^\star = -\frac{1}{2}\frac{-q+q^n}{aq^n\left(-1+q^nq\right)}\frac{k_n}{k_{n+1}}, \tag{23}$$

$$\begin{aligned} \beta_n^\star = {} & \frac{1}{2}\frac{q^n}{\left(-q^2+abcd\left(q^n\right)^2\right)\left(-1+abcd\left(q^n\right)^2\right)}\times \\ & \Big(qc+q^2adc+q^2bcd-aq^ndc-q^nbcd-qaq^ndc+aq-qb^2aq^ncd-q^2b^2aq^ncd-q^2a^2q^ndcb \\ & +qb+a^2c^2\left(q^n\right)^2qb^2d+a^2\left(q^n\right)^2qd^2b^2c-a^2q^ndcqb-qq^nbcd+ac^2\left(q^n\right)^2bd+a\left(q^n\right)^2d^2bc \\ & +qd-acq^nb-aq^ndb+q^2acb+q^2adb+a^2\left(q^n\right)^2dcb+a^2\left(q^n\right)^2d^2c^2qb+qab^2c^2d^2\left(q^n\right)^2 \\ & -ac^2q^nqbd-aq^nqd^2bc-q^2ac^2q^nbd-q^2aq^nd^2bc-qacq^nb-qaq^ndb+b^2a\left(q^n\right)^2cd\Big), \end{aligned} \tag{24}$$

$$\gamma_n^\star = -\frac{k_n}{2k_{n-1}}\frac{\left(-q+q^n dc\right)\left(q^n bd-q\right)\left(q^n bc-q\right)\left(abcdq^n-1\right)q^n a\left(-1+q^n\right)\left(baq^n-q\right)\left(acq^n-q\right)\left(-q+aq^n d\right)}{\left(-q^2+abcd\left(q^n\right)^2\right)^2\left(-q^3+abcd\left(q^n\right)^2\right)\left(-q+abcd\left(q^n\right)^2\right)}.$$

Proof. We substitute the expression of p_n given by (16) in the recurrence equation (22), and then multiply the equation obtained by $B_1(a,x)$. Next we use (7) and (9) respectively to eliminate $B_1(a,x)\mathbb{D}_x^2 B_n(a,x)$ and $x(s)B_n(a,x)$. By equating the coefficients of $B_n(a,x)$, one gets $\alpha_n^\star$. Equating the coefficients of $B_{n-1}(a,x)$ and using (23) yields $\beta_n^\star$. Similarly, equating the coefficients of $B_{n-2}(a,x)$ and using (23) and (24), we obtain $\gamma_n^\star$. □

2.3. *q-Hypergeometric Representation*

The q-hypergeometric representation of the Askey-Wilson polynomial is well known. Nevertheless we would like to recover it, by using our method to solve equation (3).

We suppose here and throughout the document that

$$p_n(x;a,b,c,d|q) = \sum_{k=0}^{n} A_k(n)B_k(a,x) \tag{25}$$

denote the basic hypergeometric representation of the Askey-Wilson polynomials ([16], p. 415), while

$$\hat{p}_n(x;a,b,c,d|q) = B_n(a,x) + k_n' B_{n-1}(a,x) + \ldots$$

are the monic Askey-Wilson polynomials represented in the basis $(B_n(a,x))_n$. Therefore, taking into account both definitions, these two families are related by

$$p_n(x;a,b,c,d|q) = \frac{(abcdq^{n-1};q)_n}{(-a)^n q^{\binom{n}{2}}}\hat{p}_n(x;a,b,c,d|q). \tag{26}$$

We have the following:

Proposition 5. *The monic Askey-Wilson orthogonal polynomial family has the following hypergeometric representation in the basis $(B_n(a,x))_n$*

$$\begin{aligned}\hat{p}_n(x;a,b,c,d|q) &= (-1)^n q^{\binom{n}{2}}\frac{(ab,ac,ad;q)_n}{(abcdq^{n-1};q)_n}\sum_{k=0}^{n}\frac{(q^{-n},abcdq^{n-1};q)_k q^k}{(ab,ac,ad;q)_k(q:q)_k}B_k(a,x),\\ &= (-1)^n q^{\binom{n}{2}}\frac{(ab,ac,ad;q)_n}{(abcdq^{n-1};q)_n}{}_4\phi_3\left(\begin{matrix} q^{-n},abcdq^{n-1},aq^s,aq^{-s}\\ ab,ac,ad\end{matrix}\middle| q;q\right).\end{aligned} \tag{27}$$

Proof. We substitute $p_n(x;a,b,c,d|q)$ given by (25) in the divided-difference equation (3). Next we multiply the equation obtained by $B_1(a,x)$. This gives using (7), (8) and (10) the following equation

$$\begin{aligned}&\phi(x(s))\sum_{k=0}^{n}A_k(n)\eta(a,k)\eta(a\sqrt{q},k-1)B_{k-1}(a,x)\\ &+\psi(x(s))\sum_{k=0}^{n}A_k(n)\eta(a,k)\left(\beta_1(a\sqrt{q},k-1)B_{k-1}(a,x)+\beta_2(k-1)B_k(a,x)\right)\\ &+\lambda_n\sum_{k=0}^{n}A_k(n)\left(\nu_1(a,k)B_k(a,x)+\nu_2(k)B_{k+1}(a,x)\right)=0.\end{aligned}$$

Using (9) and (17), we eliminate all the terms of the form $x(s)B_k(a,x)$ and $x^2(s)B_k(a,x)$. Then, we collect all the coefficients of $B_{k+1}(a,x)$, $B_k(a,x)$, $B_{k-1}(a,x)$, respectively. By an appropriate shift of index, we write the obtained equation in the form

$$\sum_{k=0}^{n+1}\Big(B_kA_{k+1}(n)+C_kA_k(n)+G_kA_{k-1}(n)\Big)B_k(a,x)=0,$$

with $A_{-1}(n) = A_{n+1}(n) = A_{n+2}(n) = 0$, where

$$\begin{aligned} B_k &= \eta(a,k+1)\Big(\eta\left(a\sqrt{q},k\right)\left(\mu_1(a,k)\right)^2\phi_2 + \beta_1\left(a\sqrt{q},k\right)\mu_1(a,k)\psi_1 + \eta\left(a\sqrt{q},k\right)\mu_1(a,k)\phi_1 \\ &\quad + \psi_0\beta_1\left(a\sqrt{q},k\right) + \eta\left(a\sqrt{q},k\right)\phi_0\Big), \end{aligned}$$

$$\begin{aligned} C_k &= \eta(a,k)\,\eta\left(a\sqrt{q},k-1\right)\mu_2(a,k-1)\left(\mu_1(a,k-1)+\mu_1(a,k)\right)\phi_2 + \Big(\eta(a,k)\,\beta_1\left(a\sqrt{q},k-1\right)\mu_2(a,k-1) \\ &\quad + \eta(a,k)\,\beta_2(k-1)\,\mu_1(a,k)\Big)\psi_1 + \eta(a,k)\,\eta\left(a\sqrt{q},k-1\right)\mu_2(a,k-1)\,\phi_1 + \eta(a,k)\,\beta_2(k-1)\,\psi_0 \\ &\quad + \lambda_n\nu_1(a,k), \end{aligned}$$

and

$$G_k = \left(\eta(a,k-1)\,\eta\left(a\sqrt{q},k-2\right)\mu_2(a,k-2)\,\mu_2(a,k-1)\,\phi_2 + \eta(a,k-1)\,\beta_2(k-2)\,\mu_2(a,k-1)\,\psi_1 + \nu_2(k-1)\,\lambda_n\right).$$

Since the family $(B_k(a,x))_k$ is linearly independent, the coefficients of the hypergeometric representation (25) satisfy the second-order recurrence equation

$$B_k A_{k+1}(n) + C_k A_k(n) + G_k A_{k-1}(n) = 0.$$

To solve this recurrence equation, we use the refined version of q-Petkovšek's algorithm published by Horn [14, 15] and implemented in Maple by Sprenger [26] (in his package `qFPS.mpl` by the command `qHypergeomsolveRE` (see also [27]) which finds the q-hypergeometric term solutions of q-holonomic recurrence equations) (see also [1, 6, 24]). This algorithm yields $A_k(n)$ up to a multiplicative constant factor (we note that all the coefficients are given in terms of the dominant coefficient $A_n(n)$). We obtain the solution for the monic family $\hat{p}_n(x;a,b,c,d|q)$ by taking $A_n(n) = 1$. We deduce the representation of $p_n(x;a,b,c,d|q)$ by using (27) and (26) to get (1) which coincides with the definition given in the book of Koekoek, Lesky, Swarttouw ([16], p. 415). □

3. Inversion Formula and Connection coefficients (Part I)

3.1. Inversion Formula

In the preceding section, we expanded $p_n(x;a,b,c,d|q)$ in the basis $(B_m(a,x))_m$. Here, we solve the inverse problem.

Since the family $(p_k(x;a,b,c,d|q))_{k=0..n}$ is a basis of polynomials of degree less than or equal to n, and since $B_n(a,x)$ is a polynomial of degree n in $x(s)$, it follows that

$$B_n(a,x) = \sum_{m=0}^{n} D_m(n)\, p_m(x;a,b,c,d|q), \tag{28}$$

which is called the inversion formula for the family $(p_n(x;a,b,c,d|q))_n$ represented in the basis $(B_n(a,x))_n$. Using an algorithmic approach, we determine the inversion coefficients $D_m(n)$, following the work done for q-classical orthogonal polynomials by Foupouagnigni *et al.* [10]:

Proposition 6. *The inversion formula of the Askey-Wilson orthogonal polynomial family is given by*

$$B_n(a,x) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{m(m-1)}{2}} \frac{(-a)^m (abq^m, acq^m, adq^m; q)_{n-m}}{(abcdq^{m-1};q)_m (abcdq^{2m};q)_{n-m}} p_m(x;a,b,c,d|q), \tag{29}$$

where $\begin{bmatrix} n \\ m \end{bmatrix}_q$ *denotes the* q*-binomial coefficient and is defined by*

$$\begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}, \quad m = 0,1,2,\cdots,n.$$

The proof of this proposition uses the following lemma:

Lemma 7. *The second derivative of the basis* $(B_n(a,x))_n$ *satisfies the recurrence relation*

$$x(s)\mathbb{D}_x^2 B_n(a,x) = \mu_1(a,n-1)\mathbb{D}_x^2 B_n(a,x) + \mu_2(a,n-1)\frac{\eta(a,n)\eta(a\sqrt{q},n-1)}{\eta(a,n+1)\eta(a\sqrt{q},n)}\mathbb{D}_x^2 B_{n+1}(a,x), \tag{30}$$

with the coefficients given in Proposition 1.

Proof. (of the lemma). From (7) we obtain $B_n(a,x)$ in terms of $\mathbb{D}_x^2 B_{n+1}(a,x)$. If we substitute this in the recurrence equation (9), the result follows. □

Proof. (of Proposition 6). We substitute $B_n(a,x)$ given by (28) in the recurrence equation (9) and replace $x(s)p_m(x;a,b,c,d|q)$ by the expression given by (13). This yields the equation

$$\mu_1(a,n)\sum_{m=0}^{n} D_m(n)p_m(x;a,b,c,d|q) + \mu_2(a,n)\sum_{m=0}^{n+1} D_m(n+1)p_m(x;a,b,c,d|q) =$$
$$\sum_{m=0}^{n} D_m(n)\Big(\alpha_m p_{m+1}(x;a,b,c,d|q) + \beta_m p_m(x;a,b,c,d|q) + \gamma_m p_{m-1}(x;a,b,c,d|q)\Big).$$

By an appropriate shift of index, and equating the coefficients of $p_m(x;a,b,c,d|q)$, we get the first cross-rule

$$\mu_1(a,n)D_m(n) + \mu_2(a,n)D_m(n+1) = \alpha_{m-1}D_{m-1}(n) + \beta_m D_m(n) + \gamma_{m+1}D_{m+1}(n). \tag{31}$$

Similarly, we substitute $B_n(a,x)$ given by (28) in (30) and replace $x(s)\mathbb{D}_x^2 p_m(x;a,b,c,d|q)$ by the expression given by (22). By an appropriate shift of index, and equating the coefficients of $p_m(x;a,b,c,d|q)$, we get the second cross-rule

$$\mu_1(a,n-1)D_m(n) + \mu_2(a,n-1)\frac{\eta(a,n)\eta(a\sqrt{q},n-1)}{\eta(a,n+1)\eta(a\sqrt{q},n)}D_m(n+1) = \alpha^\star_{m-1}D_{m-1}(n) + \beta^\star_m D_m(n) + \gamma^\star_{m+1}D_{m+1}(n). \tag{32}$$

We substitute $D_m(n+1)$ obtained from the first cross-rule (31) in the second one (32) and get a second-order recurrence equation in the variable m

$$\begin{aligned}
&q(q^n-q^m)(1-abcdq^{2m+2})(abcdq^{2m};q)_4 D_m(n) + q^m\left(1-q^m q\right)(abcdq^{2m+1};q)_3\Big(a^2q^n + \left(q^m\right)^2 qa^2dc + \\
&\left(q^m\right)^2 qa^2db + a^2\left(q^m\right)^2 q^2dc + a^2\left(q^m\right)^2 q^2bc + a^2\left(q^m\right)^2 q^2db - a^2q^m q + \left(q^m\right)^2 a^2q^n qd^2bc + q - q^m qad - \\
&q^m qab + \left(q^m\right)^2 qa^2cb - q^3\left(q^m\right)^2 abcd + q^2abcd\left(q^m\right)^2 - \left(q^m\right)^2 a^3q^n dcb - q^m a^2q^n qbc - q^m a^2q^n qbd - \\
&q^m a^2q^n qcd - \left(q^m\right)^3 a^3q^2bcd + \left(q^m\right)^4 q^3a^2b^2c^2d^2 - \left(q^m\right)^3 q^2a^2c^2bd - \left(q^m\right)^3 q^2a^2b^2cd - \left(q^m\right)^3 q^2a^2d^2bc + \\
&\left(q^m\right)^2 a^3q^n qdcb + \left(q^m\right)^2 a^2q^n qb^2cd + \left(q^m\right)^2 a^2q^n qc^2bd - q^m aq^n bcdq + a^4\left(q^m\right)^4 q^2q^n b^2c^2d^2 - q^m qac + \\
&\left(q^m\right)^2 q^2q^n a^2d^2bc - \left(q^m\right)^3 a^3bq^n q^2d^2c^2 - \left(q^m\right)^3 a^3b^2dq^n q^2c^2 - \left(q^m\right)^3 a^3b^2q^n q^2d^2c + \left(q^m\right)^2 q^2q^n a^2b^2cd + \\
&\left(q^m\right)^2 q^2q^n a^2c^2bd - \left(q^m\right)^3 q^2q^n a^2b^2c^2d^2\Big)D_{m+1}(n) + \left(q^m\right)^2 qa^2(q^{m+1};q)_2\left(q^m qdc-1\right)\left(q^m qdb-1\right)\times \\
&\left(q^m qbc-1\right)\left(q^m qad-1\right)\left(q^m qac-1\right)\left(q^m qab-1\right)\Big(abcd\left(q^m\right)^2-1\Big)\left(-1+q^m aq^n bcdq\right)D_{m+2}(n) = 0.
\end{aligned}$$

To solve this recurrence equation, we use Horn's variant of the q-Petkovšek algorithm [14, 15] using Sprenger's Maple implementation [26, 27]. With this algorithm, we get up to a multiplicative constant the solution of this recurrence equation. Equating the coefficients of $B_n(a,x)$ in (28) gives the constant and (29) follows. □

Remark 8. *The inversion formula (29) was already obtained in [2], but using (4) by Area* et al..

3.2. Connection Coefficients between $(p_n(x;a,b,c,d|q))_n$ and $(p_m(x;a,\beta,\gamma,\delta|q))_m$

Using an algorithmic approach, we determine the coefficients $C_m(n)$ (called connection coefficients) of the expansion

$$p_n(x;a,b,c,d|q)=\sum_{m=0}^{n}C_m(n)p_m(x;a,\beta,\gamma,\delta|q). \tag{33}$$

These connection coefficients are given by

Proposition 9. *The Askey-Wilson orthogonal polynomial family satisfies the following connection formula [2, 4]*

$$\begin{aligned} p_n(x;a,b,c,d|q) \quad &= \quad \sum_{m=0}^{n}\frac{a^{m-n}q^{m(m-n)}}{(q;q)_{n-m}}\frac{(q,ab,ac,ad;q)_n(abcdq^{n-1};q)_m}{(q,ab,ac,ad;q)_m(a\beta\gamma\delta q^{m-1};q)_m}\times \\ &\quad {}_5\phi_4\left(\begin{array}{c} q^{m-n},a\beta q^m,a\gamma q^m,a\delta q^m,abcdq^{m+n-1} \\ abq^m,acq^m,adq^m,a\beta\gamma\delta q^{2m}\end{array}\middle| q;q\right)p_m(x;a,\beta,\gamma,\delta|q). \end{aligned} \tag{34}$$

Proof. From (26) and (27), we have

$$p_n(x;a,b,c,d|q)=\sum_{j=0}^{n}A_j(n)B_j(a,x);$$

and it follows from (29) that

$$B_j(a,x)=\sum_{m=0}^{j}D_m(j)p_m(x;a,\beta,\gamma,\delta|q).$$

Combining the last two relations, one gets (33) with

$$C_m(n)=\sum_{j=0}^{n-m}A_{j+m}(n)D_m(j+m)\equiv\sum_{j=0}^{n-m}E_j.$$

We apply the `sum2qhyper` command of the q-version of Algorithm 2.1 of [17] to obtain the q-hypergeometric representation of $\sum\limits_{j=0}^{n-m}E_j$. This gives (34). □

4. Connection Coefficients (Part II)

We remark that in the connection formula (34), the parameter a is kept identical on both sides of the formula. We would like now to get a similar formula for different a. For this purpose, we need the following connection formula for $B_n(a,x)$.

4.1. Connection Formula between $(B_n(a,x))_n$ and $(B_n(\alpha,x))_n$

We expand $B_n(a,x)$ in terms of $B_m(\alpha,x)$

$$B_n(a,x)=\sum_{m=0}^{n}F_m(n)B_m(\alpha,x), \tag{35}$$

and obtain:

Theorem 10. *The connection between the basis $(B_n(a,x))_n$ and $(B_n(\alpha,x))_n$ is given by*

$$\begin{aligned} B_n(a,x) \quad &= \quad \sum_{m=0}^{n}(-1)^n q^{\frac{1}{2}n(n+1)}\left(\frac{a}{\alpha}\right)^n\frac{(\alpha-a)(q^{-n};q)_m(a\alpha,\frac{\alpha}{a}q^{-n};q)_nq^m}{(q;q)_m(\alpha q^m-aq^n)(a\alpha,\frac{\alpha}{a}q^{-n};q)_m}B_m(\alpha,x) \\ &= \quad \sum_{m=0}^{n}\begin{bmatrix} n \\ m\end{bmatrix}_q\left(\frac{a}{\alpha}\right)^m(a\alpha q^m;q)_{n-m}(\frac{a}{\alpha};q)_{n-m}B_m(\alpha,x). \end{aligned} \tag{36}$$

To get the coefficients $F_m(n)$ of the expansion (35), we need the following intermediate result which gives the second-order divided-difference equation for $B_n(a,x)$.

Proposition 11. *The polynomial family $(B_n(a,x))_n$ satisfies the second order divided-difference equation*

$$\begin{aligned}&\left(a^2 q^{2n-2}-1\right)\left(q-1\right)^2 B_1(a,x)\mathbb{D}_x^2 B_n(a,x)-4\,a\,\sqrt{q}q^{n-1}\left(q-1\right)B_1(a,x)\mathbb{S}_x\mathbb{D}_x B_n(a,x)\\&+4\,a^2\,\sqrt{q}\left(1+q^{n-1}\right)\left(1-q^n\right)B_n(a,x)=0.\end{aligned} \tag{37}$$

Proof. (of Proposition 11) We substitute the expression of $B_{n-1}(a,x)$ obtained from (7) in Equation (8) and the result follows. □

Proof. (of Theorem 10) We substitute $B_n(a,x)$ by the sum in (35) in the divided-difference equation (37) and multiply the obtained equation by $B_1(\alpha,x)$ for the purpose to use Equations (7), (8) and (10) (with a replaced by α). Next, we substitute $B_1(a,x)$ by its representation in terms of $x(s)$, that is $B_1(a,x)=1+a^2-2ax(s)$, and use Equation (9) with $a=\alpha$. Finally we collect the coefficients of $B_m(\alpha,x)$, $B_{m-1}(\alpha,x)$, $B_{m+1}(\alpha,x)$ and by an appropriate shift of index, we rewrite all the summands in terms of $B_m(\alpha,x)$. Since the family $(B_m(\alpha,x))_m$ is linearly independent, it follows after simplification that $F_m(n)$ satisfies the following second-order recurrence equation

$$\begin{aligned}&-a\left(q^{m+1}-1\right)\Big(\alpha\,q^3q^{m+1}-\alpha\,q^2\left(q^{m+1}\right)^2-\alpha\,qq^{m+1}a^2\left(q^n\right)^2+\alpha\left(q^{m+1}\right)^2a^2\left(q^n\right)^2+q^2aq^{m+1}\\&-q^2a\left(q^{m+1}\right)^2\alpha^2-qaq^{m+1}\left(q^n\right)^2+qa\left(q^{m+1}\right)^2\left(q^n\right)^2\alpha^2-aq^3q^n-aq^n\alpha^2\left(q^{m+1}\right)^3-aq^2q^n\\&-aqq^n\alpha^2\left(q^{m+1}\right)^3+\alpha\,q^2q^nq^{m+1}+\alpha\,qq^n\left(q^{m+1}\right)^2+\alpha\,q^2q^nq^{m+1}a^2+\alpha\,qq^n\left(q^{m+1}\right)^2a^2\Big)F_{m+1}(n)\\&+a^2q\left(-q^{m+1}+q^nq\right)\left(q^nq^{m+1}+q^2\right)F_m(n)\\&+\left(q^{m+1}-1\right)\left(-a+\alpha\,q^{m+1}\right)\left(\alpha\,q^{m+1}a-1\right)\left(q^{m+1}q-1\right)\left(aq^n-\alpha\,q^{m+1}q\right)\left(\alpha\,q^{m+1}aq^n+q\right)F_{m+2}(n)=0.\end{aligned}$$

Using again Horn's variant of the q-Petkovšek algorithm, we solve this recurrence equation and the result follows using the formulas

$$\begin{aligned}(aq^{-n};q)_n&=(a^{-1}q;q)_n(-a)^nq^{-n-\binom{n}{2}},\ a\neq 0,\\(aq^k;q)_{n-k}&=\frac{(a;q)_n}{(a;q)_k},\quad k=0,1,2,\dots,n,\\(q^{-n};q)_k&=\frac{(q;q)_n}{(q;q)_{n-k}}(-1)^kq^{\binom{k}{2}-nk},\ k=0,1,\dots,n,\end{aligned} \tag{38}$$

given respectively by Equations (1.8.12), (1.8.14), (1.8.18) of ([16], p. 12-13). □

Remark 12. *We remark that taking the limit when a tends to α in (36), we obtain $F_m(n)=0$, $m=0,\dots,n-1$ and $F_n(n)=1$.*

4.2. Connection Coefficients between $(p_n(x;a,b,c,d|q))_n$ and $(p_m(x;\alpha,\beta,\gamma,\delta|q))_m$

We will now use the connection formula between $B_n(a,x)$ and $B_n(\alpha,x)$ to derive the representation of the Askey-Wilson polynomials in the basis $(B_n(\alpha,x))_n$. From

$$p_n(x;a,b,c,d|q)=\sum_{j=0}^{n}A_j(n)B_j(a,x)\ \text{ and }\ B_j(a,x)=\sum_{m=0}^{n}F_m(j)B_m(\alpha,x),$$

we get

$$p_n(x;a,b,c,d|q)=\sum_{m=0}^{n}G_m(n)B_m(\alpha,x),$$

with

$$G_m(n)=\sum_{j=0}^{n-m}A_{j+m}(n)F_m(j+m).$$

Proceeding similarly as in Proposition 9, one gets the following hypergeometric representation.

Proposition 13. *The element $p_n(x;a,b,c,d|q)$ of the Askey-Wilson orthogonal polynomial family has the following hypergeometric representation in the basis $(B_n(\alpha,x))_n$*

$$\begin{aligned} p_n(x;a,b,c,d|q) &= \sum_{m=0}^{n}\left(-\frac{a}{\alpha}\right)^m \frac{q^{\frac{m(m-2n+1)}{2}}}{a^n(q;q)_{n-m}} \frac{(q,ab,ac,ad;q)_n(abcdq^{n-1};q)_m}{(q,ab,ac,ad;q)_m} \times \\ & {}_4\phi_3\left(\begin{matrix} q^{m-n}, a\alpha q^m, abcdq^{m+n-1}, \frac{a}{\alpha} \\ abq^m, acq^m, adq^m \end{matrix}\middle| q;q\right) B_m(\alpha,x), \end{aligned}$$

Remark 14. *Taking the limit when α tends to a in the preceding representation, we get representations* (1) *thanks to the equality* $(1;q)_j = 0,\ j \geq 1$.

From the two previous formulas, we have

$$p_n(x;a,b,c,d|q) = \sum_{j=0}^{n} G_j(n) B_j(\alpha,x) \text{ and } B_j(\alpha,x) = \sum_{m=0}^{j} D_m(j) p_m(x;\alpha,\beta,\gamma,\delta|q),$$

from which we get

$$p_n(x;a,b,c,d|q) = \sum_{m=0}^{n} C_m(n) p_m(x;\alpha,\beta,\gamma,\delta|q),$$

with

$$C_m(n) = \sum_{j=0}^{n-m} G_{j+m}(n) D_m(j+m).$$

Proceeding similarly as in Proposition 9, one gets the following

Theorem 15. *The following connection formula is satisfied by the Askey-Wilson polynomial family*

$$p_n(x;a,b,c,d|q) = \sum_{m=0}^{n} C_m(n) p_m(x;\alpha,\beta,\gamma,\delta|q), \tag{39}$$

with

$$\begin{aligned} C_m(n) &= \frac{(aq^m)^{m-n}(q;q)_n(ab,ac,ad;q)_n}{(q;q)_m(q;q)_{n-m}(\alpha\beta\gamma\delta q^{m-1};q)_m} \times \\ & \sum_{k=0}^{n-m}\left(\frac{a}{\alpha}\right)^k \frac{(q^{m-n};q)_k q^k}{(q;q)_k} \frac{(\alpha\beta q^m, \alpha\gamma q^m, \alpha\delta q^m;q)_k (abcdq^{n-1};q)_{k+m}}{(ab,ac,ad;q)_{k+m}(\alpha\beta\gamma\delta q^{2m};q)_k} \times \\ & {}_4\phi_3\left(\begin{matrix} q^{k+m-n}, a\alpha q^{k+m}, abcdq^{m+n+k-1}, \frac{a}{\alpha} \\ abq^{k+m}, acq^{k+m}, adq^{k+m} \end{matrix}\middle| q;q\right). \end{aligned}$$

Remark 16. *By taking the limit when α tends to a in the preceding relation, we obtain the connection formula of Proposition 9.*

5. Linearization Formulas

We want to determine the linearization coefficients $L_k(m,n)$ of the formula

$$p_n(x;a,b_1,c_1,d_1|q)\, p_m(x;a,b_2,c_2,d_2|q) = \sum_{k=0}^{n+m} L_k(m,n) p_k(x;a,b_2,c_2,d_2|q).$$

For this purpose we need to derive the linearization relation for the basis $(B_n(a,x))_n$.

Proposition 17. *The basis $(B_n(a,x))_n$ of the Askey-Wilson orthogonal polynomial family satisfies the following linearization formula*

$$B_n(a,x)B_m(a_1,x)=\sum_{k=0}^{m}H_{n+k}(m,n)B_{n+k}(a,x),\ m,n\in\mathbb{N},\tag{40}$$

with

$$H_{n+k}(m,n)=\begin{bmatrix} m\\ k\end{bmatrix}_q q^{-nk}\left(\frac{a_1}{a}\right)^k\left(\frac{a_1}{aq^n};q\right)_{m-k}\left(aa_1q^{n+k};q\right)_{m-k},\ k=0,1,\cdots,m.$$

Proof. We recall that

$$B_n(a,x)=B_n(a,x(s))=\prod_{j=0}^{n-1}(1-2aq^jx(s)+a^2q^{2j}).$$

For

$$x(s)=\varepsilon_j(a)\equiv\frac{1+a^2q^{2j}}{2aq^j},$$

we get

$$B_n(a,\varepsilon_j(a))=0,\ n\geq 1,\ j=0,1,\dots,n-1\tag{41}$$

and for $j=n$,

$$B_n(a,\varepsilon_n(a))\neq 0.\tag{42}$$

In the first step, we expand $B_n(a,x)B_m(a_1,x)$, $n\geq 1$ in the basis $(B_k(a,x))_k$

$$B_n(a,x)B_m(a_1,x)=\sum_{k=0}^{n+m}H_k(m,n)B_k(a,x),\tag{43}$$

and use Relation (41) to get $H_0(m,n)=B_n(a,\varepsilon_0(a))B_m(a_1,\varepsilon_0(a))=0$, $n\geq 1$.

Considering (43) for $x(s)=\varepsilon_1(a)$ and observing that $H_0(m,n)=0$, we get using again (41) that

$$H_1(m,n)B_1(a,\varepsilon_1(a))=B_n(a,\varepsilon_1(a))B_m(a_1,\varepsilon_1(a))=0,\ n\geq 2.$$

Therefore, $H_1(m,n)=0$ thanks to (42). Progressively, we obtain in a similar way for a fixed integer j using (41), (42) and (43) that

$$H_0(m,n)=H_1(m,n)=\cdots=H_j(m,n)=0,\ j\leq n-1.$$

In the second step, we rewrite Relation (43) accordingly with the previous result

$$B_n(a,x)B_m(a_1,x)=\sum_{k=0}^{m}H_{n+k}(m,n)B_{n+k}(a,x).\tag{44}$$

Using the relation

$$B_{n+k}(a,x)=B_n(a,x)B_k(aq^n,x),$$

derived from the definition in Proposition 1,

$$B_n(a,x)=\prod_{j=0}^{n-1}(1-aq^sq^j)(1-aq^{-s}q^j),$$

we get from (44) that

$$B_m(a_1,x)=\sum_{k=0}^{m}H_{n+k}(m,n)\frac{B_{n+k}(a,x)}{B_n(a,x)}=\sum_{k=0}^{m}H_{n+k}(m,n)B_k(aq^n,x).\tag{45}$$

For $x(s)=\varepsilon_0(aq^n)$, Equation (45) gives $H_n(m,n)=B_m(a_1,\varepsilon_0(aq^n))$.

Iterating Relation (6), we have for fixed l

$$\mathbb{D}_x^l B_n(a,x) = \prod_{j=0}^{l-1} \eta(aq^{\frac{j}{2}}, n-j) B_{n-l}(aq^{\frac{l}{2}}, x),\ 1 \le l \le n.$$

Applying the operator $\mathbb{D}_x^l$ to Relation (45), we gets

$$\prod_{j=0}^{l-1} \eta(a_1 q^{\frac{j}{2}}, m-j) B_{m-l}(a_1 q^{\frac{l}{2}}, x) = \sum_{k=l}^{m} H_{n+k}(m,n) \prod_{j=0}^{l-1} \eta(aq^{n+\frac{j}{2}}, k-j) B_{k-l}(aq^{n+\frac{l}{2}}, x),\ 1 \le l \le m.$$

For $k = l$ and for $x(s) = \varepsilon_0(aq^{n+\frac{l}{2}})$, the preceding relation gives, taking into account (41)

$$H_{n+l}(m,n) = \frac{\prod\limits_{j=0}^{l-1} \eta(a_1 q^{\frac{j}{2}}, m-j)}{\prod\limits_{j=0}^{l-1} \eta(aq^{n+\frac{j}{2}}, l-j)} B_{m-l}(a_1 q^{\frac{l}{2}}, \varepsilon_0(aq^{n+\frac{l}{2}})),\ l = 1, 2, \cdots, m.$$

We replace $\eta(a_1 q^{n+\frac{j}{2}}, l-j)$, $\eta(a_1 q^{\frac{j}{2}}, m-j)$ and $B_{m-l}(a_1 q^{\frac{l}{2}}, \varepsilon_0(aq^{n+\frac{l}{2}}))$ by their values obtained from Proposition 1 and from the definition $B_n(a,x) = \prod\limits_{j=0}^{n-1}(1-2aq^j x(s) + a^2 q^{2j})$. This yields

$$H_{n+k}(m,n) = \left(\frac{a_1}{a}\right)^k q^{(m-n-k)k} \frac{(q^{-m};q)_k}{(q^{-k};q)_k} \left(\frac{a_1}{aq^n}; q\right)_{m-k} \left(aa_1 q^{n+k}; q\right)_{m-k}.$$

The result follows by using the transformation (38). □

Remark 18. *1. Substituting $a_1 = a$ in Relation (40) yields*

$$B_n(a,x) B_m(a,x) = \sum_{k=0}^{m} \begin{bmatrix} m \\ k \end{bmatrix}_q q^{-nk} (q^{-n};q)_{m-k} (a^2 q^{n+k}; q)_{m-k} B_{n+k}(a,x). \tag{46}$$

2. When we take $m = 1$ in (46), we recover Relation (10). We also remark that for $n = 0$, (40) is the connection formula (36) with $a = a_1$ and $\alpha = a$.

Having derived the linearization relation for $B_n(a,x)$, we now state and prove:

Theorem 19. *The Askey-Wilson orthogonal polynomial family satisfies the linearization formulas*

$$p_n(x;a,b_1,c_1,d_1|q) p_m(x;a,b_2,c_2,d_2|q) = \sum_{r=0}^{n+m} L_r(m,n) p_r(x;a,\beta,\gamma,\delta|q), \tag{47}$$

with

$$\begin{aligned} L_r(m,n) \quad = \quad & \frac{a^{r-m-n} q^{r(r+1)} \left(ab_1, ac_1, ad_1; q\right)_n \left(ab_2, ac_2, ad_2; q\right)_m}{(q;q)_r \left(a\beta\gamma\delta q^{r-1}; q\right)_r} \times \\ & \sum_{l=0}^{n+m-r} \frac{q^{l(r+1)} \left(a\beta\gamma\delta q^{l+2r}, a\beta q^r, a\gamma q^r, a\delta q^r; q\right)_l (q^{-l-r};q)_r}{\left(a\beta\gamma\delta q^{2r}; q\right)_{2l}} \times \\ & \sum_{j=0}^{\min(n,l+r)} \frac{q^{j(j-l-r)} \left(q^{-n};q\right)_j \left(ab_1c_1d_1 q^{n-1}; q\right)_j \left(ab_2c_2d_2 q^{m-1}; q\right)_{l+r-j} \left(q^{-m};q\right)_{l+r-j}}{\left(q;q\right)_j \left(q;q\right)_{l+r-j} \left(ab_1, ac_1, ad_1; q\right)_j \left(ab_2, ac_2, ad_2; q\right)_{l+r-j}} \times \\ & \sum_{k=l+r-j}^{m} \frac{\left(q^{-m}, q^{-l-r}, a^2 q^j, ab_2c_2d_2 q^{m-1}; q\right)_k q^k}{\left(ab_2, ac_2, ad_2, q^{j+1-l-r}; q\right)_k}. \end{aligned}$$

Proof. We have

$$p_n(x;a,b_1,c_1,d_1|q)=\sum_{j=0}^{n}A_j(n)B_j(a,x) \text{ and } p_m(x;a,b_2,c_2,d_2|q)=\sum_{k=0}^{m}A_k(m)B_k(a,x).$$

Thus

$$\begin{aligned}
p_n(x;a,b_1,c_1,d_1|q)p_m(x;a,b_2,c_2,d_2|q) &= \sum_{j=0}^{n}\sum_{k=0}^{m}A_j(n)A_k(m)B_j(a,x)B_k(a,x)\\
&\overset{(46)}{=} \sum_{j=0}^{n}\sum_{k=0}^{m}A_j(n)A_k(m)(\sum_{l=0}^{k}H_{j+l}(k,j)B_{j+l}(a,x))\\
&= \sum_{l=0}^{n}\left(\sum_{j=0}^{l}\sum_{k=l-j}^{m}A_j(n)A_k(m)H_l(k,j)\right)B_l(a,x)\\
&+ \sum_{l=n+1}^{n+m}\left(\sum_{j=0}^{n}\sum_{k=l-j}^{m}A_j(n)A_k(m)H_l(k,j)\right)B_l(a,x),\\
&= \sum_{l=0}^{n+m}I_l(m,n)B_l(a,x),
\end{aligned}$$

with

$$I_l(m,n)=\sum_{j=0}^{\min(n,l)}\sum_{k=l-j}^{m}A_j(n)A_k(m)H_l(k,j).$$

Since

$$B_l(a,x)=\sum_{r=0}^{l}D_r(l)p_r(x;a,\beta,\gamma,\delta|q),$$

we get

$$p_n(x;a,b_1,c_1,d_1|q)p_m(x;a,b_2,c_2,d_2|q)=\sum_{r=0}^{n+m}L_r(m,n)p_r(x;a,\beta,\gamma,\delta|q),$$

with

$$L_r(m,n)=\sum_{l=0}^{n+m-r}I_{l+r}(m,n)D_r(l+r).$$

□

Remark 20. *Using relation (36), we get the preceding linearization formulas with all the parameters different as follows.*

Theorem 21. *The Askey-Wilson orthogonal polynomial family satisfies the linearization formulas*

$$p_n(x;a_1,b_1,c_1,d_1|q)p_m(x,a_2,b_2,c_2,d_2|q)=\sum_{r=0}^{n+m}L_r(m,n)p_r(x;\alpha,\beta,\gamma,\delta|q), \tag{48}$$

with

$$\begin{aligned}
L_r(m,n) &= \frac{q^{r(r+1)}\alpha^r\left(a_1b_1,a_1c_1,a_1d_1;q\right)_n\left(a_2b_2,a_2c_2,a_2d_2;q\right)_m}{a_1^na_2^m(q;q)_r\left(\alpha\beta\gamma\delta q^{r-1};q\right)_r}\times\\
&\sum_{i=0}^{n+m-r}\frac{q^{i(r+1)}\left(\alpha\beta\gamma\delta q^{i+2r},\alpha\beta q^r,\alpha\gamma q^r,\alpha\delta q^r;q\right)_i\left(q^{-i-r};q\right)_r\left(\frac{a_1}{\alpha}\right)^{i+r}}{(\alpha\beta\gamma\delta q^{2r};q)_{2i}(q;q)_{i+r}}\sum_{l=0}^{n+m-i-r}\frac{q^l(q;q)_{l+i+r}\left(\alpha a_1q^{i+r},\frac{a_1}{\alpha};q\right)_l}{(q;q)_l}\times\\
&\sum_{j=0}^{\min(n,l+r+i)}\frac{q^{j(j-l-r-i)}\left(\frac{a_2}{a_1}\right)^{l+i+r-j}\left(q^{-n},a_1b_1c_1d_1q^{n-1};q\right)_j\left(q^{-m},a_2b_2c_2d_2q^{m-1};q\right)_{l+r+i-j}}{(a_1b_1,a_1c_1,a_1d_1,q;q)_j(a_2b_2,a_2c_2,a_2d_2,q;q)_{l+r+i-j}}\times\\
&\sum_{k=l+r+i-j}^{m}\frac{\left(q^{-m},\frac{a_2}{a_1}q^{-l-r-i},a_1a_2q^j,a_2b_2c_2d_2q^{m-1};q\right)_kq^k}{\left(a_2b_2,a_2c_2,a_2d_2,q^{j+1-l-r-i};q\right)_k}.
\end{aligned}$$

Proof. Following the same procedure as in the preceding proof, we get

$$L_r(m,n) = \sum_{i=0}^{n+m-r} \sum_{l=0}^{n+m-i-r} \sum_{j=0}^{\min(n,l+i+r)} \sum_{k=l+i+r-j}^{m} A_j(n) A_k(m) H_{l+i+r}(k,j) F_{i+r}(l+i+r) D_r(i+r).$$

□

6. Connection Coefficients of the Continuous q-Hahn, q-Racah and Wilson Polynomials

In this section, by taking suitable limits and a specialization process [16], we obtain from the connection formula for the Askey-Wilson polynomials the connection formula for the orthogonal polynomials of the Askey and q-Askey scheme [16]. As illustration, we give the connection formula of the Continuous q-Hahn, the q-Racah and the Wilson polynomials.

6.1. Connection Coefficients of the Continuous q-Hahn polynomials

The Continuous q-Hahn polynomials given by ([16], p. 433)

$$p_n(x;a,b,c,d;q) = \frac{(abt^2, ac, ad;q)_n}{(at)^n} {}_4\phi_3\left(\begin{array}{c} q^{-n}, abcdq^{n-1}, at^2q^s, aq^{-s} \\ abt^2, ac, ad \end{array} \middle| q;q \right)$$

where

$$x = x(s) = \cos(\theta+\hat{\theta}) = \frac{t^2q^s + q^{-s}}{2t}, \text{ with } t = e^{i\hat{\theta}},$$

can be obtained from the Askey-Wilson polynomials by the substitutions [16]

$$\theta \to \theta+\hat{\theta},\ a \to ae^{i\hat{\theta}},\ b \to be^{i\hat{\theta}},\ c \to ce^{-i\hat{\theta}},\ \text{and}\ d \to de^{-i\hat{\theta}}:$$

$$p_n(\cos(\theta+\hat{\theta});a,b,c,d;q) = p_n(\cos(\theta+\hat{\theta}); ae^{i\hat{\theta}}, be^{i\hat{\theta}}, ce^{-i\hat{\theta}}, de^{-i\hat{\theta}}|q).$$

Doing the previous substitutions in (34) and (39), and taking into account the previous relation, we get:

Proposition 22. *The Continuous q-Hahn orthogonal polynomial family satisfies the following connection formulas*

$$\begin{aligned} p_n(x;a,b,c,d;q) \quad = \quad & \sum_{m=0}^{n} \frac{(at)^{m-n} q^{m(m-n)}}{(q;q)_{n-m}} \frac{(q, abt^2, ac, ad;q)_n (abcdq^{n-1};q)_m}{(q, abt^2, ac, ad;q)_m (\alpha\beta\gamma\delta q^{m-1};q)_m} \times \\ & {}_5\phi_4\left(\begin{array}{c} q^{m-n}, a\beta t^2 q^m, a\gamma q^m, a\delta q^m, abcdq^{m+n-1} \\ abt^2q^m, acq^m, adq^m, \alpha\beta\gamma\delta q^{2m} \end{array} \middle| q;q \right) p_m(x;a,\beta,\gamma,\delta;q), \end{aligned}$$

$$\begin{aligned} p_n(x;a,b,c,d;q) \quad = \quad & \sum_{m=0}^{n} \frac{(atq^m)^{m-n} (q;q)_n (abt^2, ac, ad;q)_n}{(q;q)_m (q;q)_{n-m} (\alpha\beta\gamma\delta q^{m-1};q)_m} \times \\ & \sum_{k=0}^{n-m} \left(\frac{a}{\alpha}\right)^k \frac{(q^{m-n};q)_k q^k}{(q;q)_k} \frac{(\alpha\beta t^2 q^m, \alpha\gamma q^m, \alpha\delta q^m;q)_k (abcdq^{n-1};q)_{k+m}}{(abt^2, ac, ad;q)_{k+m} (\alpha\beta\gamma\delta q^{2m};q)_k} \times \\ & {}_4\phi_3\left(\begin{array}{c} q^{k+m-n}, a\alpha t^2 q^{k+m}, abcdq^{m+n+k-1}, \frac{a}{\alpha} \\ abt^2q^{k+m}, acq^{k+m}, adq^{k+m} \end{array} \middle| q;q \right) p_m(x;\alpha,\beta,\gamma,\delta;q). \end{aligned}$$

6.2. Connection Coefficients of the q-Racah polynomials

The q-Racah polynomials given by ([16], p. 422)

$$R_n(x(s);\alpha,\beta,\gamma,\delta|q) = {}_4\phi_3\left(\begin{array}{c} q^{-n},\alpha\beta q^{n+1},q^{-s},\gamma\delta q^{s+1} \\ \alpha q,\beta\delta q,\gamma q \end{array}\middle| q;q\right),\ n=0,1,2,\ldots,N,$$

are related to the Askey-Wilson polynomials in the following way. If we substitute [16]

$$a^2=\gamma\delta q,\ b^2=\alpha^2\gamma^{-1}\delta^{-1}q,\ c^2=\beta^2\gamma^{-1}\delta q,\ d^2=\gamma\delta^{-1}q \text{ and } e^{2i\theta}=\gamma\delta q^{2s+1}$$

in the definition of the Askey-Wilson polynomials we find

$$R_n(x(s);\alpha,\beta,\gamma,\delta|q) = \frac{(\gamma\delta q)^{\frac{1}{2}n}}{(\alpha q,\beta\delta q,\gamma q;q)_n}\ p_n(\nu(s);\gamma^{\frac12}\delta^{\frac12}q^{\frac12},\alpha\gamma^{\frac{-1}{2}}\delta^{\frac{-1}{2}}q^{\frac12},\beta\gamma^{\frac{-1}{2}}\delta^{\frac12}q^{\frac12},\gamma^{\frac12}\delta^{\frac{-1}{2}}q^{\frac12}|q),$$

where

$$\nu(s)=\frac{1}{2}\gamma^{\frac12}\delta^{\frac12}q^{s+\frac12}+\frac{1}{2}\gamma^{\frac{-1}{2}}\delta^{\frac{-1}{2}}q^{-s-\frac12}.$$

For the q-Racah polynomial family, one has:

Proposition 23. *The q-Racah orthogonal polynomial family satisfies the following connection formula*

$$R_n(x(s);\alpha,\beta,\gamma,\delta|q)=\sum_{m=0}^{n}\frac{(b\delta q)^n(\beta\delta)^{-m}(\frac{\alpha q}{\delta},\frac{\beta}{b};q)_n(q^{-n},\alpha\beta q^{n+1},b\delta q,\alpha bq;q)_m(\alpha bq^3;q^2)_m}{(\beta\delta q,\alpha bq^2;q)_n(q,\frac{\alpha q}{\delta},\frac{bq}{\beta q^n},\alpha bq^{n+2};q)_m(\alpha bq;q^2)_m}R_m(x(s),\alpha,b,\gamma,\delta|q).$$

Proof. In the Askey-Wilson connection formula (34), we make the following substitutions

$$a=\gamma^{\frac12}\delta^{\frac12}q^{\frac12},\ b=\beta=\alpha\gamma^{\frac{-1}{2}}\delta^{\frac{-1}{2}}q^{\frac12},\ c=\beta\gamma^{\frac{-1}{2}}\delta^{\frac12}q^{\frac12},\ d=\delta=\gamma^{\frac12}\delta^{\frac{-1}{2}}q^{\frac12},\ \gamma=b\gamma^{\frac{-1}{2}}\delta^{\frac12}q^{\frac12},$$

and multiply the obtained coefficient $C_m(n)$ by $\dfrac{(\gamma\delta q)^{\frac{n}{2}}(\alpha q,b\delta q,\gamma q;q)_m}{(\gamma\delta q)^{\frac{m}{2}}(\alpha q,\beta\delta q,\gamma q;q)_n}$. This yields

$$\begin{aligned} R_n(x(s);\alpha,\beta,\gamma,\delta|q) \quad &= \quad \sum_{m=0}^{n}\frac{q^{m(m-n)}(q;q)_n}{(q;q)_m(q;q)_{n-m}}\frac{(b\delta q,\alpha\beta q^{n+1};q)_m}{(\beta\delta q,\alpha bq^{m+1};q)_m}\times \\ &\quad {}_3\phi_2\left(\begin{array}{c} q^{m-n},\alpha\beta q^{m+n+1},b\delta q^{m+1} \\ \beta\delta q^{m+1},\alpha bq^{2m+2} \end{array}\middle| q;q\right)R_m(x(s),\alpha,b,\gamma,\delta|q). \end{aligned}$$

Using the `qsumrecursion` command of the q-version of Zeilberger's algorithm [17], we simplify the previous expression. □

6.3. Connection Coefficients of the Wilson Polynomials

To find the Wilson polynomials given by ([16], p. 185)

$$W_n(x^2;a,b,c,d)=(a+b)_n(a+c)_n(a+d)_n\,{}_4F_3\left(\begin{array}{c} -n,n+a+b+c+d-1,a+ix,a-ix \\ a+b,a+c,a+d \end{array}\middle| 1\right)$$

from the Askey-Wilson polynomials, we set [16]

$$a\to q^a,\ b\to q^b,\ c\to q^c,\ d\to q^d, \text{ and } e^{i\theta}=q^{ix}$$

and take the limit $q\to 1$:

$$W_n(x^2;a,b,c,d)=\lim_{q\to1}\frac{p_n(\frac12(q^{ix}+q^{-ix});q^a,q^b,q^c,q^d|q)}{(1-q)^{3n}}. \tag{49}$$

We recall here that the hypergeometric function ${}_pF_q$, as well as the Pochhammer symbol $(a_1,a_2,\ldots,a_k)_n$, are defined as

$$ {}_pF_q\left(\begin{array}{c} a_1,\cdots,a_p \\ b_1,\cdots,b_q \end{array}\middle| z\right)=\sum_{k=0}^{\infty}\frac{(a_1,\cdots,a_p)_k}{(b_1,\cdots,b_q)_k}\frac{z^k}{k!}, $$

where

$$ (a_1,\cdots,a_p)_k := (a_1)_k\cdots(a_p)_k,\ \text{ with } (a_i)_k=\begin{cases}\prod_{j=0}^{k-1}(a_i+j) & \text{if } k=1,2,3.\cdots \\ 1 & \text{if } k=0\end{cases}. $$

In the case of the Wilson polynomials, we have:

Proposition 24. *The connection formulas of the Wilson polynomials are given by*

$$ \begin{aligned} W_n(x^2;a,b,c,d) &= \sum_{m=0}^{n}\binom{n}{m}\frac{(n+a+b+c+d-1)_m(m+a+b)_{n-m}(m+a+c)_{n-m}(m+a+d)_{n-m}}{(m+a+\beta+\gamma+\delta-1)_m}\times \\ &\quad {}_5F_4\left(\begin{array}{c} m-n,m+n+a+b+c+d-1,m+a+\beta,m+a+\gamma,m+a+\delta \\ 2m+a+\beta+\gamma+\delta,m+a+b,m+a+c,m+a+d \end{array}\middle|1\right)W_m(x^2;a,\beta,\gamma,\delta), \end{aligned} \tag{50} $$

$$ \begin{aligned} W_n(x^2;a,b,c,d) &= \sum_{m=0}^{n}\binom{n}{m}\frac{(n+a+b+c+d-1)_m(m+a+b,m+a+c,m+a+d)_{n-m}}{(m+\alpha+\beta+\gamma+\delta-1)_m}\times \\ &\quad \sum_{k=0}^{n-m}\frac{(m-n)_k(m+\alpha+\beta,m+\alpha+\gamma,m+\alpha+\delta)_k(m+n+a+b+c+d-1)_k}{k!(m+a+b,m+a+c,m+a+d)_k(2m+\alpha+\beta+\gamma+\delta)_k}\times \\ &\quad {}_4F_3\left(\begin{array}{c} k+m-n,k+m+a+\alpha,m+n+k+a+b+c+d-1,a-\alpha \\ m+k+a+b,m+k+a+c,m+k+a+d \end{array}\middle|1\right)W_m(x^2;\alpha,\beta,\gamma,\delta). \end{aligned} \tag{51} $$

Proof. In the Askey-Wilson connection formulas (34) , we perform the previous substitutions and limiting process and use (49). It follows that

$$ \lim_{q\to1}\frac{p_n(\frac{1}{2}(q^{ix}+q^{-ix});q^a,q^b,q^c,q^d|q)}{(1-q)^{3n}}=\sum_{m=0}^{n}C_m^1(n)\lim_{q\to1}\frac{p_n(\frac{1}{2}(q^{ix}+q^{-ix});q^a,q^b,q^c,q^d|q)}{(1-q)^{3m}}, $$

i.e.

$$ W_n(x^2;a,b,c,d)=\sum_{m=0}^{n}C_m^1(1)W_m(x^2;a,\beta,\gamma,\delta), $$

with

$$ C_m^1(n)=\lim_{q\to1}\frac{C_m(n)}{(1-q)^{3(n-m)}}. $$

We just have to calculate $C_m^1(n)$. For this purpose, we multiply and divide $C_m(n)$ by $(1-q)^m$ and use the formulas

$$ \lim_{q\to1}\frac{(q^\alpha;q)_m}{(1-q)^m}=(\alpha)_m, $$

$$ \lim_{q\to1}\begin{bmatrix} n \\ m \end{bmatrix}_q=\binom{n}{m}\ \text{ and }\ \frac{(\alpha)_n}{(\alpha)_m}=(m+\alpha)_{n-m} $$

to get (50).

For the connection formula (51), we proceed similarly after using (39) and the relation $(x;q)_{k+m}=(x;q)_m(xq^m;q)_k$.

□

Remark 25. ① *Connection formula (50) has been given by Jorge Sánchez-Ruiz and Jesús S. Dehesa in [25] using this formula derived by Fields and Wimp [7] (see also [20], p. 7)*

$$
{}_{p+r+1}F_{q+s}\left(\begin{array}{c} -n,[a_p],[c_r] \\ [b_q],[d_s] \end{array}\middle| zw\right) = \sum_{k=0}^{n}\binom{n}{k}\frac{(a_1,\dots,a_p)_k\,(\alpha_1,\dots,\alpha_t)_k}{(b_1,\dots,b_q)_k\,(\beta_1,\dots,\beta_u)_k}\frac{z^k}{(k+\lambda)_k}\times
$$
$$
{}_{p+t+1}F_{q+u+1}\left(\begin{array}{c} k-n,[k+a_p],[k+\alpha_t] \\ 2k+\lambda+1,[k+b_q],[k+\beta_u] \end{array}\middle| z\right){}_{r+u+2}F_{s+t}\left(\begin{array}{c} -k,k+\lambda,[c_r],[\beta_u] \\ [d_s],[\alpha_t] \end{array}\middle| w\right),
$$

where $[a_p]$ denotes the set $\{a_1, a_2, \cdots, a_p\}$ of complex parameters.

② *The connection and linearisation formulas for the Wilson polynomials have also been obtained by Foupouagnigni, Koepf and Njionou Sadjang [19] using similar algorithmic approach, combined with the characterization of the Wilson polynomials, by means of an appropriate basis, specific to the Wilson polynomials.*

③ *Connection formula (51) generalizes (50).*

Conclusion and Perspectives

In this work, we have used an algorithmic method to compute the connection and the linearization coefficients for the standard Askey-Wilson orthogonal polynomial families. From these results, one can deduce easily, using Relation (26), the connection and the linearization coefficients for the monic Askey-Wilson family (monic with respect to the basis $B_n(a,x)$) defined by Relation (27).

As perspectives, it would be interesting:

- to find conditions on the parameters to simplify formulas (47)and (48) appearing in triple and quadruple summations,
- to use the same approach to compute explicitly the duplication coefficients $C_m(n)$ for the relation

$$
p_n(\alpha x; a,b,c,d|q) = \sum_{m=0}^{n} C_m(n)\, p_n(x;a,b,c,d|q),
$$

of the Askey-Wilson polynomial families.

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