An extension of the Euler-Maclaurin quadrature formula using a parametric type of Bernoulli polynomials

Mohammad Masjed-Jamei, M. R. Beyki and W. Koepf
Department of Mathematics, K.N.Toosi University of Technology, P.O. Box 16315–1618, Tehran, Iran
Department of Mathematics, University of Kassel Heinrich-Plett-Str. 40, D-34132 Kassel, Germany
e-mail: *mmjamei@kntu.ac.ir, mbeyki@mail.kntu.ac.ir*
koepf@mathematik.uni-kassel.de

Abstract

In this paper, we introduce a parametric type of Bernoulli polynomials and study their basic properties in order to establish an extension of Euler-Maclaurin quadrature rules and compare them with the well-known ordinary case.

Keywords: Euler-Maclaurin quadrature rules; Bernoulli polynomials and numbers; Appell polynomials; Generating functions; Fourier expansions.

2010 Mathematics Subject Classification: 11B68, 11Y35, 11C08.

1 Introduction

The Appell polynomials $A_n(x)$ defined by

$$f(t)\mathrm{e}^{xt} = \sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!}, \tag{1}$$

where f is a formal power series in t, have found remarkable applications in different branches of mathematics, theoretical physics and chemistry [2, 15]. Two special cases of Appell polynomials are Bernoulli polynomials $B_n(x)$ and Euler polynomials $E_n(x)$ that are, respectively, generated by choosing $f(t) = \frac{t}{\mathrm{e}^t-1}$ and $f(t) = \frac{2}{\mathrm{e}^t+1}$ in (1). Also, Bernoulli numbers $B_n := B_n(0)$ and Euler numbers $E_n := 2^n E_n(\frac{1}{2})$ are of considerable importance in number theory, special functions, combinatorics and numerical analysis.

Bernoulli numbers are given by

$$\frac{t}{\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!} \quad (|t| < 2\pi),$$

or by the recurrence relation

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \;\text{ for } n \geq 1 \;\text{ and }\; B_0 = 1.$$

They are directly related to various combinatorial numbers such as Stirling, Cauchy and harmonic numbers. For example, except B_1 we have

$$B_n = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m m!}{m+1} S_2(n,m), \tag{2}$$

where

$$S_2(n,m) = \frac{1}{m!} \sum_{j=0}^{m} (-1)^j \binom{m}{j} (m-j)^n,$$

denotes the second kind of Stirling numbers [5, 7] with $S_2(n,m) = 0$ for $n < m$.

There are some algorithms for computing Bernoulli numbers. One of them is Euler's formula

$$B_{2n} = \frac{(-1)^{n-1} 2n}{2^{2n}(2^{2n}-1)} T_n,$$

where $\{T_n\}$, known as Tangent numbers, are generated by

$$\tan t = \sum_{n=1}^{\infty} T_n \frac{t^{2n-1}}{(2n-1)!}.$$

In 2001, Akiyama and Tanigawa [1] (see also [13]) found an algorithm for computing $A_{n,0} := (-1)^n B_n$ without computing Tangent numbers as

$$A_{n+1,m} = (m+1)(A_{n,m} - A_{n,m+1}),$$

where $A_{0,m} = \frac{1}{m+1}$.

Later on, a modified version of the above-mentioned algorithm was proposed by Chen [4] for computing $C_{n,0} := B_n$ as

$$C_{n+1,m} = mC_{n,m} - (m+1)C_{n,m+1}$$

where $C_{0,m} = \frac{1}{m+1}$.

Bernoulli numbers have found various extensions such as poly-Bernoulli numbers, which are somehow connected to multiple zeta values. For recent extensions of poly-Bernoulli numbers see e.g. [3, 6, 8, 9, 14]. In [12], the author has defined a new family of poly-Bernoulli numbers in terms of Gaussian hypergeometric functions and obtained its basic properties. He has also presented an algorithm for computing Bernoulli numbers and polynomials and showed that poly-Bernoulli numbers are related to the certain regular values of the Euler-Zagiers multiple zeta function at non-positive integers of depth $p \geq 1$, i.e.

$$\zeta(s_1, s_2, \ldots, s_p) = \sum_{0<n_1<n_2<\cdots<n_p} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_p^{s_p}},$$

where $s_1, s_2, \ldots, s_p$ are positive integers with $s_p > 1$.

Another combinatorial aspect of Bernoulli numbers is that they have several symmetry properties with Cauchy numbers. The first kind of Cauchy numbers is defined by [5, 11]

$$C_n = \int_0^1 t(t-1)\cdots(t-n+1)\,\mathrm{d}t = n! \int_0^1 \binom{t}{n}\,\mathrm{d}t,$$

having the generating function

$$\frac{t}{\log(1+t)} = \sum_{n=0}^{\infty} C_n \frac{t^n}{n!},$$

and the second kind is defined by

$$\hat{C}_n = \int_{-1}^0 t(t-1)\cdots(t-n+1)\,\mathrm{d}t = n! \int_{-1}^0 \binom{t}{n}\,\mathrm{d}t.$$

Both C_n and $\hat{C}_n$ can be explicitly written as

$$C_n = (-1)^n \sum_{m=0}^{n} \frac{(-1)^m S_1(n,m)}{m+1} \quad \text{and} \quad \hat{C}_n = (-1)^n \sum_{m=0}^{n} \frac{S_1(n,m)}{m+1},$$

such that $S_1(n,m)$ are the first kind of Stirling numbers given by

$$(t)_n = t(t+1)\cdots(t+n-1) = \sum_{m=0}^{n} S_1(n,m) t^m,$$

where $S_1(n,m) = 0$ for $n < m$.

This paper is organized as follows: In the next section, we introduce a parametric type of Bernoulli polynomials and present basic properties of them in section 3. We also compute the Fourier expansion of the extended polynomials in section 4. As a valuable application of the extended polynomials, we introduce in section 5 an extension of the well-known Euler-Maclaurin quadrature formula and compare it with the ordinary case in detail.

2 A Parametric Type of Bernoulli Polynomials

If $p, q \in \mathbb{R}$, it is known that the Taylor expansion of the two functions $\mathrm{e}^{pt}\cos qt$ and $\mathrm{e}^{pt}\sin qt$ are respectively as follows [10]

$$\mathrm{e}^{pt}\cos qt = \sum_{k=0}^{\infty} C_k(p,q) \frac{t^k}{k!}, \tag{3}$$

and

$$\mathrm{e}^{pt}\sin qt=\sum_{k=0}^{\infty}S_k(p,q)\frac{t^k}{k!}, \tag{4}$$

where

$$C_k(p,q)=\sum_{j=0}^{[\frac{k}{2}]}(-1)^j\binom{k}{2j}p^{k-2j}q^{2j}, \tag{5}$$

and

$$S_k(p,q)=\sum_{j=0}^{[\frac{k-1}{2}]}(-1)^j\binom{k}{2j+1}p^{k-2j-1}q^{2j+1}. \tag{6}$$

By referring to relations (3)-(6), we can introduce two kinds of bivariate Bernoulli polynomials as

$$\frac{t\mathrm{e}^{pt}}{\mathrm{e}^t-1}\cos qt=\sum_{n=0}^{\infty}B_n^{(c)}(p,q)\frac{t^n}{n!}\quad(|t|<2\pi), \tag{7}$$

and

$$\frac{t\mathrm{e}^{pt}}{\mathrm{e}^t-1}\sin qt=\sum_{n=0}^{\infty}B_n^{(s)}(p,q)\frac{t^n}{n!}\quad(|t|<2\pi). \tag{8}$$

For instance, we have

$$\begin{aligned}
B_0^{(c)}(p,q)&=1,\\
B_1^{(c)}(p,q)&=p-\frac{1}{2},\\
B_2^{(c)}(p,q)&=p^2-p-q^2+\frac{1}{6},\\
B_3^{(c)}(p,q)&=p^3-\frac{3}{2}p^2+(\frac{1}{2}-3q^2)p+\frac{3}{2}q^2,\\
B_4^{(c)}(p,q)&=p^4-2p^3+(1-6q^2)p^2+6q^2p+q^4-q^2-\frac{1}{30},\\
B_5^{(c)}(p,q)&=p^5-\frac{5}{2}p^4+(\frac{5}{3}-10q^2)p^3+15q^2p^2+(5q^4-5q^2-\frac{1}{6})p-\frac{5}{2}q^4,\\
B_6^{(c)}(p,q)&=p^6-3p^5+(\frac{5}{2}-15q^2)p^4+30q^2p^3+(15q^4-15q^2-\frac{1}{2})p^2-15q^4p\\
&\quad-q^6+\frac{5}{2}q^4+\frac{1}{2}q^2+\frac{1}{42},
\end{aligned}$$

and

$$
\begin{aligned}
&B_0^{(s)}(p,q)=0,\\
&B_1^{(s)}(p,q)=q,\\
&B_2^{(s)}(p,q)=2qp-q,\\
&B_3^{(s)}(p,q)=3qp^2-3qp-q^3+\frac{1}{2}q,\\
&B_4^{(s)}(p,q)=4qp^3-6qp^2+(2q-4q^3)p+2q^3,\\
&B_5^{(s)}(p,q)=5qp^4-10qp^3+(5q-10q^3)p^2+10q^3p+q^5-\frac{5}{3}q^3-\frac{1}{6}q,\\
&B_6^{(s)}(p,q)=6qp^5-15qp^4+(10q-20q^3)p^3+30q^3p^2+(6q^5-10q^3-q)p-3q^5.
\end{aligned}
$$

3 Some Basic Properties of $B_n^{(c)}(p,q)$ and $B_n^{(s)}(p,q)$.

3.1. *$B_n^{(c)}(p,q)$ and $B_n^{(s)}(p,q)$ can be represented in terms of Bernoulli numbers as follows*

$$B_n^{(c)}(p,q)=\sum_{k=0}^{n}\binom{n}{k}B_kC_{n-k}(p,q), \tag{9}$$

and

$$B_n^{(s)}(p,q)=\sum_{k=0}^{n}\binom{n}{k}B_kS_{n-k}(p,q). \tag{10}$$

Proof. By noting the general identity

$$\left(\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\right)\left(\sum_{k=0}^{\infty}b_k\frac{t^k}{k!}\right)=\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{k}{j}a_jb_{k-j}\right)\frac{t^k}{k!},$$

we have

$$
\begin{aligned}
\sum_{k=0}^{\infty}B_k^{(c)}(p,q)\frac{t^k}{k!}=\frac{t}{\mathrm{e}^t-1}\left(\mathrm{e}^{pt}\cos qt\right)&=\left(\sum_{k=0}^{\infty}B_k\frac{t^k}{k!}\right)\left(\sum_{k=0}^{\infty}C_k(p,q)\frac{t^k}{k!}\right)\\
&=\sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{k}{j}B_jC_{k-j}(p,q)\right)\frac{t^k}{k!},
\end{aligned}
$$

which proves (9). The proof of (10) is similar. □

3.2. *For every* $n \in \mathbb{Z}^+$ *we have*

$$B_n^{(c)}(1-p,q) = (-1)^n B_n^{(c)}(p,q), \tag{11}$$

and

$$B_n^{(s)}(1-p,q) = (-1)^{n+1} B_n^{(s)}(p,q). \tag{12}$$

Proof. Applying the generating function (7) gives

$$\sum_{n=0}^{\infty} B_n^{(c)}(1-p,q)\frac{t^n}{n!} = \frac{t\mathrm{e}^{(1-p)t}}{\mathrm{e}^t-1}\cos qt,$$

as well as

$$\sum_{n=0}^{\infty}(-1)^n B_n^{(c)}(p,q)\frac{t^n}{n!} = \frac{-t\mathrm{e}^{-pt}}{\mathrm{e}^{-t}-1}\cos(-qt) = \frac{t\mathrm{e}^{(1-p)t}}{\mathrm{e}^t-1}\cos qt.$$

Similarly, property (12) can be proved. □

Corollary 1. *Relations* (11) *and* (12) *imply that*

$$B_{2n+1}^{(c)}(\frac{1}{2},q) = 0,$$

and

$$B_{2n}^{(s)}(\frac{1}{2},q) = 0.$$

3.3. *For every* $n \in \mathbb{N}$, *the following identities hold*

$$B_n^{(c)}(1+p,q) - B_n^{(c)}(p,q) = nC_{n-1}(p,q), \tag{13}$$

and

$$B_n^{(s)}(1+p,q) - B_n^{(s)}(p,q) = nS_{n-1}(p,q). \tag{14}$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{\infty} B_n^{(c)}(1+p,q)\frac{t^n}{n!} &= \frac{t\mathrm{e}^{pt}(\mathrm{e}^t-1+1)}{\mathrm{e}^t-1}\cos qt = t\mathrm{e}^{pt}\cos qt + \frac{t\mathrm{e}^{pt}}{\mathrm{e}^t-1}\cos qt\\ &= \sum_{n=0}^{\infty} C_n(p,q)\frac{t^{n+1}}{n!} + \sum_{n=0}^{\infty} B_n^{(c)}(p,q)\frac{t^n}{n!}\\ &= \sum_{n=1}^{\infty} nC_{n-1}(p,q)\frac{t^n}{n!} + \sum_{n=0}^{\infty} B_n^{(c)}(p,q)\frac{t^n}{n!},\end{aligned}$$

which proves (13). The proof of (14) is similar. □

Corollary 2. *Relations* (13) *and* (14) *first imply that*

$$B^{(c)}_{2n+1}(1,q) - B^{(c)}_{2n+1}(0,q) = (2n+1)(-1)^n q^{2n},$$

and

$$B^{(s)}_{2n}(1,q) - B^{(s)}_{2n}(0,q) = 2n(-1)^{n+1} q^{2n-1}.$$

Hence, combining proposition 3.2 respectively yields

$$B^{(c)}_{2n+1}(1,q) = -B^{(c)}_{2n+1}(0,q) = \frac{2n+1}{2}(-1)^n q^{2n},$$

and

$$B^{(s)}_{2n}(1,q) = -B^{(s)}_{2n}(0,q) = n(-1)^{n+1} q^{2n-1}.$$

3.4. *For every* $n \in \mathbb{Z}^+$ *the following identities hold*

$$B^{(c)}_n(p+r,q) = \sum_{k=0}^{n} \binom{n}{k} B^{(c)}_k(p,q) r^{n-k}, \tag{15}$$

and

$$B^{(s)}_n(p+r,q) = \sum_{k=0}^{n} \binom{n}{k} B^{(s)}_k(p,q) r^{n-k}. \tag{16}$$

Proof. Apply (7) to obtain

$$\sum_{n=0}^{\infty} B^{(c)}_n(p+r,q)\frac{t^n}{n!} = \left(\frac{t\mathrm{e}^{pt}}{\mathrm{e}^t-1}\cos qt\right)\mathrm{e}^{rt} = \left(\sum_{n=0}^{\infty} B^{(c)}_n(p,q)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} r^n\frac{t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k} B^{(c)}_k(p,q) r^{n-k}\right)\frac{t^n}{n!},$$

which proves (15). The result (16) can be similarly proved. □

3.5. *We have*

$$\sum_{k=0}^{n} \binom{n+1}{k} B^{(c)}_k(p,q) = (n+1)C_n(p,q), \tag{17}$$

and

$$\sum_{k=0}^{n} \binom{n+1}{k} B^{(s)}_k(p,q) = (n+1)S_n(p,q). \tag{18}$$

Proof. From (15), one can conclude that

$$B_{n+1}^{(c)}(p+1,q) - B_{n+1}^{(c)}(p,q) = \sum_{k=0}^{n} \binom{n+1}{k} B_k^{(c)}(p,q).$$

Hence, by referring to (13), the result (17) is derived. The proof of (18) can be done in a similar way. □

Corollary 3. *Relations* (17) *and* (18) *imply that*

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k^{(c)}(0,q) = (n+1)q^n \cos n\frac{\pi}{2} = \begin{cases} (-1)^m(2m+1)q^{2m} & n = 2m \text{ even}, \\ 0 & n = 2m+1 \text{ odd}, \end{cases}$$

and

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k^{(s)}(0,q) = (n+1)q^n \sin n\frac{\pi}{2} = \begin{cases} 0 & n = 2m \text{ even}, \\ (-1)^m(2m+2)q^{2m+1} & n = 2m+1 \text{ odd}. \end{cases}$$

3.6. *For every* $n \in \mathbb{N}$, *the following partial differential equations hold*

$$\frac{\partial}{\partial p} B_n^{(c)}(p,q) = nB_{n-1}^{(c)}(p,q), \tag{19}$$

$$\frac{\partial}{\partial q} B_n^{(c)}(p,q) = -nB_{n-1}^{(s)}(p,q), \tag{20}$$

$$\frac{\partial}{\partial p} B_n^{(s)}(p,q) = nB_{n-1}^{(s)}(p,q), \tag{21}$$

and

$$\frac{\partial}{\partial q} B_n^{(s)}(p,q) = nB_{n-1}^{(c)}(p,q). \tag{22}$$

Proof. Relation (7) yields

$$\sum_{n=1}^{\infty} \frac{\partial B_n^{(c)}(p,q)}{\partial p} \frac{t^n}{n!} = \frac{t^2 e^{pt}}{e^t - 1} \cos qt = \sum_{n=0}^{\infty} B_n^{(c)}(p,q) \frac{t^{n+1}}{n!}$$
$$= \sum_{n=1}^{\infty} B_{n-1}^{(c)}(p,q) \frac{t^n}{(n-1)!} = \sum_{n=1}^{\infty} nB_{n-1}^{(c)}(p,q) \frac{t^n}{n!},$$

proving (19). Other equations (20), (21) and (22) can be similarly derived. □

Corollary 4. *By combining the above results and proposition 3.2 and corollary 2, we obtain*

$$\int_0^1 B_{2n}^{(c)}(p,q)\ \mathrm{d}p = (-1)^n q^{2n},$$

$$\int_0^1 B_{2n+1}^{(c)}(p,q)\ \mathrm{d}p = 0,$$

$$\int_0^1 B_{2n}^{(s)}(p,q)\ \mathrm{d}p = 0,$$

and

$$\int_0^1 B_{2n+1}^{(s)}(p,q)\ \mathrm{d}p = (-1)^n q^{2n+1}.$$

3.7. *If* $B_n^{(c)}(p,q)$ *and* $B_n^{(s)}(p,q)$ *are sorted in terms of the variable* p*, then they are polynomials of degree* n *and* $n-1$ *respectively, such that we have*

$$B_n^{(c)}(p,q) = p^n - \frac{n}{2}p^{n-1} + \cdots\ , \tag{23}$$

and

$$B_n^{(s)}(p,q) = nqp^{n-1} - \binom{n}{2}qp^{n-2} + \cdots\ . \tag{24}$$

Also, if they are sorted in terms of the variable q*, then*

$$B_n^{(c)}(p,q) = \begin{cases} (-1)^{\frac{n-1}{2}} n(p-\frac{1}{2})q^{n-1} + (-1)^{\frac{n+1}{2}}\binom{n}{3}(p^3 - \frac{3}{2}p^2 + \frac{1}{2}p)q^{n-3} + \cdots\ (n\ odd), \\ (-1)^{\frac{n}{2}} q^n + (-1)^{\frac{n+2}{2}}\binom{n}{2}(p^2 - p + \frac{1}{6})q^{n-2} + \cdots\ (n\ even), \end{cases} \tag{25}$$

and

$$B_n^{(s)}(p,q) = \begin{cases} (-1)^{\frac{n+2}{2}} n(p-\frac{1}{2})q^{n-1} + (-1)^{\frac{n}{2}}\binom{n}{3}(p^3 - \frac{3}{2}p^2 + \frac{1}{2}p)q^{n-3} + \cdots\ (n\ even), \\ (-1)^{\frac{n-1}{2}} q^n + (-1)^{\frac{n+1}{2}}\binom{n}{2}(p^2 - p + \frac{1}{6})q^{n-2} + \cdots\ (n\ odd). \end{cases} \tag{26}$$

Proof. We first prove (23) by induction. It is known from (17) that

$$B_0^{(c)}(p,q) = 1,\ B_1^{(c)}(p,q) = p - \frac{1}{2} \quad \text{and} \quad B_2^{(c)}(p,q) = p^2 - p - q^2 + \frac{1}{6}.$$

Therefore (23) holds for $n=0,1,2$. Now assume that it is valid for $n-1$. By referring to (19), we have

$$\frac{\partial}{\partial p}B_n^{(c)}(p,q)=np^{n-1}-\frac{n(n-1)}{2}p^{n-2}+\cdots.$$

To complete the proof, it is enough to integrate from the above equation with respect to the variable p to get the result (23). By referring to relation (22), the result (24) can be similarly derived.
To prove (25), suppose that it first holds for $0,1,\cdots,n-1$. If $n=2m$, then from (17) we have

$$B_{2m}^{(c)}(p,q)=-\frac{1}{2m+1}\sum_{k=0}^{2m-1}\binom{2m+1}{k}B_k^{(c)}(p,q)+\sum_{k=0}^{m}(-1)^k\binom{2m}{2k}p^{2m-2k}q^{2k}. \qquad (27)$$

Hence, the coefficient of q^{2m} in the right hand side of (27) is equal to

$$(-1)^m\binom{2m}{2m}p^{2m-2m}=(-1)^m,$$

and the coefficient of q^{2m-2} is equal to

$$-\frac{1}{2m+1}\left(\binom{2m+1}{2m-1}(-1)^{m-1}(2m-1)(p-\frac{1}{2})+\binom{2m+1}{2m-2}(-1)^{m-1}\right)$$
$$+(-1)^{m-1}\binom{2m}{2m-2}p^2=(-1)^{m+1}\binom{2m}{2}(p^2-p+\frac{1}{6}).$$

So, (25) is true for $n=2m$. In the second case, taking $n=2m+1$ in (17) gives

$$B_{2m+1}^{(c)}(p,q)=-\frac{1}{2m+2}\sum_{k=0}^{2m}\binom{2m+2}{k}B_k^{(c)}(p,q)+\sum_{k=0}^{m}(-1)^k\binom{2m+1}{2k}p^{2m+1-2k}q^{2k}. \qquad (28)$$

Hence, the coefficient of q^{2m} in the right hand side of (28) is equal to

$$\frac{-1}{2m+2}\binom{2m+2}{2m}(-1)^m+(-1)^m\binom{2m+1}{2m}p=(-1)^m(2m+1)(p-\frac{1}{2}),$$

and the coefficient of q^{2m-2} is equal to

$$-\frac{1}{2m+2}\left(\binom{2m+2}{2m}(-1)^{m+1}\binom{2m}{2}(p^2-p+\frac{1}{6})+\binom{2m+2}{2m-1}(-1)^{m-1}(2m-1)(p-\frac{1}{2})\right.$$
$$\left.+\binom{2m+2}{2m-2}(-1)^{m-1}\right)+(-1)^{m-1}\binom{2m+1}{2m-2}p^3=(-1)^{m+1}\binom{2m+1}{3}(p^3-\frac{3}{2}p^2+\frac{1}{2}p),$$

which completes the proof of (25). By combining (22) and (25), we can also obtain the result (26). □

3.8. *The following identities hold*

$$B_n^{(c)}(p,q) = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n}{2k} B_{n-2k}^{(c)}(p,0) q^{2k}, \tag{29}$$

and

$$B_n^{(s)}(p,q) = \sum_{k=0}^{[\frac{n-1}{2}]} (-1)^k \binom{n}{2k+1} B_{n-2k-1}^{(c)}(p,0) q^{2k+1}, \tag{30}$$

in which $B_{n-2k}^{(c)}(p,0) = B_{n-2k}(p)$ *and* $B_{n-2k-1}^{(c)}(p,0) = B_{n-2k-1}(p)$ *are usual Bernoulli polynomials.*

Proof. According to (20) and (22), first we have

$$\frac{\partial^{2k}}{\partial q^{2k}} B_n^{(c)}(p,q) = (-1)^k \frac{n!}{(n-2k)!} B_{n-2k}^{(c)}(p,q) \quad \text{for} \quad k = 0,1,\cdots,[\frac{n}{2}],$$

and

$$\frac{\partial^{2k+1}}{\partial q^{2k+1}} B_n^{(c)}(p,q) = (-1)^{k+1} \frac{n!}{(n-2k-1)!} B_{n-2k-1}^{(s)}(p,q) \quad \text{for} \quad k = 0,1,\cdots,[\frac{n-2}{2}],$$

because $B_n^{(c)}(p,q)$ is a polynomial of degree n for even n and of degree $n-1$ for odd n in terms of the variable q according to the proposition 3.7. The Taylor expansion of $B_n^{(c)}(p,q)$ gives

$$B_n^{(c)}(p,q+h) = \sum_{k=0}^{n} \frac{1}{k!} \frac{\partial^k}{\partial q^k} B_n^{(c)}(p,q) h^k,$$

in which $h \in \mathbb{R}$. Since $B_n^{(s)}(p,0) = 0$ for every $n \in \mathbb{Z}^+$, by replacing $q = 0$ and $h = q$, we obtain the relation (29). In a similar way, equality (30) can be derived. □

3.9. *If* $m \in \mathbb{N}$ *and* $n \in \mathbb{Z}^+$, *then we have*

$$B_n^{(c)}(mp,q) = m^{n-1} \sum_{k=0}^{m-1} B_n^{(c)}(p + \frac{k}{m}, \frac{q}{m}), \tag{31}$$

and

$$B_n^{(s)}(mp,q) = m^{n-1} \sum_{k=0}^{m-1} B_n^{(s)}(p + \frac{k}{m}, \frac{q}{m}). \tag{32}$$

Proof. To prove (31), it is enough to consider the relation

$$\sum_{n=0}^{\infty} B_n^{(c)}(p+\frac{k}{m},\frac{q}{m})\frac{t^n}{n!} = \frac{t\mathrm{e}^{(p+\frac{k}{m})t}}{\mathrm{e}^t-1}\cos(\frac{q}{m}t),$$

and then take a sum from both sides of the above equation to obtain

$$\sum_{k=0}^{m-1}\left(\sum_{n=0}^{\infty} B_n^{(c)}(p+\frac{k}{m},\frac{q}{m})\frac{t^n}{n!}\right) = \frac{t\mathrm{e}^{pt}}{\mathrm{e}^t-1}\cos(\frac{q}{m}t)\sum_{k=0}^{m-1}\left(\mathrm{e}^{\frac{t}{m}}\right)^k$$
$$= m\frac{\frac{t}{m}\mathrm{e}^{mp\frac{t}{m}}}{\mathrm{e}^{\frac{t}{m}}-1}\cos(q\frac{t}{m}) = \sum_{n=0}^{\infty} m^{1-n}B_n^{(c)}(mp,q)\frac{t^n}{n!}.$$

In a similar way, equality (32) can be proved. □

For $m=2$, relations (31) and (32) respectively yield

$$B_{2n}^{(c)}(\frac{1}{2},q) = 2^{1-2n}B_{2n}^{(c)}(0,2q) - B_{2n}^{(c)}(0,q),$$

and

$$B_{2n+1}^{(s)}(\frac{1}{2},q) = 2^{-2n}B_{2n+1}^{(s)}(0,2q) - B_{2n+1}^{(s)}(0,q).$$

3.10. *For every $n\in\mathbb{N}$ and $q\in\mathbb{R}$, the two following propositions are valid:*

$\mathcal{P}_n$ *: The function $p\mapsto(-1)^nB_{2n-1}^{(c)}(p,q)$ is positive on $(0,\frac{1}{2})$ and negative on $(\frac{1}{2},1)$. Moreover, $p=\frac{1}{2}$ is a unique simple root on $(0,1)$, i.e. the aforesaid function has no zero in the intervals $(0,\frac{1}{2})$ and $(\frac{1}{2},1)$.*

$\mathcal{Q}_n$ *: The function $p\mapsto(-1)^nB_{2n}^{(c)}(p,q)$ is strictly increasing on $[0,\frac{1}{2}]$ and strictly decreasing on $[\frac{1}{2},1]$ and always takes a positive value at $p=\frac{1}{2}$.*

Proof. The proposition $\mathcal{P}_1$ is clear, because $-B_1^{(c)}(p,q) = -(p-\frac{1}{2}) = -p+\frac{1}{2}$. Now define $f(p)=(-1)^nB_{2n}^{(c)}(p,q)$ to get $f'(p)=2n(-1)^nB_{2n-1}^{(c)}(p,q)$. By referring to $\mathcal{P}_n$, we see that f is strictly increasing on $[0,\frac{1}{2}]$ and decreasing on $[\frac{1}{2},1]$. Moreover, since $\int_0^1 f(p)\,\mathrm{d}p = q^{2n}\geq 0$ (by corollary 4) and $B_{2n}^{(c)}(1-p,q)=B_{2n}^{(c)}(p,q)$ (from proposition 3.2), one can conclude that $f(\frac{1}{2})>0$.

Finally define $g(p)=(-1)^{n+1}B_{2n+1}^{(c)}(p,q)$ to get $g'(p)=-(2n+1)(-1)^nB_{2n}^{(c)}(p,q)$. Since $B_{2n}^{(c)}(0,q)=B_{2n}^{(c)}(1,q)$, by noting $\mathcal{Q}_n$, only one of the following cases occurs:

i) $\alpha\in(0,\frac{1}{2})$ and $\beta\in(\frac{1}{2},1)$ exist such that

$$g'(\alpha)=g'(\beta)=0 \text{ and } \forall p\in(\alpha,\beta),\ g'(p)<0 \text{ and } \forall p\in[0,\alpha)\cup(\beta,1],\ g'(p)>0.$$

ii) $g'(0)=g'(1)=0$ and $\forall p\in(0,1),\ g'(p)<0$.

iii) $\forall p\in[0,1],\ g'(p)<0$.

In the first case i), by referring to corollary 2 we have

$$A=g(0)=(-1)^{n+1}B_{2n+1}^{(c)}(0,q)=\frac{2n+1}{2}q^{2n}\geq 0.$$

Therefore $g(1)=-A\leq 0$ and g takes the following table of variations

p	0			α		$\frac{1}{2}$		β			1
$g'(p)$			$+$	0		$-$		0	$+$		
$g(p)$	$A\geq 0$	$\nearrow$		$\frown$	$\searrow$	0	$\searrow$	$\smile$		$\nearrow$	$-A\leq 0$

As $g(\frac{1}{2})=0$ (by corollary 1) and $g'(\frac{1}{2})\neq 0$, $p=\frac{1}{2}$ is a simple root of g. We can similarly observe that the two other cases also hold. So the proof of $\mathcal{P}_{n+1}$ is complete. □

3.11. *For every* $n\in\mathbb{Z}^+$ *and* $q\in\mathbb{R}$ *we have*

$$\sup_{p\in[0,1]}|B_{2n}^{(c)}(p,q)|=\max\{|B_{2n}^{(c)}(0,q)|,\ |B_{2n}^{(c)}(\frac{1}{2},q)|\},\tag{33}$$

and

$$\sup_{p\in[0,1]}|B_{2n+1}^{(c)}(p,q)|\leq\frac{2n+1}{2}\max\{|B_{2n}^{(c)}(0,q)|,\ |B_{2n}^{(c)}(\frac{1}{2},q)|\}.\tag{34}$$

Proof. The result (33) is clear by referring to propositions 3.2 and 3.10. To prove (34), if $p\in[0,\frac{1}{2}]$ then we have

$$B_{2n+1}^{(c)}(p,q)=B_{2n+1}^{(c)}(p,q)-B_{2n+1}^{(c)}(\frac{1}{2},q)=(2n+1)\int_{\frac{1}{2}}^{p}B_{2n}^{(c)}(t,q)\ dt.$$

Therefore

$$\begin{aligned}|B_{2n+1}^{(c)}(p,q)|&\leq(2n+1)\int_{p}^{\frac{1}{2}}|B_{2n}^{(c)}(t,q)|\ dt\leq(2n+1)(\frac{1}{2}-p)\sup_{t\in[p,\frac{1}{2}]}|B_{2n}^{(c)}(t,q)|\\&\leq(2n+1)(\frac{1}{2}-p)\max\{|B_{2n}^{(c)}(0,q)|,\ |B_{2n}^{(c)}(\frac{1}{2},q)|\},\end{aligned}$$

which is equivalent to

$$\sup_{p\in[0,\frac{1}{2}]}|B_{2n+1}^{(c)}(p,q)|\leq\frac{2n+1}{2}\max\{|B_{2n}^{(c)}(0,q)|,\ |B_{2n}^{(c)}(\frac{1}{2},q)|\}.$$

On the other hand, $B_{2n+1}^{(c)}(1-p,q)=-B_{2n+1}^{(c)}(p,q)$ completes the proof of (34). □

3.12. *For every $n \in \mathbb{N}$ and $q > 0$, the two following propositions are valid:*

$\mathcal{P}_n$: *The function $p \mapsto (-1)^n B_{2n}^{(s)}(p,q)$ is positive on $[0, \frac{1}{2})$ and negative on $(\frac{1}{2}, 1]$. Moreover, $p = \frac{1}{2}$ is a unique simple root on $[0,1]$, i.e. the aforesaid function has no zero in the intervals $[0, \frac{1}{2})$ and $(\frac{1}{2}, 1]$.*

$\mathcal{Q}_n$: *The function $p \mapsto (-1)^n B_{2n+1}^{(s)}(p,q)$ is strictly increasing on $[0, \frac{1}{2}]$ and strictly decreasing on $[\frac{1}{2}, 1]$ and always takes a positive value at $p = \frac{1}{2}$.*

Proof. The proposition $\mathcal{P}_1$ is clear, because $-B_2^{(s)}(p,q) = -q(2p-1) = q(1-2p)$. Now define $f(p) = (-1)^n B_{2n+1}^{(s)}(p,q)$ to get $f'(p) = (2n+1)(-1)^n B_{2n}^{(s)}(p,q)$. By noting $\mathcal{P}_n$, we see that f is strictly increasing on $[0, \frac{1}{2}]$ and decreasing on $[\frac{1}{2}, 1]$. Moreover, since $\int_0^1 f(p)\, \mathrm{d}p = q^{2n+1} > 0$ (by corollary 4) and $B_{2n+1}^{(s)}(1-p,q) = B_{2n+1}^{(s)}(p,q)$ (from proposition 3.2), one can conclude that $f(\frac{1}{2}) > 0$.
Finally define $g(p) = (-1)^{n+1} B_{2n+2}^{(s)}(p,q)$ to get $g'(p) = -(2n+2)(-1)^n B_{2n+1}^{(s)}(p,q)$. Since $B_{2n+1}^{(s)}(0,q) = B_{2n+1}^{(s)}(1,q)$, by noting $\mathcal{Q}_n$, only one of the three following cases occurs:

i) $\alpha \in (0, \frac{1}{2})$ and $\beta \in (\frac{1}{2}, 1)$ exist such that

$$g'(\alpha) = g'(\beta) = 0 \text{ and } \forall p \in (\alpha, \beta),\ g'(p) < 0 \text{ and } \forall p \in [0, \alpha) \cup (\beta, 1],\ g'(p) > 0.$$

ii) $g'(0) = g'(1) = 0$ and $\forall p \in (0,1),\ g'(p) < 0.$

iii) $\forall p \in [0,1],\ g'(p) < 0.$

In the first case i), by referring to corollary 2, we have

$$A^* = g(0) = (-1)^{n+1} B_{2n+2}^{(s)}(0,q) = (n+1)q^{2n+1} > 0.$$

Therefore $g(1) = -A^* < 0$ and g takes the following table of variations

p	0		α		$\frac{1}{2}$		β		1
$g'(p)$		$+$	0		$-$		0	$+$	
$g(p)$	$A^* > 0$	$\nearrow$	$\frown$	$\searrow$	0	$\searrow$	$\smile$	$\nearrow$	$-A^* < 0$

As $g(\frac{1}{2}) = 0$ (by corollary 1) and $g'(\frac{1}{2}) \neq 0$, then $p = \frac{1}{2}$ is a simple root of function g. Similarly, we can observe that the two other cases also hold. □

Corollary 5. *For every $n \in \mathbb{N}$ and $q \in \mathbb{R}$ we have*

$$\sup_{p \in [0,1]} |B_{2n+1}^{(s)}(p,q)| = \max\{|B_{2n+1}^{(s)}(0,q)|,\ |B_{2n+1}^{(s)}(\frac{1}{2},q)|\},$$

and

$$\sup_{p \in [0,1]} |B_{2n}^{(s)}(p,q)| \leq n \max\{|B_{2n-1}^{(s)}(0,q)|,\ |B_{2n-1}^{(s)}(\frac{1}{2},q)|\}.$$

3.13. *Let m and n be two positive integers and*

$$I^{(c)} = \int_0^1 B_m^{(c)}(p,q) B_n^{(c)}(p,q)\ \mathrm{d}p.$$

If $m+n$ is odd then $I^{(c)} = 0$ and if it is even then

$$I^{(c)} = \sum_{k=0}^{m+n} \frac{1}{(k+1)!} \left(\sum_{j=A}^{B} \binom{k}{j} \frac{n!m!}{(n-j)!(m-k+j)!} B_{n-j}^{(c)}(0,q) B_{m-k+j}^{(c)}(0,q) \right),$$

where $A = \max\{0, k-m\}$ and $B = \min\{n, k\}$.

Proof. First, suppose that $m+n$ is odd. By using (11) we have

$$I^{(c)} = \int_0^1 B_m^{(c)}(1-p,q) B_n^{(c)}(1-p,q)\ \mathrm{d}p = (-1)^{m+n} \int_0^1 B_m^{(c)}(p,q) B_n^{(c)}(p,q)\ \mathrm{d}p = -I^{(c)}.$$

Now, assume that $m+n$ is even. Since $\deg_p\big(B_m^{(c)} B_n^{(c)}\big) = m+n$ (from proposition 3.7), by referring to (19) we obtain

$$\begin{aligned}
B_m^{(c)}(p,q) B_n^{(c)}(p,q) &= \sum_{k=0}^{m+n} \left(\frac{\partial^k}{\partial p^k} \big(B_m^{(c)}(p,q) B_n^{(c)}(p,q) \big) \right) \Big|_{p=0} \frac{p^k}{k!} \\
&= \sum_{k=0}^{m+n} \left(\sum_{j=0}^{k} \binom{k}{j} \left(\frac{\partial^j}{\partial p^j} B_n^{(c)}(p,q) \frac{\partial^{k-j}}{\partial p^{k-j}} B_m^{(c)}(p,q) \right) \Big|_{p=0} \right) \frac{p^k}{k!} \\
&= \sum_{k=0}^{m+n} \left(\sum_{j=A}^{B} \binom{k}{j} \frac{n!m!}{(n-j)!(m-k+j)!} B_{n-j}^{(c)}(0,q) B_{m-k+j}^{(c)}(0,q) \right) \frac{p^k}{k!},
\end{aligned}$$

which leads to the second result. □

Corollary 6. *Let m and n be two positive integers and*

$$I^{(s)} = \int_0^1 B_m^{(s)}(p,q) B_n^{(s)}(p,q)\ \mathrm{d}p.$$

If $m+n$ is odd then $I^{(s)} = 0$ and if $m+n$ is even then

$$I^{(s)} = \sum_{k=0}^{m+n-2} \frac{1}{(k+1)!} \left(\sum_{j=A}^{B} \binom{k}{j} \frac{n!m!}{(n-j)!(m-k+j)!} B_{n-j}^{(s)}(0,q) B_{m-k+j}^{(s)}(0,q) \right),$$

where $A = \max\{0, k-m\}$ and $B = \min\{n, k\}$.

4 Fourier expansions of $B_n^{(c)}(p,q)$ and $B_n^{(s)}(p,q)$

The Fourier series of a periodic function f on $[0,L]$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cos(\frac{2k\pi}{L}x)+b_k\sin(\frac{2k\pi}{L}x)\right),$$

where

$$a_0=\frac{2}{L}\int_0^L f(x)\ \mathrm{d}x,$$
$$a_k=\frac{2}{L}\int_0^L f(x)\cos(\frac{2k\pi}{L}x)\ \mathrm{d}x,$$

and

$$b_k=\frac{2}{L}\int_0^L f(x)\sin(\frac{2k\pi}{L}x)\ \mathrm{d}x,$$

which can be also extend to the complex coefficients so that we have

$$f(x)=\sum_{k=-\infty}^{\infty}c_k\mathrm{e}^{\frac{2\mathrm{i}k\pi}{L}x},$$

in which

$$c_k=\frac{1}{L}\int_0^L f(x)\mathrm{e}^{\frac{-2\mathrm{i}k\pi}{L}x}\ \mathrm{d}x.$$

By periodically extending the restrictions of the introduced parametric Bernoulli polynomials to $p\in[0,1)$, we would encounter with periodic piecewise continuous functions so that for every real p and q we can define

$$\tilde{B}_n^{(c)}(p,q)=B_n^{(c)}(\{p\},q),$$
$$\tilde{B}_n^{(s)}(p,q)=B_n^{(s)}(\{p\},q),$$

where $\{p\}=p-[p]$ is the fractional part of the real p.

Theorem 4.1. *Let $q\in\mathbb{R}$. Then for any $p\in(0,1)$*

$$B_1^{(c)}(p,q)=p-\frac{1}{2}=-\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2\pi kp)}{k}, \tag{35}$$

and for every $n\in\mathbb{N}$ we respectively have

$$B_{2n}^{(c)}(p,q)=(-1)^n q^{2n}+\sum_{k=1}^{\infty}a_{k,n}\cos(2\pi kp),\quad p\in[0,1], \tag{36}$$

where

$$a_{k,n} = 2(2n)!(-1)^{n+1}\sum_{j=1}^{n}\frac{q^{2n-2j}}{(2n-2j)!(2\pi k)^{2j}},$$

and

$$B_{2n+1}^{(c)}(p,q) = \sum_{k=1}^{\infty} b_{k,n}\sin(2\pi kp), \quad p \in (0,1), \tag{37}$$

where

$$b_{k,n} = (-1)^{n+1}(2n+1)\bigg(\frac{q^{2n}}{\pi k} + 2(2n)!\sum_{j=1}^{n}\frac{q^{2n-2j}}{(2n-2j)!(2\pi k)^{2j+1}}\bigg).$$

Proof. First, let us consider $\tilde{B}_1^{(c)}$. It is clear that

$$c_0(\tilde{B}_1^{(c)}) = \int_0^1 B_1^{(c)}(p,q)\ \mathrm{d}p = \int_0^1 (p-\frac{1}{2})\ \mathrm{d}p = 0,$$

and for $k \in \mathbb{Z}\backslash\{0\}$ we have

$$c_k(\tilde{B}_1^{(c)}) = \int_0^1 B_1^{(c)}(p,q)\mathrm{e}^{-2\mathrm{i}\pi kp}\ \mathrm{d}p = \int_0^1 (p-\frac{1}{2})\mathrm{e}^{-2\mathrm{i}\pi kp}\ \mathrm{d}p = \frac{-1}{2\mathrm{i}\pi k}. \tag{38}$$

Since $B_1^{(c)}(0,q) \neq B_1^{(c)}(1,q)$, according to Dirichlet's conditions, it can be concluded for every $p \in \mathbb{R}\backslash\mathbb{Z}$ that

$$\tilde{B}_1^{(c)}(p,q) = \sum_{k\in\mathbb{Z}} c_k(\tilde{B}_1^{(c)})\mathrm{e}^{2\mathrm{i}\pi kp} = \sum_{k\in\mathbb{Z}\backslash\{0\}}\frac{-1}{2\mathrm{i}\pi k}\mathrm{e}^{2\mathrm{i}\pi kp} = -\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2\pi kp)}{k},$$

where we use $c_{-k}(\tilde{B}_1^{(c)}) = -c_k(\tilde{B}_1^{(c)})$, which proves (35).

We now consider the case $\tilde{B}_{2n}^{(c)}$. According to corollary 4 we have

$$c_0(\tilde{B}_{2n}^{(c)}) = \int_0^1 B_{2n}^{(c)}(p,q)\ \mathrm{d}p = (-1)^n q^{2n},$$

and for $k \in \mathbb{Z}\backslash\{0\}$

$$\begin{aligned} c_k(\tilde{B}_{2n}^{(c)}) = \int_0^1 B_{2n}^{(c)}(p,q)\mathrm{e}^{-2\mathrm{i}\pi kp}\ \mathrm{d}p &= \frac{2n}{2\mathrm{i}\pi k}\int_0^1 B_{2n-1}^{(c)}(p,q)\mathrm{e}^{-2\mathrm{i}\pi kp}\ \mathrm{d}p \\ &= \frac{n}{\mathrm{i}\pi k}c_k(\tilde{B}_{2n-1}^{(c)}), \end{aligned} \tag{39}$$

where we have used $B_{2n}^{(c)}(0,q) = B_{2n}^{(c)}(1,q)$ in proposition 3.2. Similarly, we can find that

$$c_0(\tilde{B}_{2n+1}^{(c)}) = 0 \quad \text{and} \quad c_k(\tilde{B}_{2n+1}^{(c)}) = \frac{2n+1}{2\mathrm{i}\pi k}\Big((-1)^{n+1}q^{2n} + c_k(\tilde{B}_{2n}^{(c)})\Big). \tag{40}$$

Now, for every $n \in \mathbb{N}$ and $k \in \mathbb{Z}\backslash\{0\}$ we show that

$$c_k(\tilde{B}_{2n}^{(c)}) = (-1)^{n+1}(2n)!\sum_{j=1}^{n}\frac{q^{2n-2j}}{(2n-2j)!(2\pi k)^{2j}}, \tag{41}$$

and

$$c_k(\tilde{B}_{2n+1}^{(c)}) = \frac{(-1)^{n+1}(2n+1)}{\mathrm{i}}\Big(\frac{q^{2n}}{2\pi k} + (2n)!\sum_{j=1}^{n}\frac{q^{2n-2j}}{(2n-2j)!(2\pi k)^{2j+1}}\Big). \tag{42}$$

Since $c_k(\tilde{B}_1^{(c)}) = -\frac{1}{2\mathrm{i}\pi k}$ by (38), from equation (39) we obtain

$$c_k(\tilde{B}_2^{(c)}) = \frac{1}{\mathrm{i}\pi k}(-\frac{1}{2\mathrm{i}\pi k}) = \frac{2}{(2\pi k)^2}.$$

Assume that (41) is true for n. Then using (40) gives

$$\begin{aligned}
c_k(\tilde{B}_{2n+1}^{(c)}) &= \frac{2n+1}{2\mathrm{i}\pi k}\Big((-1)^{n+1}q^{2n} + (-1)^{n+1}(2n)!\sum_{j=1}^{n}\frac{q^{2n-2j}}{(2n-2j)!(2\pi k)^{2j}}\Big)\\
&= \frac{(-1)^{n+1}(2n+1)}{\mathrm{i}}\Big(\frac{q^{2n}}{2\pi k} + (2n)!\sum_{j=1}^{n}\frac{q^{2n-2j}}{(2n-2j)!(2\pi k)^{2j+1}}\Big).
\end{aligned}$$

So, (42) is satisfied for n. Now let (42) be true for n. Then for $n+1$, relation (39) gives

$$\begin{aligned}
c_k(\tilde{B}_{2n+2}^{(c)}) &= \frac{n+1}{\mathrm{i}\pi k}\frac{(-1)^{n+1}(2n+1)}{\mathrm{i}}\Big(\frac{q^{2n}}{2\pi k} + (2n)!\sum_{j=1}^{n}\frac{q^{2n-2j}}{(2n-2j)!(2\pi k)^{2j+1}}\Big)\\
&= (-1)^{n+2}(2n+2)!\Big(\frac{q^{2n}}{(2n)!(2\pi k)^2} + \sum_{j=1}^{n}\frac{q^{2n-2j}}{(2n-2j)!(2\pi k)^{2j+2}}\Big)\\
&= (-1)^{n+2}(2n+2)!\Big(\frac{q^{2n}}{(2n)!(2\pi k)^2} + \sum_{j=2}^{n+1}\frac{q^{2n-2j+2}}{(2n-2j+2)!(2\pi k)^{2j}}\Big)\\
&= (-1)^{n+2}(2n+2)!\sum_{j=1}^{n+1}\frac{q^{2(n+1)-2j}}{(2(n+1)-2j)!(2\pi k)^{2j}},
\end{aligned}$$

which approves (41) for $n+1$. From (41) and (42), it is clear that

$$c_{-k}(\tilde{B}_{2n}^{(c)}) = c_k(\tilde{B}_{2n}^{(c)}) \quad \text{and} \quad c_{-k}(\tilde{B}_{2n+1}^{(c)}) = -c_k(\tilde{B}_{2n+1}^{(c)}).$$

Since

$$B_{2n}^{(c)}(0,q) = B_{2n}^{(c)}(1,q) \quad \text{and} \quad B_{2n+1}^{(c)}(0,q) \neq B_{2n+1}^{(c)}(1,q),$$

the identities (36) and (37) can be directly obtained by Dirichlet's theorem. □

Theorem 4.2. *Let $q \in \mathbb{R}$. Then for every $p \in (0,1)$*

$$B_2^{(s)}(p,q) = 2qp - q = -\frac{2q}{\pi}\sum_{k=1}^{\infty}\frac{\sin(2\pi kp)}{k},$$

and for every $n \geq 2$ we respectively have

$$B_{2n-1}^{(s)}(p,q) = (-1)^{n-1}q^{2n-1} + \sum_{k=1}^{\infty} a'_{k,n}\cos(2\pi kp), \quad p \in [0,1], \tag{43}$$

where

$$a'_{k,n} = 2(-1)^n(2n-1)!\sum_{j=1}^{n-1}\frac{q^{2n-1-2j}}{(2n-1-2j)!(2\pi k)^{2j}},$$

and

$$B_{2n}^{(s)}(p,q) = \sum_{k=1}^{\infty} b'_{k,n}\sin(2\pi kp), \quad p \in (0,1), \tag{44}$$

where

$$b'_{k,n} = 2n(-1)^n\left(\frac{q^{2n-1}}{\pi k} + 2(2n-1)!\sum_{j=1}^{n-1}\frac{q^{2n-1-2j}}{(2n-1-2j)!(2\pi k)^{2j+1}}\right).$$

Proof. The proof of this theorem is similar to the previous one. However, note that for $k \in \mathbb{Z}\backslash\{0\}$ we have

$$c_k(\tilde{B}_{2n-1}^{(s)}) = (-1)^n(2n-1)!\sum_{j=1}^{n-1}\frac{q^{2n-1-2j}}{(2n-1-2j)!(2\pi k)^{2j}},$$

and

$$c_k(\tilde{B}_{2n}^{(s)}) = \frac{2n(-1)^n}{\mathrm{i}}\left(\frac{q^{2n-1}}{2\pi k} + (2n-1)!\sum_{j=1}^{n-1}\frac{q^{2n-1-2j}}{(2n-1-2j)!(2\pi k)^{2j+1}}\right),$$

and from corollary 4

$$c_0(\tilde{B}_{2n-1}^{(s)}) = (-1)^{n-1}q^{2n-1} \quad \text{and} \quad c_0(\tilde{B}_{2n}^{(s)}) = 0.$$

□

5 An extension of the Euler-Maclaurin quadrature formula

The Euler-Maclaurin summation formula is a suitable tool for providing a connection between integrals and sums. It gives an estimation of the sum $\sum_{k=0}^{n} f(k)$ through the integral $\int_0^n f(x)\mathrm{d}x$ with an error term which involves Bernoulli numbers. In other words, if $m, n \in \mathbb{N}$ and $f^{(2m)}$ is continuous in $[0, n]$, then [16]

$$\int_0^n f(x)\ dx = \frac{1}{2}\Big(f(0)+f(n)\Big) + \sum_{k=1}^{n-1} f(k) - \sum_{j=1}^{m} \frac{B_{2j}}{(2j)!}\Big(f^{(2j-1)}(n) - f^{(2j-1)}(0)\Big) + R_m(f), \tag{45}$$

where

$$R_m(f) = \frac{1}{(2m)!}\int_0^1 B_{2m}(x)\left(\sum_{k=0}^{n-1} f^{(2m)}(x+k)\right)\mathrm{d}x = \frac{1}{(2m)!}\int_0^n f^{(2m)}(x)B_{2m}(x-[x])\ \mathrm{d}x, \tag{46}$$

denotes the remainder term. This formula can be extended by using the integration by parts via relation (19) as follows

$$\int_0^1 f(x)\mathrm{d}x = \int_0^1 f(x)B_0^{(c)}(x,q)\mathrm{d}x, \tag{47}$$

where q is an arbitrary real number and $B_0^{(c)}(x,q)=1$.
Since $\frac{\partial}{\partial x}B_1^{(c)}(x,q) = B_0^{(c)}(x,q)$, substituting $\frac{\partial}{\partial x}B_1^{(c)}(x,q)$ into (47) and integrating by parts gives

$$\int_0^1 f(x)\mathrm{d}x = f(1)B_1^{(c)}(1,q) - f(0)B_1^{(c)}(0,q) - \int_0^1 f'(x)B_1^{(c)}(x,q)\ \mathrm{d}x. \tag{48}$$

Note that $B_1^{(c)}(1,q) = -B_1^{(c)}(0,q)$ and $B_1^{(c)}(x,q) = \frac{1}{2}\frac{\partial}{\partial x}B_2^{(c)}(x,q)$. Hence (48) reads as

$$\int_0^1 f(x)\mathrm{d}x = -B_1^{(c)}(0,q)\big(f(1)+f(0)\big) - \frac{1}{2}\int_0^1 f'(x)\frac{\partial}{\partial x}B_2^{(c)}(x,q)\ \mathrm{d}x. \tag{49}$$

Again, integrating by parts yields

$$\begin{aligned}\int_0^1 f(x)\mathrm{d}x &= -B_1^{(c)}(0,q)\big(f(1)+f(0)\big) - \frac{1}{2}\big(B_2^{(c)}(1,q)f'(1) - B_2^{(c)}(0,q)f'(0)\big)\\ &\qquad + \frac{1}{2}\int_0^1 f''(x)B_2^{(c)}(x,q)\ \mathrm{d}x\\ &= -B_1^{(c)}(0,q)\big(f(1)+f(0)\big) - \frac{B_2^{(c)}(0,q)}{2}\big(f'(1)-f'(0)\big)\\ &\qquad + \frac{1}{6}\int_0^1 f''(x)\frac{\partial}{\partial x}B_3^{(c)}(x,q)\ \mathrm{d}x, \end{aligned}\tag{50}$$

because $B_2^{(c)}(1,q) = B_2^{(c)}(0,q)$ and $B_2^{(c)}(x,q) = \frac{1}{3}\frac{\partial}{\partial x}B_3^{(c)}(x,q)$.
By using the general relations

$$B_k^{(c)}(1,q) = (-1)^k B_k^{(c)}(0,q) \quad \text{and} \quad B_k^{(c)}(x,q) = \frac{1}{k+1}\frac{\partial}{\partial x}B_{k+1}^{(c)}(x,q),$$

and continuing the process, for even m we finally obtain

$$\begin{aligned}\int_0^1 f(x)\mathrm{d}x = &-\sum_{i=0}^{\frac{m}{2}-1}\frac{B_{2i+1}^{(c)}(0,q)}{(2i+1)!}\left(f^{(2i)}(1)+f^{(2i)}(0)\right) - \sum_{i=1}^{\frac{m}{2}}\frac{B_{2i}^{(c)}(0,q)}{(2i)!}\left(f^{2i-1}(1)-f^{2i-1}(0)\right)\\ &+\frac{1}{m!}\int_0^1 f^{(m)}(x)B_m^{(c)}(x,q)\ \mathrm{d}x, \qquad (51)\end{aligned}$$

while for odd m we have

$$\begin{aligned}\int_0^1 f(x)\mathrm{d}x = &-\sum_{i=0}^{\frac{m-1}{2}}\frac{B_{2i+1}^{(c)}(0,q)}{(2i+1)!}\left(f^{(2i)}(1)+f^{(2i)}(0)\right) - \sum_{i=1}^{\frac{m-1}{2}}\frac{B_{2i}^{(c)}(0,q)}{(2i)!}\left(f^{2i-1}(1)-f^{2i-1}(0)\right)\\ &-\frac{1}{m!}\int_0^1 f^{(m)}(x)B_m^{(c)}(x,q)\ \mathrm{d}x. \qquad (52)\end{aligned}$$

On the other side, since the interval of integration in relations (51) and (52) can be shifted from $[0,1]$ to $[1,2]$ by replacing $f(x)$ by $f(x+1)$, by considering such transpositions up to the interval $[n-1,n]$ and referring to corollary 2, for every even m we obtain

$$\begin{aligned}\int_0^n f(x)\mathrm{d}x = &\frac{1}{2}\sum_{i=0}^{\frac{m}{2}-1}\frac{(-1)^i}{(2i)!}q^{2i}\left(\sum_{k=0}^{n-1}\left(f^{(2i)}(k+1)+f^{(2i)}(k)\right)\right)\\ &-\sum_{i=1}^{\frac{m}{2}}\frac{B_{2i}^{(c)}(0,q)}{(2i)!}\left(f^{(2i-1)}(n)-f^{(2i-1)}(0)\right)+R_m(f;q), \qquad (53)\end{aligned}$$

while for odd m we have

$$\begin{aligned}\int_0^n f(x)\mathrm{d}x = &\frac{1}{2}\sum_{i=0}^{\frac{m-1}{2}}\frac{(-1)^i}{(2i)!}q^{2i}\left(\sum_{k=0}^{n-1}\left(f^{(2i)}(k+1)+f^{(2i)}(k)\right)\right)\\ &-\sum_{i=1}^{\frac{m-1}{2}}\frac{B_{2i}^{(c)}(0,q)}{(2i)!}\left(f^{(2i-1)}(n)-f^{(2i-1)}(0)\right)+R_m(f;q), \qquad (54)\end{aligned}$$

where

$$R_m(f;q) = \frac{(-1)^m}{m!}\int_0^1 B_m^{(c)}(x,q)\left(\sum_{k=0}^{n-1} f^{(m)}(x+k)\right)\mathrm{d}x = \frac{(-1)^m}{m!}\int_0^n f^{(m)}(x)B_m^{(c)}(x-[x],q)\ \mathrm{d}x, \qquad (55)$$

is the remainder term. The relations (53) and (54) are indeed a parametric extension of the Euler-Maclaurin quadrature formula for $q = 0$. Let us consider the even case (53) when $m \to 2m$ as

$$\int_0^n f(x)\mathrm{d}x = \frac{1}{2}\sum_{i=0}^{m-1}\frac{(-1)^i}{(2i)!}q^{2i}\left(\sum_{k=0}^{n-1}\left(f^{(2i)}(k+1)+f^{(2i)}(k)\right)\right) - \sum_{i=1}^{m}\frac{B_{2i}^{(c)}(0,q)}{(2i)!}\left(f^{(2i-1)}(n)-f^{(2i-1)}(0)\right)+R_{2m}(f;q), \tag{56}$$

with

$$R_{2m}(f;q) = \frac{1}{(2m)!}\int_0^1 B_{2m}^{(c)}(x,q)\left(\sum_{k=0}^{n-1} f^{(2m)}(x+k)\right)\mathrm{d}x = \frac{1}{(2m)!}\int_0^n f^{(2m)}(x)B_{2m}^{(c)}(x-[x],q)\ \mathrm{d}x. \tag{57}$$

By referring to relations (46) and (57), it is clear that if $|R_{2m}(f;q)| < |R_m(f)|$ for a particular value of q, then the accuracy of the extended formula (56) is better than the standard Euler-Maclaurin formula (45). In this direction, since

$$|R_{2m}(f;q)| = \frac{1}{(2m)!}\left|\int_0^1 B_{2m}^{(c)}(x,q)\left(\sum_{k=0}^{n-1} f^{(2m)}(x+k)\right)\mathrm{d}x\right| \le \frac{n}{(2m)!}\max_{t\in[0,n]}|f^{(2m)}(t)| \times \int_0^1 |B_{2m}^{(c)}(x,q)|\ \mathrm{d}x,$$

and

$$|R_m(f)| = \frac{1}{(2m)!}\left|\int_0^1 B_{2m}(x)\left(\sum_{k=0}^{n-1} f^{(2m)}(x+k)\right)\mathrm{d}x\right| \le \frac{n}{(2m)!}\max_{t\in[0,n]}|f^{(2m)}(t)| \times \int_0^1 |B_{2m}(x)|\ \mathrm{d}x,$$

it seems that solving the polynomial type inequality

$$\int_0^1 |B_{2m}^{(c)}(x,q)|\ \mathrm{d}x \le \int_0^1 |B_{2m}(x)|\ \mathrm{d}x,$$

in terms of the variable q is a good criterion to consider formula (56) with respect to the well-known formula (45) though there might be other appropriate criterions for this purpose. In the following table, we have compared the values of $|R_{2m}(f;q)|$ and $|R_m(f)|$ for some smooth functions and found out that the absolute error of formula (57) is less than formula (46) for some specific values of q. Note that to derive these values, we have

$f(x)$	n	m	q	$\|R_m(f)\|$	$\|R_{2m}(f;q)\|$
$x\sin x$	5	1	0.1	2.14182×10^{-3}	1.55109×10^{-4}
$x^2\cos x$	20	6	0.001	1.15731×10^{-10}	1.15645×10^{-10}
e^x	20	7	0.001	1.6036×10^{-4}	1.60291×10^{-4}
e^{-x}	1	2	0.2	2.03937×10^{-5}	4.5966×10^{-6}
$x\mathrm{e}^x$	10	2	0.20159	10.6246	1.73236×10^{-4}
$x\mathrm{e}^{-x}$	10	1	0.1	4.00736×10^{-3}	9.94681×10^{-4}
x^8	3	2	0.252354	5.9	6.04417×10^{-5}
$\mathrm{e}^{-x}\sin x$	1	3	0.38	3.61361×10^{-6}	7.02988×10^{-7}

used Mathematica software.

Acknowledgments
The work of the first author has been supported by the Alexander von Humboldt Foundation under the grant number: Ref 3.4 - IRN - 1128637 - GF-E.

References

[1] S. Akiyama and Y. Tanigawa, Multiple zeta values at non-positive integers. *Ramanujan J.*, **5** (2001), 327–351.

[2] F. Avram, and M. S. Taqqu, Noncentral limit theorems and Appell polynomials. *Ann. Probab.*, **15** (1987), 767–775.

[3] A. Bayad and Y. Hamahata, Polylogarithms and poly-Bernoulli polynomials. *Kyushu J. Math.*, **65** (2011), 15–24.

[4] K. Chen, Algorithms for Bernoulli numbers and Euler numbers. *J. Integer Seq.*, **4** (2001), Article 01.1.6, 7 pp.

[5] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.

[6] M. Coppo and B. Candelpergher, The Arakawa-Kaneko zeta function. *Ramanujan J.*, **22** (2010), 153–162.

[7] R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete Mathematics, second ed.*, AddisonWesley, Reading, 1994.

[8] Y. Hamahata and H. Masubuchi, Special multi-poly-Bernoulli numbers. *J. Integer Seq.*, **10** (2007), Article 07.4.1, 6 pp.

[9] H. Jolany and R. Corcino, Explicit formula for generalization of poly-Bernoulli numbers and polynomials with a, b, c parameters. *J. Class. Anal.*, **6** (2015), 119–135.

[10] M. Masjed-Jamei and W. Koepf, Symbolic computation of some power-trigonometric series, *J. Symbolic Comput*, **80** (2017), 273–284.

[11] D. Merlini, R. Sprugnoli and M. Verri, The Cauchy numbers. *Discrete Math.*, **306** (2006), 1906–1920.

[12] M. Rahmani, On p-Bernoulli numbers and polynomials. *J. Number Theory*, **157** (2015), 350–366.

[13] M. Rahmani, The Akiyama-Tanigawa matrix and related combinatorial identities. *Linear Algebra Appl.*, **438** (2013), 219–230.

[14] Y. Sasaki, On generalized poly-Bernoulli numbers and related L-functions. *J. Number Theory*, **132** (2012), 156–170.

[15] P. Tempesta, Formal groups, Bernoulli-type polynomials and L-series. *C. R. Math. Acad. Sci. Paris*, **345** (2007), 303–306.

[16] J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis, third ed.*, Springer, 2002.