On Linearization and Connection Coefficients for Generalized Hermite Polynomials

Hamza Chaggara[*], Wolfram Koepf[†]

October 30, 2009

Abstract

We consider the problem of finding explicit formulae, recurrence relations and sign properties for both connection and linearization coefficients for generalized Hermite polynomials. The most computations are carried out by the computer algebra system *Maple* using appropriate algorithms.

Key words. Generalized Hermite polynomials; Linearization coefficients; Connection coefficients; Multiple summation; Multsum package

1 Introduction

In this paper, we deal with the connection and linearization problems which are defined as follows: Given two polynomial sets $\{S_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$ s. th. $\deg(S_n) = \deg(P_n) = n$. The so-called *connection problem* between them asks to find the coefficients $C_m(n)$ in the expression:

$$S_n(x) = \sum_{m=0}^{n} C_m(n) P_m(x) \; . \tag{1.1}$$

When $S_{i+j}(x) = Q_i(x)R_j(x)$ in (1.1), $\{Q_n\}_n$ and $\{R_n\}_n$ being two more polynomial sets with $\deg(Q_n) = \deg(R_n) = n$, we are faced with the general *linearization problem*

$$Q_i(x)R_j(x) = \sum_{k=0}^{i+j} L_{ij}(k) P_k(x). \tag{1.2}$$

A particular case of this problem is the *standard linearization problem* or *Clebsch-Gordan-type problem* if $Q_n = R_n = P_n$.
The computation of the connection and linearization coefficients plays an important role in many situations of pure and applied mathematics and also in physical and quantum chemical applications. The study of the linearization problem has gained an increasing interest in the last years. In particular, the study of positivity conditions of the connection and linearization coefficients has received special attention [2, Lecture 5]. Many problems in harmonic analysis related to nontrigonometric orthogonal expansions depend on the nonnegativity of certain connection coefficients (see [2, Lecture 7]), while the nonnegativity of the linearization coefficients gives rise to a convolution structure associated with

[*]Département de Mathématiques, École Supérieure des Sciences et de Technologie de Hammam Sousse. Rue Lamine Abassi 4011, Sousse, Tunisia, *hamza.chaggara@ipeim.rnu.tn*
[†]Fachbereich Mathematik, Universität Kassel, D-34109 Kassel, Germany, *koepf@mathematik.uni-kassel.de*

orthogonal polynomials [2,31].
The literature on this topic is extremely vast and a wide variety of methods, based on specific properties of the involved polynomials, have been devised for computing the linearization coefficients either in explicit form or by means of recursive relations (see e.g. [1,20,25,26] and the references therein).
In a series of papers [23,24], Ronveaux et al. designed the so-called *NAVIMA* algorithm which allows us to calculate recurrently the connection and linearization coefficients. There exists an alternative approach to building recurrence relations for both connection and linearization coefficients due to Lewanowicz [19,20].
In some cases (classical orthogonal polynomials), many results concerning the positivity of the connection and linearization coefficients and the recurrence relations satisfied by $C_m(n)$ and $L_{i,j}(k)$ are known. Hounkonnou et al. [26] proved that for a family of classical orthogonal polynomials the coefficients $L_{i,j}(k)$ satisfy a linear second-order recurrence relation involving only the index k. The explicit coefficients of the second order recurrence relation was obtained by Lewanowicz [19], rewriting for this purpose the fourth order differential equation for the product P_iP_j.
A further—computer algebra based—method was proposed in [17]. Using several structure formulas of the classical systems these authors derive generic recurrence equations for the connections coefficients of these systems.
If $\{P_n\}_n$ is a semi-classical orthogonal family, the corresponding standard linearization coefficients, defined in (1.2), satisfy a linear recurrence relation involving only the index k. This property also extends to the linearization coefficients arising from an arbitrary number of products of semi-classical orthogonal polynomials [25].
A general method, based on suitable operators, generating functions and a simple manipulation of formal power series, was developed to solve connection and linearization problems. The coefficients are given explicitly, very often in terms of hypergeometric terms and/or terminating hypergeometric functions [5,6,10].
For the sign properties, the nonnegativity of the connection and linearization coefficients has many important consequences. Several criteria to get sign properties for the aforementioned coefficients have been investigated by many authors. Some of them are given in terms of corresponding spectral measures [34], the others impose conditions on coefficients in the recurrence formula satisfied by the polynomials [2,3,18,31]. Moreover, alternation of signs in the $C_m(n)$ sequence is linked to the relative position of the zeros of Q_n as compared to those of P_n [14].
In a recent paper [11], we solved a special case of the connection problem, called the *duplication problem*, which asks to find the connection coefficients in

$$P_n(ax) = \sum_{m=0}^{n} C_m(n,a)P_m(x),$$

where $\{P_n\}_{n\geq 0}$ belongs to a wide class of polynomials, including the classical orthogonal polynomials (Hermite, Laguerre, Jacobi) as well as the classical discrete orthogonal polynomials (Charlier, Meixner, Krawtchouk), the latter for the specific case $a = -1$. We gave explicit expressions as well as recurrence relations satisfied by these coefficients. The essential computations were done completely automatically by some packages of the *Maple* system [16,29]. The only prerequisite is the knowledge of a suitable generating function of the involved polynomials.

In this work we extend these results to the so-called generalized Hermite polynomials (see definition below). This family does not belong to the classical families and has no hypergeometric representation.

The main aim of this paper is to study connection and linearization problems associated to generalized Hermite polynomials. We give an explicit expression of the connection and linearization coefficients as well as recurrence relations satisfied associated to these coefficients. Sign properties of both connection and linearization coefficients will be also given. As application, we consider the *classical Hermite* and the *classical Laguerre polynomials.*
The *generalized Hermite polynomial* set was introduced by Szegö [21] as a set of real polynomials

orthogonal with respect to the weight $|x|^{2\mu}e^{-x^2},\ \mu > -\frac{1}{2}$, then investigated by Chihara in his Ph.D. Thesis [12]. This family reduces to the ordinary Hermite polynomial set for $\mu = 0$.
The generalized Hermite polynomials have been mentioned in connection with the Gauss quadrature formulae by Shao et al. [28]. They were also studied by Rosenblum in connection with a Bose-like oscillator calculus [27] as a polynomial set generated by [27]

$$e^{-t^2}e_\mu(2xt) = \sum_{n=0}^{\infty} \mathcal{H}_n^\mu(x)\frac{t^n}{n!}, \quad \mu \in \mathbb{C},\ \mu \neq -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots, \tag{1.3}$$

where

$$e_\mu(x) = \sum_{n=0}^{\infty} \frac{x^n}{\gamma_\mu(n)},$$

with

$$\gamma_\mu(2m+\epsilon) = 2^{2m+\epsilon}m!\,(\mu+\frac{1}{2})_{m+\epsilon}, \quad \epsilon = 0, 1, \tag{1.4}$$

where $(a)_n = \dfrac{\Gamma(a+n)}{\Gamma(a)}$.
Many other authors investigated properties of these polynomials, using classical methods well known in the theory of special functions. For instance, some characterization problems related to this polynomial set were given in [7,13]. Recently, many characteristic properties and operational rules associated to this family were given in [9]. In particular, it was shown that $\{\mathcal{H}_n^\mu\}_n$, is a $(\frac{1}{2}\mathcal{D}_\mu)$-*Appell polynomial set of transfer power series* $A(t) = e^{-t^2}$. This means that $\mathcal{D}_\mu(f)(\mathcal{H}_n^\mu) = 2n\mathcal{H}_{n-1}^\mu$ and

$$e^{-\frac{\mathcal{D}_\mu^2}{4}}(x^n) = \frac{\gamma_\mu(n)}{2^n n!}\mathcal{H}_n^\mu(x). \tag{1.5}$$

$\mathcal{D}_\mu$ is the well-known *Dunkl operator* associated with the parameter μ on the real line [27]:

$$\mathcal{D}_\mu(f)(x) = \frac{\mathrm{d}}{\mathrm{d}x}f(x) + \frac{\mu}{x}(f(x) - f(-x)).$$

2 Connection Coefficients

In this section, we are interested in finding explicit formulas, recurrence relations and sign properties of the connection coefficients relating two generalized Hermite polynomials with different parameters. The obtained explicit connection formula (Equation (2.11) below) generalizes a well known connection formula for Laguerre polynomials and appears to be new.
We begin by recalling a result giving the connection coefficients between two σ-Appell polynomials. That is to say $\sigma P_n = nP_{n-1},\ n = 0, 1, \ldots, n$, where σ is a linear operator, not depending on n, and called lowering operator. (For more details, we refer the reader to [4,5] and the references therein).

Lemma 1. *([5, Corollary 3.4]) Let $\{P_n\}_{n\geq 0}$ and $\{Q_n\}_{n\geq 0}$ be two σ-Appell polynomial sets of transfer power series, respectively, A_1 and A_2. Then*

$$Q_n(x) = \sum_{m=0}^{n} \frac{n!}{m!}\alpha_{n-m}P_m(x), \quad \textit{where} \quad \frac{A_2(t)}{A_1(t)} = \sum_{k=0}^{\infty}\alpha_k t^k. \tag{2.1}$$

It was shown in [9], that $\{\mathcal{H}_n^\mu\}_{n\geq 0}$ and $\left\{B_n(x) = \dfrac{2^n n!}{\gamma_\mu(n)}x^n\right\}_{n\geq 0}$ are two $(\frac{1}{2}\mathcal{D}_\mu)$-Appell sets of transfer power series, respectively, e^{-t^2} and 1. As application, we obtain the following expansion formulae

$$\frac{\mathcal{H}_n^\mu(x)}{n!} = \sum_{m=0}^{[\frac{n}{2}]} \frac{(-1)^m(2x)^{n-2m}}{m!\gamma_\mu(n-2m)}, \tag{2.2}$$

$$\frac{(2x)^n}{\gamma_\mu(n)} = \sum_{m=0}^{[\frac{n}{2}]} \frac{\mathcal{H}^{\mu}_{n-2m}(x)}{m!(n-2m)!}, \tag{2.3}$$

and, by composition, we get the explicit connection relation

$$\frac{\mathcal{H}^{\mu_2}_n(x)}{n!} = \sum_{m=0}^{[\frac{n}{2}]} \left(\sum_{p=0}^{m} \frac{(-1)^{m-p}}{(m-p)!p!} \frac{\gamma_{\mu_1}(n-2m+2p)}{\gamma_{\mu_2}(n-2m+2p)} \right) \frac{\mathcal{H}^{\mu_1}_{n-2m}(x)}{(n-2m)!}. \tag{2.4}$$

To obtain a pure recurrence relation for $C_{n-2m}(n)$, with respect to m, we use Zeilberger's algorithm (see e. g. [16], Chapter 7) via the Maple `sumrecursion` command, and get, with the notation $D_m := C_{n-2m}(n)$:

• For n even,

$$(m+1)(2\mu_1 - 1 + n - 2m)D_{m+1} + 2(n-2m-1)(n-2m)(-\mu_1 + m + \mu_2)D_m = 0. \tag{2.5}$$

• For n odd,

$$(m+1)(2\mu_1 + n - 2m)D_{m+1} + 2(n-2m-1)(n-2m)(-\mu_1 + m + \mu_2)D_m = 0. \tag{2.6}$$

A unified form of the above recurrence relations can be written as follows

$$(m+1)\left(2\mu_1 + n - 2m - \theta_n\right) D_{m+1} + 2(n-2m-1)(n-2m)(-\mu_1 + m + \mu_2)D_m = 0, \tag{2.7}$$

where $\theta_n = \dfrac{1+(-1)^n}{2}$.

Using Equation (2.4) with the useful identities

$$\frac{(-1)^m}{(n-m)!} = \frac{(-n)_m}{n!} \quad \text{and} \quad (\delta)_{n+m} = (\delta)_n(\delta+m)_m, \quad 0 \le m \le n, \tag{2.8}$$

and the Chu-Vandermonde reduction formula [30],

$$ {}_2F_1\left(\begin{matrix} -k, b \\ c \end{matrix} ; 1 \right) = \frac{(c-b)_k}{(c)_k}, \; c \neq 0, -1, -2, \ldots, \tag{2.9}$$

we get the following simple and explicit form of the connection coefficients for $0 \le m \le [\frac{n}{2}]$,

$$C_{n-2m}(n) = \frac{n!}{m!(n-2m)!} \frac{4^m [\frac{n}{2}]!}{[\frac{n}{2} - m]!} \frac{\gamma_{\mu_1}(n-2m)}{\gamma_{\mu_2}(n)} (-1)^m (\mu_2 - \mu_1)_m. \tag{2.10}$$

The explicit formula (2.10) can be also obtained by induction using, for this purpose, the recurrence relation (2.7).

Putting $\widehat{\mathcal{H}}^{\mu_2}_n(x) = \dfrac{\mathcal{H}^{\mu_2}_n(x)}{[\frac{n}{2}]!n!}$, we get a simple form of the connection between two suitable generalized Hermite polynomials

$$\widehat{\mathcal{H}}^{\mu_2}_n(x) = \sum_{m=0}^{[\frac{n}{2}]} (-1)^m \frac{4^m}{m!} \frac{\gamma_{\mu_1}(n-2m)}{\gamma_{\mu_2}(n)} (\mu_2 - \mu_1)_m \widehat{\mathcal{H}}^{\mu_1}_{n-2m}(x). \tag{2.11}$$

So, when $\mu_2 > \mu_1$, the corresponding connection coefficients alternate in sign, while this coefficient is nonnegative if $\mu_2 - \mu_1$ is a negative integer. On the other hand, if $\mu_2 - \mu_1 < 0$ and it is not an integer, then the connection coefficient is always nonnegative provided that $\mu_2 - \mu_1 \geq m - 1$.

3 Linearization Problem

To study the linearization problem, we begin by recalling the following result, which gives an explicit expression of the linearization coefficients associated to three polynomial sets of *Brenke type*, generalizing a product formula associated to Appell and q-Appell polynomials given by Carlitz in [8].

Corollary 2. *([10, Corollary 2.8]) Let $\{P_n\}_{n\geq 0}$, $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ be three polynomial sets of Brenke type, i.e. polynomial sets generated, respectively, by*

$$A_1(t)B_1(xt) = \sum_{n=0}^{\infty} \frac{P_n(x)}{n!} t^n,\ A_2(t)B_2(xt) = \sum_{n=0}^{\infty} \frac{Q_n(x)}{n!} t^n \text{ and } A_3(t)B_3(xt) = \sum_{n=0}^{\infty} \frac{R_n(x)}{n!} t^n, \tag{3.1}$$

where

$$A_p(t) = \sum_{k=0}^{\infty} a_k^{(p)} t^k, \quad \text{and} \quad B_p(t) = \sum_{k=0}^{\infty} b_k^{(p)} t^k,\ a_0^{(p)} b_k^{(p)} \neq 0,\ p = 1, 2, 3. \tag{3.2}$$

Then the linearization coefficients in (1.2) are given by

$$L_{ij}(k) = \frac{i!j!}{k!} \sum_{r=0}^{i} \sum_{s=0}^{j} \frac{b_r^{(2)} b_s^{(3)}}{b_{r+s}^{(1)}} a_{i-r}^{(2)} a_{j-s}^{(3)} \widehat{a}_{r+s-k}^{(1)},\ k = 0, 1, \ldots, i+j, \tag{3.3}$$

where $\widehat{A}_1(t) = \dfrac{1}{A_1(t)} = \displaystyle\sum_{k=0}^{\infty} \widehat{a}_k^{(1)} t^k.$

According to (1.3), the generalized Hermite family is a Brenke type polynomial set.
The application of Corollary 2 allows us to solve the linearization problem for the generalized Hermite polynomials.
Taking into account the orthogonality of this family, we obtain

$$\mathcal{H}_i^{\mu}(x)\mathcal{H}_j^{\mu}(x) = \sum_{k=|i-j|}^{i+j} L_{ij}(k)\mathcal{H}_k^{\mu}(x). \tag{3.4}$$

Here the sum range is given by $k = |i-j|$ because if $k < |i-j|$ then $k+j < i$ or $k+i < j$. Hence $\deg(\mathcal{H}_k^{\mu}\mathcal{H}_j^{\mu}) < \deg\mathcal{H}_i^{\mu}$ or $\deg(\mathcal{H}_k^{\mu}\mathcal{H}_i^{\mu}) < \deg\mathcal{H}_j^{\mu}$. In both cases the coefficient

$$L_{ij}(k) = \int_{\mathbb{R}} \mathcal{H}_i^{\mu}(x)\mathcal{H}_j^{\mu}(x)\mathcal{H}_k^{\mu}(x)|x|^{2\mu} e^{-x^2} dx,$$

vanishes.
On the other hand and according to the symmetry property, $\mathcal{H}_n^{\mu}(-x) = (-1)^n \mathcal{H}_n^{\mu}(x)$, the associated standard linearization Equation (3.4) can be reduced to

$$\mathcal{H}_i^{\mu}(x)\mathcal{H}_j^{\mu}(x) = \sum_{k=0}^{\min(i,j)} L_{ij}(i+j-2k)\mathcal{H}_{i+j-2k}^{\mu}(x).$$

By virtue of (3.3) and the generating function (1.3), we obtain the explicit expression

$$L_{ij}(i+j-2k) = \frac{i!j!}{(i+j-2k)!k!} \sum_{r=0}^{[\frac{i}{2}]} \sum_{s=0}^{[\frac{j}{2}]} \frac{\gamma_\mu(i+j-2(r+s))}{\gamma_\mu(i-2r)\gamma_\mu(j-2s)} \frac{(-k)_{r+s}}{r!s!}. \tag{3.5}$$

Using the explicit formula (3.5), and a *Fasenmyer type algorithm* [16] to deduce recurrence equations for multiple hypergeometric series ([29], see also [33]) we get—using Sprenger's `multsum` *package*—the following recurrence relations (on one index) for the standard linearization coefficient of

generalized Hermite polynomials.
Denote by $S(k) := L_{ij}(i+j-2k)$ and consider three cases:
• For i even and j even:

$$\begin{aligned}
&-(k+2)\,(-2\,k+2\,\mu+i-3+j)\,S(k+2)\\
&+2\,\left(ij-2\,i-2\,ik+5\,k+3\,k^2+2-2\,j-2\,jk\right)(i+j-2\,k-3)\,S(k+1)\\
&+4\,(j-k)\,(-k+i)\,(i+j-2\,k-1)\,(i+j-2\,k-3)\,S(k) \quad = \quad 0
\end{aligned} \qquad (3.6)$$

• For i even and j odd:

$$\begin{aligned}
&-(k+3)(k-i)(2k+4-i-j-2\mu)(2k+2-i-j-2\mu)S(k+3)\\
&+2\,(-7jk-22ik+14\,k-12i-2j+17k^2+5k^3+6i^2+4jik+6\,ij\\
&-8k^2i-3jk^2+3i^2k-ji^2)(-i-j+2k+4)(2k+2-i-j-2\,\mu)S(k+2)\\
&-4(k+1-i)(-i-j+2k+4)(-i-j+2k+2)\\
&\times(-13jk-15ik+7k-5i-2j+15k^2+3j^2k+2j\mu k+8k^3+4i\mu+2j^2+3i^2+10jik\\
&+2i\mu k-2\mu k^2+8\,ij-10k^2i-10jk^2+3i^2k-2\,j\mu i-4\mu k-2j^2i-2ji^2)S(k+1)\\
&+8(k-j)(k+1-i)(k-i)-i-j+2k)\\
&\times(-i-j+2k+1)(-i-j+2k+2)(-i-j+2k+4)S(k) \quad = \quad 0
\end{aligned} \qquad (3.7)$$

• For i odd and j odd:

$$\begin{aligned}
&-(k+3)(2k+5-j-i-2\mu)S(k+3)\\
&+2(i+j-2k-5)(-ji+5k^2-18k-16+2\mu k+2\mu+3ki+6i+3jk+6j)S(k+2)\\
&-4(i+j-2k-5)(3ki+3i-2ji-4k2-7k-3+3jk+3j)(i+j-2k-3)S(k+1)\\
&+8(k-j)(k-i)(i+j-2k-3)(i+j-2k-1)(i+j-2k-5)S(k) \quad = \quad 0
\end{aligned} \qquad (3.8)$$

Note here, that the linearization problem associated to generalized Hermite polynomials was already studied by Ronveaux et al. in [25] in the context of semi-classical polynomials. In fact, it was shown that the linearization coefficients $L_{ij}(k)$ satisfy a linear recurrence relation involving only the k index. The coefficients of this recurrence relation are very complicated and can only be obtained using a symbolic manipulation system like *Maple* or *Mathematica*, the obtained coefficients filled many pages [25].

Next, we consider two interesting particular cases involving classical Hermite and Laguerre polynomials.

Classical Hermite Polynomials: The generalized Hermite polynomials reduce to the classical Hermite polynomials if $\mu = 0$.
To obtain an explicit recurrence relation satisfied by the linearization coefficients associated to classical Hermite polynomials, we use Sprenger's `multsum` *package* by applying the *multsumrecursion* command to the formula (3.5). The recurrence relation is given by :

$$\begin{aligned}
&\left(-6\,k^2-4+4\,ik-2\,ji+4\,i-10\,k+4\,j+4\,jk\right)S\,(k+1)\\
&+4\,(-j+k)\,(i-k)\,(i+j-2\,k-1)\,S\,(k)\\
&+(k+2)\,S\,(k+2) \quad = \quad 0\,,
\end{aligned} \qquad (3.9)$$

where, as usual, $S(k) := L_{ij}(i+j-2k)$.

The explicit linearization formula for Hermite polynomials is known as Feldheim formula and is given by [2]

$$H_i(x)H_j(x) = \sum_{k=0}^{\min(i,j)} \binom{i}{k}\binom{j}{k} 2^k k! H_{i+j-2k}(x). \tag{3.10}$$

Note that (3.10) follows directly from (3.9) by using the Petkovšek-van-Hoeij algorithm (see e. g. [16], Chapter 9), implemented in *Maple* by Mark van Hoeij as `LREtools[hypergeomsols]` [15].

Classical Laguerre polynomials: Now, we consider the classical Laguerre polynomials L_n^α defined by [22]

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}\, {}_1F_1\left(\begin{matrix} -n \\ \alpha+1 \end{matrix}, x\right). \tag{3.11}$$

The generalized Hermite polynomials are related to the Laguerre polynomials by the following formula

$$H^\mu_{2n+\epsilon}(x) = \frac{(-1)^n(2n+\epsilon)!}{(\mu+\frac{1}{2})_{n+\epsilon}} x^\epsilon L_n^{\mu-\frac{1}{2}+\epsilon}(x^2), \quad \epsilon = 0, 1. \tag{3.12}$$

This can be obtained according to (1.4) and the explicit formula (2.2).
The obtained expansion formulae can be used to recover some known expansions associated to Laguerre polynomials. For instance, combining (2.11) with (3.12) and using (1.4), we get the well-known connection formula [22] relating two families of Laguerre polynomials with different parameters

$$L_n^\beta(x) = \sum_{m=0}^{n} \frac{(\beta-\alpha)_m}{m!} L^\alpha_{n-m}(x). \tag{3.13}$$

For the linearization coefficients, applying Formula (3.5) to the Laguerre polynomial set, we obtain, in view of (1.4) and (3.12),

$$\begin{aligned} L_i^\beta(x)L_j^\gamma(x) = \binom{i+j}{j} \sum_{k=0}^{i+j} F^{1:2}_{2:0}\left(\begin{matrix} -k : -\beta-i, -i; -\gamma-j, -j; \\ -\alpha-i-j, -i-j : -; -; \end{matrix}\ 1, 1\right) \\ \times(-1)^k \frac{(-\alpha-i-j)_k}{k!} L^\alpha_{i+j-k}(x), \end{aligned} \tag{3.14}$$

where $F^{p:r}_{q:s}$ designates the Kampé de Fériet function defined as follows [30]:

$$F^{p:r}_{q:s}\left(\begin{matrix} (a_p) : (b_r); (c_r); \\ (\alpha_q) : (\beta_s); (\gamma_s); \end{matrix}\ x, y\right) = \sum_{n,m=0}^{\infty} \frac{[a_p]_{n+m}[b_r]_n[c_r]_m}{[\alpha_q]_{n+m}[\beta_s]_n[\gamma_s]_m} \frac{x^n}{n!}\frac{y^m}{m!}, \tag{3.15}$$

where $[a_p]_n = \prod_{j=1}^{p} (a_j)_n$.

For the standard case ($\alpha = \beta = \gamma$), using the explicit representation given by (3.14) and Sprenger's `multsum` package, we obtain the following second-order recurrence relation,

$$\begin{aligned} (-k+2\,j)\,(2\,i-k)\,(i+j-k+\alpha)\,S(k) \\ -(4\,j\,i-4\,k\,i-4\,i-4\,k\,j-4\,j+3\,k^2+5\,k+2)\,(i+j-k)\,S(k+1) \\ -2\,(k+2)\,(i+j-k)\,(i+j-k-1)\,S(k+2) &= 0 \end{aligned} \tag{3.16}$$

where $S(k) = L_{ij}(i+j-k)$.
Note here that this result corresponds to the fact that the standard linearization coefficients $L_{ij}(k)$ for classical orthogonal polynomials satisfy a second-order linear recurrence relation on the index k.

The explicit expression for the standard linearization coefficients for Laguerre polynomials was first given by Watson [32] by means of a terminating hypergeometric function ${}_3F_2$:

$$L_{ij}(i+j-k) = \frac{(-2)^k}{k!}\frac{(i+j-k)!}{(i-k)!(j-k)!}\,{}_3F_2\left(\begin{array}{c} i+j+1-k+\alpha,\ -\dfrac{k}{2},\ -\dfrac{k-1}{2} \\ \qquad\qquad\qquad\qquad\qquad ;\ 1 \\ i-k+1,\ j-k+1 \end{array}\right). \quad (3.17)$$

Unfortunately, this result cannot be automatically discovered from (3.16). However, *a posteriori*, one can prove that (3.17) is correct since it satisfies the same recurrence equation (proved by Zeilberger's algorithm) and has the same initial values.

For the general case (3.14), to deduce the recurrence relation associated to three Laguerre polynomials with arbitrary parameters we used the *Mathematica* package `MultSum` [33] to deduce for $S(k) = L_{ij}(i+j-k)$ a very complicated recurrence equation which can be found in an appendix and is put for download on `www.mathematik.uni-kassel.de/~koepf/CA/MultSumLaguerre.nb`.

Next, we will be concerned with the sign property of the linearization coefficients associated to generalized Hermite polynomials.
A generating function manipulation permits to show that the integral involving three Laguerre polynomials (with same parameters) is always nonnegative, we have [2],

$$(-1)^{i+j+k}\int_0^{+\infty} L_i^{\alpha}(x)L_j^{\alpha}(x)L_k^{\alpha}(x)x^{\alpha}e^{-x}dx \geq 0, \quad \alpha > -1. \quad (3.18)$$

This property can be useful to study the sign behavior of the linearization coefficients associated with the generalized Hermite polynomial set. This family is orthogonal with respect to the weight $|x|^{2\mu}e^{-x^2}$, $\mu > -\dfrac{1}{2}$, therefore to state the sign of the corresponding linearization coefficients it is sufficient to consider the sign behavior of

$$L_{ij}(k) = \int_{\mathbb{R}} H_i^{\mu}(x)H_j^{\mu}(x)H_k^{\mu}(x)|x|^{2\mu}e^{-x^2}dx.$$

For this end, we consider the following two cases:
• For $i = 2i'$ and $j = 2j'$, we have, in view of (3.12),

$$L_{ij}(i+j-2k) = (-1)^k\alpha_{ijk}\int_0^{\infty} L_{i'}^{\mu-\frac{1}{2}}(x)L_{i'}^{\mu-\frac{1}{2}}(x)L_{i'+j'-k}^{\mu-\frac{1}{2}}(x)x^{\mu-\frac{1}{2}}e^{-x}dx, \quad (3.19)$$

where $\alpha_{ijk} = \dfrac{i!j!(i+j-2k)!}{(\mu+\frac{1}{2})_{i'}(\mu+\frac{1}{2})_{j'}(\mu+\frac{1}{2})_{i'+j'-k}}$.
According to (3.18), we deduce that for, $\mu > -\dfrac{1}{2}$, the linearization coefficient given by Equation (3.19) is nonnegative.
• For $i = 2i'+1$ and $j = 2j'$, we have

$$L_{ij}(i+j-2k) = (-1)^k\beta_{ijk}\int_0^{\infty} L_{i'}^{\mu+\frac{1}{2}}(x)L_{j'}^{\mu-\frac{1}{2}}(x)L_{i'+j'-k}^{\mu+\frac{1}{2}}(x)x^{\mu+\frac{1}{2}}e^{-x}dx.$$

The sign of the previous integral can not be obtained directly from (3.18).
Using the connection relation (3.13) for standard Laguerre polynomials, with
$\beta = \mu - \dfrac{1}{2}$ and $\alpha = \mu + \dfrac{1}{2}$, we get

$$L_{j'}^{\mu-\frac{1}{2}}(x) = L_{j'}^{\mu+\frac{1}{2}}(x) - L_{j'-1}^{\mu+\frac{1}{2}}(x).$$

It follows that,

$$\begin{aligned} L_{ij}(i+j-2k) &= (-1)^k\beta_{ijk}\int_0^\infty L_{i'}^{\mu+\frac{1}{2}}(x)L_{j'}^{\mu+\frac{1}{2}}(x)L_{i'+j'-k}^{\mu+\frac{1}{2}}(x)x^{\mu+\frac{1}{2}}e^{-x}dx \\ &+ (-1)^{k+1}\beta_{ijk}\int_0^\infty L_{i'}^{\mu+\frac{1}{2}}(x)L_{j'-1}^{\mu+\frac{1}{2}}(x)L_{i'+j'-k}^{\mu+\frac{1}{2}}(x)x^{\mu+\frac{1}{2}}e^{-x}dx, \end{aligned} \tag{3.20}$$

where $\beta_{ijk} = \dfrac{i!j!(i+j-2k)!}{(\mu+\frac{1}{2})_{i'+1}(\mu+\frac{1}{2})_{j'}(\mu+\frac{1}{2})_{i'+j'+1-k}}$.
Then we conclude that the coefficient given by (3.20) is also nonnegative as a sum of two nonnegative integrals.
For odd values of i and j, taking into account the symmetry property $\mathcal{H}_n^\mu(-x) = (-1)^n\mathcal{H}_n^\mu(x)$ of the generalized Hermite polynomials, the corresponding linearization is always zero.
Finally, we conclude that the considered polynomials admits nonnegative linearization coefficients.
Note that sign properties of generalized Hermite polynomials have been already investigated in [31], using for this purpose, criterion based on the three term recurrence relation satisfied by the polynomials.

References

[1] R. ÁLVAREZ-NODARSE, R.J. YÁÑEZ AND J.S. DEHESA, *Modified Clebsch-Gordan-type expansions for products of discrete hypergeometric polynomials*, J. Comput. Appl. Math. **89** (1998), 171–197.

[2] R. ASKEY, *Orthogonal polynomials and Special functions*, CBMS-NSF Regional Conference Series in Appl. Math., vol. 21, SIAM, Philadelphia, Pynnsylvania, 1975.

[3] R. ASKEY AND G. GASPER, *Jacobi polynomial expansions of Jacobi polynomials with nonnegative coefficients*, Proc. Camb. Phil. Soc. **70** (1971), 243–255.

[4] Y. BEN CHEIKH, *Some results on quasi-monomiality*, Appl. Math. Comp. **141** (2003), 63–76.

[5] Y. BEN CHEIKH AND H. CHAGGARA, *Connection coefficients via lowering operators*, J. Comput. Appl. Math. **178** (2005), 45–61.

[6] ———, *Connection coefficients between Boas-Buck polynomial sets*, J. Math. Anal. Appl. **319** (2006), 665–689.

[7] Y. BEN CHEIKH AND M. GAIED, *Dunkl-Appell d-orthogonal polynomials*, Integral Transforms Spec. Funct. **18** (2007), 581–597.

[8] L. CARLITZ, *Products of Appell polynomials*, Collect. Math. **112** (1963), 133–138.

[9] H. CHAGGARA, *Operational rules and a generalized Hermite polynomials*, J. Math. Anal. Appl. **332** (2007), 11–21.

[10] H. CHAGGARA AND I. LAMIRI, *Linearization coefficients for Boas-Buck polynomial sets*, Appl. Math. Comput. **189** (2007), 1533–1549.

[11] H. CHAGGARA AND W. KOEPF, *Duplication coefficients for discrete polynomial sets via generating functions*, Complex Var. Elliptic Equ. **52** (2007), 237–249.

[12] T.S. CHIHARA, *Generalized Hermite polynomials*, Ph.D. thesis, Purdue, 1955.

[13] H. DETTE, *Characterizations of generalized Hermite and sieved ultraspherical polynomials*, Trans. Amer. Math. Soc. **348** (1996), 691–711.

[14] P. DIMITROV, *Connection coefficients and zeros of orthogonal polynomials*, J. Comput. Appl. Math. **133** (2001), 331–340.

[15] M. VAN HOEIJ, *Finite singularities and hypergeometric solutions of linear recurrence equations*, J. Pure Appl. Algebra **139** (1998), 109–131.

[16] W. KOEPF, *Hypergeometric Summation*, Vieweg, Braunschweig–Wiesbaden, 1998.

[17] W. KOEPF AND D. SCHMERSAU, *Representations of orthogonal polynomials*, J. Comput. Appl. Math. **90** (1998), 57–94.

[18] R. LASSER, *Linearization of the product of associated Legendre polynomials*, SIAM J. Math. Anal. **14** (1983), 403–408.

[19] S. LEWANOWICZ, *Second-order recurrence relations for the linearization coefficients of the classical orthogonal polynomials*, J. Comput. Appl. Math. **69** (1996), 159–160.

[20] ______, *The hypergeometric function approach to the connection problem for the classical orthogonal polynomials*, Tech. report, Inst. Computer Sci, Univ. of Wroclaw, Feb. 1998.

[21] G. POLYA AND G. SZEGÖ, *Problems and Theorems in Analysis*, vol. 1, Springer-Verlag, New York, Heidelberg and Berlin, 1972.

[22] E.D. RAINVILLE, *Special Functions*, The Macmillan Company, New York, 1960.

[23] A. RONVEAUX, *Orthogonal polynomials: Connection and linearization coefficients*, Proc. Inter. Workshop on Orthogonal Polynomials in Mathematical Physics (Leganés, Madrid, Spain) (*M. Alfano et al.*, ed.), 1996.

[24] A. RONVEAUX, A. ZARZO AND E. GODOY , *Recurrence relations for connection coefficients between two families of orthogonal polynomials*, J. Comput. Appl. Math. **62** (1995), 67–73.

[25] A. RONVEAUX, M.N. HOUNKONNOU AND S. BELMEHDI, *Generalized linearization problems*, J. Phys. A: Math. Gen. **28** (1995), 4423–4430.

[26] A. RONVEAUX, S. BELMEHDI, E. GODOY AND A. ZARZO, *Recurrence relation approach for connection coefficients. Application to discrete orthogonal polynomials*, Proc. Inter. Workshop on Symmetries and Integrability of Difference Equations, CRM Proc. and Lecture Notes Series, vol. 9, AMS, Providence R.I, 1996, pp. 321–337.

[27] M. ROSENBLUM, *Generalized Hermite polynomials and the Bose-like oscillator calculus.*, Oper. Theory Adv. Appl. **73** (1994), 369–396.

[28] T.S. SHAO, T.C. CHEN AND R.M. FRANK, *Table of zeros and gaussien weights of certain associated Laguerre polynomials and generalized Hermite polynomials.*, Math. Comp. **18** (1964), 598–616.

[29] T. SPRENGER, *Algorithmen für mehrfache Summen*, Diploma thesis at the University of Kassel (2004), 1–85.

[30] H.M. SRIVASTAVA AND H.L. MANOCHA, *A Treatise on Generating Functions*, John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 1984.

[31] R. SZWARC, *Orthogonal polynomials and a discrete boundary value problem II*, SIAM J. Math. Anal. **23** (1992), 959–964.

[32] G.N. WATSON, *A note on the polynomials of Hermite and Laguerre*, J. London Math. Soc. **13** (1938), 29–32.

[33] K. WEGSCHAIDER, *Computer generated proofs of binomial multi-sum identities*, Diploma thesis at the J. Kepler University of Linz (1997), 1–99.

[34] M.W. WILSON, *Nonnegative expansions of polynomials*, Proc. Amer. Math. Soc. **24** (1970), 100–102.

Appendix

The following is the recurrence for $S(k) = L_{ij}(i+j-k)$ in (3.14).

$$
\begin{aligned}
&-(i+j-k+\alpha)(2j-k+\alpha-\beta)(2i-k+\alpha-\gamma)(2\alpha i-2\beta i+2i-2j-2j\alpha+k\beta+2\beta+2j\gamma-k\gamma-2\gamma)S(k)\\
&-(-5\beta k^4+5\gamma k^4-5\beta^2k^3+5\gamma^2k^3-10ik^3+10jk^3-10i\alpha k^3+10j\alpha k^3+23i\beta k^3+13j\beta k^3+12\alpha\beta k^3-20\beta k^3-13i\gamma k^3\\
&-23j\gamma k^3-12\alpha\gamma k^3+20\gamma k^3-\beta^3k^2+\gamma^3k^2+26i^2k^2-26j^2k^2+24i\alpha^2k^2-24j\alpha^2k^2+21i\beta^2k^2+5j\beta^2k^2+8\alpha\beta^2k^2-17\beta^2k^2\\
&-5i\gamma^2k^2-21j\gamma^2k^2-8\alpha\gamma^2k^2+4\beta\gamma^2k^2+17\gamma^2k^2-22ik^2+22jk^2+26i^2\alpha k^2-26j^2\alpha k^2+2i\alpha k^2-2j\alpha k^2-34i^2\beta k^2-8j^2\beta k^2\\
&-9\alpha^2\beta k^2+56i\beta k^2-54ij\beta k^2+54j\beta k^2-54i\alpha\beta k^2-10j\alpha\beta k^2+41\alpha\beta k^2-26\beta k^2+8i^2\gamma k^2+34j^2\gamma k^2+9\alpha^2\gamma k^2\\
&-4\beta^2\gamma k^2-54i\gamma k^2+54ij\gamma k^2-56j\gamma k^2+10i\alpha\gamma k^2+54j\alpha\gamma k^2-41\alpha\gamma k^2+4i\beta\gamma k^2-4j\beta\gamma k^2+26\gamma k^2-16i^3k\\
&+16j^3k-18i\alpha^3k+18j\alpha^3k+3i\beta^3k+\alpha\beta^3k-3\beta^3k-3j\gamma^3k-\alpha\gamma^3k+\beta\gamma^3k+3\gamma^3k+42i^2k+40ij^2k-42j^2k-40i^2\alpha^2k+40j^2\alpha^2k\\
&+20i\alpha^2k-20j\alpha^2k-28i^2\beta^2k-3\alpha^2\beta^2k+42i\beta^2k-16ij\beta^2k+17j\beta^2k-26i\alpha\beta^2k-2j\alpha\beta^2k+22\alpha\beta^2k-16\beta^2k\\
&+28j^2\gamma^2k+3\alpha^2\gamma^2k-17i\gamma^2k+16ij\gamma^2k-42j\gamma^2k+2i\alpha\gamma^2k+26j\alpha\gamma^2k-22\alpha\gamma^2k-2i\beta\gamma^2k-13j\beta\gamma^2k\\
&-3\alpha\beta\gamma^2k+10\beta\gamma^2k+16\gamma^2k-16ik-40i^2jk+16jk-16i^3\alpha k+16j^3\alpha k+2i^2\alpha k+40ij^2\alpha k-2j^2\alpha k+22i\alpha k-40i^2j\alpha k\\
&-22j\alpha k+16i^3\beta k+2\alpha^3\beta k-42i^2\beta k+28ij^2\beta k-32j^2\beta k+41i\alpha^2\beta k-9j\alpha^2\beta k-25\alpha^2\beta k+43i\beta k+68i^2j\beta k-126ij\beta k\\
&+57j\beta k+68i^2\alpha\beta k-4j^2\alpha\beta k-91i\alpha\beta k+48ij\alpha\beta k-55j\alpha\beta k+39\alpha\beta k-13\beta k-16j^3\gamma k-2\alpha^3\gamma k-\beta^3\gamma k\\
&+32i^2\gamma k-68ij^2\gamma k+42j^2\gamma k+9i\alpha^2\gamma k-41j\alpha^2\gamma k+25\alpha^2\gamma k+13i\beta^2\gamma k+2j\beta^2\gamma k+3\alpha\beta^2\gamma k-10\beta^2\gamma k\\
&-57i\gamma k-28i^2j\gamma k+126ij\gamma k-43j\gamma k+4i^2\alpha\gamma k-68j^2\alpha\gamma k+55i\alpha\gamma k-48ij\alpha\gamma k+91j\alpha\gamma k-39\alpha\gamma k\\
&-4i^2\beta\gamma k+4j^2\beta\gamma k-9i\beta\gamma k+9j\beta\gamma k-22i\alpha\beta\gamma k+22j\alpha\beta\gamma k+13\gamma k+4i\alpha^4-4j\alpha^4-16i^3-24ij^3+16j^3\\
&+14i^2\alpha^3-14j^2\alpha^3-12i\alpha^3+12j\alpha^3-2i^2\beta^3+4i\beta^3-2i\alpha\beta^3+2\alpha\beta^3-2\beta^3+2j^2\gamma^3-4j\gamma^3+2j\alpha\gamma^3-2\alpha\gamma^3\\
&-2j\beta\gamma^3+2\beta\gamma^3+2\gamma^3+16i^2+32ij^2-16j^2+12i^3\alpha^2-12j^3\alpha^2-22i^2\alpha^2-28ij^2\alpha^2+22j^2\alpha^2-2i\alpha^2+28i^2j\alpha^2+2j\alpha^2\\
&+12i^3\beta^2-26i^2\beta^2+8i\alpha^2\beta^2-2j\alpha^2\beta^2-6\alpha^2\beta^2+18i\beta^2+12i^2j\beta^2-24ij\beta^2+12j\beta^2+18i^2\alpha\beta^2-28i\alpha\beta^2\\
&+6ij\alpha\beta^2-8j\alpha\beta^2+12\alpha\beta^2-4\beta^2-12j^3\gamma^2-12ij^2\gamma^2+26j^2\gamma^2+2i\alpha^2\gamma^2-8j\alpha^2\gamma^2+6\alpha^2\gamma^2+2i\beta^2\gamma^2\\
&-2j\beta^2\gamma^2-12i\gamma^2+24ij\gamma^2-18j\gamma^2-18j^2\alpha\gamma^2+8i\alpha\gamma^2-6ij\alpha\gamma^2+28j\alpha\gamma^2-12\alpha\gamma^2+8j^2\beta\gamma^2\\
&-8i\beta\gamma^2+6ij\beta\gamma^2-10j\beta\gamma^2-4i\alpha\beta\gamma^2+10j\alpha\beta\gamma^2-6\alpha\beta\gamma^2+4\beta\gamma^2+4\gamma^2-4i+24i^3j-32i^2j+4j-4i^3\alpha\\
&-24ij^3\alpha+4j^3\alpha-20i^2\alpha+4ij^2\alpha+20j^2\alpha+10i\alpha+24i^3j\alpha-4i^2j\alpha-10j\alpha+4i^3\beta-10i\alpha^3\beta+6j\alpha^3\beta+4\alpha^3\beta\\
&-12i^2\beta-24i^2j^2\beta+48ij^2\beta-24j^2\beta-30i^2\alpha^2\beta+8j^2\alpha^2\beta+36i\alpha^2\beta-6ij\alpha^2\beta+6j\alpha^2\beta-14\alpha^2\beta+10i\beta\\
&-24i^3j\beta+68i^2j\beta-60ij\beta+16j\beta-24i^3\alpha\beta+48i^2\alpha\beta+8j^2\alpha\beta-36i\alpha\beta-40i^2j\alpha\beta+72ij\alpha\beta-38j\alpha\beta\\
&+10\alpha\beta-2\beta+24ij^3\gamma-4j^3\gamma-6i\alpha^3\gamma+10j\alpha^3\gamma-4\alpha^3\gamma+2i\beta^3\gamma-2\beta^3\gamma+24i^2\gamma+24i^2j^2\gamma-68ij^2\gamma+12j^2\gamma-8i^2\alpha^2\gamma\\
&+30j^2\alpha^2\gamma-6i\alpha^2\gamma+6ij\alpha^2\gamma-36j\alpha^2\gamma+14\alpha^2\gamma-8i^2\beta^2\gamma+10i\beta^2\gamma-6ij\beta^2\gamma+8j\beta^2\gamma-10i\alpha\beta^2\gamma+4j\alpha\beta^2\gamma+6\alpha\beta^2\gamma\\
&-4\beta^2\gamma-16i\gamma-48i^2j\gamma+60ij\gamma-10j\gamma+24j^3\alpha\gamma-8i^2\alpha\gamma+40ij^2\alpha\gamma-48j^2\alpha\gamma+38i\alpha\gamma-72ij\alpha\gamma+36j\alpha\gamma-10\alpha\gamma+8i^2\beta\gamma\\
&-8j^2\beta\gamma+14i\alpha^2\beta\gamma-14j\alpha^2\beta\gamma-6i\beta\gamma+6j\beta\gamma+16i^2\alpha\beta\gamma-16j^2\alpha\beta\gamma-4i\alpha\beta\gamma+4j\alpha\beta\gamma+2\gamma)S(k+1)\\
&-(-9\beta k^4+9\gamma k^4-7\beta^2k^3+7\gamma^2k^3-18ik^3+18jk^3-18i\alpha k^3+18j\alpha k^3+39i\beta k^3+21j\beta k^3+15\alpha\beta k^3-54\beta k^3-21i\gamma k^3-39j\gamma k^3\\
&-15\alpha\gamma k^3+54\gamma k^3-\beta^3k^2+\gamma^3k^2+42i^2k^2-42j^2k^2+30i\alpha^2k^2-30j\alpha^2k^2+27i\beta^2k^2+7j\beta^2k^2+6\alpha\beta^2k^2-34\beta^2k^2-7i\gamma^2k^2\\
&-27j\gamma^2k^2-6\alpha\gamma^2k^2+3\beta\gamma^2k^2+34\gamma^2k^2-80ik^2+80jk^2+42i^2\alpha k^2-42j^2\alpha k^2-50i\alpha k^2+50j\alpha k^2-54i^2\beta k^2-12j^2\beta k^2-6\alpha^2\beta k^2\\
&+166i\beta k^2-78ij\beta k^2+114j\beta k^2-66i\alpha\beta k^2-8j\alpha\beta k^2+73\alpha\beta k^2-115\beta k^2+12i^2\gamma k^2+54j^2\gamma k^2+6\alpha^2\gamma k^2-3\beta^2\gamma k^2\\
&-114i\gamma k^2+78ij\gamma k^2-166j\gamma k^2+8i\alpha\gamma k^2+66j\alpha\gamma k^2-73\alpha\gamma k^2+8i\beta\gamma k^2-8j\beta\gamma k^2+115\gamma k^2-24i^3k+24j^3k-12i\alpha^3k\\
&+12j\alpha^3k+3i\beta^3k-4\beta^3k-3j\gamma^3k+4\gamma^3k+136i^2k+48ij^2k-136j^2k-44i^2\alpha^2k+44j^2\alpha^2k+86i\alpha^2k-86j\alpha^2k-32i^2\beta^2k\\
&+85i\beta^2k-20ij\beta^2k+28j\beta^2k-19i\alpha\beta^2k-j\alpha\beta^2k+22\alpha\beta^2k-52\beta^2k+32j^2\gamma^2k-28i\gamma^2k+20ij\gamma^2k-85j\gamma^2k+i\alpha\gamma^2k\\
&+19j\alpha\gamma^2k-22\alpha\gamma^2k-i\beta\gamma^2k-10j\beta\gamma^2k+10\beta\gamma^2k+52\gamma^2k-116ik-48i^2jk+116jk-24i^3\alpha k+24j^3\alpha k+92i^2\alpha k\\
&+48ij^2\alpha k-92j^2\alpha k-18i\alpha k-48i^2j\alpha k+18j\alpha k+24i^3\beta k-152i^2\beta k+36ij^2\beta k\\
&-56j^2\beta k+28i\alpha^2\beta k-8j\alpha^2\beta k-22\alpha^2\beta k+220i\beta k+84i^2j\beta k-264ij\beta k+194j\beta k+76i^2\alpha\beta k-8j^2\alpha\beta k-205i\alpha\beta k\\
&+48ij\alpha\beta k-41j\alpha\beta k+112\alpha\beta k-100\beta k-24j^3\gamma k+56i^2\gamma k-84ij^2\gamma k+152j^2\gamma k+8i\alpha^2\gamma k-28j\alpha^2\gamma k+22\alpha^2\gamma k+10i\beta^2\gamma k\\
&+j\beta^2\gamma k-10\beta^2\gamma k-194i\gamma k-36i^2j\gamma k+264ij\gamma k-220j\gamma k+8i^2\alpha\gamma k-76j^2\alpha\gamma k+41i\alpha\gamma k-48ij\alpha\gamma k+205j\alpha\gamma k\\
&-112\alpha\gamma k-8i^2\beta\gamma k+8j^2\beta\gamma k+21i\beta\gamma k-21j\beta\gamma k-18i\alpha\beta\gamma k+18j\alpha\beta\gamma k+100\gamma k-48i^3-24ij^3+48j^3+8i^2\alpha^3\\
&-8j^2\alpha^3-24i\alpha^3+24j\alpha^3-2i^2\beta^3+6i\beta^3-4\beta^3+2j^2\gamma^3-6j\gamma^3+4\gamma^3+104i^2+80ij^2-104j^2+12i^3\alpha^2-12j^3\alpha^2-72i^2\alpha^2-20ij^2\alpha^2\\
&+72j^2\alpha^2+52i\alpha^2+20i^2j\alpha^2-52j\alpha^2+12i^3\beta^2-50i^2\beta^2+62i\beta^2+12i^2\beta^2-40ij\beta^2+28j\beta^2+12i^2\alpha\beta^2-36i\alpha\beta^2+4ij\alpha\beta^2\\
&+20\alpha\beta^2-24\beta^2-12j^3\gamma^2-12ij^2\gamma^2+50j^2\gamma^2-28i\gamma^2+40ij\gamma^2-62j\gamma^2-12j^2\alpha\gamma^2-4ij\alpha\gamma^2+36j\alpha\gamma^2\\
&-20\alpha\gamma^2+6j^2\beta\gamma^2+4ij\beta\gamma^2-18j\beta\gamma^2+8\beta\gamma^2+24\gamma^2-56i+24i^3j-80i^2j+56j-36i^3\alpha-24ij^3\alpha+36j^3\alpha+24i^2\alpha+60ij^2\alpha\\
&-24j^2\alpha+20i\alpha+24i^3j\alpha-60i^2j\alpha-20j\alpha+36i^3\beta-96i^2\beta-24i^2j^2\beta+76ij^2\beta-64j^2\beta-18i^2\alpha^2\beta+6j^2\alpha^2\beta+54i\alpha^2\beta\\
&-4ij\alpha^2\beta-18j\alpha^2\beta-20\alpha^2\beta+88i\beta-24i^3j\beta+144i^2j\beta-208ij\beta+100j\beta-24i^3\alpha\beta+122i^2\alpha\beta-14j^2\alpha\beta-146i\alpha\beta\\
&-32i^2j\alpha\beta+92ij\alpha\beta-50j\alpha\beta+52\alpha\beta-28\beta+24ij^3\gamma-36j^3\gamma+64i^2\gamma+24i^2j^2\gamma-144ij^2\gamma+96j^2\gamma-6i^2\alpha^2\gamma+18j^2\alpha^2\gamma\\
&+18i\alpha^2\gamma+4ij\alpha^2\gamma-54j\alpha^2\gamma+20\alpha^2\gamma-6i^2\beta^2\gamma+18i\beta^2\gamma-4i\beta^2\gamma-8\beta^2\gamma-100i\gamma-76i^2j\gamma+208ij\gamma-88j\gamma\\
&+24j^3\alpha\gamma+14i^2\alpha\gamma+32ij^2\alpha\gamma-122j^2\alpha\gamma+50i\alpha\gamma-92ij\alpha\gamma+146j\alpha\gamma-52\alpha\gamma-14i^2\beta\gamma+14j^2\beta\gamma\\
&+10i\beta\gamma-10j\beta\gamma+12i^2\alpha\beta\gamma-12j^2\alpha\beta\gamma-36i\alpha\beta\gamma+36j\alpha\beta\gamma+28\gamma)S(k+2)\\
&-(i+j-k-2)(7\beta k^3-7\gamma k^3+3\beta^2k^2-3\gamma^2k^2+14ik^2-14jk^2+14i\alpha k^2-14j\alpha k^2-22i\beta k^2-8j\beta k^2-6\alpha\beta k^2+42\beta k^2+8i\gamma k^2\\
&+22j\gamma k^2+6\alpha\gamma k^2-42\gamma k^2-16i^2k+16j^2k-12i\alpha^2k+12j\alpha^2k\\
&-8i\beta^2k+13\beta^2k+8j\gamma^2k-13\gamma^2k+66ik-66jk-16i^2\alpha k+16j^2\alpha k+54i\alpha k-54j\alpha k+16i^2\beta k-94i\beta k+20ij\beta k-40j\beta k+20i\alpha\beta k\\
&-4j\alpha\beta k-26\alpha\beta k+75\beta k-16j^2\gamma k+40i\gamma k-20ij\gamma k+94j\gamma k+4i\alpha\gamma k-20j\alpha\gamma k+26\alpha\gamma k-4i\beta\gamma k+4j\beta\gamma k-75\gamma k-48i^2\\
&-8ij^2+48j^2+4i^2\alpha^2-4j^2\alpha^2-36i\alpha^2+36j\alpha^2+4i^2\beta^2-20i\beta^2+12\beta^2-4j^2\gamma^2+20j\gamma^2-12\gamma^2+72i+8i^2j-72j\\
&-44i^2\alpha-8ij^2\alpha+44j^2\alpha+36i\alpha+8i^2j\alpha-36j\alpha+44i^2\beta-84i\beta-8i^2j\beta+56ij\beta-48j\beta-8i^2\alpha\beta+56i\alpha\beta\\
&-16j\alpha\beta-24\alpha\beta+36\beta+8ij^2\gamma-44j^2\gamma+48i\gamma-56ij\gamma+84j\gamma+8j^2\alpha\gamma+16i\alpha\gamma-56j\alpha\gamma+24\alpha\gamma-16i\beta\gamma+16j\beta\gamma-36\gamma)S(k+3)\\
&+2(i+j-k-3)(i+j-k-2)(k+4)(2\alpha i-2\beta i+2i-2j-2j\alpha+k\beta+\beta+2j\gamma-k\gamma-\gamma)S(k+4)=0.
\end{aligned}
$$