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# On Solutions of Holonomic Divided-Difference Equations on Nonuniform Lattices

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**Abstract:** The main aim of this paper is the development of suitable bases which enable the direct series representation of orthogonal polynomial systems on nonuniform lattices (quadratic lattices of a discrete or a $q$-discrete variable). We present two bases of this type, the first of which allows to write solutions of arbitrary divided-difference equations in terms of series representations extending results given in [16] for the $q$-case. Furthermore it enables the representation of the Stieltjes function which can be used to prove the equivalence between the Pearson equation for a given linear functional and the Riccati equation for the formal Stieltjes function.

If the Askey-Wilson polynomials are written in terms of this basis, however, the coefficients turn out to be not $q$-hypergeometric. Therefore, we present a second basis, which shares several relevant properties with the first one. This basis enables to generate the defining representation of the Askey-Wilson polynomials directly from their divided-difference equation. For this purpose the divided-difference equation must be rewritten in terms of suitable divided-difference operators developed in [5], see also [6].

**Keywords:** Keywords: Askey-Wilson polynomials, Nonuniform lattices, Difference equations, Divided-difference equations, Stieltjes function.

**Classification: MSC** Primary 33D45, Secondary 39A13

## 1. Introduction

Classical orthogonal polynomials on a nonuniform lattice satisfy an equation of the type [2,6,17]

$$\left\{\phi(x(s))\frac{\Delta}{\nabla x_1(s)}\frac{\nabla}{\nabla x(s)}+\frac{\psi(x(s))}{2}\left[\frac{\Delta}{\Delta x(s)}+\frac{\nabla}{\nabla x(s)}\right]+\lambda_n\right\}P_n(x(s))=0,\ n\geq 0, \tag{1}$$

where $\phi$ and $\psi$ are polynomials of maximal degree two and one respectively, $\lambda_n$ is a constant depending on the integer $n$ and the leading coefficients $\phi_2$ and $\psi_1$ of $\phi$ and $\psi$:

$$\lambda_n=-\gamma_n\,(\phi_2\,\gamma_{n-1}+\psi_1\,\alpha_n), \tag{2}$$

and $x(s)$ is a nonuniform lattice defined by

$$x(s)=\begin{cases} c_1\,q^s+c_2\,q^{-s}+c_3 & \text{if} \quad q\neq 1\\ c_4\,s^2+c_5\,s+c_6 & \text{if} \quad q=1.\end{cases} \tag{3}$$

Here, $\Delta$ and $\nabla$ are the forward and the backward operators

$$\Delta f(x(s)):=\Delta f(s)=f(s+1)-f(s),\ \nabla f(x(s)):=\Delta f(s)=f(s)-f(s-1),$$

and

$$x_\mu(s)=x(s+\frac{\mu}{2}),\ \mu\in\mathbb{C},$$

where $\mathbb{C}$ is the set of complex numbers. The lattices (3) satisfy

$$x(s+k)-x(s) = \gamma_k\,\nabla\,x_{k+1}(s), \tag{4}$$

$$\frac{x(s+k)+x(s)}{2} = \alpha_k\,x_k(s)+\beta_k, \tag{5}$$

for $k=0,\ 1,\ \ldots,$ with

$$\alpha_0=1,\ \alpha_1=\alpha,\ \beta_0=0,\ \beta_1=\beta,\gamma_0=0,\ \gamma_1=1,$$

where the sequences $(\alpha_k),\ (\beta_k),\ (\gamma_k)$ satisfy the following relations

$$\begin{aligned}\alpha_{k+1}-2\,\alpha\,\alpha_k+\alpha_{k-1} &= 0,\\ \beta_{k+1}-2\,\beta_k+\beta_{k-1} &= 2\,\beta\,\alpha_k,\\ \gamma_{k+1}-\gamma_{k-1} &= 2\,\alpha_k,\end{aligned}$$

and are given explicitly by [2,17]

$$\alpha_n=1,\ \beta_n=\beta\,n^2,\ \gamma_n=n,\ \text{for}\ \alpha=1, \tag{6}$$

and

$$\alpha_n=\frac{q^{\frac{n}{2}}+q^{-\frac{n}{2}}}{2},\ \beta_n=\frac{\beta(1-\alpha_n)}{1-\alpha},\ \gamma_n=\frac{q^{\frac{n}{2}}-q^{-\frac{n}{2}}}{q^{\frac{1}{2}}-q^{-\frac{1}{2}}},\ \text{for}\ \alpha=\frac{q^{\frac{1}{2}}+q^{-\frac{1}{2}}}{2}. \tag{7}$$

By means of the companion operators $\mathbb{D}_x$ and $\mathbb{S}_x$ [5,6], Equation (1) can be rewritten as

$$\phi(x(s))\,\mathbb{D}_x^2 P_n(x(s))+\psi(x(s))\,\mathbb{S}_x\mathbb{D}_x P_n(x(s))+\lambda_n\,P_n(x(s))=0, \tag{8}$$

where

$$\mathbb{D}_x\, f(x(s)) = \frac{f(x_{-1}(s+1)) - f(x_{-1}(s))}{x_{-1}(s+1) - x_{-1}(s)}, \quad \mathbb{S}_x\, f(x(s)) = \frac{f(x_{-1}(s+1)) + f(x_{-1}(s))}{2}.$$

These operators fulfil important relations—called product and quotient rules—which read, taking into account the shift (compared to the definition in [6]) in the definition of the above defined companion operators as

**Theorem 1.** *[6]*

1. *The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ satisfy the product rules I*

$$\mathbb{D}_x\,(f(x(s))g(x(s))) = \mathbb{S}_x f(x(s))\,\mathbb{D}_x g(x(s)) + \mathbb{D}_x f(x(s))\,\mathbb{S}_x g(x(s)), \tag{9}$$

$$\mathbb{S}_x\,(f(x(s))g(x(s))) = U_2(x_1(s))\,\mathbb{D}_x f(x(s))\,\mathbb{D}_x g(x(s)) + \mathbb{S}_x f(x(s))\,\mathbb{S}_x g(x(s)), \tag{10}$$

*where $U_2$ is a polynomial of degree 2*

$$U_2(x(s)) = (\alpha^2-1)\,x^2(s) + 2\,\beta\,(\alpha+1)\,x(s) + \delta_x,$$

*and $\delta_x$ is a constant depending on $\alpha$, $\beta$ and the initial values $x(0)$ and $x(1)$ of $x(s)$:*

$$\delta_x = \frac{x^2(0)+x^2(1)}{4\alpha^2} - \frac{(2\alpha^2-1)}{2\alpha^2}x(0)\,x(1) - \frac{\beta\,(\alpha+1)}{\alpha^2}(x(0)+x(1)) + \frac{\beta^2\,(\alpha+1)^2}{\alpha^2}. \tag{11}$$

2. *The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ also satisfy the quotient rules*

$$\mathbb{D}_x\left(\frac{f(x(s))}{g(x(s))}\right) = \frac{\mathbb{S}_x f(x(s))\,\mathbb{D}_x g(x(s)) - \mathbb{D}_x f(x(s))\,\mathbb{S}_x g(x(s))}{U_2(x(s))\,[\mathbb{D}_x g(x(s))]^2 - [\mathbb{S}_x g(x(s))]^2};$$

$$\mathbb{S}_x\left(\frac{f(x(s))}{g(x(s))}\right) = \frac{U_2(x(s))\,\mathbb{D}_x f(x(s))\,\mathbb{D}_x g(x(s)) - \mathbb{S}_x f(x(s))\,\mathbb{S}_x g(x(s))}{U_2(x(s))\,[\mathbb{D}_x g(x(s))]^2 - [\mathbb{S}_x g(x(s))]^2},$$

*provided that $g(x(s)) \neq 0,\ s \in (a,b)$.*

3. *More generally, relations (9)-(10) remain valid if we replace $x$ and $x_1$ by $x_\mu$ and $x_{\mu+1}$ respectively, $\mu \in \mathbb{C}$. In particular, the constant $\delta_x$ remains unchanged if we replace $x$ in (11) by $x_k,\ k \in \mathbb{Z}$, i.e.,*

$$\delta_{x_k} = \delta_x := \delta,\ k \in \mathbb{Z},$$

*where $\mathbb{Z}$ is the set of integers.*

4. *The operators $\mathbb{D}_x$ and $\mathbb{S}_x$ also satisfy the product rules II*

$$\mathbb{D}_x\,\mathbb{S}_x = \alpha\,\mathbb{S}_x\,\mathbb{D}_x + U_1(s)\,\mathbb{D}_x^2; \tag{12}$$

$$\mathbb{S}_x^2 = U_1(s)\,\mathbb{S}_x\,\mathbb{D}_x + \alpha\,U_2(s)\,\mathbb{D}_x^2 + \mathbb{I}, \tag{13}$$

*where*

$$U_1(s) := U_1(x(s)) = (\alpha^2-1)\,x(s) + \beta\,(\alpha+1), \quad U_2(s) := U_2(x(s)) \tag{14}$$

For illustration, the Askey-Wilson polynomials $P_n(x;a,b,c,d|q)$ are defined by

$$P_n(x;a,b,c,d|q) = {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}\middle| q;q\right),\ x=\cos\theta. \tag{15}$$

By taking $e^{i\theta}=q^s$, the lattice reads as $x(s)=\cos\theta=\frac{q^s+q^{-s}}{2}$. By using the orthogonality relation and the Pearson-type equation satisfied by the weight of the Askey-Wilson polynomials, Foupouagnigni [6] showed that the polynomials $P_n(x;a,b,c,d|q)$ satisfy a divided-difference equation of the type (8) with

$$\begin{aligned}\phi\left(x(s)\right) &= 2\left(dcba+1\right)x^2\left(s\right)-\left(a+b+c+d+abc+abd+acd+bcd\right)x\left(s\right)\\ &\qquad +ab+ac+ad+bc+bd+cd-abcd-1,\\ \psi\left(x(s)\right) &= \frac{4\left(abcd-1\right)q^{\frac{1}{2}}\,x\left(s\right)}{q-1}+\frac{2\left(a+b+c+d-abc-abd-acd-bcd\right)\,q^{\frac{1}{2}}}{q-1}.\end{aligned} \tag{16}$$

It should be recalled that the operators $\mathbb{D}_x$ and $\mathbb{S}_x$ transform polynomials of degree $n$ in $x(s)$ into a polynomial of degree $n-1$ and $n$ in the same variable, respectively. However, the application of these operators to the monomial $x^n(s)$ produces a linear combination (with complicated coefficients) of all monomials of degree less than or equal to $n-1$ and $n$ respectively; this makes the monomial basis $(x^n(s))_n$ not appropriate for the aforementioned operators [6].

The aim of this work is:

① To provide an appropriate basis for the companion operators, that is, a basis $(F_n(x(s)))_n$ such that each $F_n(x(s))$ is a polynomial of degree $n$ in $x(s)$ fulfiling

$$\mathbb{D}_x F_n(x(s)) = a_n\, F_{n-1}(x(s)),\ \ \mathbb{S}_x F_n(x(s)) = b_n\, F_n(x(s)) + c_n\, F_{n-1}(x(s)),$$

where $a_n,\ b_n$ and $c_n$ are given constants.

② To provide an algorithmic method to solve Equation (8) as series in terms of the new basis and to extend this result to solve arbitrary linear divided-difference equations with polynomial coefficients involving only products of operators $\mathbb{D}_x$ and $\mathbb{S}_x$.

③ To use another appropriate basis for the operators $\mathbb{D}_x^2$ and $\mathbb{S}_x\,\mathbb{D}_x$ to derive representation (8) for the Askey-Wilson polynomials from the hypergeometric representation (15) without making use of the weight function.

④ To solve explicitly an equation of type (8) and to extend this result to solve arbitrary linear divided-difference equations with polynomial coefficients involving only products of operators $\mathbb{D}_x^2$ and $\mathbb{S}_x\mathbb{D}_x$.

⑤ To provide a new representation of the formal Stieltjes function of a given linear functional on a nonuniform lattice, and deduce from it various important properties connecting the functional approach and the one based on the Riccati equation for the formal Stieltjes function.

The content of this paper is organized as follows. In section 1, we recall necessary preliminaries, while in the second section, we provide the basis $(F_k)_k$ compatible with the companion operators $\mathbb{D}_x$ and $\mathbb{S}_x$. The third section deals with the algorithmic series solutions of divided-difference equations in terms of the basis $(F_k)_k$. In section 4, we give the second basis $(B_k)_k$ compatible not with the companion operators but rather with their products $\mathbb{D}_x^2$ and $\mathbb{S}_x\mathbb{D}_x$, and use this basis in the fifth section to find the algorithmic series solutions of some divided difference equations in terms of the basis $(B_k)_k$. In the last section, we apply the basis $(F_k)_k$ to provide new representation of the formal Stieltjes series and deduce its corresponding properties. Basic exponential and basic trigonometric functions have also been expanded in terms of the basis $(F_k)_k$.

## 2. A New Basis Compatible with the Companion Operators

Using the generalized basis for the lattice $x(s)$ defined by Suslov [17] (see also [2]) as

$$[x_m(z)-x_m(s)]^{(n)}=\prod_{j=0}^{n-1}[x_m(z)-x_m(s-j)],\ n\geq 1,\ m\geq 0,\ [x_m(z)-x_m(s)]^{(0)}\equiv 1, \tag{17}$$

we define the function $F_n$ by

$$F_n(x(s))=(-1)^n\,[x(z_x)-x_{n-1}(s)]^{(n)},\ n\geq 1,\ F_0(x(s))\equiv 1, \tag{18}$$

where $z_x$, which is a constant term with respect to $z$ but depending on the lattice $x$, fulfils the relations

$$q^{2z_x}=\frac{c_2}{c_1}q^{\frac{-1}{2}} \tag{19}$$

for the $q$-quadratic lattice $x(s)=c_1\,q^s+c_2\,q^{-s}+\frac{\beta}{1-\alpha},\ q\neq 1,\ c_1\,c_2\neq 0$, or the relation

$$\text{and } z_x=-\frac{1}{4}-\frac{c_5}{2c_4} \tag{20}$$

for the quadratic lattice $x(s)=c_4\,s^2+c_5\,s+c_6,\ c_4\neq 0$; and obtain the following properties:

**Theorem 2.**

$$\mathbb{D}_xF_n(x(s)) = \gamma_n\,F_{n-1}(x(s)); \tag{21}$$

$$\mathbb{S}_xF_n(x(s)) = \alpha_n\,F_n(x(s))+\frac{\gamma_n}{2}\nabla x_{n+1}(z_x)F_{n-1}(x(s)); \tag{22}$$

$$\mathbb{D}_x\frac{1}{F_n(x(s))} = -\frac{\gamma_n}{F_{n+1}(x(s))}; \tag{23}$$

$$\mathbb{S}_x\frac{1}{F_n(x(s))} = \frac{\alpha_n}{F_n(x(s))}+\frac{\gamma_n}{2}\frac{\nabla x_{n+2}(z_x)}{F_{n+1}(x(s))}; \tag{24}$$

$$F_{n+1}(x(s)) = (x(s)-x_{n+1}(z_x))\,F_n(x(s))=\prod_{j=1}^{n+1}(x(s)-x_j(z_x)),\ n\geq 0; \tag{25}$$

$$F_n(x_k(z_x)) \neq 0,\ \ \forall n\geq 0,\ \forall k>n\geq 0; \tag{26}$$

$$\frac{F_n(x(s)}{F_1(x(s)} = F_{n-1}(x(s))+\sum_{j=0}^{n-2}C_j\,F_j(x(s)),\ \textit{with}\ C_j=\prod_{i=j+2}^{n}(x(z_x)-x_i(z_x)). \tag{27}$$

In order to prove Theorem 2, we first prove the following propositions:

**Proposition 3.** *The following relations hold:*

$$[x_m(z)-x_m(s)]^{(n)} = \prod_{j=0}^{n-1}[x_{m-n+1}(z+j)-x_{m-n+1}(s)],\ n,\, m\geq 0, \tag{28}$$
$$\mathcal{E}_1(i,k,z,s) = \mathcal{E}_2(i,k,z,s),\ z\in\mathbb{C},\ i,\, k\in\mathbb{N}, \tag{29}$$

*where* $\mathbb{N}$ *is the set of nonnegative integer numbers, and*

$$\mathcal{E}_1(i,k,z,s) = [x_{-i}(z+i)-x_{-i}(s)]\,[x_{-k+1}(z+i)-x_{-k+1}(s-1)],$$
$$\mathcal{E}_2(i,k,z,s) = [x_{-k}(z+i)-x_{-k}(s)]\,[x_{i+1}(z)-x_{-i-1}(s)].$$

**Proposition 4.**

$$\mathbb{D}_x f_n(x_{n-1}(z),x_{n-1}(s)) = -\gamma_n\, f_{n-1}(x_{n-2}(z+\frac{1}{2}),x_{n-2}(s)) \tag{30}$$
$$\mathbb{S}_x f_n(x_{n-1}(z),x_{n-1}(s)) = \alpha_n\, f_n\left(x_{n-1}(z+\frac{1}{2}\right),x_{n-1}(s)) -\frac{\gamma_n}{2}\,\nabla x_{2n}(z) f_{n-1}\left(x_{n-2}\left(z+\frac{1}{2}\right),x_{n-2}(s)\right). \tag{31}$$

*Proof:* (Proof of Proposition 3)
The proof of Relation (29) is obtained by direct computation, using the expression of the lattice given by (3).

For the proof of Relation (28), we define the function $f_n$ and $g_n$ as:

$$f_n(x_m(z),x_m(s)) := \prod_{j=0}^{n-1}[x_m(z)-x_m(s-j)],\ n\geq 1,\ m\geq 0,\ f_0(x_m(z),x_m(s))\equiv 1, \tag{32}$$
$$g_n(x_m(z),x_m(s)) = \prod_{j=0}^{n-1}[x_{m-n+1}(z+j)-x_{m-n+1}(s)],\ n\geq 1,\ m\geq 0,\ g_0((x_m(z),x_m(s)))\equiv 1, \tag{33}$$

and prove by induction on $n$ that $f_n(x_m(z),x_m(s))=g_n(x_m(z),x_m(s))$, $\forall z\in\mathbb{C}$, and for a fixed $m\in\mathbb{N}$. For $n=1$, we have

$$f_1(x_m(z),x_m(s))=x_m(z)-x_m(s)=g_1(x_m(z),x_m(s)).$$

For $n=2$, we have

$$f_2(x_m(z),x_m(s))=(x_m(z)-x_m(s))(x_m(z)-x_m(s-1))=\mathcal{E}_1(0,1,z+\frac{m}{2},s+\frac{m}{2}).$$

Use of Relation (29) and relation $x_m(t)=x(t+\frac{m}{2})$, $\forall\, t\in\mathbb{C}$, transform the previous equation into

$$\begin{aligned} f_2(x_m(z),x_m(s)) &= \mathcal{E}_1(0,1,z+\frac{m}{2},s+\frac{m}{2}) \\ &= \mathcal{E}_2(0,1,z+\frac{m}{2},s+\frac{m}{2}) \\ &= (x_{m-1}(z)-x_{m-1}(s))(x_{m-1}(z+1)-x_{m-1}(s)) \\ &= g_2(x_m(z),x_m(s)). \end{aligned}$$

Now we assume the following induction's hypothesis for a fixed $m \in \mathbb{N}$ and for a fixed $n > 1$:

$$f_k(x_m(z), x_m(s)) = g_k(x_m(z), x_m(s)), \; \forall\, z \in \mathbb{C}, \; \forall\, k \in \mathbb{N}, \; with : 1 \leq k \leq n. \tag{34}$$

In the first step, we start from Relation (32) and use Relation (34), to obtain

$$\begin{aligned} f_{n+1}(x_m(z), x_m(s)) &= \prod_{j=0}^{n} (x_m(z) - x_m(s-j)) \\ &= (x_m(z) - x_m(s-n))\, f_n(x_m(z), x_m(s)) \\ &= (x_m(z) - x_m(s-n))\, g_n(x_m(z), x_m(s)). \end{aligned}$$

In the second step, we use the definition of $g_n$ given by (33) to transform the previous equation into

$$\begin{aligned} f_{n+1}(x_m(z), x_m(s)) &= (x_m(z) - x_m(s-n))\, g_n(x_m(z), x_m(s)) \\ &= (x_m(z) - x_m(s-n)) \prod_{j=0}^{n-1} (x_{m-n+1}(z+j) - x_{m-n+1}(s)) \\ &= (x_m(z) - x_m(s-n))(x_{m-n+1}(z+n-1) - x_{m-n+1}(s)) \\ &\times \prod_{j=0}^{n-2} (x_{m-n+1}(z+j) - x_{m-n+1}(s)). \end{aligned}$$

In the third step, we first use the Relation $x_{m-n+1}(t) = x_{m-n+2}(t - \frac{1}{2})$, then Relation (33), to transform the previous equation into

$$\begin{aligned} f_{n+1}(x_m(z), x_m(s)) &= (x_m(z) - x_m(s-n))(x_{m-n+1}(z+n-1) - x_{m-n+1}(s)) \\ &\quad \times \prod_{j=0}^{n-2} (x_{m-n+2}(z - \frac{1}{2} + j) - x_{m-n+2}(s - \frac{1}{2})) \\ &= (x_m(z) - x_m(s-n))(x_{m-n+1}(z+n-1) - x_{m-n+1}(s)) \\ &\quad \times g_{n-1}(x_m(z - \frac{1}{2}), x_m(s - \frac{1}{2})). \end{aligned}$$

In the fourth step, we combine Equations (29), (34) and the relation $x_\mu(z) = x(z + \frac{\mu}{2}), \; \forall \mu, \; z \in \mathbb{C}$, to obtain from the previous equation

$$\begin{aligned} f_{n+1}(x_m(z), x_m(s)) &= (x_m(z) - x_m(s-n))(x_{m-n+1}(z+n-1) - x_{m-n+1}(s)) \\ &\quad \times g_{n-1}(x_m(z - \frac{1}{2}), x_m(s - \frac{1}{2})) \\ &= \mathcal{E}_1(n-1, 2n-1, z + \frac{m}{2}, s + \frac{m}{2})\, g_{n-1}(x_m(z - \frac{1}{2}), x_m(s - \frac{1}{2})) \\ &= \mathcal{E}_2(n-1, 2n-1, z + \frac{m}{2}, s + \frac{m}{2})\, f_{n-1}(x_m(z - \frac{1}{2}), x_m(s - \frac{1}{2})) \\ &= (x_{m-2n+1}(z+n-1) - x_{m-2n+1}(s))(x_{m+n}(z) - x_{m-n}(s)) \\ &\quad \times f_{n-1}(x_m(z - \frac{1}{2}), x_m(s - \frac{1}{2})). \end{aligned}$$

Therefore, using again relation $x_\mu(z) = x(z+\frac{\mu}{2}),\ \forall \mu,\ z \in \mathbb{C}$, and Relation (32), we have

$$\begin{aligned} f_{n+1}(x_m(z), x_m(s)) &= (x_{m-1}(z) - x_{m-1}(s-n+1))(x_{m-n}(z+n) - x_{m-n}(s)) \\ &\quad \times f_{n-1}(x_{m-1}(z), x_{m-1}(s)) \\ &= (x_{m-1}(z) - x_{m-1}(s-n+1))(x_{m-n}(z+n) - x_{m-n}(s)) \\ &\quad \times \prod_{j=0}^{n-2}(x_{m-1}(z) - x_{m-1}(s-j)) \\ &= (x_{m-n}(z+n) - x_{m-n}(s)) \prod_{j=0}^{n-1}(x_{m-1}(z) - x_{m-1}(s-j)) \\ &= (x_{m-n}(z+n) - x_{m-n}(s)) f_n(x_{m-1}(z), x_{m-1}(s)) \\ &= (x_{m-n}(z+n) - x_{m-n}(s)) f_n(x_m(z-\frac{1}{2}), x_m(s-\frac{1}{2})). \end{aligned}$$

In the last step, we use the previous equation and the induction's hypothesis (34) combined with (33) to obtain

$$\begin{aligned} f_{n+1}(x_m(z), x_m(s)) &= (x_{m-n}(z+n) - x_{m-n}(s)) f_n(x_m(z-\frac{1}{2}), x_m(s-\frac{1}{2})) \\ &= (x_{m-n}(z+n) - x_{m-n}(s)) g_n(x_m(z-\frac{1}{2}), x_m(s-\frac{1}{2})) \\ &= (x_{m-n}(z+n) - x_{m-n}(s)) \prod_{j=0}^{n-1}(x_{m-n+1}(z-\frac{1}{2}+j) - x_{m-n+1}(s-\frac{1}{2})) \\ &= (x_{m-n}(z+n) - x_{m-n}(s)) \prod_{j=0}^{n-1}(x_{m-n}(z+j) - x_{m-n}(s))) \\ &= \prod_{j=0}^{n}(x_{m-n}(z+j) - x_{m-n}(s))) \\ &= g_{n+1}(x_m(z), x_m(s)). \end{aligned}$$

□

**Proof of Proposition 4**

*Proof:* We will now use the results of Proposition 3 to prove Proposition 32. In order to prove relation (30), we proceed in the following steps.

In step 1, we apply the operator $\mathbb{D}_x$ on both sides of relation (32) and multiply both sides of the equation obtained by $\Delta x_{-1}(s)$ to get

$$\begin{aligned} \Delta x_{-1}(s) \quad &\times \quad \mathbb{D}_x f_n(x_{n-1}(z), x_{n-1}(s)) \\ &= \prod_{j=0}^{n-1}(x_{n-1}(z) - x_{n-1}(s-j+\frac{1}{2})) - \prod_{j=0}^{n-1}(x_{n-1}(z) - x_{n-1}(s-j-\frac{1}{2})). \end{aligned} \tag{35}$$

In the second step, we first use the following relation, obtained by shifting the index $j$

$$\prod_{j=0}^{n-1}(x_{n-1}(z) - x_{n-1}(s-j+\frac{1}{2})) = (x_{n-1}(z) - x_{n-1}(s+\frac{1}{2})) \prod_{j=0}^{n-2}(x_{n-1}(z) - x_{n-1}(s-j-\frac{1}{2})), \tag{36}$$

then, the relation obtained by using Relation (4) and $x_\mu(z) = x(z+\frac{\mu}{2})$

$$x_{n-1}(s+\frac{1}{2}) - x_{n-1}(s-n+\frac{1}{2}) = \gamma_n \nabla x_{2n}(s-n+\frac{1}{2}) = \gamma_n \nabla x_1(s) = \gamma_n \Delta x_{-1}(s),$$

90 to transform (35) into

$$\begin{aligned}
\Delta x_{-1}(s)\, \mathbb{D}_x f_n(x_{n-1}(z), x_{n-1}(s)) &= -[x_{n-1}(s+\frac{1}{2}) - x_{n-1}(s-n+\frac{1}{2})] \prod_{j=0}^{n-2}(x_{n-1}(z) - x_{n-1}(s-j-\frac{1}{2})) \\
&= -\gamma_n \Delta x_{-1}(s) \prod_{j=0}^{n-2}(x_{n-1}(z) - x_{n-1}(s-j-\frac{1}{2})) \\
&= -\gamma_n \Delta x_{-1}(s) \prod_{j=0}^{n-2}(x_{n-2}(z+\frac{1}{2}) - x_{n-2}(s-j)) \\
&= -\gamma_n \Delta x_{-1}(s)\, f_{n-1}(x_{n-2}(z+\frac{1}{2}), x_{n-2}(s)).
\end{aligned}$$

91 The proof of relation (31), will also be done in the following steps:

92 In the first step, we apply operator $\mathbb{S}_x$ on both sides of relation (32) and use Relation (36) to obtain

$$\begin{aligned}
\mathbb{S}_x f_n(x_{n-1}(z), x_{n-1}(s)) &= \frac{1}{2}\prod_{j=0}^{n-1}(x_{n-1}(z) - x_{n-1}(s-j+\frac{1}{2})) + \frac{1}{2}\prod_{j=0}^{n-1}(x_{n-1}(z) - x_{n-1}(s-j-\frac{1}{2}) \\
&= \frac{1}{2}[x_{n-1}(z) - x_{n-1}(s+\frac{1}{2})] \prod_{j=0}^{n-2}(x_{n-1}(z) - x_{n-1}(s-j-\frac{1}{2})) \\
&+ \frac{1}{2}[x_{n-1}(z) - x_{n-1}(s-n+\frac{1}{2})] \prod_{j=0}^{n-2}(x_{n-1}(z) - x_{n-1}(s-j-\frac{1}{2}) \\
&= \frac{1}{2}[2\, x_{n-1}(z) - x_{n-1}(s-n+\frac{1}{2}) - x_{n-1}(s+\frac{1}{2})] \prod_{j=0}^{n-2}(x_{n-1}(z) - x_{n-1}(s-j-\frac{1}{2}).
\end{aligned}$$

In the second step, we use the relation

$$x_{n-1}(s-n+\frac{1}{2}) + x_{n-1}(s+\frac{1}{2}) = x_n(s-n) + x_n(s) = 2\alpha_n\, x_{2n}(s-n) + 2\beta_n = 2\alpha_n\, x(s) + 2\beta_n$$

93 obtained from (5) and the relation $x_\mu(z) = x_\mu(z+\frac{\mu}{2})$, to get

$$\begin{aligned}
\mathbb{S}_x f_n(x_{n-1}(z), x_{n-1}(s)) &= [x_{n-1}(z) - \alpha_n\, x(s) - \beta_n)] \prod_{j=0}^{n-2}(x_{n-1}(z) - x_{n-1}(s-j-\frac{1}{2}) \\
&= [x_{n-1}(z) - \alpha_n\, x(s) - \beta_n)]\, f_{n-1}(x_{n-1}(z), x_{n-1}(s-\frac{1}{2}).
\end{aligned}$$

94 In the third step, we use Relation (28) to transform the previous equation into

$$
\begin{aligned}
\mathbb{S}_x f_n(x_{n-1}(z), x_{n-1}(s)) &= [x_{n-1}(z) - \alpha_n\, x(s) - \beta_n)]\, f_{n-1}(x_{n-1}(z), x_{n-1}(s - \frac{1}{2})\\
&= [x_{n-1}(z) - \alpha_n\, x(s) - \beta_n)]\, g_{n-1}(x_{n-1}(z), x_{n-1}(s - \frac{1}{2})\\
&= [x_{n-1}(z) - \alpha_n\, x(s) - \beta_n] \prod_{j=0}^{n-2} (x_1(z+j) - x_1(s - \frac{1}{2})\\
&= \left[\alpha_n [x(z + \frac{1}{2} + n - 1) - x(s)] + x_{n-1}(z) - \alpha_n\, x(z + \frac{1}{2} + n - 1) - \beta_n\right]\\
&\times \prod_{j=0}^{n-2} (x(z + \frac{1}{2} + j) - x(s))\\
&= \alpha_n \prod_{j=0}^{n-1} (x(z + \frac{1}{2} + j) - x(s))\\
&+ \left[x_{n-1}(z) - \alpha_n\, x(z - \frac{1}{2} + n) - \beta_n\right] \prod_{j=0}^{n-2} (x(z + \frac{1}{2} + j) - x(s))\\
&= \alpha_n\, g_n(x_{n-1}(z + \frac{1}{2}), x_{n-1}(s))\\
&+ \left[x_{n-1}(z) - \alpha_n\, x(z - \frac{1}{2} + n) - \beta_n\right] g_{n-1}(x_{n-2}(z + \frac{1}{2}), x_{n-2}(s)).
\end{aligned}
$$

95 Therefore, using again Relation (28) we get

$$
\begin{aligned}
\mathbb{S}_x f_n(x_{n-1}(z), x_{n-1}(s)) &= \alpha_n\, f_n(x_{n-1}(z + \frac{1}{2}), x_{n-1}(s))\\
&+ \left[x_{n-1}(z) - \alpha_n\, x_n(z - \frac{1}{2} + \frac{n}{2}) - \beta_n\right] f_{n-1}(x_{n-2}(z + \frac{1}{2}), x_{n-2}(s)).
\end{aligned}
\tag{37}
$$

In the fourth step, we use the following relation obtained from (5)

$$
x(z - \frac{1}{2} + \frac{n}{2}) + x(z - \frac{1}{2} + \frac{n}{2} + n) = 2\alpha_n\, x_n(z - \frac{1}{2} + \frac{n}{2}) + 2\beta_n
$$

96 to transform (37) as

$$
\begin{aligned}
\mathbb{S}_x f_n(x_{n-1}(z), x_{n-1}(s)) &= \alpha_n\, f_n(x_{n-1}(z + \frac{1}{2}), x_{n-1}(s))\\
&+ \left[x_{n-1}(z) - \frac{x_n(z - \frac{1}{2}) + x_n(z - \frac{1}{2} + n)}{2}\right] f_{n-1}(x_{n-2}(z + \frac{1}{2}), x_{n-2}(s))\\
&= \alpha_n\, f_n(x_{n-1}(z + \frac{1}{2}), x_{n-1}(s))\\
&+ \left[\frac{x_n(z - \frac{1}{2}) - x_n(z - \frac{1}{2} + n)}{2}\right] f_{n-1}(x_{n-2}(z + \frac{1}{2}), x_{n-2}(s))\\
&= \alpha_n\, f_n(x_{n-1}(z + \frac{1}{2}), x_{n-1}(s))\\
&+ \left[\frac{x_{n-1}(z) - x_{n-1}(z + n)}{2}\right] f_{n-1}(x_{n-2}(z + \frac{1}{2}), x_{n-2}(s)).
\end{aligned}
$$

In the last step, we use the following relation obtained from (4)

$$
\frac{x_{n-1}(z) - x_{n-1}(z + n)}{2} = -\gamma_n\, \nabla x_{2n}(z),
$$

to obtain the equation

$$\mathbb{S}_x f_n(x_{n-1}(z), x_{n-1}(s)) = \alpha_n\, f_n(x_{n-1}(z+\frac{1}{2}), x_{n-1}(s)) - \frac{\gamma_n}{2}\,\nabla x_{2n}(z) f_{n-1}(x_{n-2}(z+\frac{1}{2}), x_{n-2}(s)),$$

and the proof is complete. □

**Proof of Theorem 2**

*Proof:* By replacing $z$ by $z - \frac{n-1}{2}$ in Equations (30) and (31), we obtain

$$\begin{aligned}
\mathbb{D}_x f_n(x(z), x_{n-1}(s)) &= -\gamma_n\, f_{n-1}(x(z), x_{n-2}(s));\\
\mathbb{S}_x f_n(x(z), x_{n-1}(s)) &= \alpha_n\, f_n(x_1(z), x_{n-1}(s)) - \frac{\gamma_n}{2}\,\nabla x_{n+1}(z) f_{n-1}(x(z), x_{n-2}(s)).
\end{aligned}$$

Therefore, for $\mathbb{S}_x f_n(x(z), x_{n-1}(s))$ to be a linear combination of $f_n(x(z), x_{n-1}(s))$ and $f_{n-1}(x(z), x_{n-2}(s))$, it is necessary for the parameter $z$ to be solution of

$$x_1(t) = x(t) \Longleftrightarrow x(t+\frac{1}{2}) = x(t). \tag{38}$$

This solution is unique, provided that the coefficients $c_j$ of (3) fulfil $c_1\, c_2 \neq 0$ or $c_4 \neq 0$ for the quadratic lattice of the $q$-discrete and discrete variable, respectively. We denote this solution by $z_x$ (which is a constant term with respect to the variable $z$ but depend on the lattice $x$), and the resulting basis by

$$F_n(x(s)) := (-1)^n\, f_n(x(z_x), x_{n-1}(s)).$$

This basis fulfils Relations (21) and (22).

The proof of Relations (23) and (24) is similar to those of Relations (21), (22).

For the proof of Relation (25), we use Relations (28), (32) and (33) for fixed non-negative integer $n$, and the fact that $x_1(z_x) = x(z_x)$ with $x_\mu(z) = x(z+\frac{\mu}{2})$, to obtain

$$\begin{aligned}
F_{n+1}(x(s)) &= (-1)^{n+1}\, f_{n+1}(x(z_x), x_n(s))\\
&= (-1)^{n+1}\, f_{n+1}(x_1(z_x), x_n(s))\\
&= (-1)^{n+1}\, f_{n+1}(x_n(z_x+\frac{1-n}{2}), x_n(s))\\
&= (-1)^{n+1}\, g_{n+1}(x_n(z_x+\frac{1-n}{2}), x_n(s))\\
&= (-1)^{n+1}\prod_{j=0}^{n}(x(z_x+\frac{1-n}{2}+j) - x(s))\\
&= ((x(s) - x_{n+1}(z_x) - x(s)))\,(-1)^n\prod_{j=0}^{n-1}(x(z_x+\frac{1-n}{2}+j) - x(s))\\
&= ((x(s) - x_{n+1}(z_x) - x(s)))\,(-1)^n\, g_n(x_{n-1}(z_x+\frac{1-n}{2}), x_{n-1}(s))\\
&= ((x(s) - x_{n+1}(z_x) - x(s)))\,(-1)^n\, f_n(x_{n-1}(z_x+\frac{1-n}{2}), x_{n-1}(s))\\
&= ((x(s) - x_{n+1}(z_x) - x(s)))\,(-1)^n\, f_n(x(z_x), x_{n-1}(s))\\
&= (x(s) - x_{n+1}(z_x))\, F_n(x(s)).
\end{aligned}$$

Relation (26) is satisfied since, for integers $n,\ j$ and $k$ such that $k \geq 0$ and $1 \leq j \leq n$, we get by direct computation using (4) that

$$x_{n+k+1}(z_x) - x_j(z_x) \neq 0.$$

Therefore, $F_n(x_k(z_x)) \neq 0,\ k > n$.

Relation (27) is proved by induction on $n$. □

Having proved Theorem 2, we would like now to give explicitly the basis $F_n$ for specific classes of the lattice $x(s)$ and recover some known results.

**Proposition 5.** ① *The basis $F_n$ is explicitly defined on the lattices $x(s) = c_1 q^{-s} + c_2 qs + c_3$ (with $c_1 c_2 \neq 0$) by*

$$F_n(x(s)) = \left(-c_1 q^{-(\frac{n}{4}+z_x+\frac{1}{4})}\right)^n \left(\frac{c_2}{c_1} q^{z_x+\frac{1}{2}} q^s; q^{1/2}\right)_n \left(q^{z_x+\frac{1}{2}} q^{-s}; q^{1/2}\right)_n \tag{39}$$

$$= \left(-c_1 q^{-z_x}\right)^n \left(q^{z_x+\frac{1-n}{2}} q^{-s}, \frac{c_2}{c_1} q^{z_x+\frac{1-n}{2}} q^s; q\right)_n, \tag{40}$$

*where $z_x$ is defined by (19) and $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$.*

② *In the particular case of the Askey-Wilson lattice $x(s) = \frac{q^s+q^{-s}}{2}$ ($c_1 = c_2 = \frac{1}{2}$, $c_3 = 0$, $z_x = -\frac{1}{4}$) the previous equations read*

$$F_n(x(s)) = \left(-\frac{q^{-\frac{n}{4}}}{2}\right)^n \left(q^{\frac{1}{4}} q^s; q^{\frac{1}{2}}\right)_n \left(q^{\frac{1}{4}} q^{-s}; q^{\frac{1}{2}}\right)_n \tag{41}$$

$$= \left(-\frac{q^{\frac{1}{4}}}{2}\right)^n \left(q^{\frac{1}{4}-\frac{n}{2}} q^s, q^{\frac{1}{4}-\frac{n}{2}} q^{-s}; q\right)_n. \tag{42}$$

*Proof:* If $x(s) = c_2 q^s + c_1 q^{-s} + c_3$ with $c_2 c_1 \neq 0$, then we obtain by direct computation using Equation (25) that

$$\begin{aligned} F_n(x(s)) &= \prod_{j=1}^{n} (x(s) - x_j(z_x)), \\ &= (-1)^n \prod_{j=1}^{n} \left(c_2 q^{z_x+\frac{1}{2}} q^{\frac{j-1}{2}} + c_1 q^{-z_x-\frac{1}{2}} q^{\frac{1-j}{2}} - c_2 q^s - c_1 q^{-s}\right) \\ &= (-1)^n \prod_{j=0}^{n-1} c_1 q^{-z_x-\frac{j}{2}} \left(\frac{c_2}{c_1} q^{2z_x+1} q^j + 1 - \frac{c_2}{c_1} q^{z_x+\frac{1}{2}} q^s q^{\frac{j}{2}} - q^{-s} q^{z_x+\frac{1}{2}} q^{\frac{j}{2}}\right) \\ &= \left(-c_1 q^{-(z_x+\frac{1}{4}+\frac{n}{4})}\right)^n \left(\frac{c_2}{c_1} q^{z_x+\frac{1}{2}} q^s; q^{1/2}\right)_n \left(q^{z_x+\frac{1}{2}} q^{-s}; q^{1/2}\right)_n. \end{aligned}$$

The proof of Relation (40) is obtained in a similar way but with Equation (25) replaced by Relation

$$F_n(x(s)) = (-1)^n \prod_{j=0}^{n} (x_{-n+1}(z_x) - x(s))$$

obtained by combination of Relations (18) and (28). □

**Remark 6.** *It should be mentioned that for the specific case of the Askey-Wilson lattice our basis $F_n$ coincides (up to a multiplicative factor) to the basis $\phi_n$ used by Ismail [10] (Equation 1.4, page 261)*

$$\phi_n(\cos\theta) = \left(q^{\frac{1}{4}} q^s; q^{\frac{1}{2}}\right)_n \left(q^{\frac{1}{4}} q^{-s}; q^{\frac{1}{2}}\right)_n = \left(-\frac{q^{\frac{-n}{4}}}{2}\right)^{-n} F_n(x),\ x = \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{q^s+q^{-s}}{2}$$

*to provide the Taylor expansion of a polynomial in terms of the basis $\phi_n$.*

**Proposition 7.** ① *The basis $F_n$ is explicitly defined on the lattices $x(s) = c_1 s^2 + c_2 s + c_3$ (with $c_1 c_2 \neq 0$) by*

$$F_n(x(s)) = \left(-\frac{c_1}{4}\right)^n \left(-2s + \frac{1}{2} - \frac{c_2}{c_1}\right)_n \left(2s + \frac{1}{2} + \frac{c_2}{c_1}\right)_n, \quad (43)$$
$$= (-c_1)^n \left(s + \frac{1}{4} - \frac{n}{2} - \frac{c_2}{2c_1}\right)_n \left(-s + \frac{1}{4} - \frac{n}{2} + \frac{c_2}{2c_1}\right)_n, \quad (44)$$

*where $(a)_n = \prod_{k=0}^{n-1}(a+k)$*

② *In the particular case of the Racah lattice $x(s) = s(s+\gamma+\delta+1)$ ($c_1 = 1,\ c_2 = \gamma+\delta+1,\ c_3 = 0,\ z_x = -\frac{1}{4} - \frac{\gamma+\delta+1}{2}$) the previous equations read*

$$F_n(x(s)) = \left(\frac{-1}{4}\right)^n \left(-2s - \gamma - \delta - \frac{1}{2}\right)_n \left(2s + \gamma + \delta + \frac{3}{2}\right)_n$$
$$= (-1)^n \left(s - \frac{1}{4} - \frac{n}{2} - \frac{\gamma+\delta}{2}\right)_n \left(-s + \frac{3}{4} - \frac{n}{2} + \frac{\gamma+\delta}{2}\right)_n .$$

*Proof:* The proof of this Proposition is similar to the one of Proposition 5. □

In the sequel we treat series representations of functions on our lattices which either converge or are considered as formal series. We will not examine convergence issues.

**Theorem 8.**
*Let $f(x(s))$ be a function of $x(s)$. Then, $f$ can be expanded in the basis $F_n(x(s))$*

$$f(x(s)) = \sum_{k=0}^{\infty} d_k\, F_k(x(s)),$$

*where*

$$d_k = \frac{\mathbb{D}_x^k f(x(z_x))}{\gamma_k!}, \quad \gamma_k! = \prod_{j=1}^{k} \gamma_j,\ k \geq 1, \quad \gamma_0! = 1.$$

*Proof:* Assume $f$ is a function of $x(s)$ and write $f_N(x(s)) = \sum_{k=0}^{N} d_k\, F_k(x(s))$. Then for $0 \leq k \leq N$,

$$\mathbb{D}_x^k f_N(x(s))\Big|_{s=z_x} = \gamma_k!\, d_k,$$

since $F_k(x(z_x)) = 0,\ \forall k \geq 1$. □

In particular, for $f(x(s)) = \frac{1}{x(z)-x(s)},\ z \neq s$, we get

$$d_k = \frac{1}{\gamma_k!}\, \mathbb{D}_x^k \frac{1}{x(z)-x(s)}\Big|_{s=z_x} = \frac{1}{F_{k+1}(x(z))},$$

by induction using (4). We therefore state the following result as consequence of the previous theorem:

**Corollary 9.**
*For $z \neq s$ the following formal expansion holds:*

$$\frac{1}{x(z)-x(s)} = \sum_{k=0}^{\infty} \frac{F_k(x(s))}{F_{k+1}(x(z))}.$$

**Remark 10.** *Theorem 8 constitutes a generalization of the Taylor expansion given by Ismail [10] (Theorem 2.1, page 262) for basis $F_n$ restricted to the Askey-Wilson case given above in Proposition 5.*

## 3. Algorithmic Series Solutions of Divided-Difference Equations I

The basis $F_n$ is relevant for the companion operators and provides a method to obtain series solutions of divided-difference equations.

**Theorem 11.**
*If*

$$y(x(s)) = \sum_{k=0}^{\infty} d_k\, F_k(x(s)) \tag{45}$$

*is a series solution of the equation*

$$\phi(x(s))\,\mathbb{D}_x^2 y(x(s)) + \psi(x(s))\,\mathbb{S}_x\mathbb{D}_x y(x(s)) + \lambda\, y(x(s)) = 0, \tag{46}$$

*where $\lambda$ is a constant, $\phi$ and $\psi$ are polynomials of degree at most two and one, respectively, and given by*

$$\phi(x(s)) = \phi_2\, F_2(x(s)) + \phi_1\, F_1(x(s)) + \phi_0, \psi(x(s)) = \psi_1\, F_1(x(s)) + \psi_0,$$

*then the coefficients $(d_n)_n$ satisfy a second-order recurrence equation*

$$A_k\, d_{k+2} + B_k\, d_{k+1} + C_k\, d_k = 0,\ k \geq 0, \tag{47}$$

*with*

$$\begin{aligned} A_k &= \left[\phi\left(x_{k+1}(z_x)\right) + \frac{\nabla x_{k+2}(z_x)}{2}\psi\left(x_{k+1}(z_x)\right)\right]\gamma_{k+1}\,\gamma_{k+2}; \\ B_k &= \left[\gamma_k\,\Theta_{z_x+\frac{k}{2}}\phi\left(x_{k+1}(z_x)\right) + \alpha_k\,\psi\left(x_{k+1}(z_x)\right) + \frac{\gamma_k \nabla x_{k+1}(z_x)}{2}\psi_1\right]\gamma_{k+1}; \\ C_k &= \gamma_k\,\gamma_{k-1}\,\phi_2 + \gamma_k\,\alpha_{k-1}\psi_1 + \lambda, \end{aligned}$$

*where*

$$\Theta_a f(x(s)) = \frac{f(x(s)) - f(x(a))}{x(s) - x(a)}. \tag{48}$$

*Proof:* In the first step, we apply the companion operators to (45) and, taking into account (21) and (22), we get

$$\mathbb{D}_x^2 y(x(s)) = \sum_{k=2}^{\infty} d_k\,\gamma_k\,\gamma_{k-1}\,F_{k-2}(x(s)); \tag{49}$$

$$\mathbb{S}_x\mathbb{D}_x y(x(s)) = \sum_{k=1}^{\infty} d_k\,\gamma_k\,\alpha_{k-1}\,F_{k-1}(x(s)) + \sum_{k=2}^{\infty} \frac{d_k\,\gamma_k\,\gamma_{k-1}\,\nabla x_k(z_x)}{2} F_{k-2}(x(s)). \tag{50}$$

In the next step, we use (49) and (50) in (46) and the following relations obtained by iterating (25):

$$\begin{aligned} F_1(x(s))\,F_n(x(s)) &= F_{n+1}(x(s)) + F_1(x_{n+1}(z_x))\,F_n(x(s)); \\ F_2(x(s))\,F_n(x(s)) &= F_{n+2}(x(s)) + \Theta_{z_x+\frac{n+1}{2}} F_2\left(x_{n+2}(z_x)\right)\ F_{n+1}(x(s)) \\ &\quad + F_2(x_{n+1}(z_x))\,F_n(x(s)), \end{aligned}$$

to get an equation of type

$$\sum_{n=0}^{\infty} A_{k-2}\,d_k\,F_{k-2}(x(s)) + B_{k-1}\,d_k\,F_{k-1}(x(s)) + C_k\,d_k\,F_k(x(s)) = 0,\ \text{with}\ A_{-j} = B_{-j} = 0,\ j \geq 1.$$

The proof is completed by transforming the previous equation into

$$\sum_{k=0}^{\infty}(A_k\, d_{k+2} + B_k\, d_{k+1} + C_k\, d_k)\, F_k(x(s)) = 0,$$

and, using the fact that $(F_k)_k$ is a basis of $\mathbb{C}[x(s)]$. □

**Remark 12.**
*If (46) has a polynomial solution of degree $n$, then the relation $d_{n+2} = d_{n+1} = 0,\ d_n \neq 0$, combined with (47) gives*

$$\lambda = \lambda_n = -\gamma_n\, \gamma_{n-1}\, \phi_2 - \gamma_n\, \alpha_{n-1}\psi_1,$$

*which coincides with the result in [6].*

The previous theorem can be extended to solve divided-difference equations of arbitrary order with polynomial coefficients. For this, we need the following results:

**Proposition 13.**

$$\begin{aligned}
\mathbb{D}_x^k F_n(x(s)) &= \left[\prod_{j=0}^{k-1}\gamma_{n-j}\right] F_{n-k}(x(s)) = \frac{\gamma_n!}{\gamma_{n-k}!}\, F_{n-k}(x(s)),\ k \le n; \qquad (51)\\
F_k(x(s))\, F_n(x(s)) &= \sum_{j=0}^{k} C_{n+j}\, F_{n+j},\ \textit{with}\\
C_{n+j} &= \Theta_{z_x+\frac{n+j}{2}} \circ \Theta_{z_x+\frac{n+j-1}{2}} \ldots \circ \Theta_{z_x+\frac{n+1}{2}}\, F_k\left(x(s)\right)|_{s=z_x+\frac{n+j+1}{2}},\ 1 \le j \le k \le n,\\
C_n &= F_k\left(x_{n+1}(z_x)\right)
\end{aligned}$$

*Proof:* The first relation is obtained by iterating (21). We split the proof of the second relation into three steps:

In the first step, for fixed $n,\ k \ge 1$, we expand $F_k\, F_n$ in the basis $F_l$

$$F_k(x(s))\, F_n(x(s)) = \sum_{l=0}^{n+k} C_l\, F_l(x(s)) \qquad (52)$$

and use the following relation due to (25)

$$F_k(x_j(z_x)) = 0,\ 1 \le j \le k, \qquad (53)$$

to get $C_0 = F_k(x_1(z_x))\, F_n(x_1(z_x)) = 0$.

Considering (52) for $x(s) = x_2(z_x)$ and $C_0 = 0$, we get using again (53) that

$$C_1 F_1(x_2(z_x)) = F_k(x_2(z_x))\, F_n(x_2(z_x)) = 0,\ n \ge 2.$$

Therefore, $C_1 = 0$ thanks to (26). Progressively, we obtain in a similar way for a fixed integer $j$ using (52), (53) and (26) that

$$C_0 = C_1 = \cdots = C_j = 0,\ n \ge j+1.$$

In the second step, we rewrite accordingly Relation (52)

$$F_k(x(s))\, F_n(x(s)) = \sum_{j=0}^{k} C_{n+j}\, F_{n+j}(x(s)),$$

and obtain using (25)

$$F_k(x(s)) = \sum_{j=0}^{k} C_{n+j}\,\frac{F_{n+j}(x(s))}{F_n(x(s))} = \sum_{j=0}^{k} C_{n+j}\,g_{n+j,n+1}(x(s)), \tag{54}$$

where

$$g_{n,j}(x(s)) = \prod_{l=j}^{n}(x(s) - x_l(z_x)),\ 1 \le j \le n,\ g_{n,n+1}(x(s)) \equiv 1,\ g_{n,n+l}(x(s)) \equiv 0,\ l > 1. \tag{55}$$

Use of Equation (54) for $x(s) = x_{n+1}(z_x)$ gives taking into account Relation (53) and the fact that $g_{n,n+1} \equiv 1$

$$C_n = F_k(x_{n+1}(z_x)).$$

In the third step, we apply the operator $\Theta_a$ (defined in (48)) on (54) and use the relation

$$\Theta_{z_x+\frac{j}{2}}\, g_{n,j}(x(s)) = g_{n,j+1}(x(s)),\ 1 \le j \le n,$$

— derived by direct computation — to obtain the relation

$$\Theta_{z_x+\frac{n+1}{2}}\, F_k(x(s)) = \sum_{j=1}^{k} C_{n+j}\, g_{n+j,n+2}(x(s)), \tag{56}$$

from which we deduce using again (53) that

$$C_{n+1} = \Theta_{z_x+\frac{n+1}{2}}\, F_k(x_{n+2}(z_x)).$$

The remaining coefficients $C_{n+l},\ l \ge 2$ are obtained in the same way by successive application of $\Theta_{z_x+\frac{n+l}{2}},\ 2 \le l \le k$ on (56) and use of the $g_{n,j}(x_j(z_x)) = 0,\ 1 \le j \le n.$ □

**Theorem 14.** *The coefficients $c_n$ of a series solution*

$$y(x(s)) = \sum_{n=0}^{\infty} c_n\, F_n(x(s)), \tag{57}$$

*of any divided-difference equation of the form*

$$\sum_{i,j=0}^{N} P_{i,j}(x(s))\, \mathbb{S}_x^i\, \mathbb{D}_x^j\, y(x(s)) = Q(x(s)), \tag{58}$$

*where $k \in \mathbb{N}$, and $P_{i,j}(x(s))$ and $Q(x(s))$ are polynomials of arbitrary (but fixed) degree in the variable $x(s)$), are solution of a linear difference equation.*

*Proof:* Equation (58) can be transformed into an equation of type

$$\sum_{i=0}^{1}\sum_{j=0}^{M} \tilde{P}_{i,j}(x(s))\, \mathbb{S}_x^i\, \mathbb{D}_x^j\, y(x(s)) = Q(x(s)), \tag{59}$$

where $M \in \mathbb{N}$, and $\tilde{P}_{i,j}(x(s))$ are polynomials of arbitrary (but fixed) degree in the variable $x(s)$ using relations (12) and (13). The proof of the theorem is completed in the same way as in Theorem 11, substituting (57) in (59) and making use of Proposition 13. □

**Remark 15.** *This method works also when the coefficients $P_{i,j}$ and $Q$ are series expansions in our new basis. Also, the previous result generalizes the one given by Atakishiyev and Suslov [3] in which they provide a method to construct particular solutions to hypergeometric-type difference equations on a nonuniform lattice.*

**Remark 16.** *Theorems 11 and 14 provide a method of expanding solutions of Askey-Wilson operator equations in terms of the basis $\phi_n$ (see Remark 6) or $F_n$, providing therefore the solution of the problem raised by Ismail ([11], page 518).*

The coefficients $(d_{n,k})_j$ of the expansion of the Askey-Wilson polynomials into the basis $(F_k)_k$ (see Proposition 25) are difficult to handle since they are not $q$-hypergeometric. In order to provide explicit and simple representation of series solutions of divided-difference equations such as (8), we provide a second basis which is compatible not with the operators $\mathbb{D}_x$ and $\mathbb{S}_x$ but rather with $\mathbb{D}_x^2$ and $\mathbb{S}_x\mathbb{D}_x$ and are therefore very useful when searching for series solutions of divided-difference equations with polynomials coefficients, involving linear combination of products of $\mathbb{D}_x^2$ and $\mathbb{S}_x\mathbb{D}_x$.

## 4. A New Basis Compatible with the Product of the Companion Operators

Expressing the Askey-Wilson polynomials (15) in terms of $q$-Pochhammer symbols

$$P_n(x;a,b,c,d|q) = \sum_{k=0}^{n} \frac{(q^{-n},q)_k\,\left(a\,b\,c\,d\,q^{n-1},q\right)_k\,(a\,q^s,q)_k\,(a\,q^{-s},q)_k}{(a\,b,q)_k\,(a\,c,q)_k\,(a\,d,q)_k}\frac{q^k}{(q,q)_k}, \tag{60}$$

and the fact that these polynomials fulfil (8) suggests the study of the action of the companion operators on the function

$$B(a,x(s),n) = (a\,q^s,q)_n\,\left(a\,q^{-s},q\right)_n,\ n\geq 1,\ B(a,x(s),0)\equiv 1, \tag{61}$$

which happens to be a polynomial of degree $n$ in $x(s)=\frac{q^s+q^{-s}}{2}$. By considering a more general situation, we get:

**Proposition 17.** *The general q-quadratic lattice*

$$x(s) = u\,q^s + v\,q^{-s}$$

*and the corresponding polynomial basis*

$$\hat{B}_n(a,u,v,x(s)) = (2a\,u\,q^s,q)_n\,\left(2a\,v\,q^{-s},q\right)_n,\ n\geq 1,\ \hat{B}_0(a,u,v,x(s))\equiv 1,$$

*which we relabel as*

$$B_n(a,s)\equiv \hat{B}_n(a,u,v,x(s))$$

*fulfil the relations*

$$\begin{aligned}
\mathbb{D}_x B_n(a,s) &= \eta(a,n)B_{n-1}(a\sqrt{q},s); && (62)\\
\mathbb{S}_x B_n(a,s) &= \beta_1(a,n)\,B_{n-1}(a\sqrt{q},s)+\beta_2(n)\,B_n(a\sqrt{q},s); && (63)\\
B_1(a,s)\mathbb{D}_x^2\,B_n(a,s) &= \eta(a,n)\,\eta(a\sqrt{q},n-1)\,B_{n-1}(a,s); && (64)\\
B_1(a,s)\mathbb{S}_x\mathbb{D}_x\,B_n(a,s) &= \eta(a,n)\,\left[\beta_1(a\sqrt{q},n-1)\,B_{n-1}(a,s)+\beta_2(n-1)\,B_n(a,s)\right]; && (65)\\
x(s)\,B_n(a,s) &= \mu_1(a,n)\,B_n(a,s)+\mu_2(n)\ \ B_{n+1}(a,s); && (66)\\
B_1(a,s)\,B_n(a,s) &= \nu_1(a,n)\,B_n(a,s)+\nu_2(n)\,B_{n+1}(a,s); && (67)\\
B_1(a,s)B_n(a\,q,s) &= B_{n+1}(a,s), && (68)
\end{aligned}$$

*where*

$$\begin{aligned}
\mu_1(a,n) &= \frac{1+4a^2uv\,q^{2n}}{2a\,q^n}, \quad \mu_2(a,n)=\frac{-1}{2a\,q^n};\\
\nu_1(a,n) &= (1-q^{-n})\,(1-4a^2uv\,q^n), \quad \nu_2(n)=q^{-n}, \quad \eta(a,n)=\frac{2a(1-q^n)}{q-1};\\
\beta_1(a,n) &= \frac{1}{2}(1-4a^2uv\;q^{2n-1})(1-q^{-n}), \quad \beta_2(n)=\frac{1}{2}+\frac{1}{2q^n}.
\end{aligned}$$

*Proof:* The proof is obtained by direct computation. □

**Remark 18.** *It should, however, be noted that for $u=v=\frac{1}{2}$ Relation (62) appears as exercise in [18], page 34. It also appears in [11], Equation (20.3.11), page 518.*

From the previous proposition, it appears clearly that because of the appearance of $a\sqrt{q}$ in Relations (62) and (63), the action of $\mathbb{D}_x$ and $\mathbb{S}_x$ on $B_n(a,s)$ cannot be written as finite (number of terms not depending on $n$) linear combination of elements of the basis $(B_k(a,s))_k$. However, this problem is solved by using the operators $B_1(a,s)\mathbb{D}_x^2$ and $B_1(a,s)\mathbb{S}_x\mathbb{D}_x$ instead, to obtain Relations (64) and (65).

Equation (8) can therefore be solved using the known coefficients $\phi$ and $\psi$ of Askey-Wilson. It can also be derived from the hypergeometric representation (60) hence obtaining the coefficients $\phi$ and $\psi$ and $\lambda_n$ of the Askey-Wilson polynomials.

## 5. Algorithmic Series Solutions of Divided-Difference Equations II

**Theorem 19.**
*If*

$$y(x(s))=\sum_{k=0}^{\infty} d_k\,B_k(a,s) \tag{69}$$

*is a series solution of the equation*

$$\phi(x(s))\,\mathbb{D}_x^2 y(x(s))+\psi(x(s))\,\mathbb{S}_x\mathbb{D}_x y(x(s))+\lambda\,y(x(s))=0, \tag{70}$$

*where $\lambda$ is a constant, $\phi$ and $\psi$ are polynomials of degree at most two and one, respectively*

$$\phi(x(s))=\phi_2\,x^2(s)+\phi_1\,x(s)+\phi_0, \quad \psi(x(s))=\psi_1\,x(s)+\psi_0, \tag{71}$$

*then the coefficients $(d_k)_n$ satisfy a second-order difference equation*

$$A_k\,d_{k+2}+B_k\,d_{k+1}+C_k\,d_k=0,\;\; k\geq 0, \tag{72}$$

*with*

$$\begin{aligned}
A_k &= \eta(a,k+2)\left[\eta(a\sqrt{q},k+1)\;\phi\left(\mu_1(a,k+1)\right)+\beta_1(a\sqrt{q},k+1)\psi\left(\mu_1(a,k+1)\right)\right];\\
B_k &= \eta(a,k+1)\;\{\eta(a\sqrt{q},k)\;\left(\phi_2\;\mu_1(a,k)\mu_2(a,k)+\phi_2\;\mu_1(a,k+1)\mu_2(a,k)+\phi_1\;\mu_2(a,k)\;\right)\\
&\quad +\beta_1(a\sqrt{q},k)\;\left(\psi_1\;\mu_2(a,k)+\beta_2(k)\left(\psi_1\;\mu_1(a,k+1)+\psi_0\right)\;\right)\}+\lambda\,\nu_1(a,k+1)\\
C_k &= \phi_2\;\mu_2(a,k-1)\;\mu_2(a,k)\;\eta(a,k)\,\eta(a\sqrt{q},k-1)+\psi_1\;\eta(a,k)\,\beta_2(k-1)\mu_2(a,k)+\lambda\,\nu_2(k).
\end{aligned}$$

*Proof:* The proof is organised in three steps:

In the first step, we assume that (8) has a series solution of the form

$$y(x(s)) = \sum_{k=0}^{\infty} d_k\, B_k(a, x(s)),$$

then apply $\mathbb{D}_x^2$ and $\mathbb{S}_x\mathbb{D}_x$ on $y(x(s))$ and use Equations (62) and (63) to get

$$\begin{aligned}
&\phi(x(s))\mathbb{D}_x^2 y(x(s)) + \psi(x(s))\,\mathbb{S}_x\,\mathbb{D}_x y(x(s)) + \lambda_k\, y(x(s)) = 0\\
\Longleftrightarrow\quad &\phi(x(s)) \sum_{k=0}^{\infty} d_k\, \eta(a,k)\, \eta(a\sqrt{q}, k-1)\, B_{k-2}(a\,q, x(s))\\
&+\psi(x(s)) \sum_{k=0}^{\infty} d_k\, \left[\eta(a,k)\, \beta_1(a\sqrt{q}, k-1)\, B_{k-2}(aq, s) + \eta(a,k)\, \beta_2(k-1)\, B_{k-1}(aq, s)\right]\\
&+\lambda_k \sum_{k=0}^{\infty} d_k\, B_k(a,s) = 0.
\end{aligned}$$

In the second step, we multiply the previous equation by $B_1(a, x(s))$ and use of (64), (65) and (67) gives

$$\begin{aligned}
&\phi(x(s)) \sum_{k=0}^{\infty} d_k\, \eta(a,k)\, \eta(a\sqrt{q}, k-1)\, B_{k-1}(a,s)\\
&+\psi(x(s)) \sum_{k=0}^{\infty} d_k\, \left[\eta(a,k)\, \beta_1(a\sqrt{q}, k-1)\, B_{k-1}(a, s) + \eta(a,k)\, \beta_2(k-1)\, B_{k}(a, s)\right]\\
&+\lambda_k \sum_{k=0}^{\infty} d_k\, \left[\nu_1(a,k)\, B_k(a,s) + \nu_2(k)\, B_{k+1}(a,s)\right] = 0.
\end{aligned}$$

In the third step, we insert (71) into the previous equation and use (66) to eliminate all occurrences of $x^j(s)\, B(a, x(s), k),\ j = 1,\, 2$ to obtain after some computation

$$\sum_{k=0}^{\infty} (A_k\, d_{k+2} + B_k\, d_{k+1} + C_k\, d_k)\, B_k(a,s) = 0,$$

where

$$\begin{aligned}
A_k &= \eta(a,k+2)\left[\eta(a\sqrt{q}, k+1)\ \phi\left(\mu_1(a,k+1)\right) + \beta_1(a\sqrt{q}, k+1)\psi\left(\mu_1(a,k+1)\right)\right];\\
B_k &= \eta(a,k+1)\ \{\eta(a\sqrt{q}, k)\ \left(\phi_2\ \mu_1(a,k)\mu_2(k) + \phi_2\ \mu_1(a,k+1)\mu_2(a,k) + \phi_1\ \mu_2(a,k)\ \right)\\
&\quad +\beta_1(a\sqrt{q}, k)\ \left(\psi_1\ \mu_2(a,k) + \psi_1\ \mu_1(a,k+1) + \psi_0\ \right)\} + \lambda\ \nu_1(a,k+1)\\
C_k &= \phi_2\ \mu_2(a,k-1)\ \mu_2(a,k)\ \eta(a,k)\, \eta(a\sqrt{q}, k-1) + \psi_1\ \eta(a,k)\, \beta_2(k-1)\mu_2(a,k) + \lambda\ \nu_2(k).
\end{aligned}$$

Therefore, $d_k$ satisfies the difference equation

$$A_k\, d_{k+2} + B_k\, d_{k+1} + C_k\, d_k = 0,\ k \geq 0, \tag{73}$$

since $(B_k(a,s))_k$ is a basis of $\mathbb{C}[x(s)]$. □

The previous theorem can be extended to divided-difference equations of arbitrary order with polynomial coefficients, involving linear combinations of powers of the operators $\mathbb{D}_x^2$ and $\mathbb{S}_x\mathbb{D}_x$. Such operators can be rewritten, using Relations (12) and (13), as linear combination of $\mathbb{D}_x^{2j}$ and $\mathbb{S}_x\mathbb{D}_x^{2j+1}$, $j \geq 0$. For this extension, we will need the following results, obtained by iteration of Relations (62)-(68).

**Proposition 20.** *The basis* $(B_n(a,s))_n$ *satisfies the following relations:*

$$\begin{aligned}
\mathbb{D}_x^{2k} B_n(a,s) &= \pi_{n,k}\, B_{n-2k}(aq^k,s),\ 0 \le 2k \le n;\\
L_k(s) B_n(aq^k,s) &= B_{n+k}(a,s);\\
L_k(s)\mathbb{D}_x^{2k} B_n(a,s) &= \pi_{n,k}\, B_{n-k}(a,s);\\
L_{k+1}(s)\mathbb{S}_x\mathbb{D}_x^{2k+1} B_n(a,s) &= I_{n,k}\, B_{n-k-1}(a,s) + J_{n,k}\, B_{n-k}(a,s),
\end{aligned}$$

*where*

$$\begin{aligned}
L_k(s) &= \prod_{j=0}^{k-1} B_1(aq^j,s),\ \pi_{n,k} = \prod_{j=0}^{2k-1} \eta(aq^{\frac{j}{2}}, n-j);\\
I_{n,k} &= \pi_{n,k}\, \eta(aq^k, n-2k)\, \beta_1(aq^{k+\frac{1}{2}}, n-2k-1);\\
J_{n,k} &= \pi_{n,k}\, \eta(aq^k, n-2k)\, \beta_2(aq^{k+\frac{1}{2}}, n-2k-1).
\end{aligned}$$

We now state the following theorem which can be proved in the same way as Theorem 19 but using instead the equations of the previous proposition.

**Theorem 21.** *If*

$$y(x(s)) = \sum_{k=0}^{\infty} d_n\, B_n(a,s)$$

*is a series solution of the divided-difference equation*

$$\left[\sum_{j=0}^{M} P_j(x(s))\mathbb{D}_x^{2j} + \sum_{j=0}^{N} Q_j(x(s))\mathbb{S}_x\mathbb{D}_x^{2j+1}\right] y(x(s)) = T(x(s)) \tag{74}$$

*where* $P_j$*,* $Q_j$ *and* $T$ *are polynomials in the variable* $x(s)$*, then the coefficients* $(d_k)_n$ *satisfy a linear difference equation of maximal order* $\max(2M, 2N+1)$.

In the following results, Theorem 19 is used to solve Equation (1) for the lattice $x(s) = \frac{q^s+q^{-s}}{2}$ and for the coefficients $\phi$ and $\psi$ given by (16) to get the representation of the Askey-Wilson polynomials given by (15). It is also used to recover the polynomials $\phi$, $\psi$ and the constant $\lambda_n$ assuming that (15) satisfies (1).

**Theorem 22.**
*The Askey-Wilson polynomials* $P_n(x,a,b,c,d|q)$ *satisfy a divided-difference equation of the form (8) if and only if the polynomial coefficients* $\phi$ *and* $\psi$ *are, up to a multiplicative factor, those of Askey-Wilson given by (16), and* $\lambda = \lambda_n$ *given by (2).*

*Proof:* The proof is organized in two steps:
In the first step, we assume that the Askey-Wilson polynomials $P_n(x;a,b,c,d|q)$ satisfy (8). This implies that

$$d_n \neq 0,\ d_{n+2} = d_{n+1} = C_n = 0,$$

and

$$d_k = \frac{(q^{-n},q)_k\, \left(a\,b\,c\,d\,q^{n-1},q\right)_k}{(a\,b,q)_k\, (a\,c,q)_k\, (a\,d,q)_k}\, \frac{q^k}{(q,q)_k} \tag{75}$$

is solution of (73). The condition $C_n = 0$ provides the constant

$$\lambda = \lambda_n = -4\, \frac{(-q + abcdq^n)\, \sqrt{q}\, (-1+q^n)}{(-1+q)^2\, q^n} \tag{76}$$

which is a special case of (2). By substituting $\lambda = \lambda_n$ and the previous expression of $d_k$ into (73), we obtain after simplification an equation of the form

$$\sum_{k=0}^{N} H_k(\phi_2, \phi_1, \phi_0, \psi_1, \psi_0)\, q^{kn} = 0,$$

where the $H_k(\phi_2, \phi_1, \phi_0, \psi_1, \psi_0)$ are linear combinations of the coefficients of $\phi$ and $\psi$. Solving the system of linear equations $H_k(\phi_2, \phi_1, \phi_0, \psi_1, \psi_0) = 0,\ 0 \leq k \leq N$ in terms of the coefficients $\phi_j$ and $\psi_j$, we obtain, up to a multiplicative factor, the coefficients of the polynomials given in (16).

In the second step, we substitute the coefficients $\phi$ and $\psi$ of (16), as well as the coefficient $\lambda_n$ of (76) in (73) to obtain the following recurrence equation for $d_k$:

$$\begin{aligned}
&4\, q^n q \left(q^{k+1} - 1\right) \left(a^2 q^{k+1} - 1\right) \left(ac q^{k+1} - 1\right) \left(ad q^{k+1} - 1\right) \left(q q^{k+1} - 1\right) \left(ab q^{k+1} - 1\right) d_{k+2} \\
&+4 \left(q^{k+1} - 1\right) \Big\{ - \left(q^{k+1}\right)^3 a^3 q^n bcd - \left(q^{k+1}\right)^3 a^3 q q^n bcd + \left(q^{k+1}\right)^2 q^n abcdq + \left(q^{k+1}\right)^2 a^2 q q^n bc \\
&+ \left(q^{k+1}\right)^2 a^3 q \left(q^n\right)^2 bcd + \left(q^{k+1}\right)^2 q^2 a^2 + \left(q^{k+1}\right)^2 a^2 q q^n cd + \left(q^{k+1}\right)^2 a^2 q q^n bd - q^{k+1} q^2 \\
&- q^{k+1} q \left(q^n\right)^2 abcd - q^{k+1} a^2 q^2 q^n - q^{k+1} a q^2 q^n c - q^{k+1} a q^2 q^n d - q^{k+1} a q^2 q^n b + q^3 q^n + q^2 q^n \Big\} d_{k+1} \\
&-4 \left(q^n q - q^{k+1}\right) q \left(q^{k+1} abcd q^n - q^2\right) d_k = 0.
\end{aligned} \tag{77}$$

In [1] and [4], see also [12], algorithms were presented to find all solutions of an arbitrary $q$-holonomic difference equation in terms of linear combinations of $q$-hypergeometric terms. This algorithm was tuned and made much more efficient in ([9], [8]), and a *Maple* implementation was made available in [16]. For the purpose to solve the second order $q$-difference equation (77) in terms of hypergeometric terms, we have used the command `qrecsolve` from the $q$sum package [4] (one could also use the command `qHypergeomSolveRE` of the $q$FPS package [16]), to obtain the coefficients $d_k$ given in (75). Details of this computation can be found in a *Maple* file made available on www.mathematik.uni-kassel.de/~koepf/Publikationen. □

## 6. Applications and Illustrations

In this section we provide two applications for the basis $F_k$: The first gives a new representation of the formal Stieltjes series in terms of the basis $F_k$, while the second gives a representation of the basic exponential and trigonometric functions in terms of the basis $F_k$.

*6.1. Series expansion of the formal Stieltjes series* Using Corollary 9 we define the formal Stieltjes series corresponding to a functional $\mathcal{L}$ as

$$S(\mathcal{L})(x(z)) = \sum_{k=0}^{\infty} \frac{\mu_k}{F_{k+1}(x(z))} \quad \text{with } \mu_k = \langle \mathcal{L}, F_k \rangle, \tag{78}$$

and obtain the following results:

**Theorem 23.** *The following results hold:*

$$\begin{aligned}
S\left(\mathbb{D}_x \mathcal{L})(s)\right) &= \mathbb{D}_x S(\mathcal{L})(s), & (79)\\
S\left(\mathbb{S}_x \mathcal{L})(s)\right) &= \alpha\, \mathbb{S}_x S\left(\mathcal{L}\right)(s) + U_1 \mathbb{D}_x S\left(\mathcal{L}\right)(s), & (80)
\end{aligned}$$

*where the actions of* $\mathbb{D}_x$ *and* $\mathbb{S}_x$ *on* $\mathcal{L}$ *are defined as*

$$\langle \mathbb{D}_x\mathcal{L}, P\rangle = -\langle \mathcal{L}, \mathbb{D}_x P\rangle,\ \langle \mathbb{S}_x\mathcal{L}, P\rangle = \langle \mathcal{L}, \mathbb{S}_x P\rangle,\ \forall P \in \mathbb{C}[x(s)],$$

*and the product of a polynomial* $f$ *by a linear functional* $\mathcal{L}$*,* $f\mathcal{L}$*, is defined by*

$$\langle f\mathcal{L}, P\rangle = \langle \mathcal{L}, fP\rangle,\ \forall P \in \mathbb{C}[x(s)].$$

*Proof:* Relation (79) is obtained by direct computation using Equations (21) and (23). The proof of (80) uses the following results:

$$\begin{aligned} S(U_1(x(s))\mathbb{D}_x\mathcal{L}) &= U_1(x(s))S(\mathbb{D}_x\mathcal{L}); && (81)\\ \mathbb{S}_xF_n + \mathbb{D}_x(U_1F_n) &= \alpha(\alpha_{n+1}F_n + \frac{\gamma_n}{2}\nabla x_{n+2}(z_x)F_{n-1}). && (82)\end{aligned}$$

Relation (81) is obtained using the well-known result by Maroni [15]

$$S\left[f\mathcal{L}\right](x) = f(x)\,S\left[\mathcal{L}\right](x) + (\mathcal{L}\theta_0 f)(x),\ f \in \mathbb{C}[x], \quad (83)$$

where

$$\theta_0 f(x) = \frac{f(x)-f(0)}{x},$$

and

$$\mathcal{L}g(x(s)) = \sum_{k=0}^{n} g_k \sum_{j=0}^{k}\langle \mathcal{L}, x^j(s)\rangle x^{k-j}(s),\ \ with\ g(x(s)) = \sum_{k=0}^{n} g_k\, x^k(s),\ n \geq 0.$$

Relation (82) is derived by direct computation using (3), (14), (19), (21) and (22).

Coming back to the proof of Relation (80), we combine (81), (82) (22), (24) and the fact that $\gamma_0 = 0$ to get:

$$\begin{aligned} S(\mathbb{S}_x\mathcal{L})(x(s)) - U_1(x(s))\mathbb{D}_x(S(\mathcal{L}))(x(s)) &= S(\mathbb{S}_x\mathcal{L}) - S(U_1(x(s))\mathbb{D}_x\mathcal{L});\\ &= S(\mathbb{S}_x\mathcal{L} - U_1(x(s))\mathbb{D}_x\mathcal{L})\\ &= \sum_{n=0}^{\infty}\frac{\langle \mathcal{L}, \mathbb{S}_xF_n + \mathbb{D}_x(U_1F_n)\rangle}{F_{n+1}(x(s))}\\ &= \alpha\sum_{n=0}^{\infty}\frac{\langle \mathcal{L}, (\alpha_{n+1}F_n + \frac{\gamma_n}{2}\nabla x_{n+2}(z_x)F_{n-1}\rangle}{F_{n+1}(x(s))}\\ &= \alpha\sum_{n=0}^{\infty}\frac{\langle \mathcal{L}, \alpha_{n+1}F_n\rangle}{F_{n+1}(x(s))} + \alpha\sum_{n=0}^{\infty}\frac{\langle \mathcal{L}, \gamma_n\nabla x_{n+2}(z_x)F_{n-1}\rangle}{2F_{n+1}(x(s))}\\ &= \alpha\sum_{n=0}^{\infty}\frac{\langle \mathcal{L}, \alpha_{n+1}F_n\rangle}{F_{n+1}(x(s))} + \alpha\sum_{n=1}^{\infty}\frac{\langle \mathcal{L}, \gamma_n\nabla x_{n+2}(z_x)F_{n-1}\rangle}{2F_{n+1}(x(s))}\\ &= \alpha\sum_{n=0}^{\infty}\frac{\langle \mathcal{L}, \alpha_{n+1}F_n\rangle}{F_{n+1}(x(s))} + \alpha\sum_{n=0}^{\infty}\frac{\langle \mathcal{L}, \gamma_{n+1}\nabla x_{n+3}(z_x)F_n\rangle}{2F_{n+2}(x(s))}\\ &= \alpha\sum_{n=0}^{\infty}\langle \mathcal{L}, F_n\rangle\left(\frac{\alpha_{n+1}}{F_{n+1}(x(s))} + \frac{\gamma_{n+1}\nabla x_{n+3}(z_x)}{2F_{n+2}(x(s))}\right)\\ &= \alpha\sum_{n=0}^{\infty}\langle \mathcal{L}, F_n\rangle\mathbb{S}_x\frac{1}{F_{n+1}(x(s))}\\ &= \alpha\mathbb{S}_x(S(\mathcal{L}))(x(s)).\end{aligned}$$

□

*6.2. Series expansion of the basic exponential function* In this sub-section, we represent the basic exponential function in terms of the basis $(F_k)_k$. The basic exponential function $\mathcal{E}_q(x(s);w)$ can be defined using the representation by Ismail and Stanton (see [18] page 21 and references therein)

$$\mathcal{E}_q(x;w)=\frac{\left(-w;q^{\frac{1}{2}}\right)_\infty}{(qw^2;q^2)_\infty}{}_2\varphi_1\left(\begin{array}{c} q^{\frac{1}{4}}e^{i\theta},q^{\frac{1}{4}}e^{-i\theta}\\ -q^{\frac{1}{2}}\end{array}\middle| q^{\frac{1}{2}};-w\right),\ x=\cos\theta,$$

where $(a,q)_\infty=\prod_{n=0}^{\infty}(1-aq^n)$.

By putting $e^{i\theta}=q^s$ and therefore

$$x=\cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2}=\frac{q^s+q^{-s}}{2}=x(s),$$

$\mathcal{E}_q(x(s);w)$ satisfies the following first-order divided-difference equation ([18], page 18)

$$\mathbb{D}_x y(x(s))=\frac{2wq^{\frac{1}{4}}}{1-q}\,y(x(s)).$$

By inserting the series expansion of $y(x(s))$ in terms of $(F_k)_k$

$$y(x(s))=\sum_{n=0}^{\infty}a_nF_n(x(s))$$

in the previous first-order divided-difference equation and following the method developed in Theorem 11, we get the following recurrence equation for $a_n$

$$a_{n+1}=\frac{2wq^{\frac{1}{4}}}{(1-q)\gamma_{n+1}}a_n,$$

from which we deduce that

$$a_n=\left(\frac{2wq^{\frac{1}{4}}}{1-q}\right)^n\frac{a_0}{\gamma_n!},\quad \gamma_0!\equiv 1.$$

Therefore we have, taking into account Equation (42), the following representation of the basic exponential function

$$\begin{aligned}\mathcal{E}_q(x(s);w) &= a_0\sum_{n=0}^{\infty}\left(\frac{2wq^{\frac{1}{4}}}{1-q}\right)^n\frac{F_n(x(s))}{\gamma_n!}\\ &= a_0\sum_{n=0}^{\infty}\left(\frac{wq^{\frac{1}{2}}}{q-1}\right)^n\frac{1}{\gamma_n!}\left(q^{\frac{1-2n}{4}}q^s;q\right)_n\left(q^{\frac{1-2n}{4}}q^{-s};q\right)_n,\end{aligned}$$

where $a_0$ is a suitable constant which from the fact that $F_n(x_1(z_x))=0,\ n\geq 1$ is given by

$$a_0=\mathcal{E}_q(x_1(z_x),w)=\frac{(-w;q^{\frac{1}{2}})_\infty}{(qw^2;q^2)_\infty},$$

with the last expression taken from [18] (Equation 2.3.10, Page 18).

### 6.3. Series expansion of the basic trigonometric functions

In this sub-section, we represent the basic trigonometric functions in terms of the basis $(F_k)_k$. The basic trigonometric cosine and sine functions are defined respectively by (see [18, page 23])

$$\begin{aligned} C_q(x;w) &= \frac{(-w^2;q^2)_\infty}{(-qw^2;q^2)_\infty} {}_2\varphi_1\left(\begin{array}{c} -qe^{2i\theta}, -qe^{-2i\theta} \\ q \end{array} \middle| q^2; -w^2\right), \\ S_q(x;w) &= \frac{(-w^2;q^2)_\infty}{(-qw^2;q^2)_\infty} \frac{2wq^{\frac14}}{1-q} \cos\theta \, {}_2\varphi_1\left(\begin{array}{c} -qe^{2i\theta}, -qe^{-2i\theta} \\ q \end{array} \middle| q^2; -w^2\right), \; x=\cos\theta, \; |w|<1. \end{aligned}$$

By putting $e^{i\theta} = q^s$, the functions $C_q(x(s);w)$ and $S_q(x(s);w)$ satisfy the following second-order divided-difference equation ([18], page 26)

$$\mathbb{D}_x^2 y(x(s)) = -\left(\frac{2wq^{\frac14}}{1-q}\right)^2 y(x(s)). \tag{84}$$

By inserting the series expansion of $y(x(s))$ in terms of $(F_k)_k$

$$y(x(s)) = \sum_{n=0}^{\infty} b_n F_n(x(s))$$

in (84) and following the method developed in Theorem 11, we get the following recurrence equation

$$b_{n+2} = -\left(\frac{2wq^{\frac14}}{1-q}\right)^2 \frac{1}{\gamma_{n+2}\gamma_{n+1}} b_n,$$

from which we deduce that

$$b_{2n} = (-1)^n \left(\frac{2wq^{\frac14}}{1-q}\right)^{2n} \frac{b_0}{\gamma_{2n}!} \text{ and } b_{2n+1} = (-1)^n \left(\frac{2wq^{\frac14}}{1-q}\right)^{2n} \frac{b_1}{\gamma_{2n+1}!}.$$

Therefore the two linearly independent solutions (84) are, taking into account the explicit representation of $F_n$ given in (42) for the Askey-Wilson lattice

$$\begin{aligned} A_q(x(s),w) &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{2wq^{\frac14}}{1-q}\right)^{2n} \frac{F_{2n}(x(s))}{\gamma_{2n}!} \\ &= \sum_{n=0}^{\infty} \left(\frac{wq^{\frac12}}{1-q}\right)^{2n} \frac{(-1)^n}{\gamma_{2n}!} \left(q^{\frac{1-4n}{4}} q^s; q\right)_{2n} \left(q^{\frac{1-4n}{4}} q^{-s}; q\right)_{2n}, \end{aligned}$$

and

$$\begin{aligned} B_q(x(s),w) &= \sum_{n=0}^{\infty} (-1)^n \left(\frac{2wq^{\frac14}}{1-q}\right)^{2n} \frac{F_{2n+1}(x(s))}{\gamma_{2n+1}!} \\ &= \sum_{n=0}^{\infty} \left(\frac{wq^{\frac12}}{1-q}\right)^{2n} \frac{(-1)^n}{\gamma_{2n+1}!} \left(q^{\frac{1-2n}{2}} q^s; q\right)_{2n+1} \left(q^{\frac{1-2n}{2}} q^{-s}; q\right)_{2n+1}. \end{aligned}$$

Since the functions $C_q$ and $S_q$ are both solutions of (84) which is a second-order linear divided-difference equation, they can be expressed as linear combination of the solutions $A_q$ and $B_q$:

$$C_q(x(s),w) = u_0 A_q(x(s),w) + u_1 B_q(x(s),w), \; S_q(x(s),w) = v_0 A_q(x(s),w) + v_1 B_q(x(s),w), \tag{85}$$

where $u_i$ and $v_i$ are constants.

Combining (85) with the following relations derived by direct computation

$$\mathbb{D}_x A_q(x(s),w) = -\left(\frac{2wq^{\frac{1}{4}}}{1-q}\right)^2 B_q(x(s),w),\ \mathbb{D}_x B_q(x(s),w) = A_q(x(s),w),$$

and using the relations (see [18], page 26)

$$\mathbb{D}_x C_q(x(s),w) = -\frac{2wq^{\frac{1}{4}}}{1-q} S_q(x(s),w),\ \mathbb{D}_x S_q(x(s),w) = \frac{2wq^{\frac{1}{4}}}{1-q} C_q(x(s),w),$$

gives

$$u_1 = -\lambda v_0,\ \ v_1 = \lambda u_0,\ \ \text{with}\ \ \lambda = \frac{2wq^{\frac{1}{4}}}{1-q}. \tag{86}$$

Use of the fact that $F_n(x_1(z_x)) = 0,\ n \geq 1$ gives the relation

$$A_q(x_1(z_x)) = 1,\ \ B_q(x_1(z_x)) = 0$$

which combined with (85) leads to

$$u_0 = C_q(x_1(z_x)),\ \ v_0 = S_q(x_1(z_x)).$$

We therefore have the following representation of the $C_q$ and $S_q$ functions:

$$\begin{aligned} C_q(x(s),w) &= C_q(x_1(z_x))A_q(x(s),w) - \lambda\, S_q(x_1(z_x))\, B_q(x(s),w),\\ S_q(x(s),w) &= S_q(x_1(z_x))A_q(x(s),w) + \lambda\, C_q(x_1(z_x))\, B_q(x(s),w),\end{aligned}$$

where the evaluation of the functions $C_q$ and $S_q$ on $x_1(z_x) = x(-\frac{1}{4}) = \frac{q^{\frac{1}{4}}+q^{\frac{-1}{4}}}{2}$ are given respectively by (see [18] page 27, equations 2.4.19 and 2.4.20)

$$C_q(x_1(z_x)) = \frac{(-iw;q^{\frac{1}{2}})_\infty + (iw;q^{\frac{1}{2}})_\infty}{2(-qw^2;q^2)_\infty}, S_q(x_1(z_x)) = \frac{(-iw;q^{\frac{1}{2}})_\infty - (iw;q^{\frac{1}{2}})_\infty}{2i(-qw^2;q^2)_\infty}.$$

*6.4. Connection coefficients between the basis $(F_k)_k$ and $(B_k(a,s))_k$* The basis basis $(F_k)_k$ and $(B_k(a,s))_k$ are connected in the following ways

**Proposition 24.**

$$F_n(x(s)) = \sum_{j=0}^{n} r_{n,j}\, B_j(a,s),\ B_n(a,s) = \sum_{j=0}^{n} s_{n,j}\, F_j(x(s)), \tag{87}$$

*where*

$$r_{n,k} = \frac{\gamma_n!}{\gamma_{n-k}!}\frac{F_{n-k}(\epsilon_{0,k})}{\prod_{l=0}^{k-1}\eta\left(aq^{\frac{l}{2}},k-l\right)},\ 0\leq k\leq n,\ n\geq 1, \tag{88}$$

$$s_{n,k} = \frac{1}{\gamma_k!}\, B_{n-k}\left(aq^{\frac{k}{2}}, z_x+\frac{1}{2}\right)\prod_{l=0}^{k-1}\eta\left(aq^{\frac{l}{2}},n-l\right),\ 0\leq k\leq n,\ n\geq 1, \tag{89}$$

*and*

$$\epsilon_{j,k} = \frac{1+4a^2u^2v^2q^{2j+k}}{4aq^{j+\frac{k}{2}}}. \tag{90}$$

*Proof:* We first apply the operator $\mathbb{D}_x^k$ on both members of (87) for fixed non-negative integers $n \geq 1$ and $k \leq n$ to get using (51) and (62)

$$\frac{\gamma_n!}{\gamma_{n-k}!} F_{n-k}(x(s)) = \sum_{j=k}^{n} r_{n,j} \left[ \prod_{l=0}^{k-1} \eta \left( aq^{\frac{l}{2}}, j-l \right) \right] B_{j-k} \left( aq^{\frac{k}{2}}, s \right), \; n \geq 1, \; 0 \leq k \leq n, \tag{91}$$

$$\left[ \prod_{l=0}^{k-1} \eta \left( aq^{\frac{l}{2}}, n-l \right) \right] B_{n-k} \left( aq^{\frac{k}{2}}, s \right) = \sum_{j=k}^{n} s_{n,j} \frac{\gamma_j!}{\gamma_{j-k}!} F_{j-k}(x(s)), \; n \geq 1, \; 0 \leq k \leq n. \tag{92}$$

Then, we write

$$B_n(a,s) = \hat{B}_n(a,u,v,x(s)) = (2auq^s;q)_n \left(2auq^{-s};q\right)_n = \prod_{j=0}^{n-1} \left(1 - 4aq^j x(s) + 4a^2 uvq^{2j}\right)$$

and deduce that

$$\hat{B}_n(a,u,v,\epsilon_{j,0}) = 0, \; \forall n \geq 1, \; \forall j \leq n,$$

where $\epsilon_{j,k}$ (which is in fact the constant $\epsilon_{j,0}$ in which $a$ is replaced by $aq^{\frac{k}{2}}$) is given by (90). We therefore obtain $r_k$ by using (91) for $x(s) = \epsilon_{0,k}$ and taking into account the previous relation. The coefficient $s_{n,k}$ is obtained in a similar way by using (92) for $x(s) = x_1(z_x)$ and taking into account the fact that

$$F_n(x_1(z_x)) = 0, \; \forall n \geq 1.$$

□

From the connection coefficients given above, one can express any polynomial given in one of the basis to another one.

**Proposition 25.** *Let $n$ be a positive integer, $P_n$ and $Q_n$ two polynomials of degree $n$ in the variable $x(s)$ such that*

$$P_n(x(s)) = \sum_{k=0}^{n} a_{n,k} F_k(x(s)), \quad Q_n(x(s)) = \sum_{k=0}^{n} b_{n,k} B_k(a,s). \tag{93}$$

*Then $P_n$ and $Q_n$ can be expanded in the basis $(B_k(a,s))_k$ and $(F_k(x(s)))_k$*

$$P_n(x(s)) = \sum_{j=0}^{n} c_{n,j} B_j(a,s), \quad Q_n(x(s)) = \sum_{j=0}^{n} d_{n,j} F_j(x(s)), \tag{94}$$

*with*

$$c_{n,j} = \sum_{k=j}^{n} a_{n,k} s_{k,j}, \quad d_{n,j} = \sum_{k,j}^{n} b_{n,k} r_{k,j},$$

*where $r_{k,j}$ and $s_{k,j}$ are defined by (88) and (89).*

*Proof:* First we use relation (87) in the expression of $P_n(x(s))$ taken from (93) to get

$$\begin{aligned} P_n(x(s)) &= \sum_{k=0}^{n} a_{n,k} F_k(x(s)) \\ &= \sum_{k=0}^{n} a_{n,k} \sum_{j=0}^{k} r_{k,j} \, B_j(a,s) \\ &= \sum_{j=0}^{n} \left( \sum_{k=j}^{n} a_{n,k} \, r_{k,j} \right) B_j(a,s). \end{aligned}$$

The expansion of polynomial $Q_n$ is obtained in the same way.

For the special case when $Q_n(x(s))$ is the Askey-Wilson polynomials given by (15)

$$b_{n,k} = \frac{(q^{-n},q)_k\,\left(a\,b\,c\,d\,q^{n-1},q\right)_k}{(a\,b,q)_k\,(a\,c,q)_k\,(a\,d,q)_k}\frac{q^k}{(q,q)_k}.$$

Therefore, we get after some computation using Relation (62) and taking care that $z_x = -\frac{1}{4}$

$$\begin{aligned} d_{n,j} &= \sum_{k=j}^{n} b_{n,k}\, s_{k,j} \\ &= \sum_{k=j}^{n} \frac{(q^{-n},q)_k\,\left(a\,b\,c\,d\,q^{n-1},q\right)_k}{(a\,b,q)_k\,(a\,c,q)_k\,(a\,d,q)_k}\frac{q^k}{(q,q)_k}\frac{1}{\gamma_j!}\;B_{k-j}\left(aq^{\frac{j}{2}},z_x+\frac{1}{2}\right)\prod_{l=0}^{j-1}\eta\left(aq^{\frac{l}{2}},k-l\right) \\ &= \sum_{k=j}^{n} \frac{(q^{-n},q)_k\,\left(a\,b\,c\,d\,q^{n-1},q\right)_k}{(a\,b,q)_k\,(a\,c,q)_k\,(a\,d,q)_k}\frac{q^k\,q^{\frac{j(j-1)}{4}}}{\gamma_j!}\frac{(2a)^j}{(q-1)^j}\left(aq^{\frac{2k+1}{4}};q\right)_{k-j}\left(aq^{\frac{2k-1}{4}};q\right)_{k-j}. \end{aligned}$$

□

## 7. Conclusion and Perspectives

In this paper, we developed suitable bases (replacing the power basis $x^n$ $(n \in \mathbb{N}_{\geq 0})$) which enable the direct series representation of orthogonal polynomial systems on nonuniform lattices (quadratic lattices of a discrete or a $q$-discrete variable). We presented two bases of this type, the first of which allows to write solutions of arbitrary divided-difference equations in terms of series representations extending results given in [16] for the $q$-case and in [3] for the quadratic case. Furthermore we used this basis to give a new representation of the Stieltjes function which we will used (see [7]) to prove the equivalence between the Pearson equation for the functional approach and the Riccati equation for the formal Stieltjes function.

When the Askey-Wilson polynomials are written in terms of this basis, we proved that the coefficients are not $q$-hypergeometric. Therefore, we presented a second basis, which shares several relevant properties with the first one. This basis enables to generate the defining representation of the Askey-Wilson polynomials directly from their divided-difference equation, and also to solve more general divided-difference equations of arbitrary order involving the linear combination of $\mathbb{D}_x^{2j}$ and $\mathbb{S}_x\mathbb{D}_x^{2j+1}$, $(j \geq 0)$.

As perspective, we mention that this paper shall lead to the characterization of orthogonal polynomials (semi-classical and Laguerre-Han classes) on quadratic and $q$-quadratic lattices by means of the functional approach, providing the link between such approach and the one developed by Magnus [13,14] using the Riccati equation for the formal Stieltjes series. It might also be used to solve specific divided-difference equations such as the $q$-wave and the $q$-heat equations [18]; and provide new identities in the domain of special functions.

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